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ADJOINTABILITY OF DENSELY DEFINED CLOSED OPERATORS AND THE MAGAJNA–SCHWEIZER THEOREM

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ABSTRACT. In this note unbounded regular operators on Hilbert C^*-modules over arbitrary C^*-algebras are discussed. A densely defined operator t possesses an adjoint operator if the graph of t is an orthogonal summand. Moreover, for a densely defined operator t the graph of t is orthogonally complemented and the range of $P_F P_{G(t)^\perp}$ is dense in its biorthogonal complement if and only if t is regular. For a given C^*-algebra $\mathcal{A}$ any densely defined $\mathcal{A}$-linear closed operator t between Hilbert C^*-modules is regular, if and only if any densely defined $\mathcal{A}$-linear closed operator t between Hilbert C^*-modules admits a densely defined adjoint operator, if and only if $\mathcal{A}$ is a C^*-algebra of compact operators. Some further characterizations of closed and regular modular operators are obtained.

KEYWORDS: *Hilbert C^*-modules, unbounded operators, regular operators, C^*-algebras of compact operators*

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INTRODUCTION

Hilbert C^*-modules are an often used tool in the study of locally compact quantum groups and their representations, in noncommutative geometry, in KK-theory, and in the study of completely positive maps between C^*-algebras.

A (left) *pre-Hilbert C^*-module* over a (not necessarily unital) C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$, which is $\mathcal{A}$-linear in the first variable and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geqslant 0 \text{ with equality if and only if } x = 0.$$

We always suppose that the linear structures of $\mathcal{A}$ and E are compatible.

A pre-Hilbert $\mathcal{A}$-module E is called a *Hilbert $\mathcal{A}$-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. If E, F are two Hilbert $\mathcal{A}$-modules then the set of all ordered pairs of elements $E \oplus F$ from E and F is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E$

$+\langle y_1, y_2\rangle_F$. It is called the direct *orthogonal sum of E and F*. A Hilbert $\mathcal{A}$-submodule of a Hilbert $\mathcal{A}$-module F is a direct orthogonal summand if E together with its orthogonal complement $E^\perp$ in F gives rise to an $\mathcal{A}$-linear isometric isomorphism of $E \oplus E^\perp$ and F. Some interesting results about orthogonally complemented submodules can be found in [4], [5], [15]. For the basic theory of Hilbert C^*-modules we refer to the book by E.C. Lance [14] and to respective chapters in the monographic publications [7], [18], [20].

As a convention, throughout the present paper we assume $\mathcal{A}$ to be an arbitrary C^*-algebra (i.e. not necessarily unital). Since we deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by lower case letters. We use the denotations $\mathrm{Dom}(\cdot)$, $\mathrm{Ker}(\cdot)$ and $\mathrm{Ran}(\cdot)$ for domain, kernel and range of operators, respectively. If $E,\ F$ are Hilbert $\mathcal{A}$-modules and W is an orthogonal summand in $E \oplus F$, P_W denotes the orthogonal projection of $E \oplus F$ onto W and P_E and P_F denote the canonical projections onto the first and second factors of $E \oplus F$.

Suppose $E,\ F$ are Hilbert $\mathcal{A}$-modules. We denote the set of all $\mathcal{A}$-linear maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that the equality

$$\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E \tag{0.1}$$

holds for any $x \in E,\ y \in F$ by $B(E,F)$. The operator T^* is called the *adjoint operator* of T. The existence of an adjoint operator T^* for some $\mathcal{A}$-linear operator $T : E \to F$ implies that each adjointable operator is necessarily bounded and $\mathcal{A}$-linear in the sense $T(ax) = aT(x)$ for any $a \in \mathcal{A}$, $x \in E$, and similarly for T^*. The reason for this is the requirement that the equality (0.1) is supposed to hold for any elements of E and F, so T has E as its domain.

In general, bounded $\mathcal{A}$-linear operators may fail to possess an adjoint operator, however, if E is a full Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, then it is known that each bounded $\mathcal{A}$-linear operator on E possesses an adjoint operator if and only if E is *orthogonally comparable*, i.e. whenever E appears as a Hilbert $\mathcal{A}$-submodule of another Hilbert $\mathcal{A}$-module F then E is an orthogonal direct summand of F (cf. Theorem 6.3 of [5]).

In several contexts where Hilbert C^*-modules arise, one also needs to study "unbounded adjointable operators", or what are now known as regular operators. These were first introduced by Baaj and Julg in [2] where they gave an interesting construction of Kasparov bimodules in KK-theory using regular operators. Later regular operators were reconsidered by Woronowicz in [21], while investigating noncompact quantum groups. The functional calculus of regular operators and the Fuglede–Putnam theorem for Hilbert C^*-modules were explained by Kustermans in [13]. Beside these works Kucerovsky gave a new approach to functional calculus of regular operators in [11], [12]. Also, Lance gave a brief indication in his book [14] about Hilbert modules and regular operators on them.

Modifying the defining equality (0.1) of adjointability for unbounded $\mathcal{A}$-linear operators $t : \mathrm{Dom}(t) \subseteq E \to \mathrm{Ran}(t) \subseteq F$ between Hilbert $\mathcal{A}$-modules E and F, the operator t is said to be adjointable if there exists another $\mathcal{A}$-linear operator $t^* : \mathrm{Dom}(t^*) \subseteq F \to \mathrm{Ran}(t^*) \subseteq E$ such that the equality

$$\langle tx, y\rangle_F = \langle x, t^*y\rangle_E \tag{0.2}$$

holds for all $x \in \mathrm{Dom}(t)$, $y \in \mathrm{Dom}(t^*)$. Despite the good properties of unbounded operators on Hilbert spaces adjointable unbounded operators on Hilbert C^*-modules may lack some good properties that are wanted in applications. So the notion of *regular operators* was introduced to provide a tractable class of unbounded C^*-linear densely defined closed operators on Hilbert C^*-modules. An operator t from a Hilbert $\mathcal{A}$-module E to another Hilbert $\mathcal{A}$-module F is said to be *regular* if

(i) t is closed and densely defined with domain $\mathrm{Dom}(t)$,
(ii) its adjoint t^* is also densely defined, and
(iii) the range of $1 + t^*t$ is dense in E.

Note that as we set $\mathcal{A} = \mathbb{C}$ i.e. if we take E, F to be a Hilbert space, then this is exactly the definition of a densely defined closed operator, except that in that case, both the second and the third condition follow from the first one. In [17] Pal considered a larger class of operators, semiregular operators, which are densely defined closable operators whose adjoints are densely defined. He proved that every closed semiregular operator (i.e. an operator that satisfies the first two conditions above) on Hilbert C^*-modules over commutative C^*-algebras as well as over subalgebras of C^*-algebras of compact operators is regular ([17], Proposition 4.1, Theorem 5.8). He also gave an example of a closed semiregular nonregular operator, and showed that regularity of its adjoint does not ensure regularity of the original operator ([17], Propositions 2.2 and 2.3).

In the present paper we prove that a densely defined operator t from a Hilbert $\mathcal{A}$-module E to another Hilbert $\mathcal{A}$-module F possesses a densely defined adjoint operator from F to E if the graph of t is orthogonally complemented in $E \oplus F$ and the range of $P_F P_{G(t)^\perp}$ is dense in its biorthogonal complement. This fact and the Magajna–Schweizer theorem show that every densely defined closed operator on Hilbert C^*-modules over C^*-algebras of compact operators is regular, that is for densely defined closed operators on such Hilbert modules, the second and the third conditions hold automatically. Magajna, Schweizer and the first author have presented nice descriptions of C^*-algebras of compact operators in [15], [19], [6]. Beside their work we give further descriptions of such C^*-algebras via some properties of densely defined closed operators.

1. PRELIMINARIES

In this section we would like to recall some definitions and present a few simple facts about regular operators on Hilbert $\mathcal{A}$-modules. For details see Chapter 9 and 10 of [14], and the paper [21]. We give a necessary and sufficient condition for closedness of the range of regular operators.

Let E, F be Hilbert $\mathcal{A}$-modules, we will use the notation $t : \mathrm{Dom}(t) \subseteq E \to F$ to indicate that t is an $\mathcal{A}$-linear operator whose domain $\mathrm{Dom}(t)$ is a dense submodule of E (not necessarily identical with E) and whose range is in F. A densely defined operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *closed* if its graph $G(t) = \{(x, tx) : x \in \mathrm{Dom}(t)\}$ is a closed submodule of the Hilbert $\mathcal{A}$-module $E \oplus F$. In accordance with the literature we give a stronger definition of adjointability of densely defined operators that extends the definition for bounded operators.

DEFINITION 1.1. A densely defined operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *adjointable* if it possesses a densely defined map $t^* : \mathrm{Dom}(t^*) \subseteq F \to E$ with the domain

$$\mathrm{Dom}(t^*) = \{y \in F : \text{there exists } z \in E \text{ such that } \langle tx, y\rangle_F = \langle x, z\rangle_E \text{ for any } x \in \mathrm{Dom}(t)\}$$

which satisfies the property $\langle tx, y\rangle_F = \langle x, t^*y\rangle_E$, for any $x \in \mathrm{Dom}(t)$ and any $y \in \mathrm{Dom}(t^*)$.

The above property implies that t^* is a closed $\mathcal{A}$-linear map. A densely defined closed $\mathcal{A}$-linear map $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. We denote the set of all regular operators from E to F by $R(E, F)$. There is an alternative definition of a regular operator between Hilbert C^*-modules (cf. Definition 1.1 of [21]), however, Lance has proved in his book [14] that both of them are equivalent. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint (cf. Corollaries 9.4, 9.6 and Proposition 9.9 of [14]). Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$, then $\mathrm{Ran}(Q_t) = \mathrm{Dom}(t)$, $0 \leqslant Q_t \leqslant 1$ in $B(E, E)$ and $F_t \in B(E, F)$ (cf. Chapter 9 of [14]). The bounded operator F_t is called the bounded transform (or z-transform) of the regular operator t. The map $t \to F_t$ defines a bijection

$$R(E, F) \to \{T \in B(E, F) : \|T\| \leqslant 1 \text{ and } \mathrm{Ran}(1 - T^*T) \text{ is dense in } F\},$$

(cf. Theorem 10.4 of [14]). This map is adjoint-preserving, i.e. $F_t^* = F_{t^*}$, and for the bounded transform $F_t = tQ_t = t(1 + t^*t)^{-1/2}$ we have $\|F_t\| \leqslant 1$ and

$$t = F_t(1 - F_t^*F_t)^{-1/2} \quad \text{and} \quad Q_t = (1 - F_t^*F_t)^{1/2}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Recall the following definitions for a regular operator $t \in R(E) := R(E, E)$:

- t is called *normal* if and only if $\mathrm{Dom}(t) = \mathrm{Dom}(t^*)$ and $\langle tx, tx\rangle = \langle t^*x, t^*x\rangle$ for all $x \in \mathrm{Dom}(t)$.
- t is called *selfadjoint* if and only if $t^* = t$.
- t is called *positive* if and only if t is normal and $\langle tx, x\rangle \geqslant 0$ for all $x \in \mathrm{Dom}(t)$.

Then there are the following transfers of properties:

- t is normal if and only if F_t is normal (cf. 1.15 of [21]).
- t is selfadjoint if and only if F_t is selfadjoint.
- t is positive if and only if F_t is positive (cf. Result 1.14 of [13]).

Let E, F be two Hilbert $\mathcal{A}$-modules and suppose that an operator T in $B(E, F)$ has closed range. We would like to consider the kernel $\mathrm{Ker}(T)$ and the range $\mathrm{Ran}(T)$ of T. Closed submodules of Hilbert modules need not to be orthogonally complemented at all, but Lance states in Theorem 3.2 of [14] under which conditions closed submodules may be orthogonally complemented (see also Theorem 15.3.8 of [20]). For the special choice of bounded operators T with closed range one has:

- $\mathrm{Ker}(T)$ is orthogonally complemented in E, with complement $\mathrm{Ran}(T^*)$,
- $\mathrm{Ran}(T)$ is orthogonally complemented in F, with complement $\mathrm{Ker}(T^*)$,
- the map $T^* \in B(F, E)$ has a closed range, too.

The collected facts, as well as Lemmata 4.1 and 4.2 from [16] lead us to the following proposition.

PROPOSITION 1.2. *Let* $t \in R(E, F)$ *and* $\mathrm{Ker}(t) = \{x \in \mathrm{Dom}(t) : tx = 0\}$. *Then*

(i) $\mathrm{Ker}(t)$ *and* $\mathrm{Ker}(t^*)$ *both are closed submodules of E and F, respectively,*

(ii) $\mathrm{Ran}(t) = \mathrm{Ran}(F_t)$ *and* $\mathrm{Ran}(t^*) = \mathrm{Ran}(F_{t^*})$,

(iii) $\mathrm{Ker}(t^*) = \mathrm{Ran}(t)^\perp$ *and* $\mathrm{Ker}(t) = \mathrm{Ran}(t^*)^\perp$,

(iv) $\mathrm{Ker}(t) = \mathrm{Ker}(F_t)$ *and* $\mathrm{Ker}(t^*) = \mathrm{Ker}(F_{t^*})$.

(v) *The regular operator t has closed range if and only if its adjoint operator t^* has closed range, and then for* $|t| := (t^*t)^{1/2}$ *the direct sum decompositions* $E = \mathrm{Ker}(t) \oplus \mathrm{Ran}(t^*) = \mathrm{Ker}(|t|) \oplus \overline{\mathrm{Ran}(|t|)}$, $F = \mathrm{Ker}(t^*) \oplus \mathrm{Ran}(t) = \mathrm{Ker}(|t^*|) \oplus \overline{\mathrm{Ran}(|t^*|)}$ *hold.*

Proof. To show (i), let $\{x_n\}$ be a sequence in $\mathrm{Ker}(t)$ which converges to $x \in E$ in norm. Then $t(x_n) = 0$ for any $n \in \mathbb{N}$. Therefore the sequence $\{t(x_n)\}$ converges to zero. Closedness of the operator t implies that $x \in \mathrm{Dom}(t)$ and therefore, $tx = 0$. So $x \in \mathrm{Ker}(t)$ and $\mathrm{Ker}(t)$ is a closed submodule. Since t is regular so t^* is regular, too, and similarly $\mathrm{Ker}(t^*)$ is also a closed submodule of F.

For the proof of (ii) recall that $F_t = tQ_t$ and $\mathrm{Ran}(Q_t) = \mathrm{Dom}(t)$. Then $\mathrm{Ran}(t) = \mathrm{Ran}(F_t)$. Since t is regular so is t^*, thus $\mathrm{Ran}(t^*) = \mathrm{Ran}(F_{t^*})$.

To demonstrate (iii) we notice that $y \in \mathrm{Ker}(t^*)$ if and only if $\langle tx, y\rangle = \langle x, 0\rangle = 0$ for all $x \in \mathrm{Dom}(t)$, or if and only if $y \in \mathrm{Ran}(t)^\perp$. Consequently, we have $\mathrm{Ker}(t^*) = \mathrm{Ran}(t)^\perp$. The second equality follows from the first equality and Corollary 9.4 of [14].

For the proof of (iv) we know that $\mathrm{Ker}(F_{t^*}) = \mathrm{Ran}(F_t)^{\perp}$, cf. Theorem 15.3.5 of [20]. Therefore,

$$\mathrm{Ker}(F_{t^*}) = \mathrm{Ran}(F_t)^{\perp} = \mathrm{Ran}(t)^{\perp} = \mathrm{Ker}(t^*)\,.$$

Similarly, we obtain $\mathrm{Ker}(F_t) = \mathrm{Ker}(t)$.

Finally, we derive (v). The bounded operator F_t has closed range if and only if its adjoint operator F_{t^*} has closed range. Hence, the regular operator t has closed range if and only if t^* has. Result 7.19 of [13] implies that

$$\overline{\mathrm{Ran}(|t|)} = \overline{\mathrm{Ran}(t^*)},\ \mathrm{Ker}(|t|) = \mathrm{Ker}(t),\ \overline{\mathrm{Ran}(|t^*|)} = \overline{\mathrm{Ran}(t)},\ \mathrm{Ker}(|t^*|) = \mathrm{Ker}(t^*)\,.$$

The equalities follow immediately from (ii), (iv) and Theorem 3.2 of [14]. ∎

PROPOSITION 1.3. *Let $t \in R(E,F)$, then t has closed range if and only if* $\mathrm{Ker}(t)$ *is orthogonally complemented in E and t is bounded below on* $(\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t)$, *i.e.* $\|tx\| \geqslant c\|x\|$, *for all* $x \in (\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t)$ *for a certain positive constant c.*

Proof. Let first $\mathrm{Ran}(t)$ be closed then the Proposition 2.2(v) implies that $\mathrm{Ker}(t)$ is orthogonally complemented in E. We define the $\mathcal{A}$-linear module map

$$\widetilde{t} : (\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t) \to \mathrm{Ran}(t)$$

by $\widetilde{t}x := tx$ for all $x \in (\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t)$. Then $\widetilde{t}$ is a bijection. The inverse of this mapping exists and is $\mathcal{A}$-linear from $\mathrm{Ran}(t)$ into $(\mathrm{Ker}(t))^{\perp}$ with closed domain $\mathrm{Dom}(\widetilde{t}^{\,-1}) = \mathrm{Ran}(t)$. Moreover $\widetilde{t}^{\,-1}$ is closed since t is. Therefore, it has to be a bounded operator by the closed graph theorem, that is there exists a positive constant c such that: $\|\widetilde{t}^{\,-1}x\| \leqslant c\|x\|$, for each $x \in \mathrm{Ran}\,t$. This implies that $\|tx\| \geqslant c^{-1}\|x\|$, for all $x \in (\mathrm{Ker}\,t)^{\perp} \cap \mathrm{Dom}(t)$.

Conversely, let t be bounded below on $(\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t)$ and $E = \mathrm{Ker}(t) \oplus (\mathrm{Ker}(t))^{\perp}$. Then if the sequence $\{y_n\} \in \mathrm{Ran}(t)$ converges to y, there exists a sequence $\{x_n\} \in (\mathrm{Ker}(t))^{\perp} \cap \mathrm{Dom}(t)$ such that $y_n = t(x_n)$. Then $(x_n - x_m) \in (\mathrm{Ker}(t))^{\perp}$, and therefore $\|x_n - x_m\| \leqslant c^{-1}\|y_n - y_m\|$ converges to zero as m, n go to infinity. This means that there exists an element $x \in (\mathrm{Ker}(t))^{\perp}$ such that the sequence $\{x_n\}$ converges to x in norm and the sequence $\{t(x_n)\}$ converges to y in norm. The closedness of t implies that $x \in \mathrm{Dom}(t)$ and $tx = y$. ∎

2. ADJOINTABILITY OF DENSELY DEFINED OPERATORS

Corollary 2.4 of [6] shows that a bounded $\mathcal{A}$-linear operator $T : E \to F$ possesses an adjoint operator $T^* : F \to E$ if and only if the graph of T is an orthogonal summand of the Hilbert $\mathcal{A}$-module $E \oplus F$. This fact motivates us to give a sufficient condition for the adjointability of densely defined operators via their graphs.

THEOREM 2.1. *Let E, F be two Hilbert $\mathcal{A}$-modules and $t : \mathrm{Dom}(t) \subseteq E \to F$ be a densely defined operator. If the graph of t is orthogonally complemented in $E \oplus F$ and the range of $P_F P_{G(t)^\perp}$ is dense in its biorthogonal complement then t is adjointable. In this case t is closed and $1 + t^*t$ is surjective.*

Proof. Consider the unitary element V of $B(E \oplus F, F \oplus E)$ defined by $V(x, y) = (y, -x)$. Then $G(t)$ is orthogonally complemented in $E \oplus F$ if and only if $V(G(t))$ is orthogonally complemented in $F \oplus E$. The orthogonal complement of $V(G(t))$ with respect to $F \oplus E$ is the closed set

$$\begin{aligned}[V(G(t))]^\perp &= \{(y,z) : y \in F, z \in E, \text{ such that } \langle (tx, -x), (y, z)\rangle = 0, \text{ for all } x \in \mathrm{Dom}(t)\} \\ &= \{(y,z) : y \in F,\ z \in E, \text{ such that } \langle tx, y\rangle = \langle x, z\rangle, \text{ for all } x \in \mathrm{Dom}(t)\}\,.\end{aligned}$$

Now we define

$$\mathrm{Dom}(t^*) := \{y \in F : \text{there exists } z \in E \text{ such that } \langle tx, y\rangle = \langle x, z\rangle \text{ for all } x \in \mathrm{Dom}(t)\}.$$

The set $\mathrm{Dom}(t^*)$ is a non-trivial submodule of F since the set $[V(G(t))]^\perp$ is a non-trivial submodule of $F \oplus E$. The domain of t is dense in E, and so we consider elements $y \in F$ such that an element z with the property

$$\langle tx, y\rangle = \langle x, z\rangle, \quad \text{for any } x \in \mathrm{Dom}(t)$$

exists and is unique. This set is not empty since $y = 0$ and $z = 0$ forms an admissible pair of elements. Collecting all such elements $y \in F$ we can define an operator $t^* : \mathrm{Dom}(t^*) \subseteq F \to E$ by $t^* y = z$. Clearly t^* is $\mathcal{A}$-linear and satisfies $\langle tx, y\rangle = \langle x, t^*y\rangle$ for all $x \in \mathrm{Dom}(t)$, $y \in \mathrm{Dom}(t^*)$. Moreover we have $[V(G(t))]^\perp = \{(y, t^*y) :\ y \in \mathrm{Dom}(t^*)\} = G(t^*)$, i.e. $F \oplus E = V(G(t)) \oplus G(t^*)$. Since $\mathrm{Ran}(P_F P_{G(t)^\perp}) = \mathrm{Dom}(t^*)$ and $(\mathrm{Ran}(P_F P_{G(t)^\perp}))^\perp = \mathrm{Ker}(P_{G(t)^\perp} P_F^*) = \{0\}$, we find $\overline{\mathrm{Dom}(t^*)} = \{0\}^\perp = F$. The graph of t is orthogonally complemented, so Lemma 15.3.4 of [20] implies that $G(t)$ is closed in $E \oplus F$. Suppose $u \in E$ is an arbitrary element then $(0, u) \in F \oplus E = V(G(t)) \oplus G(t^*)$, that is, there exist elements $x_0 \in \mathrm{Dom}(t)$ and $y_0 \in \mathrm{Dom}(t^*)$ such that $y_0 = -t(x_0)$ and $u = t^*(y_0) - x_0$, consequently, $x_0 \in \mathrm{Dom}(t^*t)$ and $u = -(1 + t^*t)x_0$. Hence, $1 + t^*t$ is surjective. ∎

The above theorem shows that the closedness and the assumption to unbounded C^*-linear densely defined operators of possessing a densely defined adjoint operator can be reduced in some results of [11], [14]. Contrary to the situation for bounded operators, the converse of the above theorem is not valid. An example of a selfadjoint densely defined closed operator whose graph is not orthogonally complemented was given by Hilsum in [9] (see also page 103 of [14]). Now we can find a necessary and sufficient condition as follows:

COROLLARY 2.2. *Let $t : \mathrm{Dom}(t) \subseteq E \to F$ be an $\mathcal{A}$-linear densely defined operator between Hilbert $\mathcal{A}$-modules E and F. Then the graph of t is orthogonally complemented in $E \oplus F$ and $\overline{\mathrm{Ran}(P_F P_{G(t)^\perp})} = \mathrm{Ran}(P_F P_{G(t)^\perp})^{\perp\perp}$ if and only if t is regular, i.e. t is adjointable, closed, and the range of $1 + t^*t$ is dense in E.*

Proof. The assertion is a direct conclusion from Theorem 9.3 of [14] and Theorem 2.1 above. ∎

The criterion found by Kucerovsky as Proposition 6 in [11] now reads as follows:

COROLLARY 2.3. *Let $t : \mathrm{Dom}(t) \subseteq E \to F$ be an $\mathcal{A}$-linear densely defined operator between Hilbert $\mathcal{A}$-modules E and F. The operator t is regular if and only if t is adjointable, closed, and for any positive real number c the operator $c1 + t^*t$ is bijective. If t is selfadjoint, then t is necessarily closed and so t is regular if and only if the operator $c\mathrm{i} \pm t$ is bijective for any non-zero real constant c.*

In [12] Kucerovsky has given a geometrical criterion for regularity of closed operators, cf. Proposition 5 of [12]. Here, by reducing some of his suppositions we can sharpen his criterion.

THEOREM 2.4. *Let E, F be two Hilbert $\mathcal{A}$-modules. Then a closed operator is regular if and only if there exists a Hilbert $\mathcal{A}$-module G and a bounded adjointable operator $S \in B(G, E \oplus F)$ such that:*

(i) *the graph of the operator is the range of S, and*
(ii) *$P_E S$ has dense range, and*
(iii) *the range of $P_F P_{\mathrm{Ker}(S^*)}$ is dense in its biorthogonal complement.*

Proof. Let a regular operator $t : \mathrm{Dom}(t) \subseteq E \to F$ be given. Suppose G is the graph of t and S is the inclusion of G into $E \oplus F$. The graph of t is orthogonally complemented in $E \oplus F$ and so S is adjointable and $\mathrm{Ker}(S^*) = \mathrm{Ran}(S)^\perp = G(t)^\perp$. Furthermore $\mathrm{Ran}(P_E S) = \mathrm{Dom}(t)$ and $\mathrm{Ran}(P_F P_{G(t)^\perp}) = \mathrm{Dom}(t^*)$ are dense in E and F, respectively.

Conversely, let $t : \mathrm{Dom}(t) \subseteq E \to F$ be closed and suppose the conditions (i) and (ii) hold. Then the range of S is closed, hence Theorem 3.2 of [14] implies that it is an orthogonal summand, with complement $\mathrm{Ker}(S^*)$. The range of $P_E S$ is dense in E, so t is a densely defined closed operator whose graph is orthogonally complemented in $E \oplus F$, that is t is regular by Corollary 2.2. ∎

Suppose that $\mathcal{A}$ is an arbitrary C^*-algebra of compact operators. It is well-known that $\mathcal{A}$ has to be of the form $\mathcal{A}$=c_0-$\oplus_{i \in I} \mathcal{K}(H_i)$, i.e. $\mathcal{A}$ is a c_0-direct sum of elementary C^*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i, $i \in I$ (cf. Theorem 1.4.5 of [1]).

Magajna and Schweizer have shown, respectively, that C^*-algebras of compact operators can be characterized by the property that every norm-closed (coinciding with its biorthogonal complement, respectively) submodule of every

Hilbert C^*-module over them is automatically an orthogonal summand, cf. [15], [19]. Recently further generic properties of the category of Hilbert C^*-modules over C^*-algebras which characterize precisely the C^*-algebras of compact operators have been found by the first author in [6]. We recall results by Magajna, Schweizer and Frank as follows:

THEOREM 2.5. *Let $\mathcal{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is an arbitrary C^*-algebra of compact operators.*

(ii) *For every Hilbert $\mathcal{A}$-module E every Hilbert $\mathcal{A}$-submodule $F \subseteq E$ is automatically orthogonally complemented, i.e. F is an orthogonal summand.*

(iii) *For every Hilbert $\mathcal{A}$-module E, every Hilbert $\mathcal{A}$-submodule $F \subseteq E$ that coincides with its biorthogonal complement $F^{\perp\perp} \subseteq E$ is automatically orthogonally complemented in E.*

(iv) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every bounded $\mathcal{A}$-linear map $T : E \to F$ possesses an adjoint bounded $\mathcal{A}$-linear map $T^* : F \to E$.*

(v) *The kernels of all bounded $\mathcal{A}$-linear operators between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

(vi) *The images of all bounded $\mathcal{A}$-linear operators with norm-closed range between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

(vii) *For every Hilbert $\mathcal{A}$-module E every Hilbert $\mathcal{A}$-submodule is automatically topologically complemented there, i.e. it is a topological direct summand.*

(viii) *For every (maximal) norm-closed left ideal I of $\mathcal{A}$ the corresponding open projection $p \in \mathcal{A}^{**}$ is an element of the multiplier C^*-algebra $M(\mathcal{A})$ of $\mathcal{A}$.*

Consider the C^*-algebra of compact operators as a Hilbert C^*-module over itself. Pal has proved in Theorem 5.8 of [17] that every closed semiregular operator (i.e. every densely defined closed operator which adjoint is densely defined) on Hilbert C^*-modules over C^*-algebras of compact operators is regular. Corollary 2.2 and the second part of the above theorem give a short proof for Pal's Theorem. Moreover we can reformulate Pal's Theorem as follows:

REMARK 2.6. In the category of all Hilbert C^*-modules over a C^*-algebra of compact operators every densely defined closed C^*-linear operator between Hilbert C^*-modules is regular.

COROLLARY 2.7. *Let $\mathcal{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is an arbitrary C^*-algebra of compact operators.*

(ix) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ possesses a densely defined adjoint operator $t^* : \mathrm{Dom}(t^*) \subseteq F \to E$.*

(x) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is regular.*

(xi) *The kernels of all densely defined closed operators between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

(xii) *The images of all densely defined closed operator with norm-closed range between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

Proof. Theorem 2.1, Pal's Theorem and condition (ii) imply (ix), (x), (xi) and (xii). To show the contrary let condition (ix) hold and let $T : E \to F$ be an arbitrary bounded $\mathcal{A}$-linear map between Hilbert $\mathcal{A}$-modules E and F. The operator $T : \mathrm{Dom}(T) = E \to F$ is a densely defined closed operator (since it is bounded), and so condition (ix) implies that there exists a (possibly unbounded) densely defined operator $T^* : \mathrm{Dom}(T^*) \subseteq F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \text{for all } x \in \mathrm{Dom}(T) = E,\ y \in \mathrm{Dom}(T^*)\,.$$

Then T^* is bounded on the pre-Hilbert module $\mathrm{Dom}(T^*)$. The domain of T^* is dense in F and E is a Hilbert module, so T^* has a unique bounded $\mathcal{A}$-linear extension $\widetilde{T}^* : F \to E$ such that $\langle\, Tx, y\rangle = \langle x, \widetilde{T}^*\, y\rangle$ for all $x \in E$, $y \in F$. Therefore every bounded $\mathcal{A}$-linear map $T : E \to F$ possesses an adjoint bounded $\mathcal{A}$-linear map, i.e. condition (iv) holds. Condition (x) implies (ix) and hence, (iv). Conditions (xi) and (xii) imply conditions (v) and (vi), respectively, since each everywhere defined bounded operator is a densely defined closed operator. ∎

COROLLARY 2.8. *Suppose $\mathcal{A}$ is any C^*-algebra which does not admit a faithful $*$-representation as a C^*-subalgebra in some C^*-algebra of compact operators. Then there exists a densely defined closed operator t between two full Hilbert C^*-modules over $\mathcal{A}$ such that t is not regular or, even more, the adjoint operator t^* of t is not densely defined (and therefore, there does not exist any adjoint operator in the strong sense of Definition* 1.1*).*

This fact directly follows from Corollary 2.7(i), (ix) and (x). In other words, Hilsum's example reflects a quite regular case for unbounded densely defined closed operators between Hilbert C^*-modules over non-compact C^*-algebras of any kind.

Let $\mathcal{K}(H)$ be the C^*-algebra of all compact operators on a Hilbert space H. Let $e \in \mathcal{K}(H)$ be an arbitrary minimal projection and E be a $\mathcal{K}(H)$-module. Suppose $E_e := eE = \{ex : x \in E\}$, then E_e is a Hilbert space with respect to the inner product $(\cdot,\cdot) = \mathrm{trace}\,(\langle\cdot,\cdot\rangle)$, which is introduced in [3]. Let $B(E)$ and $B(E_e)$ be C^*-algebras of all bounded adjointable operators on Hilbert $\mathcal{K}(H)$-module E and Hilbert space E_e, respectively. Bakić and Guljaš have shown that the map $\Phi : B(E) \to B(E_e)$, $\Phi(T) = T|_{E_e}$ is a $*$-isomorphism of C^*-algebras (cf. Theorem 5 of [3]). On the other hand, Remark 2.6 and Theorem 10.4 of [14] give adjoint-preserving bijection maps $t \to F_t = t(1+t^*t)^{-1/2}$ as follows:

$$R(E) \;\to\; \{T \in B(E) : \|T\| \leqslant 1 \text{ and } \mathrm{Ran}(1-T^*T) \text{ is dense in } E\}\,,$$
$$R(E_e) \to \{T \in B(E_e) : \|T\| \leqslant 1 \text{ and } \mathrm{Ran}(1-T^*T) \text{ is dense in } E_e\}\,.$$

These maps together with the $*$-isomorphism Φ give an adjoint-preserving bijection map between all densely defined closed operators on Hilbert $\mathcal{K}(H)$-module E and all densely defined closed operators on Hilbert space E_e. It means that all densely defined closed operators on a Hilbert $\mathcal{K}(H)$-module E are reduced by a suitable Hilbert space contained in E.

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