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CERTAIN SUBALGEBRAS OF THE TENSOR PRODUCT OF GRAPH ALGEBRAS

AMY B. CHAMBERS

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ABSTRACT. Using an action of the unit circle, we construct a conditional expectation from the tensor product of two graph algebras, $C^*(E_1) \otimes C^*(E_2)$, onto a defined subalgebra $\mathcal{B}$. In addition, we make precise the required hypotheses for this subalgebra $\mathcal{B}$ to be isomorphic to the graph algebra $C^*(\mathcal{E})$ for the graph $\mathcal{E}$ defined using the Cartesian products of the vertex and edge sets of the graphs E_1 and E_2. We study two concrete examples of the conditional expectation constructed for the general case, and we discuss the ideas of index and Paschke crossed product by an endomorphism.

KEYWORDS: *Graph algebra, conditional expectation, wavelet, Paschke crossed product by an endomorphism, Cuntz algebra, index.*

MSC (2000): 46L05.

1. INTRODUCTION

In 1977, J. Cuntz first introduced the C^*-algebras known as the Cuntz algebras $\mathcal{O}_n$ for $2 \leqslant n \leqslant \infty$ in [3]. Cuntz C^*-algebras are generated by isometries and are purely infinite, simple, separable, nuclear C^*-algebras, and these C^*-algebras have proved very important over the years in classifying all C^*-algebras of this type (see [9]). In 1980, M. Enomoto and Y. Watatani introduced the notion of C^*-algebras associated to directed graphs represented by the adjacency matrices (see [4]), and the concept of graph C^*-algebras has been developed further in intervening years by A. Kumjian, D. Pask, and I. Raeburn, among others (see [6]). In the past few years, the notion of k-graph algebras has been introduced for $k \geqslant 2$, and the general topic of graph C^*-algebras has become of such interest that I. Raeburn gave an entire CBMS lecture series on them, resulting in the reference [8].

In [3], Cuntz showed that each of the C^*-algebras $\mathcal{O}_n$ had a very natural UHF subalgebra. He constructed directly a conditional expectation from the C^*-algebra $\mathcal{O}_n$ onto its UHF subalgebra. As the theory of graph C^*-algebras developed further, it was shown that for finite n, the $\mathcal{O}_n$ were all examples of graph algebras. Moreover, the UHF algebras were special cases of so-called "core" subalgebras of graph algebras, which were AF-algebras. In this more general setting, the conditional expectation could be constructed by viewing each core subalgebra as the invariant subalgebra under the "gauge" action of the circle group $\mathbb{T}$. This was an important observation, as in general it is not always true that there exists a conditional expectation of a C^*-algebra onto a subalgebra. For an example of a case in which such a conditional expectation does not exist, see [5]. Once one has a conditional expectation of a C^*-algebra onto a subalgebra, it is possible to evaluate the "index" of the subalgebra in the original C^*-algebra, a concept for C^*-algebras generalizing the Jones index for von Neumann algebras, first defined by Y. Watatani in [10].

Cuntz showed that the algebras $\mathcal{O}_n$ were generated by the core UHF subalgebra and a single isometry that normalized the subalgebra. In [7], W. Paschke formalized the notion of the crossed product of a C^*-algebra by an endomorphism, which was further studied by many authors. In particular, in [2], M. Choda gave a necessary condition for a crossed product of a simple C^*-algebra by an endomorphism to itself to be simple.

All of the above topics aroused our interest in the embedding of a certain subalgebra isomorphic to $\mathcal{O}_{d_1 d_2}$ into the tensor product $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Some natural questions come to mind. First, is there a conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ onto this subalgebra? If so, what is the index of the subalgebra in the containing C^*-algebra? Thirdly, in our setting, can $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ be written as the crossed product of this subalgebra by an endomorphism in the sense of Paschke, and if so, is this related in any way to the work of Choda mentioned above? Finally, if the answer to any of the above questions is positive, can similar results be obtained for more general graph algebras? This article will address all of these questions.

Throughout this article, isometries and projections play a key role. Already in [1], O. Bratteli and P. Jorgensen had noticed that the isometries used to construct the Cuntz algebras $\mathcal{O}_n$ were very relevant to the study of wavelets. From their work, we know that a set of low and high-pass filters that lead to a wavelet family for dilation by the positive integer $d_1 > 1$ on $L^2(\mathbb{R})$ can be used to define d_1 isometries that satisfy the Cuntz relations for $\mathcal{O}_{d_1}$. Applying the Bratteli-Jorgensen construction to the dilation matrix $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ on $L^2(\mathbb{R} \times \mathbb{R})$ leads to a representation of $\mathcal{O}_{d_1 d_2}$, which is not isomorphic to $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. This somewhat counterintuitive result led us to search for a conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ to its subalgebra isomorphic to $\mathcal{O}_{d_1 d_2}$.

In Section 2 of this article we present a general result involving a conditional expectation from the tensor product of graph algebras to a subalgebra $\mathcal{B}$, which we define, and we apply this general result to find our desired conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ to its subalgebra isomorphic to $\mathcal{O}_{d_1 d_2}$. In Section 3 we examine the subalgebra $\mathcal{B}$ more closely and investigate when $\mathcal{B}$ might be a graph algebra itself. In Section 4 we give another example of an application of the general result, showing that $\mathcal{B}$ is not always a graph algebra of the form defined in Section 3 . In Section 5 we show that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is the crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism, and we conclude in Section 6 with a brief discussion of the index of a conditional expectation, giving the index of each of the examples discussed in this article.

2. A CONDITIONAL EXPECTATION ON THE TENSOR PRODUCT OF GRAPH ALGEBRAS

2.1. DEFINITION OF THE ALGEBRA AND SUBALGEBRA.

We begin by stating the definition of conditional expectation used in this paper.

DEFINITION 2.1. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a unital C^*-subalgebra of $\mathcal{A}$. A map ϕ from $\mathcal{A}$ onto $\mathcal{B}$ is a *projection of norm one* if ϕ is linear, $\phi(x) = x$ for all $x \in \mathcal{B}$, and $\|\phi(x)\| \leqslant \|x\|$.

DEFINITION 2.2. A *conditional expectation* from a C^*-algebra $\mathcal{A}$ onto a C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a projection of norm one from $\mathcal{A}$ onto $\mathcal{B}$.

Given an algebra $\mathcal{A}$ and a subalgebra $\mathcal{B}$, it is not always true that there is a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. However, if we have a compact group action on a C^*-algebra, it automatically follows that there is a conditional expectation onto the fixed point subalgebra. Also, we know that there is a gauge action of the unit circle on any graph algebra. With these facts in mind, it becomes of interest, given an algebra $\mathcal{A}$ and a subalgebra $\mathcal{B}$, to find, if it exists, the compact group action on $\mathcal{A}$ whose fixed point algebra is exactly $\mathcal{B}$. Then, we will have a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. In this paper we examine the case in which $\mathcal{A}$ is the tensor product of two graph algebras.

Now, given a directed graph $E = (E^0, E^1, r, s)$, the graph algebra $C^*(E)$ associated to E is the universal C^*-algebra generated by a family of partial isometries and projections satisfying certain properties, making it into a Cuntz–Krieger E-family. These properties are listed in the following definition.

DEFINITION 2.3. Associate to each $e \in E^1$ a partial isometry S_e and to each vertex $v \in E^0$ a projection P_v. If this family of partial isometries and projections $\{P_v, S_e : v \in E^0, e \in E^1\}$ satisfies the following relations, then the collection is called a *Cuntz–Krieger E-family*:

(i) ${S_e}^* S_f = 0$ for $e, f \in E^1$ with $e \neq f$.

(ii) ${S_e}^* S_e = P_{s(e)}$ where $s(e)$ is the source of the edge e.

(iii) If v is not a source, $\displaystyle\sum_{e \in r^{-1}(v)} S_e {S_e}^* = P_v$.

It has been shown that

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : s(\mu) = s(\nu)\},$$

where $\mu = \mu_1\mu_2\cdots\mu_n$ and $\nu = \nu_1\nu_2\cdots\nu_k$ are paths in E with the same source, and $S_\mu = S_{\mu_1}\cdots S_{\mu_n}$. Here we denote the length of the path μ by $|\mu| = n$. Also, any graph algebra is nuclear, as I. Raeburn remarks in Chapter 4 of [8], and so $C^*(E_1)\otimes C^*(E_2)$ is nuclear for any directed graphs E_1 and E_2. This implies that there is a unique C^*-norm on the tensor product $C^*(E_1)\otimes C^*(E_2)$. Let $\mathcal{A} = C^*(E_1)\otimes C^*(E_2)$ and let $\mathcal{B}$ be the subalgebra

$$\overline{\operatorname{span}}\{S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^* : s(\mu) = s(\nu)\ , s(\alpha) = s(\beta), |\mu| - |\nu| = |\alpha| - |\beta|\}. \tag{2.1}$$

If we can show that there is an action of the circle group $\mathbb{T}$ on $\mathcal{A}$ whose fixed point subalgebra is $\mathcal{B}$, then we will be able to construct our desired conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$.

2.2. CONSTRUCTION OF THE CONDITIONAL EXPECTATION.

LEMMA 2.4. *There exists an action of the circle group $\mathbb{T}$ on $\mathcal{A}$ whose fixed point subalgebra is $\mathcal{B}$.*

Proof. We know there exists a gauge action γ of $\mathbb{T}$ on $C^*(E_1)$ such that $\gamma_z(S_e) = zS_e$ for each edge e of E_1 and $\gamma_z(P_v) = P_v$ for each vertex v of E_1 (see [8]). Similarly, there is such a gauge action $\gamma\prime$ on $C^*(E_2)$. Form the tensor product of these actions in the usual way, obtaining an action $\gamma\otimes\gamma\prime$ of the product group $\mathbb{T}\times\mathbb{T}$ on $C^*(E_1)\otimes C^*(E_2)$. Let α_z be the action of $\mathbb{T}$ on $\mathcal{A}$ obtained by restricting $\gamma\otimes\gamma\prime$ to the closed subgroup $\{(z,\bar{z}) : z\in\mathbb{T}\}$. Our action acts on the generators of $\mathcal{A}$ as follows:

$$\alpha_z(S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*) = z^{|\mu|-|\nu|-(|\alpha|-|\beta|)} S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*.$$

Clearly the generators of $\mathcal{A}$ remaining fixed under this action are those elements $S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*$ for which $|\mu|-|\nu| = |\alpha|-|\beta|$, that is, the generators of $\mathcal{B}$. So, the fixed point subalgebra of this action is precisely $\mathcal{B}$. ∎

This action of $\mathbb{T}$ on $\mathcal{A}$ leads us to the construction of our desired conditional expectation.

THEOREM 2.5. *: There exists a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. Define $\Phi : \mathcal{A}\to\mathcal{B}$ by $\Phi(a) = \int_{\mathbb{T}} \alpha_z(a)\ \mathrm{d}z$. Then

$$\begin{aligned}
\Phi(S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*) &= \int_{\mathbb{T}} z^{|\mu|-|\nu|-(|\alpha|-|\beta|)} S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*\ \mathrm{d}z \\
&= \int_0^1 \mathrm{e}^{2\pi \mathrm{i}t[|\mu|-|\nu|-(|\alpha|-|\beta|)]} S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*\ \mathrm{d}z \\
&= \begin{cases} S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^* & \text{if } |\mu|-|\nu| = |\alpha|-|\beta|, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

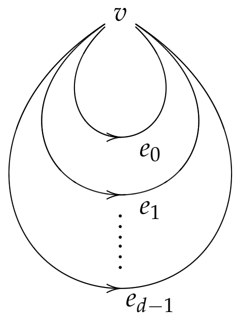

FIGURE 1. Graph Associated with $\mathcal{O}_d$

So, Φ maps $\mathcal{A}$ onto $\mathcal{B}$. ∎

2.3. EXAMPLE: A CONDITIONAL EXPECTATION FROM $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ ONTO A SUBALGEBRA ISOMORPHIC TO $\mathcal{O}_{d_1 d_2}$. One application of the above result is to the tensor product of Cuntz algebras. We know that $\mathcal{O}_d \cong C^*(E)$ where E is the graph shown in Figure 1. Then $\mathcal{O}_{d_1} \cong C^*(E_1)$ and $\mathcal{O}_{d_2} \cong C^*(E_2)$ where E_1 and E_2 are graphs similar to the one shown in Figure 1, with d_1 and d_2 edges respectively. Then by Theorem 2.5, there exists a conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ onto the subalgebra $\mathcal{B}$ defined by (2.1). In this case, $\mathcal{B}$ turns out to be isomorphic to a Cuntz algebra itself, which we know implies it is a graph algebra. This prompts us to investigate the subalgebra $\mathcal{B}$ more closely, which we will do in the next section. First, let us show that in this case $\mathcal{B}$ is isomorphic to a Cuntz algebra.

Since the set of generators of $\mathcal{B}$ is closed under the operations of multiplication and of taking stars, we have

$$\mathcal{B} = C^*(\{S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^* : s(\mu) = s(\nu)\ , s(\alpha) = s(\beta), |\mu| - |\nu| = |\alpha| - |\beta|\}). \tag{2.2}$$

Using the following lemmas, we will show that $\mathcal{B} \cong \mathcal{O}_{d_1 d_2}$.

LEMMA 2.6. *Let $\mathcal{B}$ be defined as in* (2.2)*, and let* $\mathcal{B}' = C^*(\{S_i \otimes \widetilde{S}_j : 0 \leqslant i \leqslant d_1 - 1,\ 0 \leqslant j \leqslant d_2 - 1\})$. *Then* $\mathcal{B} = \mathcal{B}'$.

Proof. Clearly $\mathcal{B}' \subseteq \mathcal{B}$. To show that $\mathcal{B} \subseteq \mathcal{B}'$ it needs to be shown that each element $S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*$ can be written as a polynomial in the generators $\{S_i \otimes \widetilde{S}_j\}$ of $\mathcal{B}'$. This can be done by first showing that generating elements of $\mathcal{B}$ of the form $S_\mu S_\nu^* \otimes \widetilde{Id}$ with $|\mu| = |\nu|$ and $Id \otimes \widetilde{S}_\alpha \widetilde{S}_\beta$ with $|\alpha| = |\beta|$ can be written as polynomials in the generating elements of $\mathcal{B}$. The argument extends to an arbitrary element of the generating set of $\mathcal{B}$. ∎

LEMMA 2.7. $\mathcal{B}' \cong \mathcal{O}_{d_1 d_2}$.

Proof. Each S_i is a generator of $\mathcal{O}_{d_1}$ and each $\widetilde{S}_j$ is a generator of $\mathcal{O}_{d_2}$. So the sets $\{S_i : 0 \leqslant i \leqslant d_1 - 1\}$ and $\{\widetilde{S}_j : 0 \leqslant j \leqslant d_2 - 1\}$ consist of isometries satisfying the Cuntz relations for $\mathcal{O}_{d_1}$ and $\mathcal{O}_{d_2}$ respectively. With this in mind, it can easily be shown that $\{S_i \otimes \widetilde{S}_j : 0 \leqslant i \leqslant d_1 - 1,\ 0 \leqslant j \leqslant d_2 - 1\}$ is a set of $d_1 d_2$ isometries satisfying the Cuntz relations for $\mathcal{O}_{d_1 d_2}$. ∎

PROPOSITION 2.8. *Let $\mathcal{A} = \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. The conditional expectation defined in Theorem 2.5 maps onto a subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ that is isomorphic to $\mathcal{O}_{d_1 d_2}$.*

Proof. This follows immediately from Theorem 2.5 and Lemmas 2.6 and 2.7. ∎

Given graphs E_1 and E_2, define the graph $\mathcal{E}$ as follows: $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ where $\mathcal{E}^0 = \{(v, w) : v \in E_1^0, w \in E_2^0\}$ and $\mathcal{E}^1 = \{(e, f) : e \in E_1^1, f \in E_2^1\}$ with $s(e, f) = (s(e), s(f))$ and $r(e, f) = (r(e), r(f))$. Since the edge set of $\mathcal{E}$ is simply the Cartesian product of the edge sets of E_1 and E_2, $\mathcal{E}$ has $d_1 d_2$ edges, all starting and ending at the same vertex.

$\mathcal{O}_{d_1 d_2} \cong C^*(\mathcal{E})$, and so with this definition of $\mathcal{E}$, we can see that our conditional expectation maps onto a subalgebra of $C^*(E_1) \otimes C^*(E_2)$ that is isomorphic to $C^*(\mathcal{E})$.

The deep results of E. Kirchberg and M. Rørdam (see [9]) show that if $gcd(d_1 - 1, d_2 - 1) = 1$, then $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is isomorphic to $\mathcal{O}_{gcd(d_1-1, d_2-1)+1}$. With this in mind, as well as Proposition 2.8, we note the following corollary.

COROLLARY 2.9. *Suppose that $gcd(d_1 - 1, d_2 - 1) = 1$. There is an embedding of $\mathcal{O}_{d_1 d_2}$ onto a subalgebra $\mathcal{B}$ of $\mathcal{O}_{gcd(d_1-1, d_2-1)+1}$ such that there exists a conditional expectation from $\mathcal{O}_{gcd(d_1-1, d_2-1)+1}$ onto $\mathcal{B}$.*

3. DISTINCTION BETWEEN $\mathcal{B}$ AND $C^*(\mathcal{E})$

Given the tensor product of graph algebras $\mathcal{A}$, we have proven that there exists a conditional expectation Φ that maps $\mathcal{A}$ onto the subalgebra $\mathcal{B}$ defined in (2.1). Let E_1 and E_2 be row-finite directed graphs such that every cycle has an entry, and form the graph $\mathcal{E}$. We would like to investigate the conditions under which $\mathcal{B}$ is isomorphic to $C^*(\mathcal{E})$.

It can be shown that the directed graph $\mathcal{E}$ is row-finite and every cycle has an entry. In addition, we assume that any Cuntz–Krieger E_1-family $\{S, P\}$ of partial isometries and projections is such that $P_v \neq 0$ for all $v \in E_1^0$. Similarly, we assume any Cuntz–Krieger E_2-family $\{\widetilde{S}, \widetilde{P}\}$ is such that $\widetilde{P}_w \neq 0$ for all $w \in E_2^0$. We need these conditions so that we can apply the Cuntz–Krieger Uniqueness Theorem in the upcoming lemmas. We state it here for reference. (See [8]).

THEOREM 3.1 (Cuntz–Krieger Uniqueness Theorem). *Suppose that E is a row-finite directed graph such that every cycle has an entry. Suppose that $\{Q_v, T_e : v \in$

$E^0, e \in E^1\}$ is a Cuntz–Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for all $v \in E^0$. Then the homomorphism $\phi : C^*(E) \to B$ is an injection.

Before we can investigate the question of when $\mathcal{B} \cong C^*(\mathcal{E})$, we need to verify that $C^*(\mathcal{E})$ is actually isomorphic to a subalgebra of $C^*(E_1) \otimes C^*(E_2)$.

LEMMA 3.2. *There is a Cuntz–Krieger $\mathcal{E}$-system in $C^*(E_1) \otimes C^*(E_2)$.*

Proof. $C^*(\mathcal{E})$ is generated by projections $P_{(v,w)}$ for $v \in E_1^0$, $w \in E_2^0$ and partial isometries $S_{(e,f)}$ for $e \in E_1^1$, $f \in E_2^1$. $C^*(E_1) \otimes C^*(E_2)$ is generated by projections $P_v \otimes \widetilde{P}_w$ and partial isometries $S_e \otimes \widetilde{S}_f$. Let π be a map from the generating set of $C^*(\mathcal{E})$ to the generating set of $C^*(E_1) \otimes C^*(E_2)$ such that $\pi(P_{(v,w)}) = P_v \otimes \widetilde{P}_w$ and $\pi(S_{(e,f)}) = S_e \otimes \widetilde{S}_f$. Extend the map π so that it preserves stars. We claim that the image of π satisfies the three relations found in Definition 2.3, making it into a Cuntz–Krieger $\mathcal{E}$-family.

We have $[\pi(S_{(e,f)})]^*[\pi(S_{(e,f)})] = [S_e{}^* \otimes \widetilde{S}_f^*][S_e \otimes \widetilde{S}_f] = S_e{}^* S_e \otimes \widetilde{S}_f^* \widetilde{S}_f = P_{s(e)} \otimes \widetilde{P}_{s(f)} = \pi(P_{(s(e),s(f))}) = \pi(P_{s(e,f)})$. Therefore, the image of π satisfies 2.3(ii).

Also

$$\begin{aligned}
\sum_{\{(e,f):r(e,f)=(v,w)\}} [\pi(S_{(e,f)})][\pi(S_{(e,f)})]^* &= \sum_{\{(e,f):r(e,f)=(v,w)\}} (S_e \otimes \widetilde{S_f})(S_e{}^* \otimes \widetilde{S}_f^*) \\
&= \sum_{\{e:r(e)=v\}} \sum_{\{f:r(f)=w\}} S_e S_e{}^* \otimes \widetilde{S}_f \widetilde{S}_f^* \\
&= \sum_{\{e:r(e)=v\}} S_e S_e{}^* \otimes \sum_{\{f:r(f)=w\}} \widetilde{S_f} \widetilde{S}_f^* \\
&= P_v \otimes \widetilde{P}_w = \pi(P_{(v,w)}).
\end{aligned}$$

Therefore, 2.3(iii) is satisfied.

Now, it remains to be shown that the projections are mutually orthogonal. If $(v_1, w_1) \neq (v_2, w_2)$ then either $v_1 \neq v_2$ or $w_1 \neq w_2$. Suppose without loss of generality that $v_1 \neq v_2$. We have $[\pi(P_{(v_1,w_1)})][\pi(P_{(v_2,w_2)})] = (P_{v_1} \otimes \widetilde{P}_{w_1})(P_{v_2} \otimes \widetilde{P}_{w_2}) = 0 \otimes \widetilde{P}_{w_1} \widetilde{P}_{w_2} = 0$ (since P_{v_1} and P_{v_2} are mutually orthogonal, being projections in the original Cuntz–Krieger E_1-family.) Then 2.3(i) is satisfied. ∎

LEMMA 3.3. *There is a subalgebra of $C^*(E_1) \otimes C^*(E_2)$ isomorphic to $C^*(\mathcal{E})$.*

Proof. By lemma 3.2, the image of π is a Cuntz–Krieger $\mathcal{E}$ system sitting inside of $C^*(E_1) \otimes C^*(E_2)$. By the universality of graph algebras, π can be extended to a homomorphism $\pi : C^*(\mathcal{E}) \to C^*(E_1) \otimes C^*(E_2)$ such that $\pi(P_{(v,w)}) = P_v \otimes \widetilde{P}_w$ and $\pi(S_{(e,f)}) = S_e \otimes \widetilde{S}_f$. Then we have an onto homomorphism $\pi : C^*(\mathcal{E}) \to \pi(C^*(\mathcal{E}))$. If the hypotheses of Theorem 3.1 are satisfied, this homomorphism π will be injective and we will have that π is an isomorphism. But, recall that $\mathcal{E}$ is a row-finite graph in which every cycle has an entry. Also, we have a Cuntz–Krieger $\mathcal{E}$-family $\{\pi(P_{(v,w)}), \pi(S_{(e,f)})\}$ such that $\pi(P_{(v,w)}) = P_v \otimes \widetilde{P}_w \neq 0$, since, by assumption, we have that $P_v \neq 0$ and $\widetilde{P}_w \neq 0$. Then, our hypotheses for the Cuntz–Krieger uniqueness theorem are satisfied, and we have that $\pi : C^*(\mathcal{E}) \to \pi(C^*(\mathcal{E}))$ is an injection. Therefore π is an isomorphism onto its image. ∎

We have a subalgebra of $C^*(E_1) \otimes C^*(E_2)$ that is isomorphic to $C^*(\mathcal{E})$. It is our desire to show that if $\mathcal{E}$ has no sources, then this subalgebra is precisely $\mathcal{B}$.

PROPOSITION 3.4. *Let E_1 and E_2 be row-finite directed graphs, and form the graph $\mathcal{E}$. If $\mathcal{E}$ has no sources and $\mathcal{B}$ is the subalgebra defined in* (2.2)*, then $\mathcal{B} \cong C^*(\mathcal{E})$.*

Proof. Recall that $C^*(\mathcal{E}) \cong \pi(C^*(\mathcal{E}))$, so if we can show that $\mathcal{B} = \pi(C^*(\mathcal{E}))$, then we are done. Clearly $\pi(C^*(\mathcal{E})) \subseteq \mathcal{B}$. To show that $\mathcal{B} \subseteq \pi(C^*(\mathcal{E}))$ it needs to be shown that $S_\mu S_\nu^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*$ can be written as a polynomial in the generators of $\pi(C^*(\mathcal{E}))$. This can be done by first showing that generating elements of $\mathcal{B}$ of the form $S_\mu S_\nu^* \otimes \widetilde{P}_v$ with $|\mu| = |\nu|$ for each $v \in E_2^0$ and $P_v \otimes \widetilde{S}_\alpha \widetilde{S}_\beta$ with $|\alpha| = |\beta|$ for each $v \in E_1^0$ can be written as polynomials in the generating elements of $\pi(C^*(\mathcal{E}))$. With these results it can be shown that this is also true for an arbitrary element of the generating set of $\mathcal{B}$. ∎

To summarize, we state the following theorem.

THEOREM 3.5. *Let E_1 and E_2 be row-finite directed graphs such that every cycle has an entry. Let $\{P_v, S_e\}$ be the Cuntz–Krieger E_1-system corresponding to $C^*(E_1)$ and let $\{\widetilde{P}_w, \widetilde{S}_f\}$ be the Cuntz–Krieger E_2-system corresponding to $C^*(E_2)$. Further, assume that $P_v \neq 0$ for every $v \in E_1^0$ and $\widetilde{P}_w \neq 0$ for every $w \in E_2^0$. If $\mathcal{E}$ has no sources, then there exists a conditional expectation from $C^*(E_1) \otimes C^*(E_2)$ to a subalgebra $\mathcal{B}$ that is isomorphic to $C^*(\mathcal{E})$. If $\mathcal{E}$ has sources, then we can still find a conditional expectation from $C^*(E_1) \otimes C^*(E_2)$ onto $\mathcal{B}$, but $\mathcal{B}$ need not be isomorphic to $C^*(\mathcal{E})$.*

The requirement that $\mathcal{E}$ has no sources is necessary for $\mathcal{B}$ to be isomorphic to $C^*(\mathcal{E})$. We give an example in the next section in which $\mathcal{B} \supsetneq C^*(\mathcal{E})$.

4. AN EXAMPLE WITH SOURCES

Let E be the graph as in Figure 2.

From Proposition 1.18 in [8] we know that $C^*(E) \cong M_4(\mathbb{C})$. We would like to find a conditional expectation from $C^*(E) \otimes C^*(E) \cong M_4(\mathbb{C}) \otimes M_4(\mathbb{C})$ to the subalgebra $\mathcal{B}$. As we typically do, let us form the graph $\mathcal{E}$, shown in Figure 3. This directed graph has sources, namely vertices v_2w, v_1w, v_0w, ww, wv_0, wv_1, and wv_2. We know that $C^*(\mathcal{E}) \cong M_{10}(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, a space that is 106-dimensional. So, is the C^*-algebra $\mathcal{B}$, the image of our conditional expectation, 106-dimensional?

By Theorem 2.5, our conditional expectation Φ maps to the subalgebra $\mathcal{B}$. Let us find the generators of $\mathcal{B}$ by brute force. Table 1 (see the end of the paper) shows the elementary tensors serving as the generators of our subalgebra $\mathcal{B}$. The

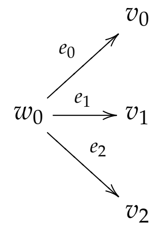

FIGURE 2. Graph E

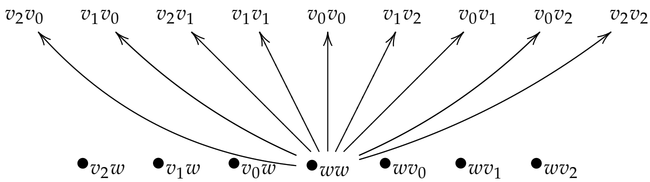

FIGURE 3. Graph $\mathcal{E}$

rows represent (μ, α) and the columns (ν, β), where each block represents the element $S_\mu S_\nu{}^* \otimes S_\alpha S_\beta{}^*$. A block is marked with an "X" if it is an element of the generating set of the subalgebra $\mathcal{B}$, that is if $|\mu| - |\nu| = |\alpha| - |\beta|$. For ease of notation we write $S_{e_i} = S_i$.

From the table, we see that our C^*-algebra $\mathcal{B}$ is isomorphic to $M_{10}(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$, which is 118-dimensional. Thus the image of our conditional expectation is 118-dimensional, but $C^*(\mathcal{E})$ is 106-dimensional. This is an example of why the requirement in Proposition 3.4 is a necessary one. We have shown explicitly that, in this case, $\mathcal{B} \not\cong C^*(\mathcal{E})$. This is because in the proof of Proposition 3.4, we use the third Cuntz–Krieger relation 2.3(iii). This relation only applies to vertices that are not sources.

It is interesting to note that $\mathcal{B}$ has precisely 12 more dimensions than $C^*(\mathcal{E})$. We already know that $\pi(C^*(\mathcal{E})) \subseteq \mathcal{B}$, and we can show that the 12 extra dimensions come exactly from the 12 entries off of the diagonal in the two 3×3 submatrices in Table 1. For notation, let Table 1 be called matrix $A = (a_{ij})$ where $1 \leqslant i, j \leqslant 16$. The extra dimensions of $\mathcal{B}$ come from the set of entries

$$S = \{a_{11,12}, a_{11,13}, a_{12,11}, a_{12,13}, a_{13,11}, a_{13,12}, a_{14,15}, a_{14,16}, a_{15,14}, a_{15,16}, a_{16,14}, a_{16,15}\}.$$

Let us show that no element of S is in $\pi(C^*(\mathcal{E}))$ by showing that the other 106 elements of the generating set of $\mathcal{B}$ are in $\pi(C^*(\mathcal{E}))$. Since $C^*(\mathcal{E})$ is 106-dimensional, that will imply that none of the elements of S are in $\pi(C^*(\mathcal{E}))$.

Vertex v_0 is not a source, and the only edge that v_0 receives is e_0. Then by 2.3(iii), $S_0 S_0{}^* \otimes \widetilde{P}_w \widetilde{P}_w^* = P_{v_0} \otimes \widetilde{P}_w \widetilde{P}_w^*$. Similarly, $S_1 S_1{}^* \otimes \widetilde{P}_w \widetilde{P}_w^* = P_{v_1} \otimes \widetilde{P}_w \widetilde{P}_w^*$, and $S_2 S_2{}^* \otimes \widetilde{P}_w \widetilde{P}_w^* = P_{v_2} \otimes \widetilde{P}_w \widetilde{P}_w^*$. Also, the elements $P_w P_w{}^* \otimes \widetilde{S}_0 \widetilde{S}_0{}^*$, $P_w P_w{}^* \otimes \widetilde{S}_1 \widetilde{S}_1{}^*$, and $P_wP_w{}^* \otimes \widetilde{S}_2\widetilde{S}_2{}^*$ reduce to $P_wP_w^* \otimes \widetilde{P}_{v_0}$, $P_wP_w^* \otimes \widetilde{P}_{v_1}$, and $P_wP_w^* \otimes \widetilde{P}_{v_2}$ respectively. Now, $\pi(C^*(\mathcal{E}))$ is generated by projections $\{P_s \otimes \widetilde{P}_t : (s,t) \in \mathcal{E}^0\}$ and partial isometries $\{S_g \otimes \widetilde{S}_h : (g,h) \in \mathcal{E}^1\}$. Then the six elements on the diagonals of the two 3×3 submatrices are in $\pi(C^*(\mathcal{E}))$. It is interesting to note that these six elements are in one-to-one correspondence with the isolated points of $\mathcal{E}$. Note also that each element in the 10×10 submatrix is either of the form $P_vP_v{}^* \otimes \widetilde{P}_v\widetilde{P}_v{}^*$ or of the form $S_iS_j{}^* \otimes \widetilde{S}_k\widetilde{S}_l{}^*$ for $0 \leqslant i,j,k,l \leqslant 2$. Clearly $P_v \otimes \widetilde{P}_v$ is a projection in $\pi(C^*(\mathcal{E}))$. Also, $S_iS_j{}^* \otimes \widetilde{S}_k\widetilde{S}_l{}^*$ is a product of generators of $\pi(C^*(\mathcal{E}))$. So, the elements of the 10×10 submatrix are in $\pi(C^*(\mathcal{E}))$. Therefore, all of the 106 elements of the generating set of $\pi(C^*(\mathcal{E}))$ have been counted. Thus, the remaining 12 elements in $S \subseteq \mathcal{B}$ must not be in $\pi(C^*(\mathcal{E}))$.

We should remark that, in this case, since $\mathcal{B}$ and $C^*(\mathcal{E})$ are finite-dimensional, there is an obvious conditional expectation Φ' from $\mathcal{B}$ onto $\pi(C^*(\mathcal{E}))$. Hence, $\Phi' \circ \Phi$ gives a conditional expectation from $C^*(E_1) \otimes C^*(E_2)$ onto $\pi(C^*(\mathcal{E}))$.

5. EXPRESSING $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ AS A CROSSED PRODUCT OF $\mathcal{O}_{d_1d_2}$ BY AN ENDOMORPHISM

In the process of developing the conditional expectation defined previously, we came across the very interesting result that it is possible to express $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ as a crossed product of $\mathcal{O}_{d_1d_2}$ by an endomorphism. The notion of the crossed product of a C^*-algebra by an endomorphism was introduced by W. Paschke in [7] and developed further by M. Choda in [2]. In [3], J. Cuntz proved that $\mathcal{O}_n$ is the crossed product of the core UHF algebra by an endomorphism.

DEFINITION 5.1 (W. Paschke). Let A be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, and let ω be a non-unitary isometry in $B(\mathcal{H})$ such that $\omega A\omega^* \subset A$ and $\omega^* A\omega \subset A$. The map $\rho : A \to A$ defined by $\rho(a) = \omega a\omega^*$ is an endomorphism of A into itself, and we call the C^*-subalgebra of $B(\mathcal{H})$ generated by A and ω the *spatial* or *Paschke crossed product of A by the endomorphism ρ*. We will use the notation for such a crossed product as is used in [2]. That is, we will write the crossed product of A by the endomorphism ρ as $A \triangleleft \langle\rho\rangle$.

For one example of such a crossed product by an endomorphism, we will cite an example pointed out by W. Paschke in [7]. He observed that Cuntz's construction showed that any representation of $\mathcal{O}_n$ on a Hilbert space is the spatial crossed product of a UHF algebra by an endomorphism, where the UHF algebra in question is $\displaystyle\bigcup_{n=1}^{\infty} C^*(\{S_\mu S_\nu{}^* : |\mu| = |\nu| = n\})$. Below we show that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is a crossed product of $\mathcal{O}_{d_1d_2}$ by an endomorphism using a similar type of argument.

THEOREM 5.2. *Let d_1 and d_2 be two positive integers greater than or equal to* 2. *Let $\pi_i : \mathcal{O}_{d_i} \to B(\mathcal{H}_i)$ be faithful representations of $\mathcal{O}_{d_i}$ on $\mathcal{H}_i$ for $i = 1,2$. Form the tensor product representation $\pi_1 \otimes \pi_2$ of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ on the Hilbert space tensor* *product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\mathcal{B}$ be the subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ isomorphic to $\mathcal{O}_{d_1 d_2}$ as in Lemma 2.7. Then $\pi_1 \otimes \pi_2(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2})$ can be expressed as the Paschke crossed product of $\pi_1 \otimes \pi_2(\mathcal{B})$ by an endomorphism.*

Proof. For simplicity of notation, we will write $\pi_1 \otimes \pi_2(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2})$ as $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ and $\pi_1 \otimes \pi_2(\mathcal{B})$ as $\mathcal{O}_{d_1 d_2}$.

Let $\rho(a) = \omega a \omega^*$ where $\omega = S_0 \otimes \mathrm{Id}$. ρ is an endomorphism on $\mathcal{O}_{d_1 d_2}$. To see this, note how ρ works on the generators of $\mathcal{O}_{d_1 d_2}$. Let $S_\mu S_\nu{}^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*$ with $|\mu| - |\nu| = |\alpha| - |\beta|$ be an element of the generating set of $\mathcal{O}_{d_1 d_2}$. (Recall that $\mathcal{O}_{d_1 d_2}$ is isomorphic to the C^*-algebra linearly generated by such elements.) Then $\rho(S_\mu S_\nu{}^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*) = (S_0 \otimes \mathrm{Id})(S_\mu S_\nu{}^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*)(S_0 \otimes \mathrm{Id})^* = S_{\mu\prime} S_{\nu\prime}{}^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^*$ where μ' is the path μ with the edge corresponding to S_0 added and similarly for ν'. Then $|\mu\prime| - |\nu\prime| = |\mu| + 1 - (|\nu| + 1) = |\mu| - |\nu| = |\alpha| - |\beta|$. Thus, the image of ρ is also in our subalgebra $\mathcal{O}_{d_1 d_2}$ and ρ is an endomorphism of $\mathcal{O}_{d_1 d_2}$ into $\mathcal{O}_{d_1 d_2}$. From this fact, the result that $\omega \mathcal{O}_{d_1 d_2} \omega^* \subseteq \mathcal{O}_{d_1 d_2}$ follows.

Also, $\omega^* \mathcal{O}_{d_1 d_2} \omega \subseteq \mathcal{O}_{d_1 d_2}$. Observe that $\omega^*(S_i \otimes \widetilde{S}_j)\omega = (S_0{}^* \otimes \mathrm{Id})(S_i \otimes \widetilde{S}_j)(S_0 \otimes \mathrm{Id}) = (S_0{}^* S_i \otimes \widetilde{S}_j)(S_0 \otimes \mathrm{Id})$. From 2.3(i), $S_0{}^* S_i = 0$ if $i \neq 0$ and $S_0{}^* S_i = \mathrm{Id}$ if $i = 0$. Then, $\omega^*(S_i \otimes \widetilde{S}_j)\omega = 0$ if $i \neq 0$ and $\omega^*(S_i \otimes \widetilde{S}_j)\omega = (\mathrm{Id} \otimes \widetilde{S}_j)(S_0 \otimes \mathrm{Id}) = S_0 \otimes \widetilde{S}_j$ if $i = 0$.

To show that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} = \mathcal{O}_{d_1 d_2} \triangleleft \langle \rho \rangle$, it needs to be shown that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} = C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id})$. $\mathcal{O}_{d_1 d_2} \subseteq \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ and $S_0 \otimes \mathrm{Id}$ is an element of the generating set of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Thus $C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id}) \subseteq \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$.

We now just need to show that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} \subseteq C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id})$. We proceed by showing that each of the generators $\{S_i \otimes \mathrm{Id} : 0 \leqslant i \leqslant d_1 - 1\} \cup \{\mathrm{Id} \otimes \widetilde{S}_j : 0 \leqslant j \leqslant d_2 - 1\}$ is an element of $C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id})$. First, consider the elements of the form $\mathrm{Id} \otimes \widetilde{S}_j$ for $0 \leqslant j \leqslant d_2 - 1$ and note that $\mathrm{Id} \otimes \widetilde{S}_j = (S_0 \otimes \mathrm{Id})^*(S_0 \otimes \widetilde{S}_j)$ by 2.3(i). But, $(S_0 \otimes \mathrm{Id})^* = \omega^*$ and $S_0 \otimes \widetilde{S}_j$ is a generator of $\mathcal{O}_{d_1 d_2}$. Therefore, $\mathrm{Id} \otimes \widetilde{S}_j$ is a product of elements in $C^*(\mathcal{O}_{d_1 d_2}, \omega)$. Next, consider the elements of the form $S_i \otimes \mathrm{Id}$ for $0 \leqslant i \leqslant d_1 - 1$. By 2.3(ii), $S_i \otimes \mathrm{Id} = S_i \otimes \sum_{k=0}^{d_2 - 1} \widetilde{S}_k \widetilde{S}_k{}^*$. This is a finite sum and so, using the properties of tensor products, we have $S_i \otimes \mathrm{Id} = \sum_{k=0}^{d_2 - 1} (S_i \otimes \widetilde{S}_k)(\mathrm{Id} \otimes \widetilde{S}_k)^*$. But, the elements $S_i \otimes \widetilde{S}_k$ are generators of $\mathcal{O}_{d_1 d_2}$ and elements of the form $(\mathrm{Id} \otimes \widetilde{S}_k)^*$ are in $C^*(\mathcal{O}_{d_1 d_2}, \omega)$ since elements of the form $\mathrm{Id} \otimes \widetilde{S}_k$ are in there by the first part. Therefore $S_i \otimes \mathrm{Id}$ is a sum of elements in $C^*(\mathcal{O}_{d_1 d_2}, \omega)$. Then, we have $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} = C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id})$, and so $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is a crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism. ∎

COROLLARY 5.3. *If $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is isomorphic to $\mathcal{O}_d$, where $d = gcd(d_1 - 1, d_2 - 1) + 1$, then $\mathcal{O}_d$ can be expressed as the crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism.*

In [2], M. Choda gives conditions on a C^*-algebra A such that the crossed product $A \triangleleft \langle\rho\rangle$ of A by an endomorphism ρ will be simple. Theorem 5.2 gives an example of a crossed product of a simple C^*-algebra by an endomorphism such that the crossed product is also simple.

6. ON THE INDEX OF A CONDITIONAL EXPECTATION

The concept of index was introduced by Y. Watatani, motivated by the success of V. Jones for von Neumann algebras. Watatani developed the concepts of a conditional expectation of a C^*-algebra onto a C^*-subalgebra and the index of a conditional expectation. Let us discuss the index of each of the examples of conditional expectations given in this paper. The definition of the index of a conditional expectation as described in [10] is stated below.

DEFINITION 6.1. Let ϕ be a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. A finite family $\{(u_i, v_i) : i = 1, \ldots, n\} \subseteq A \times A$ is a *quasi-basis* for the conditional expectation ϕ if

$$\sum_{i=1}^{n} u_i \phi(v_i a) = a = \sum_{i=1}^{n} \phi(a u_i) v_i \text{ for all } a \in A.$$

DEFINITION 6.2. A conditional expectation $\phi : A \to B$ is of *index-finite type* if there exists a quasi-basis for ϕ. Otherwise we say is is *not of index-finite type*.

DEFINITION 6.3. If a conditional expectation ϕ is of index-finite type, then the *index* of ϕ is

$$\text{Index } \phi = \sum_{i=1}^{n} u_i v_i.$$

Y. Watatani has indicated to us in a private communication, [11], that the conditional expectation $\phi : \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} \to \mathcal{B}$ is not of index-finite type. The conditional expectation in Example 4 is of index-finite type with index $3\mathrm{Id}_{16}$. In this paper we constructed a conditional expectation from a tensor product of graph algebras onto the subalgebra $\mathcal{B}$. It would be of interest to study index in this general case.

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	P_w, P_w	S_0, S_0	S_1, S_0	S_2, S_0	S_0, S_1	S_1, S_1	S_2, S_1	S_0, S_2	S_1, S_2	S_2, S_2	S_0, P_w	S_1, P_w	S_2, P_w	P_w, S_0	P_w, S_1	P_w, S_2
P_w, P_w	X	X	X	X	X	X	X	X	X	X						
S_0, S_0	X	X	X	X	X	X	X	X	X	X						
S_1, S_0	X	X	X	X	X	X	X	X	X	X						
S_2, S_0	X	X	X	X	X	X	X	X	X	X						
S_0, S_1	X	X	X	X	X	X	X	X	X	X						
S_1, S_1	X	X	X	X	X	X	X	X	X	X						
S_2, S_1	X	X	X	X	X	X	X	X	X	X						
S_0, S_2	X	X	X	X	X	X	X	X	X	X						
S_1, S_2	X	X	X	X	X	X	X	X	X	X						
S_2, S_2	X	X	X	X	X	X	X	X	X	X						
S_0, P_w											X	X	X			
S_1, P_w											X	X	X			
S_2, P_w											X	X	X			
P_w, S_0														X	X	X
P_w, S_1														X	X	X
P_w, S_2														X	X	X

TABLE 1. Generators of the Subalgebra $\mathcal{B}$ of $C^*(E_1) \otimes C^*(E_2) \cong M_4(\mathbb{C}) \otimes M_4(\mathbb{C})$

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AMY B. CHAMBERS, DEPARTMENT OF MATHEMATICS, TENNESSEE TECHNOLOGICAL UNIVERSITY, COOKEVILLE, TN, 38501, U.S.A.

E-mail address: achambers@tntech.edu

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