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CONVERGENCE OF BIMODULES OVER MAXIMAL ABELIAN SELFADJOINT ALGEBRAS

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ABSTRACT. We establish a continuity result for the map sending a masa-bimodule to its support. We characterise the convergence of a net of weakly closed convex hulls of bilattices in terms of the convergence of the corresponding supports, and prove a lower-semicontinuity result for the map sending a support to the corresponding masa-bimodule.

KEYWORDS: *Masa-bimodule, support, bilattice, convergence.*

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1. INTRODUCTION AND PRELIMINARIES

The study of collections of operator algebras from a global viewpoint was initiated by Effros in [2]. He defined a Borel structure on the set of all von Neumann algebras acting on a fixed separable Hilbert space, and showed that a number of maps defined on this set, including the commutant, are Borel. The topic has attracted considerable attention since then, see e.g. [9], [10] and [15]. In [5], Haagerup and Winslow established a continuity theorem for the commutant and obtained a number of results on the topological properties of certain collections of von Neumann algebras.

In non-selfadjoint operator algebra theory the role of the commutant is often played by the collection Lat$\mathcal{A}$ of all (closed) invariant subspaces of an operator algebra $\mathcal{A}$, known as its invariant subspace lattice. The continuity of the map sending an operator algebra $\mathcal{A}$ to Lat$\mathcal{A}$ was studied in [11] and [12]. It was shown that Lat is continuous on the collection of all von Neumann algebras as well as on the collection of all CSL algebras, a class of non-selfadjoint operator algebras introduced by Arveson in [1]. Subspaces which are bimodules over two maximal abelian selfadjoint algebras (masa-bimodules) generalise CSL algebras; these objects were extensively studied later in [4].

Associated with a masa-bimodule $\mathcal{U}$ are two objects dual to each other: its bilattice $\mathrm{Bil}\,\mathcal{U}$ [13], which generalises the notion of the invariant subspace lattice of an algebra, and its support [4], a subset κ of the direct product $X \times Y$ of two measure spaces associated in a natural way with the corresponding masas. The subject of the present paper is the convergence relation between masa-bimodules, their bilattices and their supports. More precisely, for fixed separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and masas on them, we equip the collection of all reflexive (in the sense of Loginov and Shulman [8]) masa-bimodules $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ with a convergence coming from the weak* and the strong* topologies of the space $\mathcal{B}(\mathcal{H},\mathcal{K})$ of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. Our convergence is closely related to the ones used by Tsukada [14] and Haagerup and Winslow [5]. To define a convergence on the set of supports, we use the notion of a capacity on the power set of $X \times Y$. More precisely, we equip the collection of all supports of masa-bimodules with a convergence coming from a family of capacities on $X \times Y$ that were introduced and studied by Haydon and Shulman in [7]. The main result of Section 2 is the equivalence of the convergence of a net of supports to the convergence of the net of the weakly closed convex hulls of the corresponding bilattices.

In Section 3 we establish the continuity of the mapping sending a (reflexive) masa-bimodule to its support. Our result yields a subspace version of the continuity of Lat on the collection of CSL algebras established in [12]. It naturally splits into a limsup and a liminf parts. For the limsup, we establish the equivalence of the convergence of the masa-bimodules, their supports, and their bilattices. For the liminf, we only have strict implications. The failure of equivalence motivates Section 4, where we obtain the lower semi-continuity of the map sending a support to its corresponding (minimal) masa-bimodule, in a weaker sense. This result implies that this map is lower semi-continuous for the convergence used by Haagerup and Winslow in [5].

We now introduce notation and state some preliminary results. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, we let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, and write $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. We denote by $\mathcal{B}(\mathcal{H})^+$ the set of all positive operators on $\mathcal{H}$ and by ω_x (where $x \in \mathcal{H}$) the vector functional given by $\omega_x(A) = (Ax, x)$. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, we denote by $\mathrm{Proj}(\mathcal{M})$ the set of all projections in $\mathcal{M}$, by $\mathrm{Ball}(\mathcal{M})$ the unit ball of $\mathcal{M}$ and by $\mathrm{Conv}\mathcal{M}$ the weakly closed convex hull of $\mathcal{M}$. If P is a projection, we write $P^\perp = I - P$. If $\mathcal{E} \subseteq \mathcal{H}$ we denote by $[\mathcal{E}]$ the projection onto the closed linear span of $\mathcal{E}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be maximal abelian selfadjoint algebras (masas) on $\mathcal{H}$ and $\mathcal{K}$, respectively. A $\mathcal{D}_2,\mathcal{D}_1$-bimodule (or simply a masa-bimodule if $\mathcal{D}_1$ and $\mathcal{D}_2$ are clear from the context) is a subspace $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ for which $\mathcal{D}_2\mathcal{U}\mathcal{D}_1 \subseteq \mathcal{U}$. If $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ is a masa-bimodule, let

$$\mathrm{Bil}\,\mathcal{U} = \{(P,Q) \in \mathrm{Proj}(\mathcal{D}_1) \times \mathrm{Proj}(\mathcal{D}_2) : Q\mathcal{U}P = \{0\}\}.$$

The set Bil$\mathcal{U}$ is a bilattice [13] in the sense that $(P, 0)$, $(0, Q) \in$ Bil$\mathcal{U}$ for all $P \in \text{Proj}(\mathcal{D}_1)$ and $Q \in \text{Proj}(\mathcal{D}_2)$ and $(P_1, Q_1), (P_2, Q_2) \in$ Bil$\mathcal{U}$ imply $(P_1 \wedge P_2, Q_1 \vee Q_2), (P_1 \vee P_2, Q_1 \wedge Q_2) \in$ Bil$\mathcal{U}$. Conversely, if $S \subseteq \text{Proj}(\mathcal{D}_1) \times \text{Proj}(\mathcal{D}_2)$ is a bilattice then the set

$$\text{Op}S = \{T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : QTP = 0, \text{ for all } (P, Q) \in S\}$$

is a masa-bimodule. The masa-bimodules of the form OpS for some bilattice $S \subseteq \text{Proj}(\mathcal{D}_1) \times \text{Proj}(\mathcal{D}_2)$ are precisely the masa-bimodules which are reflexive in the sense of Loginov and Shulman [8].

Let (X, m) and (Y, n) be standard (finite) measure spaces, that is, such that there exist topologies with respect to which X and Y are compact metric spaces and m and n are regular Borel measures. Let $\mathcal{H} = L^2(X, m)$, $\mathcal{K} = L^2(Y, n)$ and $\mathcal{D}_1 \equiv L^\infty(X, m)$ and $\mathcal{D}_2 \equiv L^\infty(Y, n)$ be the multiplication masas on $\mathcal{H}$ and $\mathcal{K}$, respectively. By $P(\alpha)$ we denote the projection given by multiplication by the characteristic function of a measurable set $\alpha \subseteq X$. The sets of the form $M \times Y \cup X \times N$, where $M \subseteq X$ and $N \subseteq Y$ are null sets, and their subsets, are called marginally null [1]. We say that two measurable sets κ and λ of $X \times Y$ are marginally equivalent, and write $\kappa \simeq \lambda$, if their symmetric difference is marginally null. The sets which are marginally equivalent to sets of the form $\bigcup_{i=1}^{\infty} \alpha_i \times \beta_i$, with $\alpha_i \subseteq X$ and $\beta_i \subseteq Y$ measurable, are called ω-open. The complements of ω-open sets are called ω-closed.

Let $\kappa \subseteq X \times Y$. An operator T is said to be supported on κ if $P(\beta)TP(\alpha) = 0$ whenever $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable and $(\alpha \times \beta) \cap \kappa \simeq \emptyset$. The space $\mathcal{M}_{\max}(\kappa)$ of all operators, supported on κ, is easily seen to be a reflexive masa-bimodule; indeed, $\mathcal{M}_{\max}(\kappa) = \text{Op}S_\kappa$, where S_κ is the bilattice

(1.1) $$S_\kappa = \{(P(\alpha), P(\beta)) : \alpha \subseteq X, \beta \subseteq Y \text{ measurable and } (\alpha \times \beta) \cap \kappa \simeq \emptyset\}.$$

By [1] and [3], $S_\kappa = \text{Bil}\,\mathcal{M}_{\max}(\kappa)$. It was shown in [4] that, conversely, if $\mathcal{M}$ is a reflexive masa-bimodule then there exists a unique, up to marginal equivalence, ω-closed set κ (called the support of $\mathcal{M}$) with $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$. If $\kappa \subseteq X \times Y$ is arbitrary, its ω-closure $\text{cl}_\omega(\kappa)$ is by definition the support of $\mathcal{M}_{\max}(\kappa)$. It was shown in [1] and [13] that, given a subset $\kappa \subseteq X \times Y$, there exists a minimal weak* closed masa-bimodule $\mathcal{U}$ with the property that OpBil$\mathcal{U} = \mathcal{M}_{\max}(\kappa)$; denote this masa-bimodule by $\mathcal{M}_{\min}(\kappa)$. If $\mathcal{M}_{\min}(\kappa) = \mathcal{M}_{\max}(\kappa)$, we say that κ satisfies operator synthesis [1].

Reflexive masa-bimodules are a subspace analogue of CSL algebras introduced by Arveson in [1], while their bilattices are an analogue of commutative subspace lattices, that is, strongly closed sublattices of Proj$(\mathcal{D})$, for some masa $\mathcal{D}$. The lemma that follows was established in the case of commutative subspace lattices by Arveson in [1].

LEMMA 1.1. *Let $\mathcal{D}_1 \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{D}_2 \subset \mathcal{B}(\mathcal{K})$ be masas, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a $\mathcal{D}_2, \mathcal{D}_1$-bimodule and $S = \text{Bil}\,\mathcal{M}$. Then*

(i) Conv$S$=$\{(A,B)\in\text{Ball}(\mathcal{D}_1^+)\times\text{Ball}(\mathcal{D}_2^+) : TAT^*\leqslant I-B,\ \forall\, T\in\text{Ball}(\mathcal{M})\}$, *and*

(i) *the extreme points of* Conv$S$ *are the elements of* S.

Proof. (i) We denote by $E_C(\cdot)$ the spectral measure of the selfadjoint operator C. Let $\mathcal{E}$ denote the set on the right hand side of the identity, $(A,B)\in\mathcal{E}$ and $T\in\text{Ball}(\mathcal{M})$. By Lemma 7.2 of [4], $T^*E_{I-B}[0,t]\mathcal{K}\subseteq E_A[0,t]\mathcal{H}$, for every $t\geqslant 0$. This implies

$$E_A[0,s)^{\perp}T^*E_{I-B}[0,t)=0, \quad \text{whenever } s>t.$$

Thus, $E_B[1-t,1]TE_A[s,1]=0$ whenever $s>t$ or, equivalently, $E_B[t,1]\ T\ E_A[s,1] = 0$ whenever $s+t>1$. By Lemma 3.2 of [13], $(A,B)\in$ Conv$S$.

Assume that $(P,Q)\in S$ and $T\in\text{Ball}(\mathcal{M})$. Then $PT^*Q=0$ and so $Q(TPT^*)Q=0$. It follows that $Q\mathcal{K}\subseteq\ker(TPT^*)$ and so $\text{ran}(TPT^*)\subseteq Q^{\perp}\mathcal{K}$. Since TPT^* is a positive contraction, we conclude that $TPT^*\leqslant Q^{\perp}$. Now, let $(A,B)=\sum_{i=1}^{N}\lambda_i(P_i,Q_i)$, where $\lambda_i\geqslant 0$, $\sum_{i=1}^{N}\lambda_i=1$ and $(P_i,Q_i)\in S$. We have $TP_iT^*\leqslant Q_i^{\perp}$ for each i and hence

$$T\Big(\sum_{i=1}^{N}\lambda_iP_i\Big)T^*=\sum_{i=1}^{N}\lambda_iTP_iT^*\leqslant\sum_{i=1}^{N}\lambda_iQ_i^{\perp}=I-\sum_{i=1}^{N}\lambda_iQ_i.$$

In other words, $\mathcal{E}$ contains all convex combinations of elements of S. Since $\mathcal{E}$ is weakly closed, Conv$S\subseteq\mathcal{E}$. The claim is proved.

(ii) Let $\mathcal{L}=\{P\oplus Q^{\perp} : (P,Q)\in S\}$. It is easy to see that $\mathcal{L}$ is a commutative subspace lattice and that Conv$\mathcal{L}=\{A\oplus(I-B) : (A,B)\in$ Conv$S\}$. Hence, (C,D) is an extreme point of Conv$S$ if and only if $C\oplus(I-D)$ is an extreme point of $\mathcal{L}$. The fact now follows from the corresponding result for commutative subspace lattices [1]. ∎

Let $\kappa\subseteq X\times Y$. Haydon and Shulman set [7]

$$\gamma(\kappa)=\inf\{m(\alpha)+n(\beta) : \kappa\subseteq(\alpha\times Y)\cup(X\times\beta)\},$$

where the sets α and β in the infimum are taken to be measurable. They showed that the map $\kappa\longrightarrow\gamma(\kappa)$ is a capacity on the power set of $X\times Y$ in the sense that

(a) $\gamma(\kappa)\leqslant\gamma(\kappa')$ whenever $\kappa\subseteq\kappa'$;

(b) $\lim_{n\to\infty}\gamma(\kappa_n)=\gamma(\kappa)$ whenever $\{\kappa_n\}$ is an increasing sequence of subsets of $X\times Y$ and $\kappa=\bigcup\kappa_n$, and

(c) $\gamma(\kappa)=\inf\{\gamma(U) : U$ open and $\kappa\subseteq U\}$.

Relation (c) holds with respect to any pair of topologies on X and Y which turn m and n into regular Borel measures. Moreover, κ is marginally null if and only if $\gamma(\kappa)=0$.

We next recall the general notion of the limit space structure in the set of all subsets of a topological space (Z,τ). For a net $\{E_\lambda\}$ of subsets of Z, denote by τ-$\liminf E_\lambda$ the set of all points $z\in Z$ which are τ-limits of nets $\{z_\lambda\}$ with $z_\lambda\in E_\lambda$ and by τ-$\limsup E_\lambda$ the set of all points $z\in Z$ which are τ-cluster points of such nets. If τ-$\liminf E_\lambda = \tau$-$\limsup E_\lambda = E$, we write $E = \tau$-$\lim_\lambda E_\lambda$. We will be interested in the case where $Z = \mathcal{B}(\mathcal{H}, \mathcal{K})$, equipped with the strong* and the weak* topology or $Z = \text{Proj}(\mathcal{B}(\mathcal{H})) \times \text{Proj}(\mathcal{B}(\mathcal{K}))$, equipped with the strong operator topology.

We finish this section with a general observation which will be used in the sequel.

LEMMA 1.2. *Let (Z, d) be a metric space and $\{E_\lambda\}_{\lambda \in \Lambda}$ be a net of closed subsets of Z. Then $\liminf E_\lambda$ is closed.*

Proof. It is clear that

$$\liminf E_\lambda = \{x \in Z \,:\, \lim d(x, E_\lambda) = 0\}. \tag{1.2}$$

Let $x \in \overline{\liminf E_\lambda}$, $\varepsilon > 0$ and $x' \in \liminf E_\lambda$ be such that $d(x, x') < \varepsilon$. By (1.2), there exists $\lambda_0 \in \Lambda$ such that $d(x', E_\lambda) \leqslant \varepsilon$ whenever $\lambda \geqslant \lambda_0$. Hence, if $\lambda \geqslant \lambda_0$ then

$$\begin{aligned} d(x, E_\lambda) &= \inf_{y \in E_\lambda} d(x, y) \leqslant \inf_{y \in E_\lambda} d(x, x') + d(x', y) \\ &= d(x, x') + \inf_{y \in E_\lambda} d(x', y) = d(x, x') + d(x', E_\lambda) \leqslant 2\varepsilon. \end{aligned}$$

Thus, $x \in \liminf E_\lambda$. ∎

2. CONVEX HULLS OF BILATTICES

We fix standard (finite) measure spaces (X, m) and (Y, n); let $\mathcal{H} = L^2(X, m)$, $\mathcal{K} = L^2(Y, n)$ and $\mathcal{D}_1 \equiv L^\infty(X, m)$, $\mathcal{D}_2 \equiv L^\infty(Y, n)$ be the corresponding multiplication masas. In this section, we define quantities which generalise the capacity of a subset $\kappa \subseteq X \times Y$ studied in [7], and show that they are capacities. We then show that convergence of a net of subsets of $X \times Y$ with respect to these capacities is equivalent to the convergence of the net of the weakly closed convex hulls of the bilattices corresponding to these sets via (1.1).

Let $\mathcal{Z}$ be the family of all ordered triples of the form

$$((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N),$$

where $(\alpha_i)_{i=1}^N$ (respectively $(\beta_i)_{i=1}^N$) is an (ordered) partition of X (respectively Y) into (finitely many) Borel sets, and $(\mu_i)_{i=1}^N$ is a (finite) collection of non-negative real numbers.

Fix $\kappa \subseteq X \times Y$ and let

$$\mathcal{V}_\kappa = \{A \oplus B : (A, B) \in \text{Conv} S_\kappa\};$$

$\mathcal{V}_\kappa$ is thus a (convex and weakly compact) subset of $\mathrm{Ball}(\mathcal{D}_1 \oplus \mathcal{D}_2)^+$. For $\Delta = ((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N) \in \mathcal{Z}$, let

$$\gamma_\Delta(\kappa) = \inf\Big\{ \sum_{i=1}^N \mu_i(m(\alpha \cap \alpha_i) + n(\beta \cap \beta_i)) : \kappa \subseteq (\alpha \times Y) \cup (X \times \beta) \Big\},$$

where α and β in the above infimum are taken to be measurable. It is clear that if $\mu_i = 1$ for all $i = 1, \dots, N$, then $\gamma_\Delta(\kappa) = \gamma(\kappa)$.

For a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{K})^+$ and $\xi \in \mathcal{H} \oplus \mathcal{K}$, we let

$$\Gamma_\xi(\mathcal{M}) = \sup_{C \in \mathcal{M}} \omega_\xi(C).$$

LEMMA 2.1. *Let* $\Delta = ((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N) \in \mathcal{Z}$ *and* $h = \sum_{i=1}^N \lambda_i(\chi_{\alpha_i} \oplus \chi_{\beta_i})$, *where* $\lambda_i \in \mathbb{C}$, $|\lambda_i|^2 = \mu_i$, $i = 1, \dots, N$. *Then*

(i) $\Gamma_h(\mathcal{V}_\kappa) + \gamma_\Delta(\kappa) = \sum_{i=1}^N \mu_i(m(\alpha_i) + n(\beta_i))$, *and*

(i) *the infimum in the definition of* $\gamma_\Delta(\kappa)$ *is attained.*

Proof. (i) Let $S = S_\kappa$ and $\mathcal{V} = \mathcal{V}_\kappa$. Notice that $\kappa \subseteq (\alpha \times Y) \cup (X \times \beta)$ if and only if $(\alpha^c \times \beta^c) \cap \kappa = \emptyset$, if and only if $(P(\alpha^c), P(\beta^c)) \in S$. We thus have

$$\begin{aligned}
\gamma_\Delta(\kappa) &= \inf\Big\{ \sum_{i=1}^N \mu_i(m(\alpha^c \cap \alpha_i) + n(\beta^c \cap \beta_i)) : (P(\alpha), P(\beta)) \in S \Big\} \\
&= \inf\Big\{ \sum_{i=1}^N \mu_i(m(\alpha_i) - m(\alpha \cap \alpha_i) + n(\beta_i) - n(\beta \cap \beta_i)) : (P(\alpha), P(\beta)) \in S \Big\} \qquad (2.1)\\
&= \sum_{i=1}^N \mu_i(m(\alpha_i) + n(\beta_i)) - \sup\Big\{ \sum_{i=1}^N \mu_i(m(\alpha \cap \alpha_i) + n(\beta \cap \beta_i)) : (P(\alpha), P(\beta)) \in S \Big\}.
\end{aligned}$$

For each $h \in \mathcal{H} \oplus \mathcal{K}$, the function $p : \mathcal{V} \to \mathbb{R}^+$ given by $p(C) = \omega_h(C)$ is weakly continuous and satisfies $p(\lambda C_1 + \mu C_2) = \lambda p(C_1) + \mu p(C_2)$ whenever $\lambda, \mu \geqslant 0$ and $\lambda + \mu = 1$. Since $\mathcal{V}$ is weakly compact and convex, p attains its supremum at an extreme point of $\mathcal{V}$. By Lemma 1.1 (ii), the extreme points of $\mathcal{V}$ coincide with the elements of the form $P \oplus Q$, where $(P, Q) \in S$. Hence

$$\begin{aligned}
\Gamma_h(\mathcal{V}) &= \sup\{((P \oplus Q)h, h) : (P, Q) \in S\} \\
&= \sup\Big\{ \sum_{i,j=1}^N \lambda_i \overline{\lambda}_j ((P(\alpha)\chi_{\alpha_i}, \chi_{\alpha_j}) + (P(\beta)\chi_{\beta_i}, \chi_{\beta_j})) : (P(\alpha), P(\beta)) \in S \Big\} \\
&= \sup\Big\{ \sum_{i=1}^N \mu_i((P(\alpha)\chi_{\alpha_i}, \chi_{\alpha_i}) + (P(\beta)\chi_{\beta_i}, \chi_{\beta_i})) : (P(\alpha), P(\beta)) \in S \Big\} \\
&= \sup\Big\{ \sum_{i=1}^N \mu_i(m(\alpha \cap \alpha_i) + n(\beta \cap \beta_i)) : (P(\alpha), P(\beta)) \in S \Big\}.
\end{aligned}$$

The claim follows from (2.1) and the last identity.

(ii) follows from the previous paragraph. ∎

PROPOSITION 2.2. *Let $\Delta \in \mathcal{Z}$. Then*

(i) *the function γ_Δ is a capacity;*

(ii) $\gamma_\Delta(\kappa) = \gamma_\Delta(\mathrm{cl}_\omega(\kappa))$, *for each $\kappa \subseteq X \times Y$.*

Proof. Fix $\Delta = ((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N) \in \mathcal{Z}$.

(i) Let m' (respectively n') be the measure on X (respectively Y) given by $m'(\alpha) = \sum_{i=1}^N \mu_i m(\alpha \cap \alpha_i)$ (respectively $n'(\beta) = \sum_{i=1}^N \mu_i n(\beta \cap \beta_i)$). Then $\gamma_\Delta(\kappa)$ is the capacity of κ arising from the measures m' and n' as defined in [7], and the claim follows from the Corollary of Lemma 1 and Lemma 2 of [7].

(ii) If $\kappa \subseteq (\alpha \times Y) \cup (X \times \beta)$ then

$$(P(\alpha^c), P(\beta^c)) \in S_\kappa = \mathrm{Bil}\,\mathcal{M}_{\max}(\kappa) = \mathrm{Bil}\,\mathcal{M}_{\max}(\mathrm{cl}_\omega(\kappa)) = S_{\mathrm{cl}_\omega(\kappa)},$$

and so $\mathrm{cl}_\omega(\kappa) \subseteq (\alpha \times Y) \cup (X \times \beta)$ up to a marginally null set. It follows that $\gamma_\Delta(\mathrm{cl}_\omega(\kappa)) \leqslant \gamma_\Delta(\kappa)$ and by (i) we have that $\gamma_\Delta(\mathrm{cl}_\omega(\kappa)) = \gamma_\Delta(\kappa)$. ∎

Notation. Let $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of subsets, and κ be a subset, of $X \times Y$. If $\gamma_\Delta(\kappa) \leqslant \liminf_{\lambda\in\Lambda} \gamma_\Delta(\kappa_\lambda)$ (respectively $\limsup_{\lambda\in\Lambda} \gamma_\Delta(\kappa_\lambda) \leqslant \gamma_\Delta(\kappa)$) for each $\Delta \in \mathcal{Z}$ then we will write symbolically $\kappa \leqslant \liminf_c \kappa_\lambda$ (respectively $\limsup_c \kappa_\lambda \leqslant \kappa$). If $\kappa \leqslant \liminf_c \kappa_\lambda$ and $\limsup_c \kappa_\lambda \leqslant \kappa$, we will write $\kappa = \lim_c \kappa_\lambda$, and we will say that $\{\kappa_\lambda\}$ converges to κ in capacity.

The next theorem, which is the main result of this section, characterises the convergence of the convex hulls of a net of bilattices in terms of the convergence of the corresponding ω-closed sets.

THEOREM 2.3. *Let Λ be a directed set and κ, κ_λ, $\lambda \in \Lambda$ be ω-closed sets. Let $S = S_\kappa$, $S_\lambda = S_{\kappa_\lambda}$. The following hold:*

(i) $\mathrm{Conv}S \subseteq \text{w-}\liminf \mathrm{Conv}S_\lambda$ *if and only if* $\limsup_c \kappa_\lambda \leqslant \kappa$, *and*

(ii) $\text{w-}\limsup \mathrm{Conv}S_\lambda \subseteq \mathrm{Conv}S$ *if and only if* $\kappa \leqslant \liminf_c \kappa_\lambda$.

Proof. Let $\mathcal{V} = \{A \oplus B : (A, B) \in \mathrm{Conv}S\}$ and $\mathcal{V}_\lambda = \{A \oplus B : (A, B) \in \mathrm{Conv}S_\lambda\}$, $\lambda \in \Lambda$. Let $\mathcal{F}$ be the set of vectors in $\mathcal{H} \oplus \mathcal{K}$ of the form $\sum_{i=1}^N \mu_i(\chi_{\alpha_i} \oplus \chi_{\beta_i})$, where $\mu_i \geqslant 0$ and $(\alpha_i)_{i=1}^N$ and $(\beta_i)_{i=1}^N$ are partitions of X and Y, respectively.

(i) Fix $\Delta = ((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N) \in \mathcal{Z}$ and assume that $\mathrm{Conv}S \subseteq \text{w-}\liminf \mathrm{Conv}S_\lambda$. Let $h \in \mathcal{F}$ and $C \in \mathcal{V}$. Then $C = \text{w-}\lim C_\lambda$, for some $C_\lambda \in \mathcal{V}_\lambda$, and hence $(Ch, h) = \lim(C_\lambda h, h)$. It follows that $(Ch, h) \leqslant \liminf \Gamma_h(\mathcal{V}_\lambda)$ and since this holds for each $C \in \mathcal{V}$, we conclude that $\Gamma_h(\mathcal{V}) \leqslant \liminf \Gamma_h(\mathcal{V}_\lambda)$. By Lemma 2.1, $\limsup \gamma_\Delta(\kappa_\lambda) \leqslant \gamma_\Delta(\kappa)$, for each $\Delta \in \mathcal{Z}$.

Conversely, assume that $\limsup \gamma_\Delta(\kappa_\lambda) \leqslant \gamma_\Delta(\kappa)$, for each $\Delta \in \mathcal{Z}$. Fix $(P(\alpha_0), P(\beta_0)) \in S$ and let $h = \chi_{\alpha_0} \oplus \chi_{\beta_0}$ and $\Delta = ((\alpha_0, \alpha_0^c), (\beta_0, \beta_0^c), (1, 0))$. Then $\gamma_\Delta(\kappa) = 0$ and so $\lim \gamma_\Delta(\kappa_\lambda) = 0$. Hence there exist Borel subsets $\alpha_\lambda \subseteq X$ and $\beta_\lambda \subseteq Y$ such that $(P(\alpha_\lambda), P(\beta_\lambda)) \in S_\lambda$, $m(\alpha_0 \cap \alpha_\lambda^c) \to 0$ and $n(\beta_0 \cap \beta_\lambda^c) \to 0$. It follows that $m(\alpha_0 \cap \alpha_\lambda) \to m(\alpha_0)$ and $n(\beta_0 \cap \beta_\lambda) \to n(\beta_0)$. This implies that $P(\alpha_0 \cap \alpha_\lambda) \to P(\alpha_0)$ and $P(\beta_0 \cap \beta_\lambda) \to P(\beta_0)$ in the strong operator topology. On the other hand, $(P(\alpha_0 \cap \alpha_\lambda), P(\beta_0 \cap \beta_\lambda))$ is dominated by $(P(\alpha_\lambda), P(\beta_\lambda))$ and hence belongs to S_λ. We showed that $S \subseteq$s-$\liminf S_\lambda$ and hence $S \subseteq$w-$\liminf S_\lambda$. Thus, the non-closed convex hull of S is contained in w-$\liminf \mathrm{Conv} S_\lambda$. Since the weak operator topology on the unit ball of $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ is metrisable, Lemma 1.2 shows that $\mathrm{Conv} S \subseteq$w-$\liminf \mathrm{Conv} S_\lambda$.

(ii) Assume that w-$\limsup \mathrm{Conv} S_\lambda \subseteq \mathrm{Conv} S$ and that $\gamma_\Delta(\kappa) > \delta > \liminf \gamma_\Delta(\kappa_\lambda)$, for some $\Delta = ((\alpha_i)_{i=1}^N, (\beta_i)_{i=1}^N, (\mu_i)_{i=1}^N) \in \mathcal{Z}$. Let $h = \sum_{i=1}^N \sqrt{\mu_i}\ (\chi_{\alpha_i} \oplus \chi_{\beta_i})$. By Lemma 2.1 (i), $\limsup \Gamma_h(\mathcal{V}_\lambda) > \delta_0 > \Gamma_h(\mathcal{V})$, where $\delta_0 = \sum_{i=1}^N \mu_i(m(\alpha_i) + n(\beta_i)) - \delta$. Let Λ_0 be a subnet of Λ and $\{C_\nu\}_{\nu \in \Lambda_0} \subseteq \mathcal{V}_\nu$, be such that $(C_\nu h, h) > \delta_0$, $\nu \in \Lambda_0$. Assume, without loss of generality, that $C_\nu \to C$ in the weak operator topology. It follows that $(Ch, h) \geqslant \delta$. On the other hand, $C \in \mathcal{V}$ and hence $\Gamma_h(\mathcal{V}) \geqslant \delta_0$, a contradiction.

Assume that $\gamma_\Delta(\kappa) \leqslant \liminf \gamma_\Delta(\kappa_\lambda)$, for each $\Delta \in \mathcal{Z}$. By Lemma 2.1, $\limsup \Gamma_h(\mathcal{V}_\lambda) \leqslant \Gamma_h(\mathcal{V})$ for each $h \in \mathcal{F}$. Suppose that $C =$ w-$\lim C_\nu$ for some $C_\nu \in \mathcal{V}_\nu$, where $\nu \in \Lambda_0$ for some subnet Λ_0 of Λ.

Claim. $\omega_h(C) \leqslant \Gamma_h(\mathcal{V})$, *for each* $h \in \mathcal{H} \oplus \mathcal{K}$.

Proof. Since $\omega_{|h|}(D) = \omega_h(D)$ for each $D \in \mathcal{D}$, we may assume that $h \geqslant 0$. First assume that $h \in \mathcal{F}$. Suppose that $\Gamma_h(\mathcal{V}) < \delta < \omega_h(C)$. There exists $\nu_0 \in \Lambda_0$ such that $\omega_h(C_\nu) > \delta$ whenever $\nu \geqslant \nu_0$. It follows that $\Gamma_h(\mathcal{V}_\nu) > \delta$ if $\nu \geqslant \nu_0$, and hence $\limsup \Gamma_h(\mathcal{V}_\nu) \geqslant \delta$. This implies that $\Gamma_h(\mathcal{V}) \geqslant \delta$, a contradiction.

Suppose next that $h = \xi \oplus \eta \in L^\infty(X, m) \oplus L^\infty(Y, n)$ and that $\|\xi\|_\infty \leqslant 1$ and $\|\eta\|_\infty \leqslant 1$. If $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_N = 1$, let $\alpha_j = \{x \in X : t_{j-1} \leqslant \xi(x) < t_j\}$ and $\beta_j = \{y \in Y : t_{j-1} \leqslant \eta(y) < t_j\}$. Then the vectors of the form $\sum_{i=1}^N t_i(\chi_{\alpha_i} \oplus \chi_{\beta_i})$ approximate $\xi \oplus \eta$ as $\max_{j=1,\dots,N} |t_j - t_{j-1}|$ tends to zero. Hence there exist $h_j \in \mathcal{F}$ such that $h_j \to h$. Since every non-negative L^2-function can be approximated by non-negative L^∞-functions in the L^2-norm, we may relax the assumption that $h \in L^\infty(X, m) \oplus L^\infty(Y, n)$.

We have that $\omega_{h_j} \to \omega_h$ in norm. Assume that $\omega_h(C) > \delta > \Gamma_h(\mathcal{V})$. Then there exists j_0 such that $\omega_{h_j}(C) > \delta$ if $j \geqslant j_0$. It follows that $\Gamma_{h_j}(\mathcal{V}) > \delta$ if $j \geqslant j_0$, and hence there exists $D_j \in \mathcal{V}$ such that $\omega_{h_j}(D_j) > \delta$ if $j \geqslant j_0$. Let D be a weak cluster point of $\{D_j\}$. From the inequality

$$|\omega_h(D) - \omega_{h_j}(D_j)| \leqslant \|\omega_h - \omega_{h_j}\| + |\omega_h(D) - \omega_h(D_j)|$$

it follows that $\omega_h(D) \geqslant \delta$; therefore $\Gamma_h(\mathcal{V}) \geqslant \delta$, a contradiction. ∎

We finish the proof of the theorem. Assume that $C = A \oplus B$ and fix a unit vector $\eta \in \mathcal{H}$ and $T \in \mathrm{Ball}(\mathrm{Op}S)$. Let $\xi = T^*\eta$ and $h = \xi \oplus \eta \in \mathcal{H} \oplus \mathcal{K}$. By the Claim and the weak compactness of $\mathcal{V}$, there exists $A' \oplus B' \in \mathcal{V}$ such that $(AT^*\eta, T^*\eta) + (B\eta, \eta) \leqslant (A'T^*\eta, T^*\eta) + (B'\eta, \eta)$. By Lemma 1.1,

$$(A'T^*\eta, T^*\eta) + (B'\eta, \eta) = (TA'T^*\eta, \eta) + (B\eta, \eta) \leqslant \|\eta\|^2 = 1$$

and hence $(AT^*\eta, T^*\eta) + (B\eta, \eta) \leqslant 1$. This implies that $(TAT^*\eta, \eta) \leqslant ((I - B)\eta, \eta)$ and so $TAT^* \leqslant I - B$. By Lemma 1.1 again, $A \oplus B \in \mathcal{V}$. ∎

COROLLARY 2.4. *Let κ, κ_λ, $\lambda \in \Lambda$, be ω-closed sets. The following are equivalent:*
(i) $\mathrm{Conv}S_\kappa = \text{w-lim}\,\mathrm{Conv}S_{\kappa_\lambda}$;
(ii) $\kappa = \lim_c \kappa_\lambda$.

REMARK 2.5. Easy examples show that the capacity γ [7] is not sufficient to describe the convergence of the convex hulls in Theorem 2.3. For instance, let $X = Y = [0,1]$ with the Lebesgue measure, $\mathcal{D}_1 = \mathcal{D}_2 \equiv L^\infty(0,1)$, $\kappa_n = [0, \frac{1}{2}] \times [0,1]$, $n \in \mathbb{N}$ and $\kappa = [\frac{1}{2}, 1] \times [0,1]$. Then $\gamma(\kappa) = \gamma(\kappa_n) = \frac{1}{2}$ for each $n \in \mathbb{N}$. Letting $P = P([0, \frac{1}{2}])$ we see that $S_{\kappa_n} = \{(L, M) \in \mathrm{Proj}(\mathcal{D}_1) \times \mathrm{Proj}(\mathcal{D}_2) : L \leqslant P^\perp\}$, $n \in \mathbb{N}$, while $S_\kappa = \{(L, M) \in \mathrm{Proj}(\mathcal{D}_1) \times \mathrm{Proj}(\mathcal{D}_2) : L \leqslant P\}$. Thus $(P, I) \in S_\kappa$ does not belong to $\text{w-lim sup}\,\mathrm{Conv}S_{\kappa_n}$.

3. THE CONTINUITY OF THE SUPPORT

In this section, we establish the continuity of the mapping sending a (reflexive) masa-bimodule to its support. We equip the collection of reflexive masa-bimodules with a convergence coming from the weak* and strong* topologies, and the collection of all ω-closed subsets of $X \times Y$ with the convergence arising from the capacities γ_Δ defined in Section 2.

THEOREM 3.1. *Let κ be an ω-closed subset, and $\{\kappa_\lambda\}_{\lambda \in \Lambda}$ be a net of ω-closed subsets, of $X \times Y$. The following are equivalent:*
(i) $\text{w-lim sup}\,\mathrm{Ball}(\mathcal{M}_{\max}(\kappa_\lambda)) \subseteq \mathrm{Ball}(\mathcal{M}_{\max}(\kappa))$;
(ii) $\limsup_c \kappa_n \leqslant \kappa$;
(iii) $S_\kappa \subseteq \text{s-lim inf}\,S_{\kappa_\lambda}$.

Proof. (i)⇒(iii) Set $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$, $\mathcal{M}_\lambda = \mathcal{M}_{\max}(\kappa_\lambda)$, $S = S_\kappa$ and $S_\lambda = S_{\kappa_\lambda}$. For a given $\mathcal{D}_2, \mathcal{D}_1$-bimodule $\mathcal{U}$, let $\mathcal{A}_\mathcal{U}$ be the algebra consisting of the block matrices of the form $\left(\begin{smallmatrix} B & T \\ 0 & A \end{smallmatrix}\right)$ where $A \in \mathcal{D}_1$, $B \in \mathcal{D}_2$ and $T \in \mathcal{U}$. Then

$$\mathrm{Lat}\mathcal{A}_\mathcal{U} = \{Q \oplus P \in \mathrm{Proj}(\mathcal{D}_2 \oplus \mathcal{D}_1) : (P, Q^\perp) \in \mathrm{Bil}\,\mathcal{U}\}.$$

By [12], $\mathrm{Lat}\mathcal{A}_\mathcal{M} \subseteq \text{s-lim}\,\mathrm{Lat}\mathcal{A}_{\mathcal{M}_\lambda}$ and hence $S \subseteq \text{s-lim}\,S_\lambda$.

(iii)⇒(i) Let $\{T_\mu\}_{\mu\in\Lambda_0}$ be a subnet of the net $\{T_\lambda\}_{\lambda\in\Lambda}$, where $T_\lambda \in \mathcal{M}_\lambda$, $\lambda \in \Lambda$, and assume that $T_\mu \to T$ weakly. Fix $(P,Q) \in S$. Then there exists $(P_\lambda, Q_\lambda) \in S_\lambda$ such that $P_\lambda \to P$ and $Q_\lambda \to Q$ strongly. It follows that $Q_\mu T_\mu P_\mu \to QTP$ weakly. Since $Q_\mu T_\mu P_\mu = 0$ for each $\mu \in \Lambda_0$, we conclude that $QTP = 0$; in other words, $T \in \mathcal{M}$.

(ii)⇒(iii) Was shown in the proof of Theorem 2.3 (i).

(iii)⇒(ii) As in the proof of Theorem 2.3 (i), (iii) implies $\mathrm{Conv}S \subseteq$ w-lim inf $\mathrm{Conv}S_\lambda$. The claim now follows from Theorem 2.3 (i). ∎

REMARK 3.2. Conditions (i)–(iii) of Theorem 3.1 are not equivalent to $\limsup \mathrm{Ball}\,(\mathcal{M}_{\min}\,(\kappa_\lambda)) \subseteq \mathcal{M}_{\min}(\kappa)$. To see this, let $\kappa \subseteq X \times Y$ be any ω-closed set which does not satisfy operator synthesis [1]. Assume that $\kappa^c = \bigcup\limits_{j=1}^{\infty} \alpha_j \times \beta_j$. Let $\kappa_n = \Big(\bigcup\limits_{j=1}^{n} \alpha_j \times \beta_j\Big)^c$. Then κ_n is a finite union of Borel rectangles, and it is easily seen that the sets of this form satisfy operator synthesis. We have that $\kappa_{n+1} \subseteq \kappa_n$ for each $n \in \mathbb{N}$ and that $\bigcap\limits_{n=1}^{\infty} \kappa_n = \kappa$. It follows that $\bigcap\limits_{n=1}^{\infty} \mathcal{M}_{\max}(\kappa_n) = \mathcal{M}_{\max}(\kappa)$. Hence w-lim sup $\mathcal{M}_{\max}(\kappa_n) \subseteq \mathcal{M}_{\max}(\kappa)$. However

$$\bigcap_{n=1}^{\infty} \mathcal{M}_{\min}(\kappa_n) = \bigcap_{n=1}^{\infty} \mathcal{M}_{\max}(\kappa_n) = \mathcal{M}_{\max}(\kappa) \neq \mathcal{M}_{\min}(\kappa),$$

and hence w-lim sup $\mathcal{M}_{\min}(\kappa_n) \not\subseteq \mathcal{M}_{\min}(\kappa)$.

THEOREM 3.3. *Let κ be an ω-closed subset, and $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of ω-closed subsets, of $X \times Y$. Consider the following statements:*

(i) $\mathrm{Ball}(\mathcal{M}_{\max}(\kappa)) \subseteq$ s*-lim inf $\mathrm{Ball}(\mathcal{M}_{\max}(\kappa_\lambda))$*;*

(ii) $\kappa \leqslant \liminf\limits_{c} \kappa_\lambda$*;*

(iii) s-lim sup $S_{\kappa_\lambda} \subseteq S_\kappa$.

Then (i)⇒(ii)⇒(iii).

Proof. (i)⇒(ii) Set $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$, $\mathcal{M}_\lambda = \mathcal{M}_{\max}(\kappa_\lambda)$, $S = S_\kappa$ and $S_\lambda = S_{\kappa_\lambda}$. By Theorem 2.3, it suffices to show that w-lim sup $\mathrm{Conv}S_\lambda \subseteq \mathrm{Conv}S$. Suppose that $(A_\mu, B_\mu) \in \mathrm{Conv}S_\mu$, for each $\mu \in \Lambda_0$, where Λ_0 is a subnet of Λ, and that $(A_\mu, B_\mu) \to (A, B)$ weakly. Take $T \in \mathrm{Ball}(\mathcal{M})$ and let $T_\lambda \in \mathrm{Ball}(\mathcal{M}_\lambda)$ be such that $T_\lambda \to T$ in the strong* topology. Then

$$I - B_\mu - T_\mu A_\mu T_\mu^* \to I - B - TAT^*$$

weakly. By Lemma 1.1 (i), $I - B_\mu - T_\mu A_\mu T_\mu^* \geqslant 0$ for each μ and hence $TAT^* \leqslant I - B$. By Lemma 1.1 (i) again, $(A, B) \in \mathrm{Conv}S$.

(ii)⇒(iii) Suppose that $(P(\alpha_\mu), P(\beta_\mu)) \in S_\mu$ for $\mu \in \Lambda_0$, where Λ_0 is a subnet of Λ, and that $(P(\alpha_\mu), P(\beta_\mu)) \to (P(\alpha), P(\beta))$ strongly. Then $m(\alpha \cap \alpha_\mu^c) \to 0$ and $n(\beta \cap \beta_\mu^c) \to 0$. Let $\Delta = ((\alpha, \alpha^c), (\beta, \beta^c), (1, 0))$. Since $\kappa_\mu \subseteq (\alpha_\mu^c \times Y) \cup (X \times \beta_\mu^c)$

up to a marginally null set, we have that

$$\gamma_\Delta(\kappa_\mu) \leqslant m(\alpha \cap \alpha_\mu^c) + n(\beta \cap \beta_\mu^c) \to 0$$

and hence $\gamma_\Delta(\kappa) = 0$. By Lemma 2.1 (ii), there exist Borel sets $\delta^1 \subseteq X$, $\delta^2 \subseteq Y$, such that $\kappa \subseteq (\delta^1 \times Y) \cup (X \times \delta^2)$ and $m(\alpha \cap \delta^1) = 0$ and $n(\beta \cap \delta^2) = 0$. Thus, $(\alpha \times \beta) \cap \kappa \subseteq ((\alpha \cap \delta^1) \times Y) \cup (X \times (\beta \cap \delta^2))$ which implies that $(\alpha \times \beta) \cap \kappa \simeq \emptyset$ and hence $(P(\alpha), P(\beta)) \in S_\kappa$. ∎

PROPOSITION 3.4. *The converse implications in Theorem* 3.3 *do not hold.*

Proof. We first show that implication (ii)⇒(i) in Theorem 3.3 does not hold. By [4], there exists an ω-closed set $\kappa \subseteq X \times Y$ such that κ is the ω-closure of an ω-open set $\kappa_0 = \bigcup_{j=1}^{\infty} \alpha_j \times \beta_j$, and $\mathcal{M}_{\min}(\kappa) \neq \mathcal{M}_{\max}(\kappa)$. Let $\kappa_n = \bigcup_{j=1}^{n} \alpha_j \times \beta_j$, $n \in \mathbb{N}$, and $\mathcal{U} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_{\max}(\kappa_n)}^{w^*}$. Since the support $\widetilde{\kappa}$ of $\mathcal{U}$ contains κ_n for each n, we have that $\widetilde{\kappa} \supseteq \kappa$, up to a marginally null set, and so $\mathcal{M}_{\max}(\kappa) \subseteq \mathcal{M}_{\max}(\widetilde{\kappa})$. On the other hand, clearly $\mathcal{U} \subseteq \mathcal{M}_{\max}(\kappa)$ and hence $\kappa = \widetilde{\kappa}$. Since the sets κ_n satisfy operator synthesis, we conclude that $\mathcal{U} = \mathcal{M}_{\min}(\kappa)$. Clearly, s*-lim inf Ball$(\mathcal{M}_{\max}(\kappa_n)) \subseteq \mathcal{U}$. Since $\mathcal{U} \neq \mathcal{M}_{\max}(\kappa)$, we conclude that

$$\mathrm{Ball}(\mathcal{M}_{\max}(\kappa)) \not\subseteq \text{s}^*\text{-}\liminf \mathrm{Ball}(\mathcal{M}_{\max}(\kappa_n)).$$

It is obvious that $\mathrm{Conv}S_\kappa \subseteq \bigcap \mathrm{Conv}S_{\kappa_n}$ and easy to see that $S_\kappa = \bigcap S_{\kappa_n}$. Assume that $(A, B) \in \mathrm{Conv}S_{\kappa_n}$ for each n. By Lemma 3.2 of [13], $(E_A[s,1], E_B[t,1]) \in S_{\kappa_n}$ whenever $s + t > 1$ and hence $(E_A[s,1], E_B[t,1]) \in S_\kappa$ whenever $s + t > 1$. By Lemma 3.1 of [13] again, $(A, B) \in \mathrm{Conv}S_\kappa$. We thus have that $\mathrm{Conv}S_\kappa = \bigcap \mathrm{Conv}S_{\kappa_n}$. In particular, w-lim sup $\mathrm{Conv}S_n = \mathrm{Conv}S$. By Theorem 2.3 (ii), $\kappa \leqslant \liminf \kappa_n$. We showed that (ii) does not imply (i).

We next show that implication (iii)⇒(ii) in Theorem 3.3 does not hold. Let $\mathcal{H} = L^2(0,1)$ and $\mathcal{D} \equiv L^\infty(0,1)$. Let $\{L_n\} \subseteq \mathcal{D}$ be a sequence of projections which converges weakly to an element $A \in \mathcal{D}$ such that $\|A\| = \|I - A\| = 1$ and $\ker A = \ker(I - A) = \{0\}$. Let $S = \{(P, 0), (0, Q) : P, Q \in \mathrm{Proj}(\mathcal{D})\}$ and $S_n = \{(P, Q) \in \mathrm{Proj}(\mathcal{D}) \times \mathrm{Proj}(\mathcal{D}) : P \leqslant L_n \leqslant Q^\perp\} \cup S$. It is easy to see that $S =$s-lim S_n.

We have that $\mathrm{Conv}S$ consists of the weak limits of the pairs of the form $\left(\sum_{i=1}^{k} \lambda_i P_i, \sum_{j=1}^{m} \mu_j Q_j\right)$, where $P_i, Q_j \in \mathrm{Proj}(\mathcal{D})$ and $\sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{m} \mu_j = 1$, $\lambda_i, \mu_j \geqslant 0$. It follows that if $(A, B) \in \mathrm{Conv}S$ then $\|A\| + \|B\| \leqslant 1$. On the other hand, $(A, I - A) \in$ w-lim sup $\mathrm{Conv}S_n$ and hence $(A, I - A) \notin \mathrm{Conv}S$. By Theorem 2.3 (ii), condition (ii) of Theorem 3.3 does not hold. ∎

Theorems 3.1 and 3.3 have the following immediate corollary. Implication (i)⇒(ii) establishes the continuity of the mapping sending a masa-bimodule to its support.

COROLLARY 3.5. *Let κ be an ω-closed subset, and $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of ω-closed subsets, of $X \times Y$. Consider the following statements:*
(i) w-$\limsup \mathrm{Ball}(\mathcal{M}_{\max}(\kappa_\lambda) = \mathrm{Ball}(\mathcal{M}_{\max}(\kappa)) =$ s*-$\liminf \mathrm{Ball}\ (\mathcal{M}_{\max}\ (\kappa_\lambda)$*;*
(ii) $\kappa = \lim_c \kappa_\lambda$*;*
(iii) $S =$ s-$\lim S_\lambda$.
Then (i)⇒(ii)⇒(iii).

In the case where the limit masa-bimodule is trivial, the continuity result takes a simpler form which we state in the next proposition.

PROPOSITION 3.6. *Let $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of ω-closed subsets of $X \times Y$. The following statements are equivalent:*
(i) w-$\lim \mathrm{Ball}(\mathcal{M}_{\max}(\kappa_\lambda)) = \{0\}$*;*
(ii) $\lim \gamma(\kappa_\lambda) = 0$.

Proof. (i)⇒(ii) follows from Corollary 3.5 by taking $\Delta = ((X),(Y),(1))$.
(ii)⇒(i) Assume that $\gamma(\kappa_\lambda) \to 0$ and let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a weakly convergent net with $T_\lambda \in \mathrm{Ball}(\mathcal{M}_{\max}(\kappa_\lambda))$. There exist Borel sets $\alpha_\lambda \subseteq X$ and $\beta_\lambda \subseteq Y$ with $(P(\alpha_\lambda^c), P(\beta_\lambda^c)) \in S_{\kappa_\lambda}$ such that $\lim m(\alpha_\lambda) = \lim n(\beta_\lambda) = 0$. We have

$$T_\lambda = P(\beta_\lambda)T_\lambda P(\alpha_\lambda) + P(\beta_\lambda^c)T_\lambda P(\alpha_\lambda) + P(\beta_\lambda)T_\lambda P(\alpha_\lambda^c).$$

It follows that $T_\lambda \to 0$ weakly and hence $\limsup \mathcal{M}_{\max}(\kappa_\lambda) = \{0\}$. ∎

4. LOWER SEMI-CONTINUITY OF $\mathcal{M}_{\min}$

Our next aim is to establish a partial converse of the implication (ii)⇒(i) of Theorem 3.3. Namely, we show that if Λ is a directed set, κ and κ_λ, $\lambda \in \Lambda$, are ω-closed sets, $\kappa \leqslant$ c-$\liminf \kappa_\lambda$ and, moreover κ satisfies operator synthesis, then every $T \in \mathcal{M}_{\max}(\kappa)$ can be approximated by a net $\{T_\lambda\}_{\lambda\in\Lambda}$, where $T_\lambda \in \mathcal{M}_{\max}(\kappa_\lambda)$, on every countable set of vectors. This is a consequence of a more general result on the lower semi-continuity of the map sending an ω-closed set κ to the masa-bimodule $\mathcal{M}_{\min}(\kappa)$.

We will need the notion of a semistrong limit of a net of projections introduced by Halmos in [6]. If $\{P_\lambda\}_{\lambda\in\Lambda}$ is a net of projections on a Hilbert space $\mathcal{H}$ then we say that P_λ converge semistronlgy to a projection P on $\mathcal{H}$ (and write $P =$ss-$\lim P_\lambda$) if for every $x = Px$ there exist $x_\lambda = P_\lambda x_\lambda$ such that $x_\lambda \to x$, and whenever y is a cluster point of a net $\{y_\lambda\}_{\lambda\in\Lambda}$ with $y_\lambda = P_\lambda y_\lambda$, we have $y = Py$. In other words, $P =$ss-$\lim P_\lambda$ if the space $P\mathcal{H}$ is the limit of the subspaces $P_\lambda\mathcal{H}$ in the power set of $\mathcal{H}$, when $\mathcal{H}$ is equipped with its norm topology.

We will use the following fact proved in [6].

LEMMA 4.1. *If $T =$ w-$\lim P_\lambda$ then* $[\ker(I-T)] =$ ss-$\lim P_\lambda$.

LEMMA 4.2. *Let Λ be a directed set and $\mathcal{U},\mathcal{U}_\lambda \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ be reflexive subspaces. Suppose that if $Q_\lambda \mathcal{U}_\lambda P_\lambda = \{0\}$ for each $\lambda \in \Lambda$, and $(P, Q^\perp)$ is a semistrong cluster point of $\{(P_\lambda, Q_\lambda^\perp)\}_{\lambda\in\Lambda}$ then $Q\mathcal{U}P = \{0\}$ (where P, Q, P_λ, Q_λ are projections). Then for each $T \in \mathcal{U}$ and each $x \in \mathcal{H}$ there exists a net $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $T_\lambda \in \mathcal{U}_\lambda$ and $T_\lambda x \to Tx$.*

Proof. Let $T \in \mathcal{U}$, $x \in \mathcal{H}$ and P be the projection onto the one dimensional space generated by x. Suppose that $\operatorname{dist}(Tx, \mathcal{U}_\lambda x) \nrightarrow 0$. Then there exists $\varepsilon > 0$ and a cofinal subset $\Lambda_0 \subseteq \Lambda$ such that $\operatorname{dist}(Tx, \mathcal{U}_\lambda x) > \varepsilon$ for each $\lambda \in \Lambda_0$.

Let Q_λ be the projection onto $(\mathcal{U}_\lambda x)^\perp$. We have that $Q_\lambda \mathcal{U}_\lambda P = \{0\}$. Let Λ_1 be a subnet of Λ_0 such that $A = \text{w-}\lim_{\mu\in\Lambda_1} Q_\mu^\perp$. By Lemma 4.1, if $Q = [\ker(I-A)]^\perp$ then $Q^\perp = \text{ss-}\lim_{\mu\in\Lambda_1} Q_\mu^\perp$.

By the assumption, $QTP = 0$. Thus, $QTx = 0$, that is,

$$Tx \in \ker(I-A) = \text{ss-}\lim_{\mu\in\Lambda_1} Q_\mu^\perp.$$

Hence, $\lim_{\mu\in\Lambda_1} \operatorname{dist}(Tx, Q_\mu^\perp \mathcal{K}) = 0$, a contradiction. ∎

THEOREM 4.3. *Let κ be an ω-closed subset, and $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of ω-closed subsets, of $X \times Y$. If $\kappa \leqslant \liminf_c \kappa_\lambda$ then for each $T \in \mathcal{M}_{\min}(\kappa)$ and each countable collection of vectors $\mathcal{E} \subseteq \mathcal{H}$ there exists a net $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $T_\lambda \in \mathcal{M}_{\min}(\kappa_\lambda)$ and $T_\lambda x \to Tx$, for each $x \in \mathcal{E}$.*

Proof. Let $S = S_\kappa$, $S_\lambda = S_{\kappa_\lambda}$,

$$\widehat{S} = \{(P,Q) \in \operatorname{Proj}(\mathcal{B}(l^2 \otimes \mathcal{H})) \times \operatorname{Proj}(\mathcal{B}(l^2 \otimes \mathcal{K})) : Q(1 \otimes \mathcal{M}_{\min}(\kappa))P = \{0\}\}$$

and

$$\widehat{S}_\lambda = \{(P,Q) \in \operatorname{Proj}(\mathcal{B}(l^2 \otimes \mathcal{H})) \times \operatorname{Proj}(\mathcal{B}(l^2 \otimes \mathcal{K})) : Q(1 \otimes \mathcal{M}_{\min}(\kappa_\lambda))P = \{0\}\},$$

where $1 \otimes \mathcal{U} = \{I \otimes T : T \in \mathcal{U}\}$. Assume that P and Q are projections on $l^2 \otimes \mathcal{H}$ and $l^2 \otimes \mathcal{K}$, respectively, such that $(P, Q^\perp)$ is a semistrong cluster point of a net $\{(P_\lambda, Q_\lambda^\perp)\}_{\lambda\in\Lambda}$, where $(P_\lambda, Q_\lambda) \in \widehat{S}_\lambda$, $\lambda \in \Lambda$. Then there exist subnets $\{P_\mu\}_{\mu\in\Lambda_0}$ and $\{Q_\mu\}_{\mu\in\Lambda_0}$ such that $P_\mu \to_{\mu\in\Lambda_0} P$ and $Q_\mu^\perp \to_{\mu\in\Lambda_0} Q^\perp$ semistrongly. We may assume that $P_\mu \to_{\mu\in\Lambda_0} A$ and $Q_\lambda^\perp \to_{\mu\in\Lambda_0} I - B$ weakly, for some operators $A \in \mathcal{B}(l^2 \otimes \mathcal{H})$ and $B \in \mathcal{B}(l^2 \otimes \mathcal{K})$. By Lemma 4.1, $P_\mu \to_{\mu\in\Lambda_0} [\ker(I-A)]$ and $Q_\lambda^\perp \to_{\mu\in\Lambda_0} [\ker B]$ semistrongly and so $P = [\ker(I-A)]$ and $Q = [\operatorname{im} B]$.

For each state φ of $\mathcal{B}(l^2)$, let

$$L_\varphi : \mathcal{B}(l^2 \otimes \mathcal{H}, l^2 \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H},\mathcal{K})$$

be the slice map given by $L_\varphi(A \otimes T) = \varphi(A)T$, $A \in \mathcal{B}(l^2)$, $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. We have that

$$L_\varphi(P_\mu) \to_{\mu\in\Lambda_0} L_\varphi(A) \quad \text{and} \quad L_\varphi(Q_\lambda) \to_{\mu\in\Lambda_0} L_\varphi(B)$$

in the weak operator topology. By Lemma 4.1 and Corollary 4.4 of [13], $(L_\varphi(P_\mu), L_\varphi(Q_\mu)) \in \mathrm{Conv}S_\mu$ for each $\mu \in \Lambda_0$. Theorem 2.3 (ii) now implies that $(L_\varphi(A), L_\varphi(B)) \in \mathrm{Conv}S$. By Lemma 5.1 of [13], $(P, Q) \in \widehat{S}$. We showed that the condition of Lemma 4.2 is satisfied for $1 \otimes \mathcal{M}_{\min}(\kappa)$ and $1 \otimes \mathcal{M}_{\min}(\kappa_\lambda)$, $\lambda \in \Lambda$.

Let $\{x_j\}_{j=1}^\infty \subseteq \mathcal{H}$ be a countable set of non-zero vectors and

$$\xi = \sum_{j=1}^{\infty} \frac{1}{n\|x_j\|} e_j \otimes x_j,$$

where $\{e_j\}$ is the standard basis of l^2. Lemma 4.2 implies that there exists a net $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $T_\lambda \in \mathcal{M}_{\min}(\kappa_\lambda)$ and $(I \otimes T)\xi = \lim_{\lambda\in\Lambda}(I \otimes T_\lambda)\xi$. This implies that $T_\lambda x_j \to_{\lambda\in\Lambda} Tx_j$, for each j. The proof is complete. ∎

We have the following immediate corollary.

COROLLARY 4.4. *Let κ be an ω-closed subset, and $\{\kappa_\lambda\}_{\lambda\in\Lambda}$ be a net of ω-closed subsets, of $X \times Y$, and assume that κ satisfies operator synthesis. If $\kappa \leqslant \liminf_c \kappa_\lambda$ then for each $T \in \mathcal{M}_{\max}(\kappa)$ and each countable collection of vectors $\mathcal{E} \subseteq \mathcal{H}$ there exists a net $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $T_\lambda \in \mathcal{M}_{\max}(\kappa_\lambda)$ and $T_\lambda x \to Tx$, for each $x \in \mathcal{E}$.*

REMARK 4.5. Haagerup and Winslow [5] have studied a different kind of limes inferior for von Neumann algebras: if τ is a topology on $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ is a net of subspaces of $\mathcal{B}(\mathcal{H},\mathcal{K})$, let $\liminf^\tau \mathcal{U}_\lambda$ be the set of all operators $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with the property that for each τ-neighborhood Ω of T there exists $\lambda_0 \in \Lambda$ such that $\Omega \cap \mathcal{U}_\lambda \neq \emptyset$ if $\lambda \geqslant \lambda_0$. Theorem 4.3 implies that if $\kappa \leqslant \liminf_c \kappa_\lambda$ then $\mathcal{M}_{\min}(\kappa) \subseteq \liminf^{\mathrm{SOT}} \mathcal{M}_{\min}(\kappa_\lambda)$, where SOT denotes the strong operator topology.

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