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# CONCAVE FUNCTIONS OF POSITIVE OPERATORS, SUMS, AND CONGRUENCES

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ABSTRACT. Let $A$, $B$, $Z$ be positive semidefinite matrices of same size and suppose $Z$ is expansive, i.e., $Z \geqslant I$. Two remarkable inequalities are

$$\|f(A+B)\| \leqslant \|f(A)+f(B)\| \quad \text{and} \quad \|f(ZAZ)\| \leqslant \|Zf(A)Z\|$$

for all non-negative concave function $f$ on $[0,\infty)$ and all symmetric norms $\|\cdot\|$ (in particular for all Schatten $p$-norms). In this paper we survey several related results and we show that these inequalities are two aspects of a unique theorem. For the operator norm, our result also holds for operators on an infinite dimensional Hilbert space.

KEYWORDS: *Hermitian operators, symmetric norms, operator inequalities.*

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## INTRODUCTION

A good part of Matrix Analysis consists in establishing results for Hermitian operators considered as generalized real numbers or functions. In Section 1 we recall two recent norm inequalities which are matrix versions of the obvious scalars inequalities

$$f(za) \leqslant zf(a) \quad \text{and} \quad f(a+b) \leqslant f(a)+f(b)$$

for non-negative concave functions $f$ on $[0,\infty)$ and scalars $a$, $b \geqslant 0$ and $z \geqslant 1$. In Section 2 we unify and generalize these norm inequalities. The norms considered are the symmetric (or unitarily invariant) norms. Such norms satisfy $\|A\| = \|UAV\|$ for all $A$ and all unitaries $U$, $V$. Here and in the sequel capital letters $A, B, \dots, Z$ mean $n$-by-$n$ complex matrices, or operators on an $n$-dimensional Hilbert space. If $A$ is positive (semi-definite), respectively positive definite, we write $A \geqslant 0$, respectively $A > 0$. If $Z^*Z$ dominates the identity $I$, we say that $Z$ is expansive. An $m$-tuple of operators $\{X_i\}_{i=1}^m$ is said positive (respectively, expansive) if $X_i$ is positive (respectively, expansive) for all $i = 1, \dots, m$. As a corollary of Section 2 we have:

THEOREM 0.1. *Let $\{A_i\}_{i=1}^m$ be positive and let $\{Z_i\}_{i=1}^m$ be expansive. Then, for all symmetric norms and all $p > 1$,*

$$\left\| \sum Z_i^* A_i^p Z_i \right\| \leqslant \left\| \left( \sum Z_i^* A_i Z_i \right)^p \right\|.$$

The sum is over the first $m$ integers. If $Z_i = I$ for all $i$, this is a famous result of Ando and Zhan [1] and of Bhatia and Kittaneh [3] in case of integer exponents. The very special case $\mathrm{Tr}\,(A_1^p + A_2^p) \leqslant \mathrm{Tr}\,(A_1 + A_2)^p$ is McCarthy's inequality ([13], p. 20).

## 1. TWO NORM INEQUALITIES

The cone of positive operators is invariant under congruences $A \longrightarrow S^*AS$ and there are several inequalities involving congruences with a contraction $Z$ and concave functions $f : [0, \infty) \to [0, \infty)$. Brown–Kosaki's trace inequality states

$$\mathrm{Tr}\, f(Z^*AZ) \geqslant \mathrm{Tr}\, Z^* f(A) Z. \tag{1.1}$$

Actually a stronger result holds [8] (see also [4]), there exists a unitary $V$ such that

$$f(Z^*AZ) \geqslant VZ^* f(A) ZV^*. \tag{1.2}$$

This means that the eigenvalues of $f(Z^*AZ)$ dominate those of $Z^*f(A)Z$. Further, if $f$ operator concave (equivalently operator monotone [10]), then Hansen's operator inequality holds [9], [10],

$$f(Z^*AZ) \geqslant Z^* f(A) Z^*. \tag{1.3}$$

What happens to (1.1), (1.2), (1.3) when $Z$ is no longer contractive but, in an opposite way, is expansive ? It is obvious that (1.3) is reversed, meanwhile (1.2) can not be reversed though a non-trivial proof [4] shows that (1.1) is reversed:

$$\mathrm{Tr}\, f(Z^*AZ) \leqslant \mathrm{Tr}\, Z^* f(A) Z. \tag{1.4}$$

However a quite unexpected phenomena occurs: (1.1) can be extended to all Hermitians $A$ and all concave functions $f$ on the real line with $f(0) \geqslant 0$, but in (1.4) the assumption $A \geqslant 0$ can not be dropped. The good statement for expansive congruences requires positivity and involves symmetric (or unitarily invariant) norms $\|\cdot\|$. We have [5]:

THEOREM 1.1. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Let $A \geqslant 0$ and let $Z$ be expansive. Then, for all symmetric norms,*

$$\|f(Z^*AZ)\| \leqslant \|Z^* f(A) Z\|.$$

Besides these inequalities for congruences, there are nice subadditivity results for concave functions $f : [0, \infty) \longrightarrow [0, \infty)$ and sums of positive operators.

The most elementary one is a trace inequality companion to (1.1) credited to Rotfel'd [12]: For $A$, $B \geqslant 0$,

$$\operatorname{Tr} f(A+B) \leqslant \operatorname{Tr} f(A) + f(B). \tag{1.5}$$

The operator inequality $f(A+B) \leqslant f(A) + f(B)$ may not hold even if $f$ is operator concave. However, a remarkable subadditivity inequality related to (1.2) is shown in [2]: There exist unitaries $U$, $V$ such that

$$f(A+B) \leqslant Uf(A)U^* + Vf(B)V^*. \tag{1.6}$$

From this, it follows that the map

$$X \longrightarrow \|f(|X|)\|$$

is subadditive, see [7]. This was first noted by Uchiyama [14]. In case of the trace norm, this extension of (1.5) is Rotfel'd's theorem [12]. Of course (1.6) considerably strenghtens (1.5). Another improvement of (1.5), companion to Theorem 0.1, is shown in [7]:

THEOREM 1.2. *Let $A$, $B \geqslant 0$ and let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Then, for all symmetric norms,*

$$\|f(A+B)\| \leqslant \|f(A) + f(B)\|.$$

In case of the operator norm, Kosem [11] gave a short proof. The general case is considerably more difficult: When $f$ is operator concave, Theorem 1.3 had first been proved by Ando and Zhan [1] by using integral representation of operator concave functions and a delicate process. The proof given in [7] is much more elementary.

## 2. COMBINED RESULT AND PROOF

We can naturally embody Theorem 1.1 and Theorem 1.2 (and its version for sums of several operators) in a unique statement:

THEOREM 2.1. *Let $\{A_i\}_{i=1}^m$ be positive, let $\{Z_i\}_{i=1}^m$ be expansive and let $f(t)$ be a non-negative concave function on $[0, \infty)$. Then, for all symmetric norms,*

$$\left\| f\Big(\sum Z_i^* A_i Z_i\Big) \right\| \leqslant \left\| \sum Z_i^* f(A_i) Z_i \right\|.$$

Of course the sum is over the first $m$ integers. For $m = 1$ we get Theorem 1.1 and if $Z_i = I$ for all $i$ we get Theorem 1.2. There is no obvious way to derive Theorem 2.1 from Theorems 1.1, 1.2. However our proof is adapted from the proof of Theorem 1.1 which is itself partially based on the proof of Theorem 1.2. Therefore our proof is rather elementary. We use Hansen's inequality (1.3) for a quite elementary case equivalent to the fact that $t \to 1/t$ is operator convex on $(0, \infty)$. We also use some basic facts about symmetric norms. In particular we need the following two facts for arbitrary $X, Y \geqslant 0$.

(1) If $\|X\| \leqslant \|Y\|$ for all symmetric norms, then we also have $\|g(X)\| \leqslant \|g(Y)\|$ for all symmetric norms and all increasing convex functions $g : [0,\infty) \longrightarrow [0,\infty)$.

(2) If $\|X\|_k \leqslant \|Y\|_k$ for all Ky Fan $k$-norms, then we also have $\|X\| \leqslant \|Y\|$ for all symmetric norms (Ky Fan's principle). Recall that the Ky Fan $k$-norms are the sum of the $k$ largest singular values.

For this background we refer to any expository text such as [13].

*Proof of Theorem* 2.1. The proof for an arbitrary $m$ is the same as the proof for $m = 2$, so we consider the case of two expansive operators $X$, $Y$ and two positive operators $A$, $B$. The proof is divided in four steps.

*Step 1.* If $f$ is operator concave, the proof immediately follows from Theorem 1.1 and (1.3):

$$\|f(X^*AX+Y^*BY)\| \leqslant \|f(X^*AX)+f(Y^*BY)\| \leqslant \|X^*f(A)X+Y^*f(B)Y\|. \tag{2.1}$$

*Step 2.* Now consider a one to one convex function $g : [0,\infty) \longrightarrow [0,\infty)$ whose inverse function $f$ is operator concave. Since $g$ is onto there exist $A'$, $B' \geqslant 0$ such that $A = g(A')$ and $B = g(B')$; moreover $A'$ and $B'$ can be chosen arbitrarily so that (2.1) can be read as

$$\|f(X^*g(A)X + Y^*g(B)Y)\| \leqslant \|X^*AX + Y^*BY\|.$$

Since $g$ is convex increasing we infer

$$\|X^*g(A)X + Y^*g(B)Y\| \leqslant \|g(X^*AX + Y^*BY)\|. \tag{2.2}$$

*Step 3.* Now we extend (2.2) to the class of all non-negative convex functions on $[0,\infty)$ vanishing at 0. It suffices to consider the Ky Fan $k$-norms $\|\cdot\|_k$. Suppose that $g_1$ and $g_2$ both satisfy (2.2). Using the triangle inequality and the fact that $g_1$ and $g_2$ are non-decreasing,

$$\begin{aligned}
&\|X^*(g_1+g_2)(A)X+Y^*(g_1+g_2)(B)Y\|_k \\
&\qquad \leqslant \|X^*g_1(A)X + Y^*g_1(B)Y\|_k + \|X^*g_2(A)X + Y^*g_2(B)Y\|_k \\
&\qquad \leqslant \|g_1(X^*AX + Y^*BY)\|_k + \|g_2(X^*AX + Y^*BY)\|_k \\
&\qquad = \|(g_1+g_2)(X^*AX + Y^*BY)\|_k,
\end{aligned}$$

hence the set of functions satisfying to (2.2) is a cone. It is also closed for pointwise convergence. Since any positive convex function vanishing at 0 can be approached by a positive combination of angle functions at $a > 0$,

$$\gamma(t) = \frac{1}{2}\{|t-a| + t - a\},$$

it suffices to prove (2.2) for such a $\gamma$. By Step 2 it suffices to approach $\gamma$ by functions whose inverses are operator concave. We take (with $r > 0$)

$$h_r(t) = \frac{1}{2}\left\{\sqrt{(t-a)^2+r} + t - \sqrt{a^2+r}\right\},$$

whose inverse

$$t - \frac{r/2}{2t + \sqrt{a^2+r} - a} + \frac{\sqrt{a^2+r}+a}{2}$$

is operator concave since $1/t$ is operator convex on the positive half-line. Clearly, as $r \to 0$, $h_r(t)$ converges uniformly to $\gamma$.

*Step 4.* Proof for any concave function $f : [0,\infty) \longrightarrow [0,\infty)$. Again, it suffices to consider the Ky Fan $k$-norms. This shows that we may and do assume $f(0) = 0$. Note that $f$ is necessarily non-decreasing. Hence, there exists a rank $k$ spectral projection $E$ for $X^*AX + Y^*BY$, corresponding to the $k$-largest eigenvalues $\lambda_1(X^*AX + Y^*BY), \dots, \lambda_k(X^*AX + Y^*BY)$ of $X^*AX + Y^*BY$, such that

$$\|f(X^*AX + Y^*BY)\|_k = \sum_{j=1}^{k} \lambda_j(f(X^*AX + Y^*BY)) = \operatorname{Tr} Ef(X^*AX + Y^*BY)E.$$

Therefore, using a well-known property of Ky Fan norms, it suffices to show that

$$\operatorname{Tr} Ef(X^*AX + Y^*BY)E \leqslant \operatorname{Tr} E(X^*f(A)X + Y^*f(B)Y)E.$$

This is the same as requiring that

$$\operatorname{Tr} E(X^*g(A)X + Y^*g(B)Y)E \leqslant \operatorname{Tr} Eg(X^*AX + Y^*BY)E \tag{2.3}$$

for all non-positive convex functions $g$ on $[0,\infty)$ with $g(0) = 0$. Any such function can be approached by a combination of the type

$$g(t) = \lambda t + h(t)$$

for a scalar $\lambda < 0$ and some non-negative convex function $h$ vanishing at 0. Hence, it suffices to show that (2.3) holds for $h(t)$. We have

$$\begin{aligned}
\operatorname{Tr} E(X^*h(A)X+Y^*h(B)Y)E &= \sum_{j=1}^{k} \lambda_j(E(X^*h(A)X + Y^*h(B)Y)E) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(X^*h(A)X + Y^*h(B)Y) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(h(X^*AX + Y^*BY)) \quad \text{(by Step 3)} \\
&= \sum_{j=1}^{k} \lambda_j(Eh(X^*AX+Y^*BY)E) = \operatorname{Tr} Eh(X^*AX+Y^*BY)E
\end{aligned}$$

where the second equality follows from the fact that $h$ is non-decreasing and hence $E$ is also a spectral projection of $h(X^*AX + Y^*BY)$ corresponding to the $k$ largest eigenvalues. ∎

COROLLARY 2.2. *Let $g : [0,\infty) \longrightarrow [0,\infty)$ be a convex function with $g(0) = 0$. Let $\{A_i\}_{i=1}^m$ be positive and let $\{Z_i\}_{i=1}^m$ be expansive. Then, for all symmetric norms,*

$$\left\| \sum Z_i^* g(A_i) Z_i \right\| \leqslant \left\| g\Big(\sum Z_i^* A_i Z_i\Big) \right\|.$$

This corollary is proved in Step 3. It can also be derived from Theorem 2.1 by using the first fact recalled before the proof. When $Z_i = I$ for all $i$ this is a remarkable result of Kosem [11]. Note that Theorem 0.1 is a special case of Corollary 2.2. There are several well-known results involving sums of congruences. But, in contrast with Theorem 2.1 and its corollary, these results deal with strong contractive assumptions. We give an example generalizing (1.2):

*Let $\{A_i\}_{i=1}^m$ be positive and $\{Z_i\}_{i=1}^m$ such that $\sum Z_i^* Z_i \leqslant I$. If $f$ is a monotone concave function on $[0,\infty)$, $f(0) \geqslant 0$, then,*

$$f\Big(\sum Z_i^* A_i Z_i\Big) \geqslant V\Big(\sum Z_i^* f(A_i) Z_i\Big)V^*.$$

*for some unitary $V$.*

Such an inequality is connected to Jensen type inequalities for compressions or positive unital linear maps, see [4], [2]. We conclude with a discussion of the extension of our results for operators on an infinite dimensional Hilbert space $\mathcal{H}$. In the proof of Theorem 2.1, (2.1) is derived from a version of Hansen's inequality (1.3) involving congruences with expansive operators $X, Y$. This Hansen's inequality remains valid in the infinite dimensional setting if one requires that $X, Y$ are both expansive and invertible. Consequently we have the following result for the usual operator norm $\|\cdot\|_\infty$.

*Theorem* 2.1 *and its corollaries are still valid for $\|\cdot\|_\infty$ and operators on $\mathcal{H}$ when $Z$ and $\{Z_i\}_{i=1}^m$ are expansive and invertible.*

This statement is meaningful. The original proof ([5]) of Theorem 1.1 was unsuccessful to cover the infinite dimensional setting. Concerning inequality (1.3), the original statement is in the framework of the spectral order in a semi-finite von Neumann algebra. It is also possible to give a version for operators on $\mathcal{H}$ by adding a $rI$ term in the RHS with $r > 0$ arbitrarily small, see pp. 11–15 of [6]. By arguing as in [2] we then obtain:

*Let $A, B \geqslant 0$ on $\mathcal{H}$ and let $r > 0$. If $f$ is a monotone concave function on $[0,\infty)$, $f(0) \geqslant 0$, then*

$$f(A+B) \leqslant Uf(A)U^* + Vf(B)V^* + rI$$

*for some unitaries $U$, $V$.*

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