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DILATION THEORY FOR RANK 2 GRAPH ALGEBRAS

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ABSTRACT. An analysis is given of $*$-representations of rank 2 single vertex graphs. We develop dilation theory for the non-selfadjoint algebras $\mathcal{A}_\theta$ and $\mathcal{A}_u$ which are associated with the commutation relation permutation θ of a 2-graph and, more generally, with commutation relations determined by a unitary matrix u in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. We show that a defect free row contractive representation has a unique minimal dilation to a $*$-representation and we provide a new simpler proof of Solel's row isometric dilation of two u-commuting row contractions. Furthermore it is shown that the C^*-envelope of $\mathcal{A}_u$ is the generalised Cuntz algebra $\mathcal{O}_{X_u}$ for the product system X_u of u; that for $m \geqslant 2$ and $n \geqslant 2$ contractive representations of $\mathcal{A}_\theta$ need not be completely contractive; and that the universal tensor algebra $\mathcal{T}_+(X_u)$ need not be isometrically isomorphic to $\mathcal{A}_u$.

KEYWORDS: *Higher rank graph, atomic $*$-representation, dilation, C^*-envelope.*

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1. INTRODUCTION

Kumjian and Pask [19] have introduced a family of C^*-algebras associated with higher rank graphs. In [18], Kribs and Power examined the corresponding non-selfadjoint operator algebras and recently Power [25] has presented a detailed analysis of the single vertex case, with particular emphasis on rank 2-graphs. Already this case contains many new and intriguing algebras. In this paper, we continue this investigation by beginning a study of the representation and dilation theory of these algebras as well as more general algebras determined by unitary commutation relations.

In the 2-graph case the C^*-algebras are the universal C^*-algebras of unital discrete semigroups which are given concretely in terms of a finite set of generators and relations of a special type. Given a permutation θ of $m \times n$, form a unital semigroup $\mathbb{F}_\theta^+$ with generators $e_1, \dots, e_m, f_1, \dots, f_n$ which is free in the e_i's and free in the f_j's, and has the commutation relations $e_i f_j = f_{j'} e_{i'}$ where

$\theta(i,j) = (i',j')$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. This is a cancellative semigroup with unique factorization [19], [25].

Consider the left regular representation λ of these relations on $\ell^2(\mathbb{F}_\theta^+)$ given by $\lambda(w)\xi_x = \xi_{wx}$. The norm closed unital operator algebra generated by these operators is denoted by $\mathcal{A}_\theta$. In line with Arveson's approach pioneered in [1], we are interested in understanding the completely contractive representations of this algebra. The message of two recent papers on the Shilov boundary of a unital operator algebra, Dritschel and McCullough [11] and Arveson [2], is that a representation should be dilated to a maximal dilation; and these maximal dilations extend uniquely to $*$-representations of the generated C^*-algebra that factor through the C^*-envelope. Thus a complete description of maximal dilations will lead to the determination of the C^*-envelope.

Kumjian and Pask define a $*$-representation of the semigroup $\mathbb{F}_\theta^+$ to be a representation π of $\mathbb{F}_\theta^+$ as isometries with the following property which we call the *defect free* property:

$$\sum_{i=1}^{m} \pi(e_i)\pi(e_i)^* = I = \sum_{j=1}^{n} \pi(f_j)\pi(f_j)^*.$$

The universal C^*-algebra determined by this family of representations is denoted by $C^*(\mathbb{F}_\theta^+)$. We shall show that every completely contractive representation of $\mathcal{A}_\theta$ dilates to a $*$-representation. This allows us in particular to deduce that the C^*-envelope of $\mathcal{A}_\theta$ is $C^*(\mathbb{F}_\theta^+)$. This identification is due to Katsoulis and Kribs [17] who show, more generally, that the universal C^*-algebra of a higher rank graph (Λ, d) is the enveloping C^*-algebra of the associated left regular representation algebra $\mathcal{A}_\Lambda$.

The left regular representation of $\mathbb{F}_\theta^+$ is not a $*$-representation. It is important though that it dilates (in many ways) to a $*$-representation.

A significant class of representations which play a key role in our analysis are the *atomic* $*$-representations. These row isometric representations have an orthonormal basis which is permuted, up to unimodular scalars, by each of the generators. They have a rather interesting structure, and in a sequel to this paper [8], we shall completely classify them in terms of families of explicit partially isometric representations. In this paper, we see the precursors of that analysis. The dilation theory for partial isometry representations that we develop will be crucial to our later analysis.

These atomic representations allow us to describe the C^*-algebra $C^*(\mathbb{F}_\theta^+)$. Such a description relies on an understanding of the Kumjian–Pask aperiodicity condition. The periodic case is characterized in [9], leading to the structure of $C^*(\mathbb{F}_\theta^+)$.

An important tool for us will be Solel's generalisation of Ando's dilation theorem to the case of a pair of row contractions $[A_1 \cdots A_m]$, $[B_1 \cdots B_n]$ that satisfy the commutation relations

$$A_i B_j = \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} B_{j'} A_{i'}$$

where $u = u_{(i,j),(i',j')}$ is a unitary matrix in $M_{mn}(\mathbb{C})$. Solel obtained this result as part of his analysis of the representation theory for the tensor algebra $\mathcal{T}_+(X)$ associated with a product system of correspondences X. We obtain a new simple proof which is based on the Frahzo–Bunce–Popescu dilation theory of row contractions and the uniqueness of minimal dilations.

The relevant tensor algebra, as defined in [29], arises as a universal algebra associated with a product system of correspondences,

$$X_u = \{E_{k,l} = (\mathbb{C}^m)^{\otimes k} \otimes (\mathbb{C}^n)^{\otimes l} : k, l \in \mathbb{Z}_+\},$$

where the composition maps

$$E_{k,l} \otimes E_{r,s} \to E_{k+r,l+s}$$

are unitary equivalences determined naturally by u. An equivalent formulation which fits well with our perspectives is to view $\mathcal{T}_+(X_u)$ as the universal operator algebra for a certain class of representations (row contractive ones) of the norm closed operator algebra $\mathcal{A}_u$ generated by creation operators $\lambda(e_i), \lambda(f_j)$ on the Fock space of X_u. These unitary relation algebras generalise the 2-graph algebras $\mathcal{A}_\theta$. While the atomic representation theory of these algebras remains to be exposed we can analyse C^*-envelopes, C^*-algebra structure and dilation theory in this wider generality and so we do so. Also we prove, as one of the main results, that a defect free row contractive representation of $\mathcal{A}_u$ has a unique minimal row isometric defect free dilation.

Prior to Solel's study [29], the operator algebra theory of product systems centered on C^*-algebra considerations. In particular Fowler [12], [13] has defined and analyzed the Cuntz algebras $\mathcal{O}_X$ associated with a discrete product system X of finite dimensional Hilbert spaces. Such an algebra is the universal C^*-algebra for certain $*$-representations satisfying the defect free property. We shall prove that the C^*-algebra envelope of $\mathcal{A}_u$ is $\mathcal{O}_{X_u}$.

The atomic representations of the 2-graph semigroups $\mathbb{F}_\theta^+$ give many insights to the general theory. For example we note contrasts with the representation theory for the bidisc algebra, namely that row contractive representations of $\mathcal{A}_u$ need not be contractive, and that contractive representations of $\mathcal{A}_u$ need not be completely contractive.

We remark that the structure of automorphisms of the algebras $\mathcal{A}_u$ and a classification up to isometric isomorphism has been given in [26]. In fact we make use of such automorphisms and the failure of contractivity of row contractive representations to show that $\mathcal{T}_+(X_u)$ and $\mathcal{A}_u$ may fail to be isometrically isomorphic.

2. RANK 2 GRAPHS, SEMIGROUPS AND REPRESENTATIONS

Let $\theta \in S_{m\times n}$ be a permutation of $m \times n$. The semigroup $\mathbb{F}_\theta^+$ is generated by $e_1, \dots, e_m$ and $f_1, \dots, f_n$. The identity is denoted as $\varnothing$. There are no relations among the e's, so they generate a copy of the free semigroup on m letters, $\mathbb{F}_m^+$; and there are no relations on the f's, so they generate a copy of $\mathbb{F}_n^+$. There are *commutation relations* between the e's and f's given by

$$e_i f_j = f_{j'} e_{i'} \quad \text{where } \theta(i,j) = (i',j').$$

A word $w \in \mathbb{F}_\theta^+$ has a fixed number of e's and f's regardless of the factorization; and the *degree* of w is (k,l) if there are k e's and l f's. The *length* of w is $|w| = k + l$. The commutation relations allow any word $w \in \mathbb{F}_\theta^+$ to be written with all e's first, or with all f's first, say $w = e_u f_v = f_{v'} e_{u'}$. Indeed, one can factor w with any prescribed pattern of e's and f's as long as the degree is (k,l). It is straightforward to see that the factorization is uniquely determined by the pattern and that $\mathbb{F}_\theta^+$ has the unique factorization property. See also [19], [18], [25].

We do not need the notion of a k-graph (Λ, d), in which Λ is a countable small category with functor $d : \Lambda \to \mathbb{Z}_+^k$ satisfying a unique factorisation property. However, in the single object (i.e. single vertex) rank 2 case, with $d^{-1}(1,0)$, $d^{-1}(0,1)$ finite, the small category Λ, viewed as a semigroup, is isomorphic to $\mathbb{F}_\theta^+$ for some θ and d is equal to the degree map.

EXAMPLE 2.1. With $n = m = 2$ we note that the relations

$$e_1 f_1 = f_2 e_1, \quad e_1 f_2 = f_1 e_2, \quad e_2 f_1 = f_1 e_1, \quad e_2 f_2 = f_2 e_2,$$

arise from the permutation θ in S_4 which is the 3-cycle $\big((1,1),(1,2),(2,1)\big)$. We refer to $\mathbb{F}_\theta^+$ as the forward 3-cycle semigroup. The reverse 3-cycle semigroup is the one arising from the 3-cycle $\big((1,1),(2,1),(1,2)\big)$.

It can be shown that the 24 permutations of S_4 give rise to 9 isomorphism classes of semigroups $\mathbb{F}_\theta^+$, where we allow isomorphisms to exchange the e_i's for f_j's. The forward and reverse 3-cycles give non-isomorphic semigroups [25].

EXAMPLE 2.2. With $n = m = 2$ the relations

$$e_1 f_1 = f_1 e_1, \quad e_1 f_2 = f_1 e_2, \quad e_2 f_1 = f_2 e_1, \quad e_2 f_2 = f_2 e_2,$$

are those arising from the 2-cycle permutation $((1,2),(2,1))$. We refer $\mathbb{F}_\theta^+$ in this case as the flip semigroup and $\mathcal{A}_\theta$ as the flip algebra. The generated C^*-algebra is identified in Example 3.6 and an illuminating atomic representation is given in Example 4.1.

Consider the left regular representation λ of these relations. This is defined on $\ell^2(\mathbb{F}_\theta^+)$ with the orthonormal basis $\{\xi_x : x \in \mathbb{F}_\theta^+\}$ by $\lambda(w)\xi_x = \xi_{wx}$. The norm closed unital operator algebra generated by these operators is denoted by $\mathcal{A}_\theta$.

DEFINITION 2.3. A representation of $\mathbb{F}_\theta^+$ is a semigroup homomorphism σ of $\mathbb{F}_\theta^+$ into $\mathcal{B}(\mathcal{H})$. If it extends to a continuous representation of the algebra $\mathcal{A}_\theta$, then it is said to be *contractive* or *completely contractive* if the extension to $\mathcal{A}_\theta$ has this property.

A representation of $\mathbb{F}_\theta^+$ is *partially isometric* if the range consists of partial isometries on the Hilbert space $\mathcal{H}$ and is *isometric* if the range consists of isometries.

A partially isometric representation is *atomic* if there is an orthonormal basis which is permuted, up to scalars, by each partial isometry. That is, π is atomic if there is a basis $\{\xi_k : k \geqslant 1\}$ so that for each $w \in \mathbb{F}_\theta^+$, $\pi(w)\xi_k = \alpha\xi_l$ for some l and some $\alpha \in \mathbb{T} \cup \{0\}$.

A representation σ is *row contractive* if $[\sigma(e_1) \cdots \sigma(e_m)]$ and $[\sigma(f_1) \cdots \sigma(f_n)]$ are row contractions, and is *row isometric* if these row operators are isometries. A row contractive representation is *defect free* if

$$\sum_{i=1}^{m} \sigma(e_i)\sigma(e_i)^* = I = \sum_{j=1}^{n} \sigma(f_j)\sigma(f_j)^*.$$

A row isometric defect free representation is called a *$*$-representation* of $\mathbb{F}_\theta^+$. We reserve the term defect free for row contractive representations.

The row isometric condition is equivalent to saying that the $\sigma(e_i)$'s are isometries with pairwise orthogonal range; and the same is true for the $\sigma(f_j)$'s. In a defect free, isometric representation, the $\sigma(e_i)$'s generate a copy of the Cuntz algebra $\mathcal{O}_m$ (respectively the $\sigma(f_j)$'s generate $\mathcal{O}_n$) rather than a copy of the Cuntz–Toeplitz algebra $\mathcal{E}_m$ (respectively $\mathcal{E}_n$) as is the case for the left regular representation. The left regular representation λ is row isometric, but is not defect free.

There is a universal C^*-algebra $C^*(\mathbb{F}_\theta^+)$ which can be described by taking a direct sum π_u of all $*$-representations on a fixed separable Hilbert space, and forming the C^*-algebra generated by $\pi_u(\mathbb{F}_\theta^+)$. It is the unique C^*-algebra generated by a $*$-representation of $\mathbb{F}_\theta^+$ with the property that given any $*$-representation σ, there is a $*$-homomorphism $\pi : C^*(\mathbb{F}_\theta^+) \to C^*(\sigma(\mathbb{F}_\theta^+))$ so that $\sigma = \pi\pi_u$. This C^*-algebra is a higher rank graph C^*-algebra in the sense of Kumjian and Pask [19] for the rank two single vertex graph determined by θ.

EXAMPLE 2.4. *Inductive representations.* We now define an important family of atomic $*$-representations of $\mathbb{F}_\theta^+$. These representations are called type 3a in the classification obtained in [8].

Start with an arbitrary *infinite word* or *tail* $\tau = e_{i_0} f_{j_0} e_{i_1} f_{j_1} \cdots$. Let $\mathcal{G}_s = \mathcal{G} := \mathbb{F}_\theta^+$, for $s = 0, 1, 2, \ldots$, viewed as a discrete set on which the generators of $\mathbb{F}_\theta^+$ act as injective maps by right multiplication, namely,

$$\rho(w)g = gw \quad \text{for all } g \in \mathcal{G}.$$

Consider $\rho_s = \rho(e_{i_s} f_{j_s})$ as a map from $\mathcal{G}_s$ into $\mathcal{G}_{s+1}$. Define $\mathcal{G}_\tau$ to be the injective limit set

$$\mathcal{G}_\tau = \varinjlim(\mathcal{G}_s, \rho_s);$$

and let ι_s denote the injections of $\mathcal{G}_s$ into $\mathcal{G}_\tau$. Thus $\mathcal{G}_\tau$ may be viewed as the union of $\mathcal{G}_0, \mathcal{G}_1, \dots$ with respect to these inclusions.

The left regular action λ of $\mathbb{F}_\theta^+$ on itself induces corresponding maps on $\mathcal{G}_s$ by $\lambda_s(w)g = wg$. Observe that $\rho_s \lambda_s(w) = \lambda_{s+1}(w)\rho_s$. The injective limit of these actions is an action λ_τ of $\mathbb{F}_\theta^+$ on $\mathcal{G}_\tau$. Let λ_τ also denote the corresponding representation of $\mathbb{F}_\theta^+$ on $\ell^2(\mathcal{G}_\tau)$. Let $\{\xi_g : g \in \mathcal{G}_\tau\}$ denote the basis. A moment's reflection shows that this provides a defect free, isometric representation of $\mathbb{F}_\theta^+$; i.e. it is a $*$-representation.

Davidson and Pitts [7] classified the atomic $*$-representations of $\mathbb{F}_m^+$ and showed that the irreducibles fall into two types, known as ring representations and infinite tail representations. The 2-graph situation analysed in [8] turns out to be considerably more complicated and in particular it is shown that the irreducible atomic $*$-representations of $\mathbb{F}_\theta^+$ fall into six types.

We now define the more general unitary relation algebras $\mathcal{A}_u$ which are associated with a unitary matrix $u = (u_{(i,j),(k,l)})$ in $M_{mn}(\mathbb{C})$. Also we define the (universal) tensor algebra $\mathcal{T}_+(X_u)$ considered by Solel [29] and the generalised Cuntz algebra $\mathcal{O}(X_u)$, both of which are associated with a product system X_u for u.

Let $e_1, \dots, e_m$ and $f_1, \dots, f_n$ be viewed as bases for the vector spaces $E = \mathbb{C}^m$ and $F = \mathbb{C}^n$ respectively. Then u provides an identification $u : E \otimes F \to F \otimes E$ such that

$$e_i \otimes f_j = \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} f_{j'} \otimes e_{i'}$$

or, equivalently,

$$f_l \otimes e_k = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{u}_{(i,j),(k,l)} e_i \otimes f_j.$$

Moreover, for each pair (k,l) in $\mathbb{Z}_+^2$ with $k+l=r$, u determines an unambiguous identification $G_1 \otimes \cdots \otimes G_r \to H_1 \otimes \cdots \otimes H_r$, whenever each G_i and H_i is equal to E or F and is such that the multiplicity of E and F in each product is k and l respectively. Thus these different patterns of multiple tensor products of E and F are identified with $E^{\otimes k} \otimes F^{\otimes l}$. The family $X_u = \{E^{\otimes k} \otimes F^{\otimes l}\}$ together with the associative multiplication $\otimes$ induced by u, as above, is an example of a *product system over* $\mathbb{Z}_+^2$, consisting of finite dimensional Hilbert spaces.

Let $\mathcal{H}_u$ be the $\mathbb{Z}_+^2$-graded Fock space $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\oplus(E^{\otimes k}\otimes F^{\otimes l})$ with the convention $E^{\otimes 0}=F^{\otimes 0}=\mathbb{C}$. The left creation operators L_{e_i}, L_{f_j} are defined on $\mathcal{H}_u$ in the usual way. Thus

$$L_{f_i}(e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes\cdots\otimes f_{j_l})=f_i\otimes(e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes\cdots\otimes f_{j_l}).$$

As in [26] we define the *unitary relation algebra* $\mathcal{A}_u$ to be the norm closed algebra generated by these shift operators. Note that for $\mathbb{F}_\theta^+$ we have $\mathcal{A}_\theta=\mathcal{A}_u$ where the unitary is the permutation matrix u with $u_{(i,j),(i',j')}=1$ if $\theta(i,j)=(i',j')$ and $u_{(i,j),(i',j')}=0$ otherwise. In consistency with the notation for the left regular representation of $\mathbb{F}_\theta^+$ we shall write $\xi_{e_uf_v}$ for the basis element $e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes\cdots\otimes f_{j_l}$ where (with tolerable notation ambiguity) $u=i_1\cdots i_k$ and $v=j_1\cdots j_l$.

We define $\mathbb{F}_u^+$ to be the semigroup generated by the left creation operators. Moreover we are concerned with representations of this semigroup that satisfy the unitary commutation relations, that is, with representations that extend to the complex algebra $\mathbb{C}[\mathbb{F}_u^+]$ generated by the creation operators. This will be an implicit assumption henceforth. Thus a unital representation σ of $\mathbb{F}_u^+$ is determined by two row operators $A=[A_1\cdots A_m]$, $B=[B_1\cdots B_n]$ that satisfy the commutation relations

$$A_iB_j=\sum_{i'=1}^{m}\sum_{j'=1}^{n}u_{(i,j),(i',j')}B_{j'}A_{i'}.$$

The terms row contractive, row isometric, and partially isometric are defined as before, and we say that σ is contractive or completely contractive if the extension of σ to $\mathcal{A}_u$ exists with this property.

In [29], Solel defines the universal non-selfadjoint tensor algebra $\mathcal{T}_+(X)$ of a general product system X of correspondences. In the present context it is readily identifiable with the universal operator algebra for the family of row contractive representations $\pi_{A,B}$ and we take this as the definition of the *tensor algebra* $\mathcal{T}_+(X_u)$.

On the C^*-algebra side the *generalised Cuntz algebra* $\mathcal{O}_X$ associated with a product system X is the universal algebra for a natural family of $*$-representation of X. See [12], [13], [14]. In the present context this C^*-algebra is the same as the universal operator algebra for the family of defect free row isometric representations $\pi_{S,T}$ and we take this as the definition of $\mathcal{O}_{X_u}$.

We shall not need the general framework of correspondences, for which the associated C^*-algebras are the Cuntz–Pimsner algebras. See [27] for an overview of this. However, let us remark that the direct system X_u is a direct system of correspondences over $\mathbb{C}$. The universality in [29] entails that $\mathcal{T}_+(X_u)$ is the completion of $\mathbb{C}[\mathbb{F}_u^+]$ with respect to representations $\pi_{A,B}$ for which each restriction $\pi_{A,B}|E^{\otimes k}\otimes F^{\otimes l}$ is completely contractive with respect to the matricial norm structure arising from the left regular inclusions $E^{\otimes k}\otimes F^{\otimes l}\subseteq\mathcal{A}_u$. These matricial spaces are row Hilbert spaces and so, taking $(k,l)=(1,0)$ and $(0,1)$ we see that

A and B are necessarily row contractions. This necessary condition is also sufficient. Indeed, each restriction $\pi_{A,B}|E^{\otimes k} \otimes F^{\otimes l}$ is determined by a single row contraction $[T_1 \cdots T_N]$ (which is a tensor power of A and B) and these maps are completely contractive and are of the form

$$(\alpha_1, \dots, \alpha_N) \to [\alpha_1 T_1, \dots, \alpha_N T_N].$$

EXAMPLE 2.5. We now show that as in the case of the permutation algebras $\mathcal{A}_\theta$, the algebra $\mathcal{A}_u$ has a defect free row isometric representation λ_τ associated with each infinite tail τ. In particular there are nontrivial $*$-representations (Cuntz representations) for the product system X_u and $\mathcal{O}_{X_u}$ is nontrivial.

Consider, once again, an *infinite word* or *tail* $\tau = e_{i_0} f_{j_0} e_{i_1} f_{j_1} \cdots$. Let $\mathcal{H}_t = \mathcal{H}_u$, for $t = 0, 1, 2, \dots$, and for $s = 0, 1, \dots$, define isometric Hilbert space injections $\rho_s : \mathcal{H}_s \to \mathcal{H}_{s+1}$ with $\rho_s(\xi) = \xi \otimes e_{i_s} f_{j_s}$ for each $\xi \in E^{\otimes k} \otimes F^{\otimes l}$ and all k, l. Let $\mathcal{H}_\tau$ be the Hilbert space $\varinjlim \mathcal{H}_s$, with each $\mathcal{H}_s$ identified as a closed subspace and let λ_τ denote the induced isometric representation of $\mathbb{F}_u^+$ on $\mathcal{H}_\tau$.

It follows readily that λ_τ is a row isometric representation. Moreover, it is a $*$-representation of $\mathbb{F}_u^+$, that is, λ_τ has the defect free property. To see this, let $\xi^s_{e_u f_v}$ denote the basis element of $\mathcal{H}$ equal to $\xi_{e_u f_v}$ in $\mathcal{H}_s$ where e_u and f_v are words as before with lengths $|u| = k \geqslant 0, |v| = l \geqslant 0$. Then $\xi^s_{e_u f_v} = \xi^{s+1}_{e_u f_v e_{i_s} f_{j_s}}$. The commutation relations show that this vector lies both in the subspace of $\mathcal{H}_{s+1}$ spanned by the spaces $\lambda_\tau(e_i) E^k \otimes F^{l+1}$, $i = 1, \dots, m$, and in the subspace spanned by the spaces $\lambda_\tau(f_j) E^{k+1} \otimes F^l$, $j = 1, \dots, n$. It follows that the range projections of the isometries $\lambda_\tau(e_i)$, and also those of $\lambda_\tau(f_j)$, sum to the identity.

3. $C^*(\mathcal{A}_u)$ AND THE C^*-ENVELOPE

There are three natural C^*-algebras associated with $\mathcal{A}_u$ namely the generated C^*-algebra $C^*(\mathcal{A}_u)$, the universal C^*-algebra $\mathcal{O}_{X_u}$, and the C^*-envelope $C^*_{\mathrm{env}}(\mathcal{A}_u)$. By its universal property the latter algebra is the smallest C^*-algebra containing $\mathcal{A}_u$ completely isometrically. In the case of $\mathcal{A}_\theta$ the generated C^*-algebra is simply the C^*-algebra generated by the left regular representation of the semigroup $\mathbb{F}_\theta^+$.

In this section we show that $C^*_{\mathrm{env}}(\mathcal{A}_\theta) = C^*(\mathbb{F}_\theta^+)$ and more generally that $C^*_{\mathrm{env}}(\mathcal{A}_u) = \mathcal{O}_{X_u}$. Also we analyse ideals and show how this algebra is a quotient of $C^*(\mathcal{A}_u)$.

LEMMA 3.1. *Let λ_τ be any inductive representation of $\mathbb{F}_\theta^+$. Then the imbedding of $\mathcal{A}_\theta$ into $C^*(\lambda_\tau(\mathbb{F}_\theta^+))$ is a complete isometry. Also, if λ_τ is a tail representation of $\mathcal{A}_u$ then the imbedding of $\mathcal{A}_u$ into $C^*(\lambda_\tau(\mathcal{A}_u))$ is a complete isometry.*

Proof. Let $\mathcal{A}$ be the norm closed subalgebra of $C^*(\lambda_\tau(\mathbb{F}_\theta^+))$ generated by $\lambda_\tau(\mathbb{F}_\theta^+)$. We showed in Example 2.4 that λ_τ is an inductive limit of copies of λ.

That is, $\ell^2(\mathcal{G}_\tau)$ is the closure of an increasing union of subspaces $\ell^2(\mathcal{G}_s)$, each is invariant under $\mathcal{A}$, and the restriction of λ_τ to $\ell^2(\mathcal{G}_s)$ is unitarily equivalent to λ. The norm of any matrix polynomial is thus determined by its restrictions to these subspaces, and the norm on each one is precisely the norm in $\mathcal{A}_\theta$. It follows that $\mathcal{A}$ is completely isometrically isomorphic to $\mathcal{A}_\theta$. The same argument applies to an inductive representation of the unitary relation algebra $\mathcal{A}_u$. ∎

COROLLARY 3.2. *There is a canonical quotient map from $C^*(\mathbb{F}_\theta^+)$ onto $C^*(\lambda_\tau(\mathbb{F}_\theta^+))$ and, more generally, from $\mathcal{O}_{X_u}$ onto $C^*(\lambda_\tau(\mathcal{A}_u))$. Also there is a canonical quotient map from $C^*(\lambda_\tau(\mathcal{A}_u))$ onto $C^*_{\mathrm{env}}(\mathcal{A}_u)$.*

Proof. That there are canonical quotient maps from the universal C^*-algebras $C^*(\mathbb{F}_\theta^+)$ and $\mathcal{O}_{X_u}$ follows from the fact that λ_τ is a $*$-representation.

By Lemma 3.1, $\mathcal{A}_u$ imbeds completely isometrically in $C^*(\lambda_\tau(\mathcal{A}_u))$. Hence there is a canonical quotient map of $C^*(\lambda_\tau(\mathcal{A}_u))$ onto $C^*_{\mathrm{env}}(\mathcal{A}_u)$ which is the identity on $\mathcal{A}_u$. ∎

3.1. GAUGE AUTOMORPHISMS. First we consider the graph C^*-algebra $C^*(\mathbb{F}_\theta^+)$. It will be convenient in this subsection to consider a faithful representation π, or equivalently a $*$-representation π of $\mathbb{F}_\theta^+$, so that $C^*(\mathbb{F}_\theta^+) = C^*(\pi(\mathbb{F}_\theta^+))$. The universal property of $C^*(\mathbb{F}_\theta^+)$ yields a family of *gauge automorphisms* $\gamma_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{T}$ determined by

$$\gamma_{\alpha,\beta}(\pi(e_i)) = \alpha\pi(e_i) \quad \text{and} \quad \gamma_{\alpha,\beta}(\pi(f_j)) = \beta\pi(f_j).$$

Integration around the 2-torus yields a faithful expectation

$$\Phi(X) = \int_{\mathbb{T}^2} \gamma_{\alpha,\beta}(X)\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

It is easy to check on monomials that the range is spanned by words of degree $(0,0)$ (where $\pi(e_i)^*$ and $\pi(f_j)^*$ count as degree $(-1,0)$ and $(0,-1)$ respectively).

Kumjian and Pask identify this range as an AF C^*-algebra. In our case, the analysis is simplified. To recap, the first observation is that any monomial in e's, f's and their adjoints can be written with all of the adjoints on the right. Clearly the row isometric condition means that

$$\pi(f_i)^*\pi(f_j) = \delta_{ij} = \pi(e_i)^*\pi(e_j).$$

Also, observe that if $f_j e_k = e_{k'} f_{j_k}$, for $1 \leqslant k \leqslant m$, then

$$\begin{aligned}\pi(e_i)^*\pi(f_j) &= \pi(e_i)^*\pi(f_j)\Big(\sum_k \pi(e_k)\pi(e_k)^*\Big)\\ &= \sum_k \pi(e_i)^*\pi(e_{k'})\pi(f_{j_k})\pi(e_k)^* = \sum_k \delta_{ik'}\pi(f_{j_k})\pi(e_k)^*.\end{aligned}$$

So, in the universal representation, every word in the generators and their adjoints can be expressed as a sum of words of the form xy^* for $x, y \in \mathbb{F}_\theta^+$.

Next, observe that for each integer $s \geqslant 1$, the words W_s of degree (s,s) determine a family of degree $(0,0)$ words, namely $\{\pi(x)\pi(y)^* : x, y \in W_s\}$. It is clear that

$$\pi(x_1)\pi(y_1)^*\pi(x_2)\pi(y_2)^* = \delta_{y_1,x_2}\pi(x_1)\pi(y_2)^*.$$

Thus these operators form a family of matrix units that generate a unital copy $\mathfrak{F}_s$ of the matrix algebra $M_{(mn)^s}(\mathbb{C})$. Moreover, these algebras are nested because the identity

$$\pi(x)\pi(y)^* = \pi(x)\sum_i \pi(e_i)\pi(e_i)^* \sum_j \pi(f_j)\pi(f_j)^* \ \pi(y)^*$$

allows one to write elements of $\mathfrak{F}_s$ in terms of the basis for $\mathfrak{F}_{s+1}$.

It follows that the range of the expectation Φ is the $(mn)^\infty$-UHF algebra $\mathfrak{F} = \overline{\bigcup_{s\geqslant 1} \mathfrak{F}_s}$. This is a simple C^*-algebra.

An almost identical argument is available for the C^*-algebra $\mathcal{O}_{X_u}$. (See also Proposition 2.1 of [12].) As above there is an abelian group of gauge automorphisms $\gamma_{\alpha,\beta}$ and the map $\Phi : \mathcal{O}_{X_u} \to \mathcal{O}_{X_u}$ is a faithful expectation onto its range. Moreover the range is equal to the fixed point algebra, $\mathcal{O}_{X_u}^\gamma$, of the automorphism group and this can be identified with a UHF C^*-algebra, $\mathfrak{F}_{X_u}$ say. To see this, note that in the universal representation, we have

$$e_i^* f_j = e_i^* f_j\Big(\sum_k e_k e_k^*\Big) = \sum_k \sum_{i'=1}^m \sum_{j'=1}^n \overline{u}_{(i',j'),(k,j)} e_i^* e_{i'} f_{j'} e_k^* = \sum_k \sum_{j'=1}^n \overline{u}_{(i,j'),(k,j)} f_{j'} e_k^*.$$

This, as before, leads to the fact that the operators $\pi(x)\pi(y^*)$, for $x, y \in X_u$, span a dense $*$-algebra in $\mathcal{O}_{X_u}$. Moreover, the span of

$$\{\pi(x)\pi(y^*) :\ x, y \in E^{\otimes s} \otimes F^{\otimes t},\ (s,t) \in \mathbb{Z}_+^2\}$$

has closure equal to the range of Φ and, as before, this is a UHF C^*-algebra.

LEMMA 3.3. *Let λ_τ be a tail representation of $\mathcal{A}_u$. Then the C^*-algebras $C^*(\lambda_\tau(\mathcal{A}_u))$ and $C^*_{\rm env}(\mathcal{A}_u)$ carry gauge automorphisms which commute with the natural quotient maps*

$$\mathcal{O}_{X_u} \to C^*(\lambda_\tau(\mathcal{A}_u)) \to C^*_{\rm env}(\mathcal{A}_u).$$

In the case of $C^*(\lambda_\tau(\mathcal{A}_u))$, the gauge automorphisms are unitarily implemented.

Proof. We use the notation of Example 2.5. Thus $\xi^{s+1}_{we_{i_s}f_{j_s}} = \xi^s_w$ and

$$\xi^s_{e_u f_v} = \xi^{s+k}_w = \xi^{s+k}_{e_{u'} f_{v'}},$$

where

$$w = e_u f_v e_{i_s} f_{j_s} \cdots e_{i_s+k-1} f_{j_s+k-1} = e_{u'} f_{v'};$$

moreover $|u'| = |u| + k$ and $|v'| = |v| + k$.

Thus we may define a well-defined diagonal unitary $U_{\alpha,\beta}$ on $\mathcal{H}_\tau$ such that, for $s \geqslant 0$,

$$U_{\alpha,\beta}\xi^s_{e_u f_v} = \alpha^{|u|-s}\beta^{|v|-s}\xi^s_{e_u f_v}.$$

Now

$$U_{\alpha,\beta}\lambda_\tau(e_i)U^*_{\alpha,\beta}\xi^s_{e_u f_v} = \alpha\xi^s_{e_i e_u f_v} = \alpha\lambda_\tau(e_i)\xi^s_{e_u f_v}$$

and

$$U_{\alpha,\beta}\lambda_\tau(f_j)U^*_{\alpha,\beta}\xi^s_{e_u f_v} = \beta\xi^s_{f_j e_u f_v} = \beta\lambda_\tau(f_j)\xi^s_{e_u f_v}.$$

It follows that $\operatorname{Ad} U_{\alpha,\beta}$ determines an automorphism of $\lambda_\tau(\mathcal{A}_u)$, denoted also by $\gamma_{\alpha,\beta}$ in view of the gauge action.

These automorphisms are completely isometric, since they are restrictions of $*$-automorphisms. So by the universal property of the C^*-envelope, each automorphism has a unique completely positive extension to $C^*_{\mathrm{env}}(\mathcal{A}_u)$ and the extension is a $*$-isomorphism. In this way a gauge action is determined on $C^*_{\mathrm{env}}(\mathcal{A}_u)$. That the maps commute with the quotients is evident. ∎

The next lemma follows a standard technique in graph C^*-algebra. See Theorem 3.4 of [19] for example.

LEMMA 3.4. *Let $\pi : \mathcal{O}_{X_u} \to B$ be a homomorphism of C^*-algebras and let $\delta : \mathbb{T}^2 \to \mathrm{Aut}(B)$ be an action such that $\pi \circ \gamma_{\alpha,\beta} = \delta_{\alpha,\beta} \circ \pi$ for all (α,β) in $\mathbb{T}^2$. Suppose that π is nonzero on the* UHF *subalgebra $\mathfrak{F}_{X_u}$. Then π is faithful.*

Proof. As before let Φ be the expectation map on $\mathcal{O}_{X_u}$, and let Φ_δ the expectation on B induced by δ. If $\pi(x) = 0$, then

$$0 = \Phi_\delta(\pi(x^*x)) = \pi(\Phi(x^*x)).$$

Since $\mathfrak{F}_{X_u}$ is simple and the restriction of π to it is non zero by assumption it follows that the restriction is faithful. Thus $\Phi(x^*x) = 0$ and now the faithfulness of Φ implies $x = 0$. ∎

THEOREM 3.5. *The C^*-envelope of the unitary relation algebra $\mathcal{A}_u$ is the generalised Cuntz algebra $\mathcal{O}_{X_u}$ of the product system X_u for the unitary matrix u. In particular the C^*-envelope of $\mathcal{A}_\theta$ is $C^*(\mathbb{F}^+_\theta)$. Also each inductive representation λ_τ extends to a faithful representation of $\mathcal{O}_{X_u}$.*

Proof. This is immediate from the lemma in view of the fact that there is a quotient map q of $\mathcal{O}_{X_u}$ onto $C^*_{\mathrm{env}}(\mathcal{A}_u)$ which commutes with gauge automorphisms on both algebras. ∎

EXAMPLE 3.6. Consider the flip graph semigroup $\mathbb{F}^+_\theta$ of Example 2.2. Pask and Kumjian observed that $C^*(\mathbb{F}^+_\theta) \simeq \mathcal{O}_2 \otimes \mathrm{C}(\mathbb{T})$. To see this in an elementary way, consider the relations

$$e_i f_j = f_i e_j \quad \text{for all } 1 \leqslant i,j \leqslant 2.$$

Suppose that $\sigma(e_i) = E_i$ and $\sigma(f_j) = F_j$ is a $*$-representation. Then E_i and F_i have the same range for $i = 1,2$. Therefore there are unitaries U_i so that $F_i = E_iU_i$. Then the commutation relations show that

$$E_1^2U_1 = E_1U_1E_1, \quad E_1E_2U_2 = E_1U_1E_2, \quad E_2E_1U_1 = E_2U_2E_1, \quad E_2^2U_2 = E_2U_2E_2.$$

Therefore

$$E_1 U_1 = U_1 E_1 = U_2 E_1 \quad \text{and} \quad E_2 U_2 = U_2 E_2 = U_1 E_2.$$

It follows that $U_1 = U_2 =: U$ on $\operatorname{Ran} E_1 + \operatorname{Ran} E_2 = \mathcal{H}$; and that U commutes with $C^*(E_1, E_2) \simeq \mathcal{O}_2$.

Consequently an irreducible $*$-representation π of $C^*(\mathbb{F}_\theta^+)$ sends U to a scalar tI, and the restriction of π to $C^*(e_1, e_2)$ is a $*$-representation of $\mathcal{O}_2$. All representations of $\mathcal{O}_2$ are $*$-equivalent because $\mathcal{O}_2$ is simple. Therefore we obtain $\pi(f_i) = t\pi(e_i)$ and $C^*(\pi(\mathbb{F}_\theta^+)) \simeq \mathcal{O}_2$. It is now easy to see that

$$C^*(\mathbb{F}_\theta^+) \simeq \mathcal{O}_2 \otimes \mathrm{C}(\mathbb{T}) \simeq \mathrm{C}(\mathbb{T}, \mathcal{O}_2).$$

By Theorem 3.5, this is also the C^*-envelope $C^*_{\text{env}}(\mathcal{A}_\theta)$. The structure of $C^*(\mathcal{A}_\theta)$ will now follow from Lemmas 3.8 and 3.9.

We can use Theorem 3.5 and the theory of C^*-envelopes and maximal dilations to identify the completely contractive representations of $\mathcal{A}_u$ with those that have dilations to defect free isometric representations, that is, to $*$-representations. As we note in the next section, the contractive representations of $\mathcal{A}_u$ form a wider class. First we recap the significance of maximal dilations.

Recall that a representation π of an algebra $\mathcal{A}$, or semigroup, on a Hilbert space $\mathcal{K}$ is a *dilation* of a representation σ on a Hilbert space $\mathcal{H}$ if there is an injection J of $\mathcal{H}$ into $\mathcal{K}$ so that $J\mathcal{H}$ is a semi-invariant subspace for $\pi(\mathcal{A})$ (i.e. there is a $\pi(\mathcal{A})$-invariant subspace $\mathcal{M}$ orthogonal to $J\mathcal{H}$ so that $\mathcal{M} \oplus J\mathcal{H}$ is also invariant) so that $J^*\pi(\cdot)J = \sigma(\cdot)$.

A dilation π of σ is *minimal* if the smallest reducing subspace containing $J\mathcal{H}$ is all of $\mathcal{K}$. This minimal dilation is called unique if for any two minimal dilations π_i on $\mathcal{K}_i$, there is a unitary operator U from $\mathcal{K}_1$ to $\mathcal{K}_2$ such that $J_2 = UJ_1$ and $\pi_2 = \operatorname{Ad} U \pi_1$.

Generally we are interested in dilations within the same class, such as row contractive representations of semigroups which are generated by two free families, or completely contractive representations of algebras. A representation σ within a certain class of representations is called *maximal* if every dilation π of σ has the form $\pi \simeq \sigma \oplus \pi'$, or equivalently $J\mathcal{H}$ always reduces π. It is possible for a dilation to be both minimal and maximal.

In his seminal paper [1], Arveson showed how to understand non-self-adjoint operator algebras in terms of dilation theory. He defined the C^*-envelope of an operator algebra $\mathcal{A}$ to be the unique C^*-algebra $C^*_{\text{env}}(\mathcal{A})$ containing a completely isometrically isomorphic copy of $\mathcal{A}$ which generates it, but any proper quotient is no longer completely isometric on $\mathcal{A}$. It was not shown that this object always exists, but that was later established by Hamana [16]. For background on C^*-envelopes, see Paulsen [22].

A completely contractive unital representation of an operator algebra $\mathcal{A} \subset C^*(\mathcal{A})$ has the *unique extension property* if there is a unique completely positive extension to $C^*(\mathcal{A})$ and this extension is a $*$-representation. If this $*$-representation is irreducible, it is called a *boundary representation*.

There is a new proof of the existence of the C^*-envelope. Dritschel and McCullough [11] showed that the C^*-envelope can be constructed by exhibiting sufficiently many representations with the unique extension property. Arveson [2] completed his original program by then showing that it suffices to use irreducible representations.

The insight of Dritschel and McCullough, based on ideas of Agler, was that the maximal completely contractive dilations coincide with dilations with the unique extension property. Therefore maximal dilations factor through the C^*-envelope. In particular, a maximal representation σ which is completely isometric yields the C^*-envelope: $C^*_{\text{env}}(\mathcal{A}) = C^*(\sigma(\mathcal{A}))$.

From a different viewpoint, this was also observed by Muhly and Solel [20]. They show that a completely contractive unital representation factors through the C^*-envelope if and only if it is orthogonally injective and orthogonally projective. While we do not define these notions here, we point out that it is easy to see that these two properties together are equivalent to being a maximal representation.

The upshot of the theory of C^*-envelopes and maximal dilations is the following consequence. Recall that a $*$-representation of $\mathbb{F}_u^+$ is a representation satisfying the unitary commutation relations which is isometric and defect free.

THEOREM 3.7. *Let σ be a unital representation $\mathbb{F}_u^+$ satisfying the unitary commutation relations. Then the following are equivalent:*

(i) *σ dilates to a $*$-representation of $\mathbb{F}_u^+$.*

(ii) *σ is completely contractive, that is, σ extends to a completely contractive representation of $\mathcal{A}_u$.*

In particular a unital representation of the semigroup $\mathbb{F}_\theta^+$ dilates to a $*$-representation if and only if it is completely contractive.

Proof. Suppose that σ dilates to a $*$-dilation π. By the definition of $\mathcal{O}_{X_u}$, π extends to a $*$-representation of $\mathcal{O}_{X_u}$. By Theorem 3.5, $\mathcal{A}_u$ sits inside $\mathcal{O}_{X_u}$ completely isometrically. As $*$-representations are completely contractive, it follows that π restricts to a completely contractive representation of $\mathcal{A}_u$. By compression to the original space, we see that σ is also completely contractive on $\mathcal{A}_u$.

Conversely, any completely contractive representation σ of $\mathcal{A}_u$ has a maximal dilation π. Thus it has the unique extension property, and so extends to a $*$-representation of $C^*_{\text{env}}(\mathcal{A}_u)$. By Theorem 3.5, $C^*_{\text{env}}(\mathcal{A}_u) = \mathcal{O}_{X_u}$. Therefore π restricts to a $*$-representation of $\mathbb{F}_u^+$. ∎

3.2. IDEALS OF THE C^*-ALGEBRA $C^*(\mathcal{A}_u)$. We shall show that $\mathcal{O}_{X_u}$ is a quotient of $C^*(\mathcal{A}_u)$. Indeed, there are several ideals that are evident:

$$\mathcal{K} := \Big\langle \Big(I - \sum_i \lambda(e_i)\lambda(e_i)^*\Big)\Big(I - \sum_j \lambda(f_j)\lambda(f_j)^*\Big) \Big\rangle,$$

$$\mathcal{I} := \Big\langle \Big(I - \sum_i \lambda(e_i)\lambda(e_i)^*\Big) \Big\rangle, \quad \mathcal{J} := \Big\langle \Big(I - \sum_j \lambda(f_j)\lambda(f_j)^*\Big) \Big\rangle,$$

$$\mathcal{I} + \mathcal{J} = \Big\langle \Big(I - \sum_i \lambda(e_i)\lambda(e_i)^*\Big), \Big(I - \sum_j \lambda(f_j)\lambda(f_j)^*\Big) \Big\rangle.$$

Note that the projections

$$P = I - \sum_i \lambda(e_i)\lambda(e_i)^* \quad \text{and} \quad Q = I - \sum_j \lambda(f_j)\lambda(f_j)^*$$

are the projections onto the subspaces

$$\sum_{l=0}^{\infty} \mathbb{C} \otimes F^{\otimes l} \quad \text{and} \quad \sum_{k=0}^{\infty} E^{\otimes k} \otimes \mathbb{C}$$

and $PQ = QP$ is the rank one projection $\xi_\varnothing \xi_\varnothing^*$. Note that $\lambda(e_u f_v)\xi_\varnothing \xi_\varnothing^* \lambda(e_s f_t)^*$ is the rank one operator $\xi_{e_u f_v}\xi_{e_s f_t}^*$ mapping basis element $\xi_{e_s f_t}$ to basis element $\xi_{e_u f_v}$. Thus a complete set of matrix units for $\mathcal{L}(\mathcal{H}_u)$ is available in $\mathcal{K}$, and so $\mathcal{K} = \mathfrak{K}$, the ideal of compact operators.

The projection P generates a copy of $\mathfrak{K}$ in $C^*(\{\lambda(e_i)\}) \simeq \mathcal{E}_m$, where the matrix units permute the subspaces

$$\xi_{e_u} \otimes \Big(\sum_{l=0}^{\infty} \mathbb{C} \otimes F^{\otimes l} \Big) = \operatorname{span}\{\xi_{e_u f_v} : f_v \in \mathbb{F}_n^+\}.$$

Also it is clear that $P\mathcal{A}_u P$ is a copy of $\mathcal{A}_n$, the noncommutative disk algebra generated by $f_1, \dots, f_n$ and so it generates a copy of the Cuntz–Toeplitz algebra $\mathcal{E}_n$ acting on $P\mathcal{H}_u$. It is now easy to see that $\mathcal{I}$ is $*$-isomorphic to $\mathfrak{K} \otimes \mathcal{E}_n$.

Similarly, $\mathcal{J}$ is isomorphic to $\mathcal{E}_m \otimes \mathfrak{K}$. The intersection of these two ideals is $\mathcal{I} \cap \mathcal{J} = \mathcal{K}$; and $\mathcal{K}$ is isomorphic to $\mathfrak{K} \otimes \mathfrak{K}$ sitting inside both $\mathcal{I}$ and $\mathcal{J}$. Then $\mathcal{I} + \mathcal{J}$ is also an ideal by elementary C^*-algebra theory.

LEMMA 3.8. *The quotient $C^*(\mathcal{A}_u)/(\mathcal{I}+\mathcal{J})$ is isomorphic to $\mathcal{O}_{X_u}$.*

Proof. The quotient $C^*(\mathcal{A}_u)/(\mathcal{I}+\mathcal{J})$ yields a representation of $\mathbb{F}_u^+$ as isometries. It is defect free by construction, and thus $C^*(\mathcal{A}_u)/(\mathcal{I}+\mathcal{J})$ is a quotient of $\mathcal{O}_{X_u}$. It is easy to see that the gauge automorphisms leave $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ invariant; and so $C^*(\mathcal{A}_u)/(\mathcal{I}+\mathcal{J})$ has a compatible family of gauge automorphisms. Thus the quotient is again isomorphic to $C^*(\mathbb{F}_u^+)$ as in the proof of Theorem 3.5. In particular, this quotient is completely isometric on $\mathcal{A}_u$. ∎

LEMMA 3.9. *The only proper ideals of $\mathcal{I} + \mathcal{J}$ are $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$.*

Proof. It is a standard result that if a C^*-algebra of operators acting on a Hilbert space contains $\mathfrak{K}$, then $\mathfrak{K}$ is the unique minimal ideal. So $\mathcal{K}$ is the unique minimal ideal of $C^*(\mathcal{A}_u)$.

Suppose that $\mathcal{M}$ is an ideal of $\mathcal{I} + \mathcal{J}$ properly containing $\mathcal{K}$. Then $\mathcal{M}/\mathcal{K}$ is an ideal of

$$(\mathcal{I}+\mathcal{J})/\mathcal{K} \simeq \mathcal{O}_m \otimes \mathfrak{K} \oplus \mathfrak{K} \otimes \mathcal{O}_n.$$

The two ideals $\mathcal{I}/\mathcal{K} \simeq \mathcal{O}_m \otimes \mathfrak{K}$ and $\mathcal{J}/\mathcal{K} \simeq \mathfrak{K} \otimes \mathcal{O}_n$ are mutually orthogonal and simple. So the ideal $\mathcal{M}/\mathcal{K}$ either contains one or the other or both. ∎

Kumjian and Pask define a notion called the *aperiodicity condition* for higher rank graphs. In our context, for the algebra $\mathcal{A}_\theta$ it means that there is an *irreducible* inductive representation. They show ([19], Proposition 4.8) that aperiodicity implies the simplicity of $C^*(\mathbb{F}_\theta^+)$. The converse is established by Robertson and Sims [28]. In [9], this is examined carefully. Aperiodicity seems to be typical, but there are periodic 2-graphs such as the flip algebra of Example 2.2.

When $C^*(\mathbb{F}_\theta^+)$ is simple, we have described the complete ideal structure of $C^*(\mathcal{A}_\theta)$. For the general case, see [9].

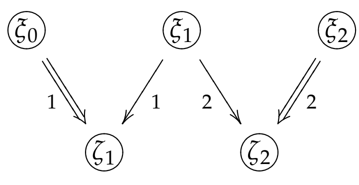
4. ROW CONTRACTIVE DILATIONS

Now we turn to dilation theory. We saw in Theorem 3.7 that maximal completely contractive representations of $\mathcal{A}_u$ correspond to the $*$-representations of $\mathbb{F}_u^+$. In the next section, we will show that defect free contractive representations of $\mathbb{F}_u^+$ are completely contractive, and therefore dilate to $*$-representations. Here we consider row contractive representations and give a simple proof of Solel's result that they dilate to row isometric representations. Despite such favourable dilation we give examples of contractive representations that contrast significantly with the defect free case. In particular we show that contractive representations of $\mathbb{F}_\theta^+$ need not be completely contractive.

EXAMPLE 4.1. Consider the flip graph of Examples 2.2 and 3.6. Define the representation of $\mathcal{A}_\theta$ on a basis $\xi_0, \xi_1, \xi_2, \zeta_1, \zeta_2$ given by

$$\pi(e_i) = \zeta_i \xi_1^*, \quad \pi(f_1) = \zeta_1 \xi_0^*, \quad \text{and} \quad \pi(f_2) = \zeta_2 \xi_2^*.$$

Note that π is row contractive.



However π *does not dilate to a defect free isometric representation.* To see this, suppose that π has a dilation σ that is isometric and defect free. The path from ξ_0

to ξ_2 is given by $\pi(f_2^* e_2 e_1^* f_1)$. However in any defect free dilation,

$$\begin{aligned}\sigma(f_2^* e_2 e_1^* f_1) &= \sigma(f_2^* e_2 e_1^* f_1)\sigma(e_1e_1^* + e_2e_2^*) = \sigma(f_2^* e_2 e_1^*(e_1 f_1 e_1^* + e_1 f_2 e_2^*)) \\ &= \sigma(f_2^* e_2(f_1e_1^* + f_2e_2^*)) = \sigma(f_2^* f_2(e_1e_1^* + e_2e_2^*)) = \sigma(1) = I.\end{aligned}$$

Hence $\xi_2 = \sigma(f_2^* e_2 e_1^* f_1)\xi_0 = \xi_0$, contrary to fact.

Next we show that π *is contractive on* $\mathcal{A}_\theta$. We need to show that $\|\pi(x)\| \leqslant \|\lambda(x)\|$ for $x \in \mathcal{A}_\theta$. Let

$$x = a + b_1e_1 + b_2e_2 + c_1f_1 + c_2f_2 + \text{higher order terms}.$$

Then

$$\pi(x) = \left[\begin{array}{ccc|cc} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ \hline c_1 & b_1 & 0 & a & 0 \\ 0 & b_2 & c_2 & 0 & a \end{array}\right] = \begin{bmatrix} aI_3 & 0 \\ X & aI_2 \end{bmatrix}.$$

Now the 5×5 corner of $\lambda(x)$ on $\operatorname{span}\{\xi_\varnothing, \xi_{e_1}, \xi_{e_2}, \xi_{f_1}, \xi_{f_2}\}$ has the form

$$\left[\begin{array}{c|cccc} a & 0 & 0 & 0 & 0 \\ \hline b_1 & a & 0 & 0 & 0 \\ b_2 & 0 & a & 0 & 0 \\ c_1 & 0 & 0 & a & 0 \\ c_2 & 0 & 0 & 0 & a \end{array}\right] = \begin{bmatrix} a & 0 \\ y & aI_4 \end{bmatrix}.$$

Note that $\|X\| \leqslant \|X\|_2 = \|y\|_2$. So

$$\|\pi(x)\| \leqslant \left\| \begin{bmatrix} |a| & 0 \\ \|X\| & |a| \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} |a| & 0 \\ \|y\|_2 & |a| \end{bmatrix} \right\| \leqslant \|\lambda(x)\|.$$

Nevertheless, we show that π *is not completely contractive.* Let $B_1 = B_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_1 = -C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$; and consider the matrix polynomial $X = B_1e_1 + B_2e_2 + C_1f_1 + C_2f_2$. Then

$$\|\pi(X)\| = \left\| \left[\begin{array}{cc|cc|cc} 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \right\| = \left\| \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right\| = \sqrt{3}.$$

By Example 3.6, the C^*-envelope of $\mathcal{A}_\theta$ is $\mathcal{O}_2 \otimes \mathrm{C}(\mathbb{T})$. As shown there, an irreducible representation σ is determined by its restriction to $\mathrm{C}^*(e_1, e_2)$ and a scalar $t \in \mathbb{T}$ so that $\sigma(f_i) = t\sigma(e_i)$. Since $C^*(e_1, e_2) \simeq \mathcal{O}_2$ is simple, it does not matter which representation is used, as all are faithful. Let $S_i = \sigma(e_i)$ be Cuntz isometries. Then the norm $\lambda(X)$ is determined as the supremum over $t \in \mathbb{T}$ of these representations.

$$\begin{aligned}\|\lambda(X)\| &= \sup_{t\in\mathbb{T}} \|(B_1 + tC_1) \otimes S_1 + (B_2 + tC_2) \otimes S_2\| \\ &= \sup_{t\in\mathbb{T}} \left\| \begin{bmatrix} B_1 + tC_1 \\ B_2 + tC_2 \end{bmatrix} \right\| = \sup_{t\in\mathbb{T}} \left\| \begin{bmatrix} 1 & t \\ 1 & -t \end{bmatrix} \right\| = \sqrt{2}.\end{aligned}$$

An alternative proof is obtained by noting that by Theorem 3.7, if π were completely contractive on $\mathcal{A}_\theta$, then one could dilate it to a $*$-representation of $\mathbb{F}_\theta^+$, which was already shown to be impossible.

A related example shows that a row contractive representation may not even be contractive.

EXAMPLE 4.2. Take any $\mathbb{F}_\theta^+$ for which there are indices i_0 and j_0 so that there is no solution to $e_{i_0} f_j = f_{j_0} e_i$. The flip graph is such an example, with $i_0 = 1$ and $j_0 = 2$. Consider the two dimensional representation π of $\mathbb{F}_\theta^+$ on $\mathbb{C}^2$ with basis $\{\xi_1, \xi_2\}$ given by

$$\pi(e_{i_0}) = \pi(f_{j_0}) = \xi_2 \xi_1^* \quad \text{and} \quad \pi(e_i) = \pi(f_j) = 0 \text{ otherwise.}$$

The product $\pi(e_i f_j) = 0$ for all i, j; so this is a representation. Evidently it is row contractive.

$$\begin{tikzcd} \xi_1 \arrow[d, bend right, "i_0"'] \arrow[d, Rightarrow, bend left, "j_0"] \\ \xi_2 \end{tikzcd}$$

However $\pi(e_{i_0} + f_{j_0}) = 2\xi_2 \xi_1^*$ has norm 2. The hypothesis guarantees that no word beginning with e_{i_0} coincides with any word beginning with f_{j_0}. Thus in the left regular representation, $\lambda(e_{i_0})$ and $\lambda(f_{j_0})$ are isometries with orthogonal ranges. Hence $\|\lambda(e_{i_0} + f_{j_0})\| = \sqrt{2}$.

So this row contractive representation does not extend to a contractive representation of $\mathcal{A}_\theta$.

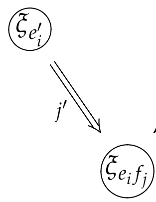
Another problem with dilating row contractive representations is that the minimal row isometric dilation need not be unique. Consider the following illustrations.

EXAMPLE 4.3. Let π be the 2-dimensional trivial representation of $\mathbb{F}_\theta^+$, $\pi(\varnothing) = I_2$ and $\pi(w) = 0$ for $w \neq \varnothing$. Evidently this dilates to the row isometric representation $\lambda \oplus \lambda$; and this is clearly minimal.

Now pick any i, j and factor $e_i f_j = f_{j'} e_{i'}$. Inside of the left regular representation, identify $\mathbb{C}^2$ with $\mathcal{M}_0 := \operatorname{span}\{\xi_{e_{i'}}, \xi_{f_j}\}$. Note that the compression of λ to $\mathcal{M}_0$ is unitarily equivalent to π. The invariant subspace that $\mathcal{M}_0$ determines is $\mathcal{M} = \overline{\mathcal{A}_\theta \xi_{e_{i'}} + \mathcal{A}_\theta \xi_{f_j}}$. The restriction σ of λ to $\mathcal{M}$ is therefore a minimal row isometric dilation of π. However

$$\sigma(e_i)\xi_{f_j} = \xi_{e_i f_j} = \xi_{f_{j'} e_{i'}} = \sigma(f_{j'})\xi_{e_{i'}}.$$

For any non-zero vector $\zeta = a\xi_{e_{i'}} + b\xi_{f_j}$ in $\mathcal{M}_0$, either $\sigma(e_i)^*\sigma(f_{j'})\zeta = a\xi_{f_j}$ or ζ itself is a non-zero multiple of ξ_{f_j}; and similarly $\xi_{e_{i'}}$ belongs to the reducing subspace containing ζ. Therefore σ is irreducible.



So these two minimal row isometric dilations are not unitarily equivalent.

EXAMPLE 4.4. Here is another example where the original representation is irreducible. Consider $\mathbb{F}_\theta^+$ where $m = 2$, $n = 3$ and the permutation θ has cycles

$$\big((1,2),(2,1)\big) \quad \text{and} \quad \big((2,2),(2,3),(1,3)\big).$$

Let π be the representation on $\mathbb{C}^3$ with basis $\zeta_1, \zeta_2, \zeta_3$ given by

$$\pi(e_1) = \zeta_3\zeta_1^* \quad \text{and} \quad \pi(f_1) = \zeta_3\zeta_2^*$$

and all other generators are sent to 0. We show that this may be dilated to a subrepresentation of λ in two different ways.

First identify ζ_1 with ξ_{f_1}, ζ_2 with ξ_{e_1} and ζ_3 with $\xi_{e_1 f_1} = \xi_{f_1 e_1}$. Then a minimal row isometric dilation is obtained by $\sigma_1 = \lambda|_{\mathcal{M}_1}$ where $\mathcal{M}_1 = \overline{\mathcal{A}_\theta \xi_{e_1} + \mathcal{A}_\theta \xi_{f_1}}$. A second dilation is obtained from the identification of ζ_1 with ξ_{f_2}, ζ_2 with ξ_{e_2} and ζ_3 with $\xi_{e_1 f_2} = \xi_{f_1 e_2}$. Then $\sigma_2 = \lambda|_{\mathcal{M}_2}$ where $\mathcal{M}_2 = \overline{\mathcal{A}_\theta \xi_{e_2} + \mathcal{A}_\theta \xi_{f_2}}$.

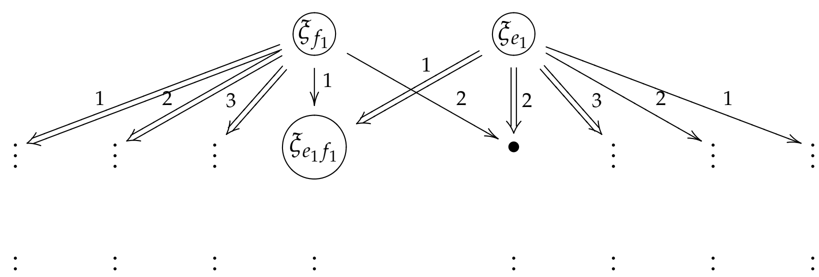
These two dilations are different because

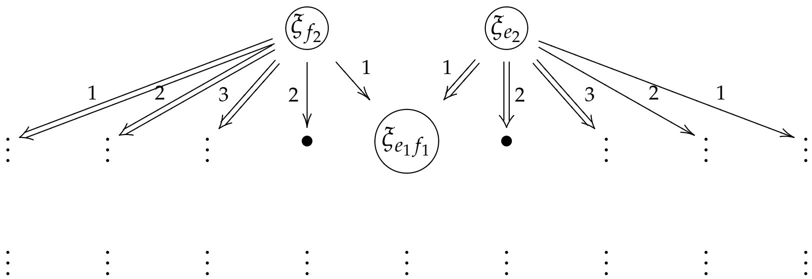
$$\sigma_1(e_2)\xi_{f_1} = \xi_{e_2 f_1} = \xi_{f_2 e_1} = \sigma_1(f_2)\xi_{e_1}$$

while

$$\sigma_2(e_2)\xi_{f_2} = \xi_{e_2 f_2} \neq \xi_{f_2 e_2} = \sigma_2(f_2)\xi_{e_2}.$$

So the two dilations are not equivalent.





With these examples as a caveat, we provide a simple proof of Solel's result ([29], Corollary 4.5). Our proof is based on the much more elementary result of Frahzo [15], Bunce [3] and Popescu [23] that every contractive n-tuple has a unique minimal dilation to a row isometry.

First we recall some details of Bunce's proof. Consider a row contraction $A = \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}$. Following Schaeffer's proof of Sz. Nagy's isometric dilation theorem, let $D_A = (I_{\mathbb{C}^m \otimes \mathcal{H}} - A^*A)^{1/2}$. Observe that $\begin{bmatrix} A \\ D_A \end{bmatrix}$ is an isometry. Hence the columns $\begin{bmatrix} A_i \\ D_A^{(i)} \end{bmatrix}$ are isometries with pairwise orthogonal ranges in $\mathcal{B}(\mathcal{H}, \mathcal{H} \oplus (\mathcal{V} \otimes \mathcal{H}))$ where $\mathcal{V} = \mathbb{C}^m$. Now consider $\mathcal{K} = \mathcal{V} \otimes \mathcal{H} \otimes \ell^2(\mathbb{F}_m^+)$ where we identify $\mathcal{V} \otimes \mathcal{H}$ with $\mathcal{V} \otimes \mathcal{H} \otimes \mathbb{C}\xi_\varnothing$ inside $\mathcal{K}$. Let λ be the left regular representation of $\mathbb{F}_m^+$ on $\ell^2(\mathbb{F}_m^+)$, and set $L_i = \lambda(e_i)$. Define isometries on $\mathcal{H} \oplus \mathcal{K}$ by $S_i = \begin{bmatrix} A_i & 0 \\ \begin{bmatrix} D_A^{(i)} \\ 0 \end{bmatrix} & I_{\mathcal{V}\otimes\mathcal{H}} \otimes L_i \end{bmatrix}$. These isometries have the desired properties except minimality. One can then restrict to the invariant subspace $\mathcal{M}$ generated by $\mathcal{H}$. Popescu establishes the uniqueness of this minimal dilation in much the same way as for the classical case.

LEMMA 4.5. *Let $S = \begin{bmatrix} S_1 & \cdots & S_m \end{bmatrix}$ be a row isometry, where each S_i is an isometry in $\in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ that leaves $\mathcal{K}$ invariant. Suppose that there is a Hilbert space $\mathcal{W}$ so that $\mathcal{K} \simeq \mathcal{W} \otimes \ell^2(\mathbb{F}_m^+)$ and $S_i|_{\mathcal{K}} \simeq I_{\mathcal{W}} \otimes L_i$ for $1 \leqslant i \leqslant m$. Let $\mathcal{M}$ be the smallest invariant subspace for $\{S_i\}$ containing $\mathcal{H}$. Then $\mathcal{M}$ reduces $\{S_i\}$ and there is a subspace $\mathcal{W}_0 \subset \mathcal{W}$ so that $\mathcal{M}^\perp \simeq \mathcal{W}_0 \otimes \ell^2(\mathbb{F}_m^+)$.*

Proof. Clearly $\mathcal{M} = \bigvee_{w \in \mathbb{F}_m^+} S_w\mathcal{H}$. For any non-trivial word $w = iw'$ in $\mathbb{F}_m^+$, $S_j^* S_w \mathcal{H} = \delta_{ij} S_{w'}\mathcal{H}$; and $S_j^*\mathcal{H} \subset \mathcal{H}$ because $\mathcal{K} = \mathcal{H}^\perp$ is invariant for S_j. So $\mathcal{M}$ reduces each S_j.

Thus $\mathcal{M}^\perp \subset \mathcal{K} \simeq \mathcal{W} \otimes \ell^2(\mathbb{F}_m^+)$ reduces each $S_i|_{\mathcal{K}} \simeq I_{\mathcal{W}} \otimes L_i$. But $W^*(L_1, \dots, L_m) = \mathcal{B}(\ell^2(\mathbb{F}_m^+))$ because $C^*(L_1, \dots, L_m)$ contains the compact operators. Hence $W^*(\{S_i|_{\mathcal{K}}\}) \simeq \mathbb{C}I_{\mathcal{W}} \otimes \mathcal{B}(\ell^2(\mathbb{F}_m^+))$. Therefore a reducing subspace is equivalent to one of the form $\mathcal{W}_0 \otimes \ell^2(\mathbb{F}_m^+)$. ∎

THEOREM 4.6 (Solel). *Let σ be a row contractive representation of $\mathbb{F}_u^+$ on $\mathcal{H}$. Then σ has a dilation to a row isometric representation π on a Hilbert space $\mathcal{H} \oplus \mathcal{K}$.*

Proof. Start with a Hilbert space $\mathcal{W} = \mathcal{V} \otimes \mathcal{H}$, where $\mathcal{V}$ is a separable, infinite dimensional Hilbert space, and set $\mathcal{K} = \mathcal{W} \otimes \mathcal{H}_u$. Let λ denote the left regular representation of $\mathbb{F}_u^+$ on $\mathcal{H}_u$. Note that the restriction to $\mathbb{F}_m^+ = \langle e_1, \dots, e_m \rangle$ yields a multiple of the left regular representation of $\mathbb{F}_m^+$.

Following Bunce's argument, set $A_i = \sigma(e_i)$ and define isometries on $\mathcal{H} \oplus \mathcal{K}$ by $S_i = \begin{bmatrix} A_i & 0 \\ \begin{bmatrix} D_A^{(i)} \\ 0 \end{bmatrix} & I_{\mathcal{V}\otimes\mathcal{H}} \otimes \lambda(e_i) \end{bmatrix}$. However, note that the increased size of $\mathcal{V}$ means that the m element column $D_A^{(i)}$ must be extended by zeros even within the subspace $\mathcal{W} \otimes \mathbb{C}\xi_\varnothing$. Thus there is always a subspace orthogonal to the minimal invariant subspace $\mathcal{M}$ containing $\mathcal{H}$ on which S_i acts like a multiple of the left regular representation with multiplicity at least $\max\{\aleph_0, \dim \mathcal{H}\}$.

Similarly, set $B_j = \sigma(f_j)$ for $1 \leqslant j \leqslant n$, and define the defect operator $D_B = (I_{\mathbb{C}^n\otimes\mathcal{H}} - B^*B)^{1/2}$. Then define isometries on $\mathcal{H} \oplus \mathcal{K}$ by

$$T_j = \begin{bmatrix} B_j & 0 \\ \begin{bmatrix} D_B^{(j)} \\ 0 \end{bmatrix} & I_{\mathcal{V}\otimes\mathcal{H}} \otimes \lambda(f_j) \end{bmatrix}.$$

Now notice that in $\mathbb{C}[\mathbb{F}_u^+]$ the semigroup generated by $e_1f_1, \dots, e_mf_n$ is the free semigroup $\mathbb{F}_{mn}^+$. Indeed, if $e_if_jw = e_kf_lw'$, then by cancellation, it follows that $i = k$, $j = l$ and $w = w'$. So, with successive cancellation, the alternating products e_if_jw, e_kf_lw' are equal in $\mathbb{C}[\mathbb{F}_u^+]$ only if they are identical.

We will consider two row isometric representations of $\mathbb{F}_{mn}^+$:

$$\pi_1(e_if_j) = S_iT_j \quad \text{and} \quad \pi_2(e_if_j) = \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} T_{j'} S_{i'}$$

for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. The reason that π_2 has the desired properties is that a p-tuple of isometries with orthogonal ranges spans a subspace isometric to a Hilbert space consisting of scalar multiples of isometries. So the fact that u is a unitary matrix ensures that the mn operators $\pi_2(e_if_j)$ are indeed isometries with orthogonal ranges.

Since σ is a representation of $\mathbb{F}_u^+$, we see that π_1 and π_2 both compress to σ on $\mathcal{H}$. So both are dilations of the same row contractive representation of $\mathbb{F}_{mn}^+$. By Lemma 4.5, for both $k = 1, 2$, we have $\pi_k(e_if_j) \simeq \mu(e_if_j) \oplus (I_{\mathcal{W}_k} \otimes \lambda(e_if_j))$ where μ is the minimal row isometric dilation of $\sigma|\mathbb{F}_{mn}^+$ and $\dim \mathcal{W}_k = \max\{\aleph_0, \dim \mathcal{H}\}$. The two minimal dilations are unitarily equivalent via a unitary which is the identity on $\mathcal{H}$, and the multiples of the left regular representation are also unitarily equivalent. So π_1 and π_2 are unitarily equivalent on $\mathcal{H} \oplus \mathcal{K}$ via a unitary W

which fixes $\mathcal{H}$, i.e.

$$\pi_2(e_i f_j) = W\pi_1(e_i f_j)W^* \quad \text{for all } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n.$$

Now set

$$\pi(e_i) = S_i W \quad \text{and} \quad \pi(f_j) = W^* T_j \quad \text{for } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n.$$

This provides a row isometric dilation of $\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$ and $\begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$. Moreover,

$$\begin{aligned}\pi(e_i)\pi(f_j) &= S_i W W^* T_j = S_i T_j = \pi_1(e_i f_j) \\ &= W^* \pi_2(e_i f_j) W = \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} W^* T_{j'} S_{i'} W \\ &= \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} \pi(f_{j'}) \pi(e_{i'}).\end{aligned}$$

So π yields a representation of $\mathbb{F}_u^+$. ∎

We remark that the case $n = 1$ of Solel's theorem was obtained earlier by Popescu [24]; and the special case of this for commutant lifting is due to Muhly and Solel [21].

Our discussion of the flip algebra in Examples 4.1 and 4.2 show that a row contractive representation of the algebra $\mathcal{A}_u$ need not be contractive. As a consequence, the natural map $\mathcal{T}_+(X_u) \to \mathcal{A}_u$ from the tensor algebra is not isometric.

In fact in this case, using results from [26], we can show that there is no map which is an isometric isomorphism. Firstly, note that the explicit unitary automorphisms of $\mathcal{A}_u$ given there may be readily defined on the tensor algebra. Secondly, the character space $M(\mathcal{A}_u)$ of $\mathcal{A}_u$ and its core subset (which is definable in terms of nest representations) identify with the character space and core of $\mathcal{T}_+(X_u)$. Suppose now that $\Gamma : \mathcal{A}_u \to \mathcal{T}_+(X_u)$ is an isometric isomorphism. Composing with an appropriate automorphism of $\mathcal{T}_+(X_u)$, we may assume that the induced character space map γ maps the origin to the origin (in the realisation of $M(\mathcal{A}_u)$ in $\mathbb{C}^{n+m}$ [18]). By the generalized Schwarz inequality in [26], it follows that the biholomorphic map γ is simply a rotation automorphism, defined by a pair of unitaries $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$. Composing Γ with the inverse of the associated gauge automorphism $\pi_{A,B}$ of $\mathcal{T}_+(X_u)$, we may assume that γ is the identity map. Since Γ is isometric it follows, as in [26], that Γ is the natural map, which is a contradiction.

5. DILATION OF DEFECT FREE REPRESENTATIONS

We now show the distinctiveness of the defect free contractive representations in that they are completely contractive and have *unique* minimal $*$-dilations. Moreover, we show that atomic contractive defect free representations of $\mathbb{F}_\theta^+$ have unique minimal atomic representations. This is an essential tool for the representation theory of 2-graph semigroups developed in [8] because we frequently describe $*$-representations by their restriction to a cyclic coinvariant subspace.

THEOREM 5.1. *Let σ be a defect free, row contractive representation of $\mathbb{F}_u^+$. Then σ has a unique minimal $*$-dilation.*

The proof follows from Theorem 4.6 and the next two lemmas and the fact that a defect free row isometric representation is a $*$-dilation.

LEMMA 5.2. *Let σ be a defect free, row contractive representation. Then any minimal row isometric dilation is defect free.*

Proof. Let π be a minimal row isometric dilation acting on $\mathcal{K}$. Set $\mathcal{M} = \Big(I - \sum\limits_i \pi(e_i)\pi(e_i)^*\Big)\mathcal{K}$. We first show that $\mathcal{M}$ is coinvariant. Indeed, if $x \in \mathcal{M}$ and $y \in \mathcal{K}$, then plainly

$$\langle \pi(e_i)^* x, \pi(e_k)y\rangle = \langle x, \pi(e_i)\pi(e_k)y\rangle = 0$$

for each i and k, while, using the commutation relations,

$$\langle \pi(f_l)^* x, \pi(e_k)y\rangle = \langle x, \pi(f_l e_k)y\rangle = \sum_{i=1}^m \sum_{j=1}^n u_{(i,j),(k,l)} \langle x, \pi(e_i)\pi(f_j)y\rangle = 0.$$

So $\sigma(w)^* x$ belongs to $\mathcal{M} = \Big(\sum\limits_i \pi(e_i)\mathcal{K}\Big)^\perp$ for any word w.

If we write each $\pi(e_i)$ as a matrix with respect to $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, we have $\pi(e_i) = \begin{bmatrix} \sigma(e_i) & 0 \\ * & * \end{bmatrix}$. Therefore

$$\sum_i \pi(e_i)\pi(e_i)^* = \begin{bmatrix} \sum\limits_i \sigma(e_i)\sigma(e_i)^* & * \\ * & * \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & * \\ * & * \end{bmatrix}.$$

This is a projection, and thus

$$\sum_i \pi(e_i)\pi(e_i)^* = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & * \end{bmatrix} \geqslant P_{\mathcal{H}}.$$

Thus $\mathcal{M}$ is orthogonal to $\mathcal{H}$. It now follows that for any $x \in \mathcal{M}$, $h \in \mathcal{H}$ and $w \in \mathbb{F}_u^+$,

$$\langle \pi(w)h, x\rangle = \langle h, \pi(w)^* x\rangle = 0$$

because $\pi(w)^* x \in \mathcal{M}$. But the vectors of the form $\pi(w)h$ span $\mathcal{K}$, and therefore $\mathcal{M} = \{0\}$. ∎

An immediate consequence of this lemma and Theorem 3.7 is:

COROLLARY 5.3. *Every defect free, row contractive representation π of $\mathbb{F}_u^+$ extends to a completely contractive representation of $\mathcal{A}_u$.*

LEMMA 5.4. *The minimal row isometric dilation of a defect free, row contractive representation of $\mathbb{F}_u^+$ is unique up to a unitary equivalence that fixes the original space.*

Proof. Let π be a minimal row isometric dilation of σ on the Hilbert space $\mathcal{K}$. Let $\mathcal{W}$ be the set of words $w = e_u f_v$ in $\mathbb{F}_u^+$. By minimality and the commutation relations, a dense set in $\mathcal{K}$ is given by the vectors of the form $\sum_k \pi(w_k)h_k$ where this is a finite sum, each $h_k \in \mathcal{H}$ and $w_k \in \mathcal{W}$. We first show that given any two such vectors, $\sum_k \pi(w_k)h_k$ and $\sum_l \pi(w_l')h_l'$, we may suppose that each w_k and w_k' has the same degree.

To this end, let $d(w_k) = (m_k, n_k)$ and $d(w_l') = (m_l', n_l')$, and set

$$m_0 = \max\{m_k, m_l'\} \quad \text{and} \quad n_0 = \max\{n_k, n_l'\}.$$

For each w_k, let $a_k = m_0 - m_k$ and $b_k = n_0 - n_k$. Then because π is defect free by Lemma 5.2,

$$\pi(w_k)h_k = \pi(w_k)\Big(\sum_{d(v)=(a_k,b_k)} \pi(v)\pi(v)^* \Big) h_k = \sum_{d(v)=(a_k,b_k)} \pi(w_k v)(\sigma(v)^* h_k).$$

The second line follows because $\mathcal{H}$ is coinvariant for $\pi(\mathbb{F}_u^+)$; and consequently, $\pi(v)^* h_k = \sigma(v)^* h_k$ belongs to $\mathcal{H}$. Using the commutation relations we may write the original sum with new terms, each of which has degree (m_0, n_0). Combine terms if necessary so that the words w_k are distinct. Then we obtain a sum of the form $\sum_{d(w)=(m_0,n_0)} \pi(w)h_w$. We similarly rewrite

$$\sum_l \pi(w_l')h_l' = \sum_{d(w)=(m_0,n_0)} \pi(w)h_w'.$$

Now the isometries $\pi(w)$ for distinct words of degree (m_0, n_0) have pairwise orthogonal ranges. Therefore we compute

$$\begin{aligned} \Big\langle \sum_{d(w)=(m_0,n_0)} \pi(w)h_w, \sum_{d(w)=(m_0,n_0)} \pi(w)h_w' \Big\rangle &= \sum_{d(w)=(m_0,n_0)} \langle \pi(w)h_w, \pi(w)h_w' \rangle \\ &= \sum_{d(w)=(m_0,n_0)} \langle h_w, h_w' \rangle. \end{aligned}$$

Now suppose that π' is another minimal row isometric dilation of σ on a Hilbert space $\mathcal{K}'$. The same computation is valid for it. Thus we may define a map from the dense subspace $\operatorname{span}\{\pi(\mathbb{F}_u^+)\mathcal{H}\}$ of $\mathcal{K}$ to the dense subspace $\operatorname{span}\{\pi'(\mathbb{F}_u^+)\mathcal{H}\}$ of $\mathcal{K}'$ by

$$U \sum_{d(w)=(m_0,n_0)} \pi(w)h_w = \sum_{d(w)=(m_0,n_0)} \pi'(w)h_w.$$

The calculation of the previous paragraph shows that U preserves inner products, and thus is well defined and isometric. Hence it extends by continuity to a unitary operator of $\mathcal{K}$ onto $\mathcal{K}'$. Moreover, each vector in h has the form $h = \pi(\varnothing)h$; and thus $Uh = h$. That is, U fixes the subspace $\mathcal{H}$. Finally, it is evident from its definition that $\pi'(w) = U\pi(w)U^*$ for all $w \in \mathbb{F}_u^+$. So π' is equivalent to π. ∎

For our applications in [8], we need the following refinement for atomic representations.

THEOREM 5.5. *If σ is an atomic defect free partially isometric representation of $\mathbb{F}_\theta^+$, then the unique minimal $*$-dilation π is also atomic.*

This follows from the theorem above and the next lemma.

LEMMA 5.6. *Let σ be an atomic, defect free, partially isometric representation of $\mathbb{F}_\theta^+$. Then any minimal $*$-dilation of σ is atomic.*

Proof. Let π be a minimal row isometric dilation of σ acting on $\mathcal{K}$. Consider the standard basis $\{\xi_k : k \geqslant 1\}$ for $\mathcal{H}$ with respect to which σ is atomic. Let $\dot{\xi}_k$ denote $\mathbb{C}^*\xi = \{\alpha\xi_k : \alpha \in \mathbb{C}\backslash\{0\}\}$. We claim that the set $\{\pi(x)\dot{\xi}_k : k \geqslant 1, x \in \mathbb{F}_\theta^+\}$ forms an orthonormal family of 1-dimensional subsets spanning $\mathcal{K}$, with repetitions. Indeed, $\mathcal{H}$ is coinvariant and cyclic; so these sets span $\mathcal{K}$. It suffices to show that any two such sets, say $\pi(x_1)\dot{\xi}_1$ and $\pi(x_2)\dot{\xi}_2$, either coincide or are orthogonal.

Let $d(x_k) = (m_k, n_k)$ for $k = 1, 2$; and set

$$(m_0, n_0) = (m_1, n_1) \vee (m_2, n_2) = (\max\{m_1, m_2\}, \max\{n_1, n_2\}).$$

Since σ is defect free, there are unique basis vectors ζ_k and words y_k with $d(y_k) = (m_0 - m_k, n_0 - n_k)$ so that $\sigma(y_k)\dot{\zeta}_k = \dot{\xi}_k$. Thus using $\dot{\zeta}_k$ and the word $x_k y_k$, we may suppose that the two words have the same degree. For convenience of notation, we suppose that this has already been done.

Write $x_k = e_{u_k} f_{v_k}$. As noted in the proof of Theorem 4.6, two distinct words of the same degree have pairwise orthogonal ranges. Thus if $x_1 \neq x_2$, then $\pi(x_1)\dot{\xi}_1$ and $\pi(x_2)\dot{\xi}_2$ are orthogonal. On the other hand, if $x_1 = x_2$, then if $\dot{\xi}_1 = \dot{\xi}_2$, the images are equal; while if $\dot{\xi}_1$ and $\dot{\xi}_2$ are orthogonal, they remain orthogonal under the action of the isometry $\pi(x_1)$. ∎

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