J. OPERATOR THEORY
63:2(2010), 455–482

SPACES OF INFINITE MEASURE AND POINTWISE CONVERGENCE OF THE BILINEAR HILBERT AND ERGODIC AVERAGES DEFINED BY L^p-ISOMETRIES

EARL BERKSON and CIPRIAN DEMETER

Communicated by William Arveson

ABSTRACT. We generalize the respective "double recurrence" results of Bourgain and of the second author, which established for pairs of L^∞ functions on a finite measure space the a.e. convergence of the discrete bilinear ergodic averages and of the discrete bilinear Hilbert averages defined by invertible measure-preserving point transformations. Our generalizations are set in the context of arbitrary sigma-finite measure spaces and take the form of a.e. convergence of such discrete averages, as well as of their continuous variable counterparts, when these averages are defined by Lebesgue space isometries and act on $L^{p_1} \times L^{p_2}$ $(1 < p_1, p_2 < \infty,\ p_1^{-1} + p_2^{-1} < \frac{3}{2})$. In the setting of an arbitrary measure space, this yields the a.e. convergence of these discrete bilinear averages when they act on $L^{p_1} \times L^{p_2}$ and are defined by an invertible measure-preserving point transformation.

KEYWORDS: *Measure, a.e. convergence, bilinear, Hilbert averages, ergodic averages.*

MSC (2000): 28A20, 28A65, 37A05, 42A50, 42B20, 42B25, 46E30.

1. INTRODUCTION

For an arbitrary measure space (X, σ), we shall denote by $\mathcal{A}(\sigma)$ the algebra under pointwise operations consisting of all complex-valued σ-measurable functions on X (identified modulo equality σ-a.e. on X). The class of all real-valued functions belonging to $\mathcal{A}(\sigma)$ will be denoted by $\Re(\mathcal{A}(\sigma))$. In this setting we shall use the following terminology and notation.

DEFINITION 1.1. Let T be a linear bijection of $\mathcal{A}(\sigma)$ onto $\mathcal{A}(\sigma)$. For each real number $r \geqslant 1$, we shall denote the integer part of r by $[r]$, and for $f \in \mathcal{A}(\sigma)$, $g \in \mathcal{A}(\sigma)$, we define the corresponding discrete bilinear ergodic average $A_{r,T}(f,g)$ and the corresponding discrete bilinear Hilbert average $H_{r,T}(f,g)$ by writing pointwise on X,

$$A_{r,T}(f,g) = \frac{1}{[r]} \sum_{n=0}^{[r]-1} (T^n f)(T^{-n} g); \tag{1.1}$$

$$H_{r,T}(f,g) = \sum_{0<|n|\leqslant[r]} \frac{(T^n f)(T^{-n} g)}{n}. \tag{1.2}$$

Our principal concern will be to generalize the double recurrence theorem of Bourgain for discrete bilinear ergodic averages [3] and its counterpart for discrete bilinear Hilbert averages (recently established in Theorem 1.2 of [4]), whose statements are reproduced as the following theorem.

THEOREM 1.2. *Suppose that $(\mathfrak{X},\rho)$ is a finite measure space, and ϕ is an invertible measure-preserving point transformation of $(\mathfrak{X},\rho)$ onto $(\mathfrak{X},\rho)$. Let $\mathcal{T} : \mathcal{A}(\rho) \to \mathcal{A}(\rho)$ denote composition with ϕ. Then for every $f \in L^\infty(\rho)$, and every $g \in L^\infty(\rho)$, each of the sequences $\{A_{k,\mathcal{T}}(f,g)\}_{k=1}^\infty$ and $\{H_{k,\mathcal{T}}(f,g)\}_{k=1}^\infty$ converges ρ-a.e. on $\mathfrak{X}$ to a corresponding function belonging to $\mathcal{A}(\rho)$.*

Our main result, which is stated as follows, generalizes Theorem 1.2 in the direction of L^p-isometries for sigma-finite measure spaces. (See Theorem 5.2 in Section 5 below for the continuous variable version of this generalization.)

THEOREM 1.3. *Suppose that (Ω,μ) is a sigma-finite measure space, and let U be a bijective linear mapping of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$ such that the following two conditions hold:*

(i) *Whenever $\{g_k\}_{k=1}^\infty \subseteq \mathcal{A}(\mu)$, $g \in \mathcal{A}(\mu)$, and $g_k \to g$ μ-a.e. on Ω, it follows that as $k \to \infty$, $U(g_k) \to U(g)$ μ-a.e. on Ω, and $U^{-1}(g_k) \to U^{-1}(g)$ μ-a.e. on Ω.*

(ii) *The restriction $U|L^p(\mu)$ is a surjective linear isometry of $L^p(\mu)$ onto $L^p(\mu)$ for $0 < p \leqslant \infty$.*

Suppose further that

$$1 < p_1, p_2 < \infty; \tag{1.3}$$

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} < \frac{3}{2}. \tag{1.4}$$

Then for every $f \in L^{p_1}(\mu)$, and every $g \in L^{p_2}(\mu)$, each of the sequences

$$\{A_{k,U}(f,g)\}_{k=1}^\infty, \{H_{k,U}(f,g)\}_{k=1}^\infty$$

converges μ-a.e. on Ω and in the metric topology of $L^{p_3}(\mu)$ to a corresponding function belonging to $\mathcal{A}(\mu)$.

Although the proof of Theorem 1.3 will be deferred to Section 4, the hypotheses of Theorem 1.3 on (Ω,μ) and U will remain in effect henceforth. Theorem 1.3 has the following corollary, which is valid for arbitrary measure spaces, and which likewise obviously implies Theorem 1.2, since in the setting of any finite measure space $(\mathfrak{X},\rho)$, the inclusion $L^\infty(\rho) \subseteq L^2(\rho)$ holds. (A variant of this result, likewise valid for all measure spaces, is described below for the continuous variable averages in Corollary 5.3.)

COROLLARY 1.4. *Suppose that (X,σ) is an arbitrary measure space, and let τ be an invertible measure-preserving point transformation of (X,σ) onto (X,σ). Suppose also that p_1, p_2, p_3 satisfy* (1.3) *and* (1.4), *and let $f \in L^{p_1}(\sigma)$, $g \in L^{p_2}(\sigma)$. Then each of the sequences*

$$\left\{\frac{1}{k}\sum_{n=0}^{k-1} f(\tau^n)g(\tau^{-n})\right\}_{k=1}^{\infty}, \left\{\sum_{0<|n|\leqslant k} \frac{f(\tau^n)g(\tau^{-n})}{n}\right\}_{k=1}^{\infty}$$

converges σ-a.e. on X and in the metric topology of $L^{p_3}(\sigma)$ to a corresponding function belonging to $\mathcal{A}(\sigma)$.

Proof. Since $f \in L^{p_1}(\sigma)$, we can write $\{x \in X : |f(x)| > 0\} = \bigcup_{j=1}^{\infty} E_j$, where $\sigma(E_j) < \infty$, for each $j \in \mathbb{N}$. Putting

$$Y = \bigcup_{n\in\mathbb{Z}} \bigcup_{j=1}^{\infty} \tau^n(E_j),$$

we see that for all $n \in \mathbb{Z}$: $\tau^n(Y) = Y$, and $f(\tau^n(x)) = 0$ for all $x \in X \setminus Y$. So in order to establish the desired σ-a.e. convergence on X it suffices to prove that for σ-almost all $x \in Y$, each of the sequences $\left\{\frac{1}{k}\sum_{n=0}^{k-1} f(\tau^n(x))g(\tau^{-n}(x))\right\}_{k=1}^{\infty}$ and $\left\{\sum_{0<|n|\leqslant k} \frac{f(\tau^n(x))g(\tau^{-n}(x))}{n}\right\}_{k=1}^{\infty}$ is convergent. But this follows immediately upon application of Theorem 1.3 to the sigma-finite measure space (Y,σ) and the composition operator corresponding to the restriction $\tau\,|Y$.

Likewise for convergence in $L^{p_3}(X,\sigma)$, which can also be seen as follows. By combining the reasoning regarding Y with the maximal estimates in the setting of sigma-finite measure spaces of Theorems 9 and 10 in [2] (whose statements are reproduced in Theorem 2.5 below), we see without difficulty that by dominated convergence each of the following sequences converges in the metric of $L^{p_3}(\sigma)$:

$$\left\{\frac{1}{k}\sum_{n=0}^{k-1} f(\tau^n)g(\tau^{-n})\right\}_{k=1}^{\infty}, \left\{\sum_{0<|n|\leqslant k} \frac{f(\tau^n)g(\tau^{-n})}{n}\right\}_{k=1}^{\infty}. \quad \blacksquare$$

REMARK 1.5. While our main concern will be with extensions of Theorem 1.2 to spaces of infinite measure, we briefly comment here on aspects in which Theorem 1.3 generalizes Theorem 1.2, when Theorem 1.3 is restricted to the finite measure space setting. If (Ω_0,μ_0) is a finite measure space, then a bijective linear mapping U of $\mathcal{A}(\mu_0)$ onto $\mathcal{A}(\mu_0)$ satisfying conditions (i) and (ii) of Theorem 1.3 need not have any of its odd powers implemented by an invertible measure-preserving point transformation of Ω_0 onto Ω_0. Such an example is furnished by Section 343J of [8], which, taken in conjunction with standard considerations about measure-preserving set transformations (see pages 452–454 of [6]), furnishes a complete, non-atomic, finite measure space $(\widetilde{\Omega}_0, \widetilde{\mu}_0)$ whose measure algebra is a separable metric space, together with a self-inverse algebra automorphism U_0 of $\mathcal{A}(\widetilde{\mu}_0)$ onto $\mathcal{A}(\widetilde{\mu}_0)$ satisfying conditions (i) and (ii) of Theorem 1.3, but is such that $U_0|L^\infty(\widetilde{\mu}_0)$ cannot be expressed by composition with an invertible measure-preserving point transformation of the measure space $(\widetilde{\Omega}_0, \widetilde{\mu}_0)$ onto $(\widetilde{\Omega}_0, \widetilde{\mu}_0)$. So Theorem 1.2 does not directly apply here, although in this particular example, since U_0 is self-inverse, the $\widetilde{\mu}_0$-a.e. convergence from Theorem 1.3 will hold trivially — in fact, whenever $f \in \mathcal{A}(\widetilde{\mu}_0)$ and $g \in \mathcal{A}(\widetilde{\mu}_0)$. (For information about the general relationships between measure-preserving set transformations and measure-preserving point transformations of non-atomic finite measure spaces satisfying separability conditions, see Section 41 of [10] and [11].) In the special case of Theorem 1.3 where the setting is an arbitrary finite measure space (Ω_0, μ_0) and $(f, g) \in L^\infty(\mu_0) \times L^\infty(\mu_0)$, the convergence μ_0 -a.e. of the averages $\{A_{k,U}(f,g)\}_{k=1}^\infty$ and $\{H_{k,U}(f,g)\}_{k=1}^\infty$ can be deduced directly from Theorem 1.2 in conjunction with maximal results from [2] (which are quoted as Theorem 2.5 below) by following G.-C. Rota's "dilation" theory for measure spaces [7],[15] in a spirit similar to that in Chapter IV, Section 4 of [16]. We omit the details of this line of reasoning for the finite measure space setting, since we will be proving Theorem 1.3 in full generality by following a different path.

The reasoning used in [4] to deduce the a.e. convergence of the discrete averages $H_{k,\mathcal{T}}(f,g)$ under the conditions of Theorem 1.2 above also furnished a new proof of the result of [3] for the a.e. convergence of the discrete averages $\{A_{k,\mathcal{T}}(f,g)\}_{k=1}^\infty$ in the same circumstances. Our strategy (particularly as regards infinite measures) for treating the wider scope of Theorem 1.3 and for obtaining its continuous variable counterpart will be to combine suitable modifications of the unified coverage of $\{A_{k,\mathcal{T}}(f,g)\}_{k=1}^\infty$ and $\{H_{k,\mathcal{T}}(f,g)\}_{k=1}^\infty$ in [4] with the treatment of the relevant bisublinear maximal operators in Theorems 9, 10, and 13 of [2]. Accordingly, the remaining four sections of this article will be organized as follows. In Section 2 we collect some background items that furnish key structural tools for the demonstration in Section 4 of Theorem 1.3. In particular, Section 2 includes an expanded version of Lemma 3.1 of [4] aimed at providing, in the form of a suitable oscillation estimate, an abstract sufficiency criterion for μ-a.e. convergence (see Lemma 2.3 below). The way is then opened for proceeding from Lemma 2.3 and the maximal bisublinear theorems of [2] to derive Theorem 1.3 in Section 4 after the development in Section 3 of the relevant oscillation estimates, which will take the form of general discretization and transference results. We close by treating the continuous variable model in Section 5, which, in particular, establishes the counterpart of Theorem 1.3 for one-parameter groups of Lebesgue space isometries associated with the arbitrary sigma-finite measure space (Ω, μ) (see Theorem 5.2 below).

Henceforth, the following notation will be in effect. If A is a subset of a given set Y, then, except where otherwise indicated, the characteristic function of A, defined on Y, will be designated by χ_A, and the restriction to A of a function F defined on Y will be written $F|A$. The collection of all mappings of a set E into a set W will be denoted by W^E. Lebesgue measure on $\mathbb{R}$ (respectively, counting measure on $\mathbb{Z}$) will be symbolized by $\mathfrak{m}_{\mathbb{R}}$ (respectively, by $\mathfrak{m}_{\mathbb{Z}}$). Given an arbitrary measure space (X, σ), and a function $f \in \mathcal{A}(\sigma)$, we shall denote by $\lambda(f, \sigma; (\cdot))$ the *distribution function* of f specified by:

$$\lambda(f, \sigma; y) = \sigma(\{x \in X : |f(x)| > y\}), \quad \text{for each real number } y > 0, \tag{1.5}$$

and we shall follow standard notation by writing

$$\|f\|_{L^{1,\infty}(\sigma)} = \sup\{y\lambda(f, \sigma; y) : y \in \mathbb{R}, y > 0\}.$$

Given a positive real number ξ and a function $f : \mathbb{R} \to \mathbb{C}$, we shall symbolize by $\delta_\xi f$ the dilation of f by ξ, which is defined by

$$(\delta_\xi f)(x) = \frac{1}{\xi} f\left(\frac{x}{\xi}\right), \quad \text{for all } x \in \mathbb{R}. \tag{1.6}$$

The letter "C" with a (possibly empty) set of subscripts will signify a constant which depends only on those subscripts, and which can change its value from one occurrence to another.

2. BACKGROUND ITEMS

The following proposition, a version of Corollary 3.1 in [14] (see also Proposition 5 and Remark 5(i) in [2]), describes the structure of the operator U in the hypotheses of Theorem 1.3.

PROPOSITION 2.1. *In the setting of the arbitrary sigma-finite measure space (Ω, μ), let U be a bijective linear mapping of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$ such that the conditions* (i) *and* (ii) *in the hypotheses of Theorem* 1.3 *hold. Then there are unique sequences $\{h_j\}_{j=-\infty}^{\infty}$ and $\{\Phi_j\}_{j=-\infty}^{\infty}$ such that for each $j \in \mathbb{Z}$:*

(i) *$h_j \in \mathcal{A}(\mu)$, with $|h_j| = 1$ on Ω, and Φ_j is an algebra automorphism of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$;*

(ii) *for every $f \in \mathcal{A}(\mu)$, $U^j f$ is expressed by the pointwise product on Ω of the functions h_j and $\Phi_j(f)$;*

(iii) *whenever $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{A}(\mu)$, $f \in \mathcal{A}(\mu)$, and $f_k \to f$ μ-a.e. on Ω, it follows that as $k \to \infty$, $\Phi_j(f_k) \to \Phi_j(f)$ μ-a.e. on Ω. This unique sequence $\{\Phi_j\}_{j=-\infty}^{\infty}$ has the* *property that*

$$\mu(E) = \int_{\Omega} \Phi_j(\chi_E)\,\mathrm{d}\mu, \quad \textit{for each } j \in \mathbb{Z}, \textit{ and each } \mu\textit{-measurable set } E.$$

The sequences $\{h_j\}_{j=-\infty}^{\infty}$ *and* $\{\Phi_j\}_{j=-\infty}^{\infty}$ *also enjoy the following properties for arbitrary* $j \in \mathbb{Z}, k \in \mathbb{Z}$*:*

$$\Phi_j(g) \geqslant 0 \quad \textit{for each } g \geqslant 0 \textit{ belonging to } \mathcal{A}(\mu); \tag{2.1}$$

$$|\Phi_j(f)|^{\alpha} = \Phi_j(|f|^{\alpha}), \quad \textit{for } f \in \mathcal{A}(\mu), \textit{ and } 0 < \alpha < \infty; \tag{2.2}$$

$$\Phi_{j+k}(f) = \Phi_j(\Phi_k(f)), \quad \textit{for every } f \in \mathcal{A}(\mu); \tag{2.3}$$

$$h_{j+k}(x) = h_j(x)(\Phi_j h_k)(x), \quad \textit{for } \mu\textit{-almost all } x \in \Omega. \tag{2.4}$$

REMARK 2.2. (i) It is clear that for each $j \in \mathbb{Z}$ the restriction $\Phi_j | L^p(\mu)$. is a surjective linear isometry of $L^p(\mu)$ onto $L^p(\mu)$, for $0 < p \leqslant \infty$.

(ii) By (2.1), $\Phi_j(g)$ is real-valued for each $j \in \mathbb{Z}$, and each real-valued g belonging to $\mathcal{A}(\mu)$. Moreover, from (2.3) we see that $\Phi_{-j} = \Phi_j^{-1}$. Hence application of (2.1) to Φ_j and to Φ_{-j} shows that the restriction of Φ_j to the class $\Re(\mathcal{A}(\mu))$ consisting of the real-valued μ -measurable functions on Ω is a surjective order isomorphism. It follows that for each finite sequence $\{g_k\}_{k=1}^{N} \subseteq \Re(\mathcal{A}(\mu))$,

$$\Phi_j\Big(\sup_{1 \leqslant k \leqslant N} g_k\Big) = \sup_{1 \leqslant k \leqslant N} \Phi_j(g_k). \tag{2.5}$$

(iii) It is readily seen from the foregoing considerations that each Φ_j preserves distribution functions. That is, for $f \in \mathcal{A}(\mu)$, each positive real number y, and each $j \in \mathbb{Z}$, we have, in the notation of (1.5),

$$\lambda(\Phi_j(f), \mu; y) = \lambda(f, \mu; y). \tag{2.6}$$

The following expanded version of the sufficiency criterion for a.e. convergence in Lemma 3.1 of [4] will play a pivotal role in our considerations.

LEMMA 2.3. *Let* (Ω, μ) *be a sigma-finite measure space, and suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of complex-valued* μ*-measurable functions defined on* Ω *which has the following property: there is a positive real constant* Θ *such that for every positive integer* $J \geqslant 2$*, and every sequence of positive integers* $u_1 < u_2 < \cdots < u_J$*, we have*

$$\Big\| \Big\{ \sum_{j=1}^{J-1} \sup_{\substack{n \in \mathbb{N}, \\ u_j \leqslant n < u_{j+1}}} |f_n - f_{u_{j+1}}|^2 \Big\}^{1/2} \Big\|_{L^{1,\infty}(\mu)} \leqslant \Theta J^{1/4}. \tag{2.7}$$

Then there is a μ*-measurable function* $f : \Omega \to \mathbb{C}$ *such that* $\{f_n\}_{n=1}^{\infty}$ *converges to* f μ *-a.e. on* Ω*.*

Proof. In view of the sigma-finiteness of (Ω, μ) and the form of the hypothesis (2.7), without loss of generality we can and shall assume henceforth in the proof of this lemma that $\mu(\Omega) < \infty$. Putting $L(x) = \limsup_{n \to \infty} |f_n(x)|$, for all $x \in \Omega$, and writing $A = \{x \in \Omega : L(x) = \infty\}$, we claim that $\mu(A) = 0$. For each $j \in \mathbb{N}$, let

$$E_j = \{x \in \Omega : |f_1(x)| \leqslant j\}.$$

For the claim that $\mu(A) = 0$, it clearly suffices to show that $\mu(A \cap E_j) = 0$, for each $j \in \mathbb{N}$. Assume to the contrary that $\mu(A \cap E_{j_0}) > 0$, for some $j_0 \in \mathbb{N}$. Observe that by Egoroff's Theorem there is $B \subseteq A \cap E_{j_0}$ such that $\mu(B) > \frac{\mu(A \cap E_{j_0})}{2}$, and such that the sequence

$$\Big\{ \min_{1 \leqslant k \leqslant n} \exp(-|f_k|) \Big\}_{n=1}^{\infty}$$

converges to the zero function uniformly on B. Accordingly, for each $m \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\sup_{1 \leqslant k \leqslant n} |f_k| > m + j_0 \quad \text{on } B.$$

For $1 \leqslant \nu \leqslant N$, we have on Ω,

$$|f_\nu - f_1| \leqslant |f_\nu - f_{N+1}| + |f_{N+1} - f_1| \leqslant 2 \sup_{1 \leqslant k < N+1} |f_k - f_{N+1}|, \tag{2.8}$$

and it now follows that

$$\mu(B) \leqslant \mu\Big\{x \in \Omega : \sup_{1 \leqslant k < N+1} |f_k - f_1| > m\Big\} \leqslant \mu\Big\{x \in \Omega : \sup_{1 \leqslant k < N+1} |f_k - f_{N+1}| > \frac{m}{2}\Big\}.$$

This, together with an application of (2.7) (for $J = 2$, $u_1 = 1$, $u_2 = N + 1$), shows that

$$\frac{m\mu(A \cap E_{j_0})}{2} < m\mu(B) \leqslant \Theta 2^{5/4}, \quad \text{for all } m \in \mathbb{N}.$$

Since $m \in \mathbb{N}$ is arbitrary, we can let $m \to \infty$ in this to contradict the supposition that $\mu(A \cap E_{j_0}) > 0$, thereby establishing the claim that $\mu(A) = 0$. Hence

$$\mu\Big(\Omega \setminus \bigcup_{k=1}^{\infty} \Big\{x \in \Omega : \sup_{n \in \mathbb{N}} |f_n(x)| \leqslant k\Big\}\Big) = 0,$$

and so by confining attention to each set $\Big\{x \in \Omega : \sup_{n \in \mathbb{N}} |f_n(x)| \leqslant k\Big\}$ separately, we can assume (in addition to (2.7) and $\mu(\Omega) < \infty$) that $\{f_n\}_{n=1}^{\infty} \subseteq L^{\infty}(\Omega, \mu)$, with

$$\sup_{n \in \mathbb{N}} \|f_n\|_{L^{\infty}(\Omega,\mu)} < \infty.$$

From this point onwards, the proof of Lemma 2.3 is a straightforward adaptation of the proof sketched in [4] for Lemma 3.1 therein. ∎

REMARK 2.4. Given a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{A}(\mu)$, a positive integer $J \geqslant 2$, and any sequence of positive integers $u_1 < u_2 < \cdots < u_J$, we can adapt the elementary reasoning of (2.8) to infer readily that for $1 \leqslant j \leqslant J-1$,

$$\sup_{\substack{n\in\mathbb{N},\\ u_j \leqslant n < u_{j+1}}} |f_n - f_{u_{j+1}}| \leqslant 2 \sup_{\substack{n\in\mathbb{N},\\ u_j < n \leqslant u_{j+1}}} |f_n - f_{u_j}|.$$

Consequently,

$$\Big\|\Big\{\sum_{j=1}^{J-1} \sup_{\substack{n\in\mathbb{N},\\ u_j \leqslant n < u_{j+1}}} |f_n - f_{u_{j+1}}|^2\Big\}^{1/2}\Big\|_{L^{1,\infty}(\mu)} \leqslant 2 \Big\|\Big\{\sum_{j=1}^{J-1} \sup_{\substack{n\in\mathbb{N},\\ u_j < n \leqslant u_{j+1}}} |f_n - f_{u_j}|^2\Big\}^{1/2}\Big\|_{L^{1,\infty}(\mu)}.$$

Similarly we also have

$$\Big\|\Big\{\sum_{j=1}^{J-1} \sup_{\substack{n\in\mathbb{N},\\ u_j < n \leqslant u_{j+1}}} |f_n - f_{u_j}|^2\Big\}^{1/2}\Big\|_{L^{1,\infty}(\mu)} \leqslant 2 \Big\|\Big\{\sum_{j=1}^{J-1} \sup_{\substack{n\in\mathbb{N},\\ u_j \leqslant n < u_{j+1}}} |f_n - f_{u_{j+1}}|^2\Big\}^{1/2}\Big\|_{L^{1,\infty}(\mu)}. \tag{2.9}$$

Hence the minorant in (2.7) can be equivalently replaced by the minorant in (2.9).

In order to cover the μ-a.e. convergence for all $f \in L^{p_1}(\mu)$ and all $g \in L^{p_2}(\mu)$ in the conclusion of Theorem 1.3, we shall require the following bisublinear maximal theorem for this setting (a combination of the statements of Theorems 9 and 10 in [2], which were obtained by discretizing and transferring Michael Lacey's bisublinear maximal theorems for the classical setting [13]).

THEOREM 2.5. *Assume the hypotheses of Theorem* 1.3 *on* (Ω, μ), U, p_1, p_2, *and* p_3. *For* $f \in L^{p_1}(\mu)$ *and* $g \in L^{p_2}(\mu)$, *define the maximal functions* $\mathcal{H}_U(f,g)$ *and* $\mathcal{M}_U(f,g)$ *by writing pointwise on* Ω,

$$\mathcal{H}_U(f,g) = \sup_{j\in\mathbb{N}} |H_{j,U}(f,g)| = \sup_{j\in\mathbb{N}} \Big| \sum_{0<|n|\leqslant j} \frac{(U^n f)\,(U^{-n} g)}{n} \Big|; \tag{2.10}$$

$$\mathcal{M}_U(f,g) = \sup_{j\in\mathbb{N}} \frac{1}{2j+1} \sum_{n=-j}^{j} |U^n f|\,|U^{-n} g|. \tag{2.11}$$

Then there are positive constants $\mathfrak{M}_{p_1,p_2}$ *and* $\mathfrak{N}_{p_1,p_2}$, *each depending only on* p_1 *and* p_2, *such that for all* $f \in L^{p_1}(\mu)$ *and all* $g \in L^{p_2}(\mu)$:

$$\|\mathcal{H}_U(f,g)\|_{L^{p_3}(\mu)} \leqslant \mathfrak{M}_{p_1,p_2} \, \|f\|_{L^{p_1}(\mu)} \, \|g\|_{L^{p_2}(\mu)}; \tag{2.12}$$

$$\|\mathcal{M}_U(f,g)\|_{L^{p_3}(\mu)} \leqslant \mathfrak{N}_{p_1,p_2} \, \|f\|_{L^{p_1}(\mu)} \, \|g\|_{L^{p_2}(\mu)}. \tag{2.13}$$

3. DISCRETIZED AND TRANSFERRED OSCILLATION ESTIMATES

In this section, we develop general discretization and transference results that will implement the derivation of Theorem 1.3 by setting up suitable applications of Lemma 2.3 and Theorem 2.5. The first step in this process will be to discretize the following oscillation theorem for the real line (Theorem 1.4 of [4]), which plays a seminal role in our considerations.

THEOREM 3.1. *Let $K : \mathbb{R} \to \mathbb{R}$ belong to $L^2(\mathbb{R})$, and suppose that its Fourier transform $\widehat{K}$ satisfies the following conditions:*

$$\widehat{K} \in C^\infty(\mathbb{R} \setminus \{0\});$$

$$\sup_{y \in \mathbb{R}\setminus\{0\}} \left(|\widehat{K}(y)| \max\{1, |y|\}\right) < \infty;$$

$$\sup_{y \in \mathbb{R}\setminus\{0\}} \left(\left|\frac{\mathrm{d}^n \widehat{K}(y)}{\mathrm{d}y^n}\right| \max\{|y|^{n-1}, |y|^{n+1}\}\right) < \infty, \quad \textit{for each } n \in \mathbb{N}.$$

Suppose that $m \in \mathbb{N}$, and let $d_m = 2^{1/m}$. Then there is a positive constant $\gamma_{K,m}$, depending only on K and m, such that for every pair of compactly supported functions f and g belonging to $L^\infty(\mathbb{R})$, for each positive integer $J \geqslant 2$, and for each sequence $\{u_j\}_{j=1}^J \subseteq \mathbb{Z}$ such that $u_1 < u_2 < \cdots < u_J$, we have, in terms of the notation (1.6) *for dilations,*

$$\left\| \left\{ \sum_{j=1}^{J-1} \sup_{k \in \mathbb{Z}, u_j \leqslant k < u_{j+1}} \left| \int_{\mathbb{R}} f(x+y) g(x-y) \left((\delta_{d_m^k} K)(y) - (\delta_{d_m^{u_{j+1}}} K)(y) \right) \mathrm{d}y \right|^2 \right\}^{1/2} \right\|_{L_x^{1,\infty}}$$

$$\leqslant \gamma_{K,m} J^{1/4} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \tag{3.1}$$

The discretization of Theorem 3.1 will be accomplished by making suitable use of a general discretization tool for maximal functions (Lemma 3 of [2], whose statement is reproduced below in Lemma 3.2). Before embarking on this course, we introduce some auxiliary notions and notation in order to avoid digressions later on. The closed interval $[-\frac{1}{4}, \frac{1}{4}]$ in $\mathbb{R}$ will be designated by $\mathcal{I}$, and for each $n \in \mathbb{Z}$, we denote by $\mathcal{I}_n$ the interval $\mathcal{I} + n = [n - \frac{1}{4}, n + \frac{1}{4}]$. For each $\phi \in L^1(\mathbb{R})$ such that the support of ϕ is a subset of $\mathcal{I}$, we define the linear mapping $P_\phi : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{R}}$ by putting

$$(P_\phi(\{a_n\}_{n=-\infty}^\infty))(x) = \sum_{n \in \mathbb{Z}} a_n \phi(x-n), \quad \text{for all } \{a_n\}_{n=-\infty}^\infty \in \mathbb{C}^{\mathbb{Z}}, \text{and all } x \in \mathbb{R}. \tag{3.2}$$

When ϕ is specialized to be $\chi_{\mathcal{I}}$, the characteristic function, defined on $\mathbb{R}$, of $\mathcal{I}$, we shall write P rather than P_ϕ. Clearly, if $0 < p \leqslant \infty$, and if $\phi \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ with support contained in $\mathcal{I}$, then

$$\|P_\phi(\{a_n\}_{n=-\infty}^\infty)\|_{L^p(\mathbb{R})} = \|\phi\|_{L^p(\mathbb{R})} \|\{a_n\}_{n=-\infty}^\infty\|_{\ell^p(\mathbb{Z})}, \quad \text{for all } \{a_n\}_{n=-\infty}^\infty \in \ell^p(\mathbb{Z}). \tag{3.3}$$

Notice also that if $\phi \in L^1(\mathbb{R})$ is a non-negative function with support contained in $\mathcal{I}$, if $N \in \mathbb{N}$, if, for $1 \leqslant j \leqslant N$, $a^{(j)} \equiv \{a_n^{(j)}\}_{n=-\infty}^\infty$ is a sequence of real numbers, and if we put

$$a_n^\# = \sup_{1 \leqslant j \leqslant N} a_n^{(j)}, \quad \text{for all } n \in \mathbb{Z},$$

then pointwise on $\mathbb{R}$ we have

$$P_\phi(\{a_n^\#\}_{n=-\infty}^\infty) = \sup_{1\leqslant j\leqslant N} P_\phi(\{a_n^{(j)}\}_{n=-\infty}^\infty). \tag{3.4}$$

For a given $F \in L^1(\mathbb{R})$, we shall denote by S_F the bilinear mapping of $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ specified by

$$(S_F(f,g))(x) = \int_{\mathbb{R}} f(x+y)g(x-y)F(y)\,\mathrm{d}y. \tag{3.5}$$

Similarly, for a given $\mathcal{W} \equiv \{\mathcal{W}_n\}_{n=-\infty}^\infty \in \ell^1(\mathbb{Z})$, we define the bilinear mapping $\mathfrak{S}_\mathcal{W} : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ by writing for all $\mathfrak{a} \in \ell^2(\mathbb{Z})$, all $\mathfrak{b} \in \ell^2(\mathbb{Z})$, and all $n \in \mathbb{Z}$,

$$(\mathfrak{S}_\mathcal{W}(\mathfrak{a},\mathfrak{b}))(n) = \sum_{k=-\infty}^\infty \mathfrak{a}_{n+k}\mathfrak{b}_{n-k}\mathcal{W}_k. \tag{3.6}$$

Using the foregoing notation, we can reproduce the statement of Lemma 3 from [2] as follows.

LEMMA 3.2. *Let $N \in \mathbb{N}$, and suppose that $\{F_j\}_{j=1}^N \subseteq L^1(\mathbb{R})$ is such that for $1 \leqslant j \leqslant N$, and each $n \in \mathbb{Z}$, the restriction $(F_j|\mathcal{I}_n)$ belongs to $C^1(\mathcal{I}_n)$, and denote its derivative by $(F_j|\mathcal{I}_n)'$. For each $n \in \mathbb{Z}$, let*

$$\Lambda_n = \sup\{|(F_j|\mathcal{I}_n)'(x)| : 1 \leqslant j \leqslant N, x \in \mathcal{I}_n\},$$

and assume that the sequence $\Lambda \equiv \{\Lambda_n\}_{n=-\infty}^\infty \in \ell^1(\mathbb{Z})$. Let

$$(S^{(N)}(f,g))(x) = \sup_{\substack{j\in\mathbb{N},\\ 1\leqslant j\leqslant N}} |(S_{F_j}(f,g))(x)|, \quad \text{for all } f,g \in L^2(\mathbb{R}), \text{ and all } x \in \mathbb{R}, \tag{3.7}$$

and put

$$(\mathfrak{S}^{(N)}(\mathfrak{a},\mathfrak{b}))(m) = \sup_{1\leqslant j\leqslant N}\Big|\sum_{n=-\infty}^\infty \mathfrak{a}_{m+n}\mathfrak{b}_{m-n}F_j(n)\Big|, \tag{3.8}$$

$$\text{for all } \mathfrak{a},\mathfrak{b} \in \ell^2(\mathbb{Z}), \text{ and all } m \in \mathbb{Z}.$$

Further, let $\phi_0 \geqslant 0$ and $\phi_1 \geqslant 0$ be the functions defined on $\mathbb{R}$ by writing for each $u \in \mathbb{R}$,

$$\phi_0(u) = 2\Big(\frac{1}{4} - |u|\Big)\chi_\mathcal{I}(u); \tag{3.9}$$

$$\phi_1(u) = \Big(\frac{1}{4} - |u|\Big)^2\chi_\mathcal{I}(u). \tag{3.10}$$

Then for every pair a, b of finitely supported complex-valued sequences defined on $\mathbb{Z}$, the following inequality holds pointwise on $\mathbb{R}$:

$$P_{\phi_0}(\mathfrak{S}^{(N)}(a,b)) \leqslant S^{(N)}(Pa,Pb) + P_{\phi_1}(\mathfrak{S}_\Lambda(|a|,|b|)). \tag{3.11}$$

This discussion has paved the way for our discretization of Theorem 3.1 in the following form. (n.b. In Section 4, we shall need to modify the proof of this discretization theorem in order to obtain a variant of its conclusion for a certain kernel lacking compact support. For the sake of this subsequent modification we shall keep careful track of the constants involved in the present proof — particularly as regards the parameters each of them depends on.)

THEOREM 3.3. *Suppose that $K : \mathbb{R} \to \mathbb{R}$ belongs to $C^\infty(\mathbb{R})$ and has compact support. We put*

$$\alpha_K = \min\{n \in \mathbb{N} : K(x) = 0 \text{ whenever } |x| > n\}; \quad \beta_K = \sup_{x\in\mathbb{R}} |K'(x)|. \tag{3.12}$$

Let $m \in \mathbb{N}$, and put $d_m = 2^{1/m}$. For each $j \in \mathbb{N}$, let $K_{j,m} \in C^\infty(\mathbb{R})$ be the compactly supported function specified by writing (in accordance with the notation for dilations defined in (1.6)*) $K_{j,m} = \delta_{d_m^j} K$, and let $Q_{j,K,m} : \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ be the bilinear mapping defined for all $\mathfrak{v} \in \mathbb{C}^{\mathbb{Z}}$ and all $\mathfrak{w} \in \mathbb{C}^{\mathbb{Z}}$ by*

$$(Q_{j,K,m}(\mathfrak{v},\mathfrak{w}))(n) = \sum_{k=-\infty}^{\infty} \mathfrak{v}(n+k)\mathfrak{w}(n-k)K_{j,m}(k), \quad \text{for all } n \in \mathbb{Z}. \tag{3.13}$$

(Notice, in particular, that if $\mathfrak{v}$ and $\mathfrak{w}$ are finitely supported, then so is the sequence $Q_{j,K,m}(\mathfrak{v},\mathfrak{w})$.) Then for every pair of finitely supported sequences $a \in \mathbb{C}^{\mathbb{Z}}$ and $b \in \mathbb{C}^{\mathbb{Z}}$, for every integer $R \geqslant 2$, and for each sequence of positive integers $u_1 < u_2 < \cdots < u_R$, we have:

$$\Big\| \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r \leqslant j < u_{r+1}}} |Q_{j,K,m}(a,b) - Q_{u_{r+1},K,m}(a,b)|^2 \Big\}^{1/2} \Big\|_{\ell^{1,\infty}(\mathbb{Z})} \tag{3.14}$$
$$\leqslant \mathfrak{c}_m (\gamma_{K,m} + \alpha_K \beta_K) R^{1/4} \, \|a\|_{\ell^2(\mathbb{Z})} \|b\|_{\ell^2(\mathbb{Z})},$$

where $\mathfrak{c}_m$ is a positive constant depending only on m, and $\gamma_{K,m}$ is the positive constant depending only on K and m that occurs in (3.1).

Proof. Until further notice we now fix $r \in \mathbb{N}$ such that $1 \leqslant r \leqslant R-1$. For each $j \in \mathbb{N}$ such that $u_r \leqslant j < u_{r+1}$, define the compactly supported $C^\infty(\mathbb{R})$ function $F_{j,r}$ by writing

$$F_{j,r} = K_{j,m} - K_{u_{r+1},m}. \tag{3.15}$$

Continuing with the notation used in Lemma 3.2, we now define the bilateral sequence $\Lambda^{(r,K,m)} \equiv \{\Lambda_n^{(r,K,m)}\}_{n=-\infty}^{\infty}$ by writing for each $n \in \mathbb{Z}$,

$$\Lambda_n^{(r,K,m)} = \sup\{|(F_{j,r}|\mathcal{I}_n)'(x)| : u_r \leqslant j < u_{r+1}, x \in \mathcal{I}_n\}. \tag{3.16}$$

Notice that $\{\Lambda_n^{(r,K,m)}\}_{n=-\infty}^{\infty}$ is finitely supported, since K has compact support, and so we can apply the conclusion (3.11) of Lemma 3.2 to the finite sequence of functions $\{F_{j,r}\}_{j=u_r}^{u_{r+1}-1}$ and its corresponding sequence $\{\Lambda_n^{(r,K,m)}\}_{n=-\infty}^{\infty}$ specified by (3.16). This furnishes our present circumstances with the following pointwise inequality on $\mathbb{R}$:

$$P_{\phi_0}\Big(\sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}}|Q_{j,K,m}(a,b)-Q_{u_{r+1},K,m}(a,b)|\Big) \leqslant \sup_{\substack{j\in\mathbb{N}\\ u_r\leqslant j<u_{r+1}}}|S_{F_{j,r}}(Pa,Pb)|+P_{\phi_1}(\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|)), \tag{3.17}$$

where the bilinear form $\mathfrak{S}_{\Lambda^{(r,K,m)}}:\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})\to\ell^1(\mathbb{Z})$ is defined in accord with (3.6), and thus satisfies

$$(\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|))=\Big\{\sum_{k=-\infty}^{\infty}|a_{n+k}b_{n-k}|\Lambda_k^{(r,K,m)}\Big\}_{n=-\infty}^{\infty}. \tag{3.18}$$

Before letting $r\in\mathbb{N}$ run through all values $1\leqslant r\leqslant R-1$ as on the left side of (3.14), we first estimate $\|\Lambda^{(r,K,m)}\|_{\ell^1(\mathbb{Z})}$ for a fixed r in this range. For $j\in\mathbb{N}$ satisfying $u_r\leqslant j<u_{r+1}$, we have $F_{j,r}(x)=0$, whenever $|x|>\alpha_K d_m^{u_{r+1}}$, and consequently

$$\Lambda_n^{(r,K,m)}=0,\quad \text{for all } n\in\mathbb{Z} \text{ such that } |n|>\alpha_K d_m^{u_{r+1}}+\frac{1}{4}. \tag{3.19}$$

Suppose that $s\in\mathbb{N}$ with $u_r\leqslant s<u_{r+1}$. Then for all $j\in\mathbb{N}$ such that $s\leqslant j<u_{r+1}$, we have for all $x\in\mathbb{R}$,

$$|K'_{j,m}(x)-K'_{u_{r+1},m}(x)|=\Big|\frac{1}{d_m^{2j}}K'\Big(\frac{x}{d_m^j}\Big)-\frac{1}{d_m^{2u_{r+1}}}K'\Big(\frac{x}{d_m^{u_{r+1}}}\Big)\Big| \leqslant\beta_K\Big(\frac{1}{d_m^{2s}}+\frac{1}{d_m^{2u_{r+1}}}\Big). \tag{3.20}$$

Moreover, if $n\in\mathbb{Z}$ and satisfies

$$|n|\geqslant\alpha_K d_m^s+\frac{1}{4}, \tag{3.21}$$

then for all $x\in\mathcal{I}_n$ and for all $j\in\mathbb{N}$ such that $j<s$,

$$K'_{j,m}(x)=0. \tag{3.22}$$

Combining this with (3.20) we infer that for each $s\in\mathbb{N}$ satisfying $u_r\leqslant s<u_{r+1}$, the following estimate is valid:

$$\Lambda_n^{(r,K,m)}\leqslant\beta_K\Big(\frac{1}{d_m^{2s}}+\frac{1}{d_m^{2u_{r+1}}}\Big),\quad \text{for all } n\in\mathbb{Z} \text{ such that } |n|\geqslant\alpha_K d_m^s+\frac{1}{4}. \tag{3.23}$$

Consequently, we see with the aid of (3.19) that

$$\begin{aligned}(3.24)\qquad &\sum\Big\{\Lambda_n^{(r,K,m)} : n \in \mathbb{Z}, |n| \geqslant \alpha_K d_m^{u_r} + \frac{1}{4}\Big\} \\ &\leqslant \sum_{\substack{s\in\mathbb{N} \\ u_r \leqslant s < u_{r+1}}} \sum\Big\{\Lambda_n^{(r,K,m)} : n \in \mathbb{Z}, \alpha_K d_m^{s+1} + \frac{1}{4} \geqslant |n| \geqslant \alpha_K d_m^{s} + \frac{1}{4}\Big\} \\ &\leqslant 2\alpha_K\beta_K d_m \sum_{\substack{s\in\mathbb{N} \\ u_r \leqslant s < u_{r+1}}} \Big(\frac{1}{d_m^{2s}} + \frac{1}{d_m^{2u_{r+1}}}\Big) d_m^s \\ &\leqslant \Big(\frac{2\alpha_K\beta_K d_m^2}{d_m - 1}\Big)\frac{1}{d_m^{u_r}} + 2\alpha_K\beta_K d_m \frac{u_{r+1}}{d_m^{u_{r+1}}}.\end{aligned}$$

By specializing the value of s in (3.20) to be u_r, we see that for all $n \in \mathbb{Z}$,

$$\Lambda_n^{(r,K,m)} \leqslant \beta_K\Big(\frac{1}{d_m^{2u_r}} + \frac{1}{d_m^{2u_{r+1}}}\Big).$$

Hence

$$\begin{aligned}(3.25)\qquad &\sum\Big\{\Lambda_n^{(r,K,m)} : n \in \mathbb{Z}, |n| \leqslant \alpha_K d_m^{u_r} + \frac{1}{4}\Big\} \\ &\leqslant \beta_K\Big(\frac{1}{d_m^{2u_r}} + \frac{1}{d_m^{2u_{r+1}}}\Big)\Big(2\alpha_K d_m^{u_r} + \frac{3}{2}\Big) \\ &\leqslant 2\alpha_K\beta_K\Big(\frac{1}{d_m^{u_r}} + \frac{1}{d_m^{u_{r+1}}}\Big) + \frac{3}{2}\beta_K\Big(\frac{1}{d_m^{2u_r}} + \frac{1}{d_m^{2u_{r+1}}}\Big).\end{aligned}$$

Upon combining (3.24) and (3.25), we deduce that for each $r \in \mathbb{N}$ such that $1 \leqslant r \leqslant R - 1$,

$$(3.26)\qquad \|\Lambda^{(r,K,m)}\|_{\ell^1(\mathbb{Z})} \leqslant \alpha_K\beta_K C_m\Big(\frac{1}{d_m^{u_r}} + \frac{u_{r+1}}{d_m^{u_{r+1}}}\Big).$$

We next square both sides of (3.17). In view of the definitions of ϕ_0 and ϕ_1 in (3.9) and (3.10) this gives us the following inequality, valid pointwise on $\mathbb{R}$.

$$\begin{aligned}(3.27)\qquad &4P_{\phi_1}\Big(\sup_{\substack{j\in\mathbb{N}, \\ u_r \leqslant j < u_{r+1}}} |Q_{j,K,m}(a,b) - Q_{u_{r+1},K,m}(a,b)|^2\Big) \\ &\leqslant \Big\{\sup_{\substack{j\in\mathbb{N} \\ u_r \leqslant j < u_{r+1}}} |S_{F_{j,r}}(Pa, Pb)| + P_{\phi_1}(\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|, |b|))\Big\}^2.\end{aligned}$$

After summing this inequality for $1 \leqslant r \leqslant R-1$, we deduce with the aid of Minkowski's inequality for ℓ^2 that the following holds pointwise on $\mathbb{R}$.

$$P_{\phi_0}\Big(\Big\{\sum_{r=1}^{R-1}\Big(\sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}}|Q_{j,K,m}(a,b)-Q_{u_{r+1},K,m}(a,b)|^2\Big)\Big\}^{1/2}\Big) \leqslant \Big\{\sum_{r=1}^{R-1}\sup_{\substack{j\in\mathbb{N}\\ u_r\leqslant j<u_{r+1}}}|S_{F_{j,r}}(Pa,Pb)|^2\Big\}^{1/2}+P_{\phi_1}\Big(\sum_{r=1}^{R-1}\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|)\Big). \tag{3.28}$$

For each $n \in \mathbb{Z}$, $\phi_0(x-n) \geqslant \frac{1}{4}$ provided $|x-n| \leqslant \frac{1}{8}$. It follows readily that, in terms of distribution functions (taken with respect to Lebesgue measure $\mathfrak{m}_{\mathbb{R}}$ on $\mathbb{R}$ and counting measure $\mathfrak{m}_{\mathbb{Z}}$ on $\mathbb{Z}$), we have for each positive real number y,

$$\frac{1}{4}\lambda\Big(\Big\{\sum_{r=1}^{R-1}\Big(\sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}}|Q_{j,K,m}(a,b)-Q_{u_{r+1},K,m}(a,b)|^2\Big)\Big\}^{1/2},\mathfrak{m}_{\mathbb{Z}};y\Big) \leqslant \lambda\Big(P_{\phi_0}\Big(\Big\{\sum_{r=1}^{R-1}\Big(\sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}}|Q_{j,K,m}(a,b)-Q_{u_{r+1},K,m}(a,b)|^2\Big)\Big\}^{1/2}\Big),\mathfrak{m}_{\mathbb{R}};\frac{y}{4}\Big). \tag{3.29}$$

From (3.28) and (3.29) we see at once that

$$\begin{aligned}\frac{1}{4}\lambda\Big(\Big\{\sum_{r=1}^{R-1}\Big(\sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}}|Q_{j,K,m}(a,b)-Q_{u_{r+1},K,m}(a,b)|^2\Big)\Big\}^{1/2},\mathfrak{m}_{\mathbb{Z}};y\Big) &\leqslant \lambda\Big(\Big\{\sum_{r=1}^{R-1}\sup_{\substack{j\in\mathbb{N}\\ u_r\leqslant j<u_{r+1}}}|S_{F_{j,r}}(Pa,Pb)|^2\Big\}^{1/2},\mathfrak{m}_{\mathbb{R}};\frac{y}{8}\Big)\\ &\quad+\lambda\Big(P_{\phi_1}\Big(\sum_{r=1}^{R-1}\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|)\Big),\mathfrak{m}_{\mathbb{R}};\frac{y}{8}\Big).\end{aligned} \tag{3.30}$$

Moreover, an application of Theorem 3.1 shows that

$$\lambda\Big(\Big\{\sum_{r=1}^{R-1}\sup_{\substack{j\in\mathbb{N}\\ u_r\leqslant j<u_{r+1}}}|S_{F_{j},r}(Pa,Pb)|^2\Big\}^{1/2},\mathfrak{m}_{\mathbb{R}};\frac{y}{8}\Big) \leqslant \frac{4\gamma_{K,m}\,R^{1/4}\|a\|_{\ell^2(\mathbb{Z})}\,\|b\|_{\ell^2(\mathbb{Z})}}{y}. \tag{3.31}$$

Applying Chebychev's inequality to the function $P_{\phi_1}\Big(\sum_{r=1}^{R-1}\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|)\Big)$, we find with the aid of (3.26) that

$$\lambda\Big(P_{\phi_1}\Big(\sum_{r=1}^{R-1}\mathfrak{S}_{\Lambda^{(r,K,m)}}(|a|,|b|)\Big),\mathfrak{m}_{\mathbb{R}};\frac{y}{8}\Big) \leqslant \frac{\alpha_K\beta_K C_m\|a\|_{\ell^2(\mathbb{Z})}\,\|b\|_{\ell^2(\mathbb{Z})}}{y}. \tag{3.32}$$

The desired conclusion (3.14) is an immediate consequence of (3.30), (3.31), and (3.32). ∎

The next theorem and its included corollary of Theorem 3.3 (see Corollary 3.5 below) furnish key applications of Lemma 2.3 by using the "isometric" transformation U in the hypotheses of Theorem 1.3 to transfer discrete oscillation estimates such as (3.14) of Theorem 3.3 to the setting of the arbitrary sigma-finite measure space (Ω, μ). (Compare the transference reasoning in [4] and [5] that targeted averages defined by the invertible measure-preserving point transformations of finite measure spaces.) The notation of Proposition 2.1 will now be in effect, and it will be convenient to observe that since each Φ_k, $k \in \mathbb{Z}$, is multiplicative on $\mathcal{A}(\mu)$, it follows from Proposition 2.1(ii), together with (2.3) and (2.4), that for all $F \in \mathcal{A}(\mu)$, $G \in \mathcal{A}(\mu)$, $n_1 \in \mathbb{Z}$, and $n_2 \in \mathbb{Z}$, we have μ -a.e. on Ω,

$$\begin{aligned} \Phi_k((U^{n_1}F)(U^{n_2}G)) &= \Phi_k(h_{n_1})\,\Phi_{k+n_1}(F)\,\Phi_k(h_{n_2})\,\Phi_{k+n_2}(G) \\ &= h_k^{-2}\,(U^{k+n_1}F)\,(U^{k+n_2}G). \end{aligned} \tag{3.33}$$

THEOREM 3.4. *Suppose that (Ω, μ) is a sigma-finite measure space, and let U be a bijective linear mapping of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$ such that conditions* (i) *and* (ii) *in the hypotheses of Theorem* 1.3 *hold. For each $j \in \mathbb{N}$, let $\mathfrak{s}_j : \mathbb{Z} \to \mathbb{C}$ be finitely supported, and define the bilinear mappings $T_j : \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ and $\mathfrak{T}_j : \mathcal{A}(\mu) \times \mathcal{A}(\mu) \to \mathcal{A}(\mu)$ as follows:*

$$(T_j(\mathfrak{v},\mathfrak{w}))(k) = \sum_{n=-\infty}^{\infty} \mathfrak{v}(k+n)\mathfrak{w}(k-n)\mathfrak{s}_j(n), \quad \textit{for each } \mathfrak{v} \in \mathbb{C}^{\mathbb{Z}}, \textit{ each } \mathfrak{w} \in \mathbb{C}^{\mathbb{Z}}, \textit{ and all } k \in \mathbb{Z};$$

$$\mathfrak{T}_j(F,G) = \sum_{n=-\infty}^{\infty} (U^n F)\,(U^{-n}G)\mathfrak{s}_j(n), \quad \textit{for all } F \in \mathcal{A}(\mu), \textit{ and all } G \in \mathcal{A}(\mu).$$

Suppose that there is a positive real constant ζ such that for every pair of finitely supported sequences $a \in \mathbb{C}^{\mathbb{Z}}$ and $b \in \mathbb{C}^{\mathbb{Z}}$, for every integer $R \geqslant 2$, and for each sequence of positive integers $u_1 < u_2 < \cdots < u_R$, we have:

$$\left\| \left\{ \sum_{r=1}^{R-1} \sup_{\substack{j \in \mathbb{N}, \\ u_r \leqslant j < u_{r+1}}} |T_j(a,b) - T_{u_{r+1}}(a,b)|^2 \right\}^{1/2} \right\|_{\ell^{1,\infty}(\mathbb{Z})} \leqslant \zeta R^{1/4}\, \|a\|_{\ell^2(\mathbb{Z})} \|b\|_{\ell^2(\mathbb{Z})}. \tag{3.34}$$

Then for every $f \in L^2(\mu)$, every $g \in L^2(\mu)$, every integer $R \geqslant 2$, and each sequence of positive integers $u_1 < u_2 < \cdots < u_R$, we have:

$$\left\| \left\{ \sum_{r=1}^{R-1} \sup_{\substack{j \in \mathbb{N}, \\ u_r \leqslant j < u_{r+1}}} |\mathfrak{T}_j(f,g) - \mathfrak{T}_{u_{r+1}}(f,g)|^2 \right\}^{1/2} \right\|_{L^{1,\infty}(\mu)} \leqslant \zeta R^{1/4}\, \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}. \tag{3.35}$$

Hence by Lemma 2.3, *for every $f \in L^2(\mu)$ and every $g \in L^2(\mu)$ the sequence $\{\mathfrak{T}_j(f,g)\}_{j=1}^{\infty}$ converges μ -a.e. on Ω to a corresponding function belonging to $\mathcal{A}(\mu)$.*

Proof. For convenience, put

$$\Delta = \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} |\mathfrak{T}_j(f,g) - \mathfrak{T}_{u_{r+1}}(f,g)|^2 \Big\}^{1/2}. \tag{3.36}$$

Applying (2.6) to the distribution function of Δ, we see that for each $k \in \mathbb{Z}$, and each real number $y > 0$,

$$\lambda(\Delta,\mu;y) = \lambda(\Phi_k(\Delta),\mu;y). \tag{3.37}$$

From the properties of Φ_k in (2.1), (2.2), and (2.5) we deduce that

$$\begin{aligned}\Phi_k(\Delta) &= \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} |\Phi_k(\mathfrak{T}_j(f,g)) - \Phi_k(\mathfrak{T}_{u_{r+1}}(f,g))|^2 \Big\}^{1/2} \\ &= \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} |h_k^2\Phi_k(\mathfrak{T}_j(f,g)) - h_k^2\Phi_k(\mathfrak{T}_{u_{r+1}}(f,g))|^2 \Big\}^{1/2}.\end{aligned} \tag{3.38}$$

For every $\nu \in \mathbb{N}$, we have by virtue of (3.33),

$$h_k^2\Phi_k(\mathfrak{T}_v(f,g)) = \sum_{n=-\infty}^{\infty} (U^{k+n}f)\,(U^{k-n}g)\mathfrak{s}_v(n). \tag{3.39}$$

Using (3.39) to substitute in (3.38), we find that for each $k \in \mathbb{Z}$,

$$\begin{aligned}\Phi_k(\Delta) = \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} \Big| &\sum_{n=-\infty}^{\infty} (U^{k+n}f)\,(U^{k-n}g)\mathfrak{s}_j(n) - \cdots \\ &- \sum_{n=-\infty}^{\infty} (U^{k+n}f)\,(U^{k-n}g)\mathfrak{s}_{u_{r+1}}(n)\Big|^2 \Big\}^{1/2}.\end{aligned} \tag{3.40}$$

Now let N_0 be the least positive integer N such that $\mathfrak{s}_j(n) = 0$ whenever $1 \leqslant j \leqslant u_R$ and $|n| > N$, temporarily fix an arbitrary $L \in \mathbb{N}$, and let $\mathfrak{C}_{L,N_0}$ denote the characteristic function, defined on $\mathbb{Z}$, of

$$\{n \in \mathbb{Z} : |n| \leqslant L + N_0\}.$$

For each $x \in \Omega$, we define the finitely supported sequences $\phi_x : \mathbb{Z} \to \mathbb{C}$ and $\psi_x : \mathbb{Z} \to \mathbb{C}$ by writing for each $n \in \mathbb{Z}$,

$$\phi_x(n) = \mathfrak{C}_{L,N_0}(n)\,((U^n f)(x)); \quad \psi_x(n) = \mathfrak{C}_{L,N_0}(n)\,((U^n g)(x)).$$

In terms of this notation, we can use (3.40) to write for each $k \in \mathbb{Z}$ such that $-L \leqslant k \leqslant L$, and for each $x \in \Omega$,

$$(\Phi_k(\Delta))(x) = \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} |(T_j(\phi_x,\psi_x))(k) - (T_{u_{r+1}}(\phi_x,\psi_x))(k)|^2 \Big\}^{1/2}. \tag{3.41}$$

From (3.37) we have for each real number $y > 0$,

$$(2L+1)\lambda(\Delta,\mu;y) = \sum_{k=-L}^{L} \lambda(\Phi_k(\Delta),\mu;y). \tag{3.42}$$

Temporarily fix an arbitrary positive real number y, and for each $k \in \mathbb{Z}$ with $-L \leqslant k \leqslant L$, denote by χ_k the characteristic function, defined on Ω, of the set E_k specified by

$$E_k = \{x \in \Omega : (\Phi_k(\Delta))(x) > y\}.$$

This permits us to rewrite (3.42) in the form

$$(2L+1)\lambda(\Delta,\mu;y) = \int_\Omega \Big(\sum_{k=-L}^{L} \chi_k(x)\Big)\, \mathrm{d}\mu(x). \tag{3.43}$$

With the aid of (3.41) we see that at each $x \in \Omega$, the integrand in (3.43) can be expressed in terms of counting measure $\mathfrak{m}_{\mathbb{Z}}$ on $\mathbb{Z}$ by:

$$\begin{aligned}\sum_{k=-L}^{L} \chi_k(x) &= \mathfrak{m}_{\mathbb{Z}}\{k \in \mathbb{Z} : -L \leqslant k \leqslant L, \text{ and } x \in E_k\}\\ &\leqslant \mathfrak{m}_{\mathbb{Z}}\Big\{k \in \mathbb{Z} : \Big\{\sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r\leqslant j<u_{r+1}}} |(T_j(\phi_x,\psi_x))(k) - (T_{u_{r+1}}(\phi_x,\psi_x))(k)|^2\Big\}^{1/2} > y\Big\}.\end{aligned}$$

Application to this of the hypothesis (3.34) shows that for each $x \in \Omega$,

$$\begin{aligned}\sum_{k=-L}^{L} \chi_k(x) &\leqslant \frac{\zeta R^{1/4} \|\phi_x\|_{\ell^2(\mathbb{Z})}\, \|\psi_x\|_{\ell^2(\mathbb{Z})}}{y}\\ &= \frac{\zeta R^{1/4}}{y}\Big\{\sum_{n=-L-N_0}^{L+N_0} |(U^n f)(x)|^2\Big\}^{1/2} \Big\{\sum_{n=-L-N_0}^{L+N_0} |(U^n g)(x)|^2\Big\}^{1/2}.\end{aligned}$$

Using this on the right of (3.43) and then invoking Cauchy–Schwarz, we find, since $U|L^2(\mu)$ is a surjective linear isometry, that:

$$\begin{aligned}\lambda(\Delta,\mu;y) &\leqslant \frac{\zeta R^{1/4}}{y(2L+1)} \int_\Omega \Big\{\sum_{n=-L-N_0}^{L+N_0} |(U^n f)|^2\Big\}^{1/2}\Big\{\sum_{n=-L-N_0}^{L+N_0} |(U^n g)|^2\Big\}^{1/2} \mathrm{d}\mu\\ &\leqslant \frac{\zeta R^{1/4}}{y}\Big(\frac{2L+2N_0+1}{2L+1}\Big)\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}.\end{aligned} \tag{3.44}$$

In view of the definition of Δ in (3.36), we can immediately arrive at (3.35) by letting $L \to \infty$ on the right of (3.44). ∎

The following corollary results directly from Theorem 3.3 and Theorem 3.4.

COROLLARY 3.5. *Suppose that (Ω,μ) is a sigma-finite measure space, and let U be a bijective linear mapping of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$ such that conditions* (i) *and* (ii) *in the hypotheses of Theorem* 1.3 *hold. Suppose that $K : \mathbb{R} \to \mathbb{R}$ belongs to $C^\infty(\mathbb{R})$ and has* *compact support. Let $m \in \mathbb{N}$, and put $d_m = 2^{1/m}$. For each $j \in \mathbb{N}$, let $K_{j,m} \in C^\infty(\mathbb{R})$ be the compactly supported function given by $K_{j,m} = \delta_{d_m^j} K$, and define the bilinear mapping $\mathfrak{A}_{j,K,m,U}$ of $\mathcal{A}(\mu) \times \mathcal{A}(\mu)$ into $\mathcal{A}(\mu)$ by writing for all $F \in \mathcal{A}(\mu)$, and all $G \in \mathcal{A}(\mu)$,*

$$\mathfrak{A}_{j,K,m,U}(F,G) = \sum_{n=-\infty}^{\infty} (U^n F)(U^{-n} G) K_{j,m}(n) = \frac{1}{d_m^j} \sum_{n=-\infty}^{\infty} (U^n F)(U^{-n} G) K\Big(\frac{n}{d_m^j}\Big). \tag{3.45}$$

Then for every $f \in L^2(\mu)$, every $g \in L^2(\mu)$, every integer $R \geqslant 2$, and each sequence of positive integers $u_1 < u_2 < \cdots < u_R$, we have:

$$\Big\| \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j \in \mathbb{N}, \\ u_r \leqslant j < u_{r+1}}} |\mathfrak{A}_{j,K,m,U}(f,g) - \mathfrak{A}_{u_{r+1},K,m,U}(f,g)|^2 \Big\}^{1/2} \Big\|_{L^{1,\infty}(\mu)} \leqslant \Gamma_{K,m} R^{1/4} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}, \tag{3.46}$$

where $\Gamma_{K,m}$ denotes the constant $\mathfrak{c}_m(\gamma_{K,m} + \alpha_K \beta_K)$ that occurs in (3.14). *Hence for every $f \in L^2(\mu)$ and every $g \in L^2(\mu)$, the sequence $\{\mathfrak{A}_{j,K,m,U}(f,g)\}_{j=1}^\infty$ converges μ-a.e. on Ω to a corresponding function belonging to $\mathcal{A}(\mu)$.*

4. PROOF OF THEOREM 1.3

In view of Theorem 2.5 and dominated convergence, we need only demonstrate the conclusions of Theorem 1.3 regarding the existence of μ-a.e. limits. By Theorem 2.5 and the Multilinear Banach Principle (see, e.g., Proposition 1 of [2] for the latter), it suffices for the demonstration of Theorem 1.3 to show that each of the sequences $\{A_{k,U}(f,g)\}_{k=1}^\infty$ and $\{H_{k,U}(f,g)\}_{k=1}^\infty$ (as defined by (1.1) and (1.2)) converges μ-a.e. on Ω when we specialize f and g to be μ-integrable simple functions such that $\|f\|_{L^\infty(\mu)} = \|g\|_{L^\infty(\mu)} = 1$. This will be carried out in two parts. (In the ensuing discussion, we shall use without explicit mention the convenient fact that, since $U|L^\infty(\mu)$ is a surjective linear isometry, we have for all $n \in \mathbb{Z}$,

$$\|U^n f\|_{L^\infty(\mu)} = \|U^n g\|_{L^\infty(\mu)} = 1.$$

Part (i). We first prove the μ-a.e. convergence of $\{A_{k,U}(f,g)\}_{k=1}^\infty$ for such f and g. Let M be an arbitrary positive real number, and choose a real-valued function $K^{(M)} \in C^\infty(\mathbb{R})$ such that: $K^{(M)}$ vanishes on $(-\infty, -\frac{1}{M}] \cup [1 + \frac{1}{M}, \infty)$; $K^{(M)} = 1$ on $[0,1]$; $K^{(M)}$ is increasing on $[-\frac{1}{M}, 0]$; $K^{(M)}$ is decreasing on $[1, 1 + \frac{1}{M}]$. By Corollary 3.5, for each $m \in \mathbb{N}$, the sequence $\{\mathfrak{A}_{j,K^{(M)},m,U}(f,g)\}_{j=1}^\infty$ corresponding to f, g, $K^{(M)}$, and m in accordance with (3.45) converges μ-a.e. on Ω. For each

$j \in \mathbb{N}$, we have

$$\begin{aligned}(4.1)\quad & \mathfrak{A}_{j,K^{(M)},m,U}(f,g)\\ &= \sum\Big\{(U^n f)(U^{-n}g)K^{(M)}_{j,m}(n) : -\frac{2^{j/m}}{M} < n < 2^{j/m}\Big(1+\frac{1}{M}\Big)\Big\}\\ &= \sum\Big\{(U^n f)(U^{-n}g)K^{(M)}_{j,m}(n) : -\frac{2^{j/m}}{M} < n < 0\Big\} + \frac{1}{2^{j/m}}\sum_{n=0}^{[2^{j/m}]}(U^n f)(U^{-n}g)\\ &\qquad + \sum\Big\{(U^n f)(U^{-n}g)K^{(M)}_{j,m}(n) : 2^{j/m} < n < 2^{j/m}\Big(1+\frac{1}{M}\Big)\Big\}.\end{aligned}$$

But for each $n \in \mathbb{Z}$ such that $-\frac{2^{j/m}}{M} < n < 0$ or $2^{j/m} < n < 2^{j/m}(1+\frac{1}{M})$,

$$0 \leqslant K^{(M)}_{j,m}(n) \leqslant \frac{1}{2^{j/m}},$$

and so

$$\begin{aligned}(4.2)\quad & \Big|\sum\Big\{(U^n f)(U^{-n}g)K^{(M)}_{j,m}(n) : -\frac{2^{j/m}}{M} < n < 0\Big\}\Big| \leqslant \frac{1}{2^{j/m}}\Big(\frac{2^{j/m}}{M}\Big) = \frac{1}{M};\\ & \Big|\sum\Big\{(U^n f)(U^{-n}g)K^{(M)}_{j,m}(n) : 2^{j/m} < n < 2^{j/m}\Big(1+\frac{1}{M}\Big)\Big\}\Big| \leqslant \Big(\frac{1}{M} + \frac{1}{2^{j/m}}\Big).\end{aligned}$$

Since $\{\mathfrak{A}_{j,K^{(M)},m,U}(f,g)\}_{j=1}^{\infty}$ converges μ-a.e. on Ω for arbitrary $m \in \mathbb{N}$, and arbitrary positive $M \in \mathbb{R}$, it follows readily from (4.1) and (4.2) that for each $m \in \mathbb{N}$ the sequence $\{\widetilde{A}_{2^{j/m},U}(f,g)\}_{j=1}^{\infty}$ specified by

$$\widetilde{A}_{2^{j/m},U}(f,g) = \frac{1}{2^{j/m}}\sum_{n=0}^{[2^{j/m}]-1}(U^n f)(U^{-n}g), \quad \text{for each } j \in \mathbb{N},$$

converges μ-a.e. on Ω. So for Part (i) of the proof it remains to show that the μ-a.e. convergence of $\{\widetilde{A}_{2^{j/m},U}(f,g)\}_{j=1}^{\infty}$ for each fixed $m \in \mathbb{N}$ can be converted into μ-a.e. convergence of the sequence $\{A_{k,U}(f,g)\}_{k=1}^{\infty}$. Given $m \in \mathbb{N}$, $k \in \mathbb{N}$, with $k \geqslant 2$, let $j = j(k,m) \in \mathbb{N}$ satisfy

$$(4.3)\qquad 2^{j/m} \leqslant k < 2^{(j+1)/m}.$$

Hence for some absolute constant η we have

$$(4.4)\qquad 0 \leqslant \frac{k-2^{j/m}}{k} \leqslant \frac{k-2^{j/m}}{2^{j/m}} < 2^{1/m} - 1 \leqslant \frac{\eta}{m},$$

and consequently we have pointwise on Ω,

$$\begin{aligned}(4.5)\quad & |A_{k,U}(f,g) - \widetilde{A}_{2^{j/m},U}(f,g)|\\ &\leqslant \Big(\frac{k-2^{j/m}}{2^{j/m}k}\Big)\sum_{n=0}^{[2^{j/m}]-1}|(U^n f)|\,|U^{-n}g| + \frac{1}{k}\sum_{n=[2^{j/m}]}^{k-1}|(U^n f)|\,|U^{-n}g|\\ &\leqslant 2\Big(\frac{k-2^{j/m}}{k}\Big) + \frac{2^{j/m}-[2^{j/m}]}{k} \leqslant \frac{2\eta}{m} + \frac{1}{k}.\end{aligned}$$

It follows from (4.5) and the μ-a.e. convergence for each $m \in \mathbb{N}$ of the sequence $\{\widetilde{A}_{2^{j/m},U}(f,g)\}_{j=1}^{\infty}$ that the sequence $\{A_{k,U}(f,g)\}_{k=1}^{\infty}$ is pointwise Cauchy μ-a.e on Ω.

Part (ii). To complete the demonstration of Theorem 1.3, it will suffice (as noted above) to establish the μ-a.e. convergence of the averages $\{H_{k,U}(f,g)\}_{k=1}^{\infty}$ for μ-integrable simple functions f, g such that $\|f\|_{L^\infty(\mu)} = \|g\|_{L^\infty(\mu)} = 1$. For this purpose, we shall follow the main outlines of the proof for Theorem 1.2 of [4]. We start by letting M be an arbitrary integer such that $M \geqslant 2$, and then choosing, as the relevant kernel for applying Theorem 3.1, an odd $C^\infty(\mathbb{R})$ function $\mathfrak{K}^{\langle M\rangle} : \mathbb{R} \to \mathbb{R}$ such that:

$$\mathfrak{K}^{\langle M\rangle}(x) = \frac{1}{x}, \quad \text{for } |x| \geqslant 1; \tag{4.6}$$

$$\mathfrak{K}^{\langle M\rangle}(x) = 0, \quad \text{for } |x| \leqslant 1 - \frac{1}{M}; \tag{4.7}$$

$$|\mathfrak{K}^{\langle M\rangle}(x)| \leqslant 2, \quad \text{for } |x| \leqslant 1. \tag{4.8}$$

Notice that by virtue of (4.6), the successive derivatives $\frac{d^n \mathfrak{K}^{\langle M\rangle}(x)}{dx^n}$, for $n \in \mathbb{N}$, all belong to $L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, and this fact is helpful in seeing by elementary considerations that $\mathfrak{K}^{\langle M\rangle}$ satisfies the hypotheses of Theorem 3.1, which thereby furnishes us with the following oscillation estimate, valid for each integer $M \geqslant 2$, each $m \in \mathbb{N}$, each integer $J \geqslant 2$, each sequence of positive integers $u_1 < u_2 < \cdots < u_J$, and every pair of compactly supported functions F and G belonging to $L^\infty(\mathbb{R})$.

$$\begin{aligned}\Big\|\Big\{ \sum_{j=1}^{J-1} \sup_{\substack{k\in\mathbb{N},\\ u_j \leqslant k < u_{j+1}}} \Big| \int_{\mathbb{R}} F(x+y)G(x-y)\big((\delta_{d_m^k}\mathfrak{K}^{\langle M\rangle})(y) \\ - (\delta_{d_m^{u_{j+1}}}\mathfrak{K}^{\langle M\rangle})(y)\big)\,\mathrm{d}y\Big|^2\Big\}^{1/2}\Big\|_{L_x^{1,\infty}} \\ \leqslant C_{M,m}\, J^{1/4}\, \|F\|_{L^2(\mathbb{R})}\, \|G\|_{L^2(\mathbb{R})}.\end{aligned} \tag{4.9}$$

However, since the kernel $\mathfrak{K}^{\langle M\rangle}$ lacks compact support and does not belong to $L^1(\mathbb{R})$, its exploitation of the oscillation estimate (4.9) will require extra care. In this regard, it is convenient to observe that because of (4.6) $\mathfrak{K}^{\langle M\rangle}$ has the following "quasi-stability" under dilations: for each positive real number ξ,

$$(\delta_\xi \mathfrak{K}^{\langle M\rangle})(x) = \frac{1}{x}, \quad \text{whenever } |x| \geqslant \xi\,.$$

Hence if $0 < \xi_1 \leqslant \xi_2$, then $(\delta_{\xi_1}\mathfrak{K}^{\langle M\rangle})(x) = (\delta_{\xi_2}\mathfrak{K}^{\langle M\rangle})(x) = \frac{1}{x}$, for $|x| \geqslant \xi_2$. In particular, for $1 \leqslant j \leqslant J-1$, and $u_j \leqslant k < u_{j+1}$,

$$(\delta_{d_m^k}\mathfrak{K}^{\langle M\rangle} - \delta_{d_m^{u_{j+1}}}\mathfrak{K}^{\langle M\rangle})(x) = 0, \quad \text{whenever } |x| > d_m^{u_{j+1}}. \tag{4.10}$$

Consequently, although the $C^{\infty}(\mathbb{R})$ kernel $\mathfrak{K}^{\langle M\rangle}$ itself lacks compact support (and so is not automatically covered by the discretization result in Theorem 3.3), the discretization methods in the proof of Theorem 3.3 can nevertheless go forward from (4.9) by straightforward adjustments which rely on the compact supports of the relevant difference kernels in accordance with (4.10). This procedure discretizes (4.9) by yielding the following result. For each integer $M \geqslant 2$, for each $m \in \mathbb{N}$, for every pair of finitely supported sequences $a \in \mathbb{C}^{\mathbb{Z}}$ and $b \in \mathbb{C}^{\mathbb{Z}}$, for every integer $R \geqslant 2$, and for each sequence of positive integers $u_1 < u_2 < \cdots < u_R$,

$$\Big\| \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r \leqslant j < u_{r+1}}} |\mathcal{K}_{j,M,m}(a,b) - \mathcal{K}_{u_{r+1},M,m}(a,b)|^2 \Big\}^{1/2} \Big\|_{\ell^{1,\infty}(\mathbb{Z})} \leqslant C_{M,m}\, R^{1/4}\, \|a\|_{\ell^2(\mathbb{Z})} \|b\|_{\ell^2(\mathbb{Z})}, \tag{4.11}$$

where for each $\nu \in \mathbb{N}$, and each $k \in \mathbb{Z}$,

$$(\mathcal{K}_{\nu,M,m}(a,b))(k) = \sum_{n=-\infty}^{\infty} \frac{a(k+n)\, b(k-n)}{d_m^{\nu}} \mathfrak{K}^{\langle M\rangle}\Big(\frac{n}{d_m^{\nu}}\Big). \tag{4.12}$$

(Since a and b are finitely supported, the sum on the right of (4.12) has only finitely many non-zero terms, and also $\mathcal{K}_{\nu,M,m}(a,b)$ is finitely supported.)

In order to obtain a transferred counterpart of (4.11) to which we can apply Theorem 3.4, we shall first recast (4.11) so that it becomes completely expressed in terms of finitely supported discrete kernels (rather than the present discrete kernels of the form $(\delta_{d_m^{\nu}} \mathfrak{K}^{\langle M\rangle})|\mathbb{Z})$. The method for doing so will be taken from the proof for Theorem 1.2 in [4]. Specifically, for each $j \in \mathbb{N}$, we define $\mathbb{A}_{j,M,m} : \mathbb{Z} \to \mathbb{R}$, $\mathfrak{H}_{j,m} : \mathbb{Z} \to \mathbb{R}$ and $\mathfrak{D}_{j,M,m} : \mathbb{Z} \to \mathbb{R}$ by writing:

$$\mathbb{A}_{j,M,m}(n) = \begin{cases} \frac{1}{d_m^j} \mathfrak{K}^{\langle M\rangle}\Big(\frac{n}{d_m^j}\Big) & \text{if } |n| \leqslant d_m^j\,, \\ 0 & \text{otherwise;} \end{cases} \tag{4.13}$$

$$\mathfrak{H}_{j,m}(n) = \begin{cases} \frac{1}{n} & \text{if } 0 < |n| \leqslant d_m^j, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathfrak{D}_{j,M,m}(n) = \mathbb{A}_{j,M,m}(n) - \mathfrak{H}_{j,m}(n), \quad \text{for all } n \in \mathbb{Z}.$$

Then it is easy to verify from definitions that whenever $j \in \mathbb{N}$ and $\nu \in \mathbb{N}$, we have for all $n \in \mathbb{Z}$,

$$\frac{1}{d_m^j} \mathfrak{K}^{\langle M\rangle}\Big(\frac{n}{d_m^j}\Big) - \frac{1}{d_m^{\nu}} \mathfrak{K}^{\langle M\rangle}\Big(\frac{n}{d_m^{\nu}}\Big) = \mathfrak{D}_{j,M,m}(n) - \mathfrak{D}_{\nu,M,m}(n).$$

For each $\nu \in \mathbb{N}$, we define the bilinear mapping $D_{\nu,M,m} : \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by writing for all $\mathfrak{v} \in \mathbb{C}^{\mathbb{Z}}$, all $\mathfrak{w} \in \mathbb{C}^{\mathbb{Z}}$, and all $k \in \mathbb{Z}$,

$$(D_{\nu,M,m}(\mathfrak{v},\mathfrak{w}))(k) = \sum_{n=-\infty}^{\infty} \mathfrak{v}(k+n)\, \mathfrak{w}(k-n)\, \mathfrak{D}_{\nu,M,m}(n)\,.$$

We can now use the finitely supported discrete kernels $\mathfrak{D}_{j,M,m}$ to rewrite the inequality (4.11) in the following form, valid for each integer $M \geqslant 2$, each $m \in \mathbb{N}$, every pair of finitely supported sequences $a \in \mathbb{C}^{\mathbb{Z}}$ and $b \in \mathbb{C}^{\mathbb{Z}}$, every integer $R \geqslant 2$, and each sequence of positive integers $u_1 < u_2 < \cdots < u_R$.

$$\Big\| \Big\{ \sum_{r=1}^{R-1} \sup_{\substack{j\in\mathbb{N},\\ u_r \leqslant j < u_{r+1}}} |D_{j,M,m}(a,b) - D_{u_{r+1},M,m}(a,b)|^2 \Big\}^{1/2} \Big\|_{\ell^{1,\infty}(\mathbb{Z})} \leqslant C_{M,m}\, R^{1/4}\, \|a\|_{\ell^2(\mathbb{Z})} \|b\|_{\ell^2(\mathbb{Z})}. \tag{4.14}$$

After applying Theorem 3.4 to (4.14), we infer that for each integer $M \geqslant 2$, for each $m \in \mathbb{N}$, and for the above-described μ-integrable simple functions f, g, the sequence

$$\Big\{ \sum_{|k| \leqslant d_m^n} (U^k f)\,(U^{-k} g)\mathfrak{D}_{n,M,m}(k) \Big\}_{n=1}^{\infty} \quad \text{converges } \mu\text{-a.e. on } \Omega. \tag{4.15}$$

For each $n \in \mathbb{N}$, it is clear from definitions and the notation of (1.2) that the following identity holds pointwise on Ω.

$$\sum_{|k| \leqslant d_m^n} (U^k f)(U^{-k} g)\mathfrak{D}_{n,M,m}(k) = \sum_{|k| \leqslant d_m^n} (U^k f)(U^{-k} g)\mathbb{A}_{n,M,m}(k) - H_{d_m^n,U}(f,g). \tag{4.16}$$

Moreover, (4.13), taken in conjunction with (4.7) and (4.8), shows that for each integer $M \geqslant 2$, each $m \in \mathbb{N}$, and each $n \in \mathbb{N}$,

$$\sum_{|k| \leqslant d_m^n} (U^k f)(U^{-k} g)\mathbb{A}_{n,M,m}(k) = \sum \Big\{ (U^k f)\,(U^{-k} g) \frac{1}{d_m^n} \mathfrak{K}^{\langle M \rangle}\Big(\frac{k}{d_m^n}\Big) : \Big(1 - \frac{1}{M}\Big) d_m^n < |k| \leqslant d_m^n \Big\},$$

and so we have pointwise on Ω,

$$\Big| \sum_{|k| \leqslant d_m^n} (U^k f)\,(U^{-k} g)\mathbb{A}_{n,M,m}(k) \Big| \leqslant 4\Big(\frac{1}{M} + \frac{1}{d_m^n}\Big) = 4\Big(\frac{1}{M} + \frac{1}{2^{n/m}}\Big).$$

Since the integer $M \geqslant 2$ is arbitrary, it follows from this, (4.15), and (4.16) that for each $m \in \mathbb{N}$,

$$\{H_{2^{n/m},U}(f,g)\}_{n=1}^{\infty} \quad \text{converges } \mu\text{-a.e. on } \Omega. \tag{4.17}$$

The μ-a.e. convergence of the averages $\{H_{k,U}(f,g)\}_{k=1}^{\infty}$ can now be deduced from (4.17) by reasoning analogous to that used to complete Part (i) of the proof.

5. THE CONTINUOUS VARIABLE COUNTERPART OF THEOREM 1.3

This brief final section features Theorem 5.2 below, which is the counterpart of Theorem 1.3 for averages defined by one-parameter groups of Lebesgue space isometries associated with the arbitrary sigma-finite measure space (Ω, μ).

These continuous variable averages do not require any discretization for oscillation estimates based on Theorem 3.1, and in this respect their treatment is simpler than that for the discrete averages. Moreover, Theorem 5.2 below can be established by techniques which, though at times involving measure-theoretic technicalities, are transparently analogous to those used in the preceding sections for the discrete averages. In particular, for the relevant transferred maximal estimates (repectively, relevant transferred oscillation estimates) in the present setting, we need only replace the role of Theorem 2.5 (respectively, Theorem 3.4) by suitable reasoning based on Section 6 of [2] so as to transfer from $\mathbb{R}$ to the (Ω, μ) context Michael Lacey's classical estimates [13] for the bisublinear Hardy–Littlewood maximal operator and the bisublinear maximal Hilbert transform (respectively, oscillation estimates of the form (3.1) of Theorem 3.1). In view of this state of affairs, the discussion below will, for expository reasons, omit detailed arguments.

In order to formulate the results for the continuous variable setting, we begin by describing the main ingredients of the discussion. Our transference vehicle for defining the relevant averages on the measure space side will be a one-parameter group $\mathcal{U} \equiv \{U_t : t \in \mathbb{R}\}$ consisting of linear bijections of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$. Thus,

$$U_{s+t}(f) = U_s(U_t f), \quad \text{for all } s \in \mathbb{R}, t \in \mathbb{R}, f \in \mathcal{A}(\mu). \tag{5.1}$$

The one-parameter group $\mathcal{U} \equiv \{U_t : t \in \mathbb{R}\}$ will be required to satisfy the following conditions:

(1) For each $t \in \mathbb{R}$, $\lim_{k\to\infty} (U_t g_k) = U_t g$ μ-a.e. on Ω, whenever $\{g_k\}_{k=1}^{\infty} \subseteq \mathcal{A}(\mu)$, $g \in \mathcal{A}(\mu)$, and $\lim_{k\to\infty} g_k = g$ μ-a.e. on Ω.

(2) For $0 < p < \infty$, and each $s \in \mathbb{R}$, $L^p(\mu)$ is invariant under U_s, and the restrictions $\{U_t | L^p(\mu) : t \in \mathbb{R}\}$ form a strongly continuous one-parameter group of surjective linear isometries of $L^p(\mu)$ onto $L^p(\mu)$.

(3) For each $f \in \mathcal{A}(\mu)$, the expression $(U_t f)(x)$, where (t, x) runs through $\mathbb{R} \times \Omega$, can be regarded as being a jointly measurable version with respect to the product of linear Lebesgue measure $\mathfrak{m}_{\mathbb{R}}$ and the measure μ. In other words, there exists a complex-valued $(\mathfrak{m}_{\mathbb{R}} \times \mu)$-measurable function $\mathfrak{F}_f$ on $\mathbb{R} \times \Omega$ such that for each $t \in \mathbb{R}$, $\mathfrak{F}_f(t, \cdot)$ belongs to the equivalence class (modulo equality μ -a.e. on Ω) of $U_t f$. (For convenience, we shall denote such a function $\mathfrak{F}_f$ by $(U_t f)(x)$.)

REMARK 5.1. (i) We observe here that for $f \in \mathcal{A}(\mu)$, any two jointly measurable versions $\mathfrak{F}_f^{(1)}$ and $\mathfrak{F}_f^{(2)}$ representing $(U_t f)(x)$ on $\mathbb{R} \times \Omega$ in accordance with condition (3) would automatically have the additional property that for μ-almost all $x \in \Omega$,

$$\mathfrak{F}_f^{(1)}(t, x) = \mathfrak{F}_f^{(2)}(t, x), \quad \text{for } \mathfrak{m}_{\mathbb{R}}\text{-almost all } t \in \mathbb{R}. \tag{5.2}$$

For this reason among other obvious reasons, the particular choice of jointly measurable version of $(U_t f)(x)$ on $\mathbb{R} \times \Omega$ will be immaterial in all our considerations below.

(ii) As is well-known (see, e.g., Proposition 5 in [2]), the above conditions (1) and (2) can be shown to imply that for each $t \in \mathbb{R}$, $U_t|L^\infty(\mu)$ is a surjective linear isometry of $L^\infty(\mu)$ onto $L^\infty(\mu)$. Hence, in view of condition (3), for every $f \in L^\infty(\mu)$, we have for μ-almost all $x \in \Omega$,

$$\|(U_{(\cdot)}f)(x)\|_{L^\infty(\mathbb{R})} \leqslant \|f\|_{L^\infty(\mu)}. \tag{5.3}$$

In terms of the foregoing notation, our continuous variable version of Theorem 1.3 takes the following form. (In particular, the variable limits of integration of the indefinite integrals occurring below are continuous rather than discrete variables.)

THEOREM 5.2. *Let (Ω, μ) be a sigma-finite measure space, let $\mathcal{U} \equiv \{U_t : t \in \mathbb{R}\}$ be a one-parameter group of linear bijections of $\mathcal{A}(\mu)$ onto $\mathcal{A}(\mu)$ satisfying the above conditions* (1), (2), *and* (3), *and let p_1, p_2, p_3 satisfy* (1.3) *and* (1.4). *Then for every pair of functions $f \in L^{p_1}(\mu) \cap L^2(\mu)$ and $g \in L^{p_2}(\mu) \cap L^2(\mu)$, the following assertions hold:*

(i) *Each of the following two limits exists μ-a.e. on Ω, as well as with respect to the metric topology of the space $L^{p_3}(\mu)$:*

$$(\mathbb{E}_{\mathcal{U},\infty}(f,g))(x) \equiv \lim_{r\to\infty} \frac{1}{r}\int_0^r (U_t f)(x)(U_{-t}g)(x)\,\mathrm{d}t; \tag{5.4}$$

$$(\mathbb{E}_{\mathcal{U},0}(f,g))(x) \equiv \lim_{r\to 0^+} \frac{1}{r}\int_0^r (U_t f)(x)(U_{-t}g)(x)\,\mathrm{d}t. \tag{5.5}$$

Moreover,

$$\max\{\|\mathbb{E}_{\mathcal{U},0}(f,g)\|_{L^{p_3}(\mu)}, \|\mathbb{E}_{\mathcal{U},\infty}(f,g)\|_{L^{p_3}(\mu)}\} \leqslant C_{p_1,p_2}\|f\|_{L^{p_1}(\mu)}\,\|g\|_{L^{p_2}(\mu)}. \tag{5.6}$$

(ii) *For μ-almost all $x \in \Omega$, the (Cauchy principal value) improper integral*

$$\int_{\varepsilon\leqslant|t|} \frac{(U_t f)(x)(U_{-t}g)(x)}{t}\,\mathrm{d}t \equiv \lim_{\varsigma\to\infty}\int_{\varepsilon\leqslant|t|\leqslant\varsigma} \frac{(U_t f)(x)(U_{-t}g)(x)}{t}\,\mathrm{d}t \tag{5.7}$$

exists in $\mathbb{C}$ for every $\varepsilon > 0$, and $(\mathbb{H}_{\varepsilon,\mathcal{U}}(f,g))(x) \equiv \int_{\varepsilon\leqslant|t|} \frac{(U_t f)(x)(U_{-t}g)(x)}{t}\,\mathrm{d}t$ approaches a limit $(\mathbb{H}_{\mathcal{U}}(f,g))(x) \in \mathbb{C}$, as $\varepsilon \to 0^+$. We also have the following two limit relations with respect to convergence in the metric topology of the space $L^{p_3}(\mu)$:

$$\mathbb{H}_{\varepsilon,\mathcal{U}}(f,g) = \lim_{\varsigma\to\infty}\int_{\varepsilon\leqslant|t|\leqslant\varsigma} \frac{(U_t f)(\cdot)(U_{-t}g)(\cdot)}{t}\,\mathrm{d}t, \quad \textit{for each } \varepsilon > 0; \tag{5.8}$$

$$\mathbb{H}_{\mathcal{U}}(f,g) = \lim_{\varepsilon\to 0^+} \mathbb{H}_{\varepsilon,\mathcal{U}}(f,g). \tag{5.9}$$

Moreover,

$$\|\mathbb{H}_{\mathcal{U}}(f,g)\|_{L^{p_3}(\mu)} \leqslant \Big\| \sup_{\varepsilon>0} |\mathbb{H}_{\varepsilon,\mathcal{U}}(f,g)| \Big\|_{L^{p_3}(\mu)} \leqslant C_{p_1,p_2} \|f\|_{L^{p_1}(\mu)} \|g\|_{L^{p_2}(\mu)}. \tag{5.10}$$

We come now to a discussion of the convergence properties exhibited by the one-parameter (continuous variable) averages in the setting of an arbitrary measure space (X,σ). Although the measure σ need not be sigma-finite, this context does offer a notion of joint measurability with respect to the measurable spaces of $\mathfrak{m}_{\mathbb{R}}$ and σ for complex-valued functions defined on $\mathbb{R} \times X$ (as in, e.g., Section 33 of [10]), but a product measure of $\mathfrak{m}_{\mathbb{R}}$ and σ (in the sense of the abstract Fubini's theorem, as in, e.g., Section Section 35, 36 of [10]) is lacking, and this lack imposes technical constraints on attempts to mirror the general measure space results for the a.e. convergence of the discrete averages induced by measure-preserving point transformations (Corollary 1.4). For example, we no longer have a route to compatibility conditions like (5.2), and so the framing of σ-a.e. convergence questions in the continuous variable framework can become refractory. In view of these circumstances we shall, for convenience, forgo discussion of σ-a.e. convergence for the one-parameter averages in favor of studying the convergence in $L^p(\sigma)$ of their Bochner integral formulations.

The transference vehicle for the present framework will be a one-parameter group (under composition of mappings) $\mathcal{P} \equiv \{\psi_t : t \in \mathbb{R}\}$ consisting of invertible measure-preserving point transformations of (X,σ) — thus, for all $x \in X$, all $s \in \mathbb{R}$, and all $t \in \mathbb{R}$, $\psi_{s+t}(x) = \psi_s(\psi_t(x))$. In this setup, we postulate the following two properties for $\mathcal{P} \equiv \{\psi_t : t \in \mathbb{R}\}$, thereby endowing $\mathcal{V} \equiv \{V_t : t \in \mathbb{R}\}$, the one-parameter group of corresponding composition operators on $\mathcal{A}(\sigma)$, with the counterpart of the conditions (1), (2), and (3) that were imposed on $\mathcal{U} \equiv \{U_t : t \in \mathbb{R}\}$ at the outset of this section.

(A) $\lim\limits_{t\to t_0} \sigma((\psi_t(E))\Delta(\psi_{t_0}(E))) = 0$, for each set $E \subseteq X$ such that $\sigma(E) < \infty$, and each $t_0 \in \mathbb{R}$, where Δ denotes the symmetric difference of sets.

(B) For each $f \in \mathcal{A}(\sigma)$, there exists a complex-valued function $\mathfrak{E}_f$ on $\mathbb{R} \times X$ such that $\mathfrak{E}_f$ is jointly measurable with respect to the measurable spaces of $\mathfrak{m}_{\mathbb{R}}$ and σ, and for each $t \in \mathbb{R}$, $\mathfrak{E}_f(t,\bullet)$ belongs to the equivalence class (modulo equality σ-a.e. on X) of $V_t f$. (For convenience, we shall denote such a function $\mathfrak{E}_f$ by $(V_t f)(x) \equiv f(\psi_t(x))$.)

In view of (A) and Cauchy–Schwarz, for each $f \in L^2(\sigma)$ and each $g \in L^2(\sigma)$, the pointwise product $f(\psi_t)g(\psi_{-t})$ *qua* function of $t \in \mathbb{R}$ moves continuously in $L^1(\sigma)$. So for each $F \in L^1(\mathbb{R})$, the $L^1(\sigma)$-valued Bochner integral $\int_{\mathbb{R}} f(\psi_t)g(\psi_{-t})F(t)\,\mathrm{d}t$ exists and clearly satisfies

$$\Big\| \int_{\mathbb{R}} f(\psi_t)g(\psi_{-t})F(t)\,\mathrm{d}t \Big\|_{L^1(\sigma)} \leqslant \|F\|_{L^1(\mathbb{R})} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}.$$

With due attention to technical details arising in this context, one can deduce, as a corollary of Theorem 5.2, the following continuous variable variant of Corollary 1.4.

COROLLARY 5.3. *Suppose that* (X, σ) *is an arbitrary measure space, and let* $\mathcal{P} \equiv \{\psi_t : t \in \mathbb{R}\}$ *be a one-parameter group of invertible measure-preserving point transformations of* (X, σ) *onto* (X, σ) *which has the properties* (A) *and* (B) *listed above. Let* p_1, p_2, p_3 *satisfy* (1.3) *and* (1.4). *Then for each pair of functions* $f \in L^{p_1}(\sigma) \cap L^2(\sigma)$ *and* $g \in L^{p_2}(\sigma) \cap L^2(\sigma)$, *the following assertions are valid:*

(i) *The* $L^1(\sigma)$*-valued Bochner integrals* $\int_0^r f(\psi_t) g(\psi_{-t})\, \mathrm{d}t$ *(*$r > 0$*) belong to* $L^{p_3}(\sigma)$, *and have the property that both the following limits exist with respect to the metric topology of the space* $L^{p_3}(\sigma)$.

$$\mathbb{E}_{\mathcal{P},\infty}(f, g) \equiv \lim_{r \to \infty} \frac{1}{r} \int_0^r f(\psi_t) g(\psi_{-t})\, \mathrm{d}t; \tag{5.11}$$

$$\mathbb{E}_{\mathcal{P},0}(f, g) \equiv \lim_{r \to 0^+} \frac{1}{r} \int_0^r f(\psi_t) g(\psi_{-t})\, \mathrm{d}t. \tag{5.12}$$

Moreover,

$$\max\{\|\mathbb{E}_{\mathcal{P},0}(f, g)\|_{L^{p_3}(\sigma)}, \|\mathbb{E}_{\mathcal{P},\infty}(f, g)\|_{L^{p_3}(\sigma)}\} \leqslant C_{p_1,p_2} \|f\|_{L^{p_1}(\sigma)} \|g\|_{L^{p_2}(\sigma)}. \tag{5.13}$$

(ii) *For each* $\varepsilon > 0$, *the* $L^1(\sigma)$*-valued Bochner integrals* $\int_{\varepsilon \leqslant |t| \leqslant \varsigma} f(\psi_t) g(\psi_{-t}) t^{-1}\, \mathrm{d}t$ *(*$\varepsilon < \varsigma$*) belong to* $L^{p_3}(\sigma)$, *and have the property that, with respect to the metric topology of the space* $L^{p_3}(\sigma)$,

$$\mathbb{H}_{\varepsilon,\mathcal{P}}(f, g) \equiv \lim_{\varsigma \to \infty} \int_{\varepsilon \leqslant |t| \leqslant \varsigma} f(\psi_t) g(\psi_{-t}) t^{-1}\, \mathrm{d}t$$

exists. We also have, with respect to the metric topology of the space $L^{p_3}(\sigma)$, *the existence of*

$$\mathbb{H}_{\mathcal{P}}(f, g) \equiv \lim_{\varepsilon \to 0^+} \mathbb{H}_{\varepsilon,\mathcal{P}}(f, g).$$

Moreover,

$$\|\mathbb{H}_{\mathcal{P}}(f, g)\|_{L^{p_3}(\sigma)} \leqslant \sup_{\varepsilon > 0} \|\mathbb{H}_{\varepsilon,\mathcal{P}}(f, g)\|_{L^{p_3}(\sigma)} \leqslant C_{p_1,p_2} \|f\|_{L^{p_1}(\sigma)} \|g\|_{L^{p_2}(\sigma)}.$$

We close with the following example, which illustrates Theorem 5.2 in the realm of harmonic analysis on groups.

EXAMPLE 5.4. The context of this example will be Helson's classic theory of generalized analyticity and invariant subspaces, which we first describe in order to set the stage. (For a full discussion of this context and its generalizations, we refer the reader to [1], [12].) Let Γ be a dense subgroup of the additive group $\mathbb{R}$ of all real numbers. Endow Γ with the discrete topology and the order it inherits from $\mathbb{R}$, and let K be the dual group of Γ. (Equivalently, K can be characterized as a compact abelian group other than $\{0\}$ or the unit circle $\mathbb{T}$ such that the dual group of K is archimedean ordered.) Denote the normalized Haar measure of K by $\mathfrak{m}_K$, and for each $t \in \mathbb{R}$, let $\mathfrak{e}_t \in K$ be specified by writing for all $\gamma \in \Gamma$, $\mathfrak{e}_t(\gamma) = \mathrm{e}^{\mathrm{i}t\gamma}$. A cocycle on K is a Borel measurable function $A : \mathbb{R} \times K \to \mathbb{T}$ such that

$$A(t+u,x) = A(t,x)A(u,x+\mathfrak{e}_t), \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in \mathbb{R}, \text{ and all } x \in K.$$

Every cocycle A on K automatically has the property that the mapping $t \to A(t,\cdot)$ is continuous from $\mathbb{R}$ into $L^p(\mathfrak{m}_K)$ for $0 < p < \infty$ (see Lemma VII.12.1 of [9]). We denote by $\mathcal{C}$ the class of all cocycles on K (identified modulo equality $(\mathfrak{m}_\mathbb{R} \times \mathfrak{m}_K)$-a.e. on $\mathbb{R} \times K$), and we associate with each $A \in \mathcal{C}$ the following one-parameter group $\mathcal{U}^{(A)} \equiv \{U_t^{(A)} : t \in \mathbb{R}\}$ of linear bijections of $\mathcal{A}(\mathfrak{m}_K)$: for each $t \in \mathbb{R}$ and each $f \in \mathcal{A}(\mathfrak{m}_K)$,

$$(U_t^{(A)} f)(x) = A(t,x)\, f(x+\mathfrak{e}_t), \quad \text{for } \mathfrak{m}_K \text{-almost all } x \in K.$$

It is readily seen that Theorem 5.2 applies to $\mathcal{U}^{(A)} \equiv \{U_t^{(A)} : t \in \mathbb{R}\}$. Briefly put, Helson's classic theory of generalized analyticity and invariant subspaces uses the spectral decomposability of the one-parameter unitary groups $\{U_t^{(A)}|L^2(\mathfrak{m}_K) : t \in \mathbb{R}\}$, $A \in \mathcal{C}$, to establish a one-to-one correspondence between the cocycles A on K and the normalized simply invariant subspaces of $L^2(\mathfrak{m}_K)$. This state of affairs has been generalized to $L^p(\mathfrak{m}_K)$, $1 \leqslant p < \infty$, as follows (see Sections 2 and 3 of [1]): the one-parameter group $\mathcal{U}^{(A)} \equiv \{U_t^{(A)} : t \in \mathbb{R}\}$ transfers the Hilbert transform for $\mathbb{R}$ to a weak type $(1,1)$ operator $\mathcal{H}^{(A)} : L^1(\mathfrak{m}_K) \to \mathcal{A}(\mathfrak{m}_K)$ which is specified by taking

$$(\mathcal{H}^{(A)} f)(x) = \lim_{n\to\infty} \int\limits_{n^{-1}\leqslant|t|\leqslant n} \frac{(U_{-t}^{(A)} f)(x)}{\pi t}\,\mathrm{d}t, \quad \text{for } \mathfrak{m}_K\text{-almost all } x \in K,$$

and $\mathcal{H}^{(A)}$ furnishes, via generalized Hardy spaces, a cocycle characterization of the normalized simply invariant subspaces of $L^p(\mathfrak{m}_K)$. For $1 < p < \infty$, $\mathcal{H}^{(A)}|L^p(\mathfrak{m}_K)$ is a continuous linear mapping of $L^p(\mathfrak{m}_K)$ into itself. Clearly, Theorem 5.2 above, when specialized to $\{U_t^{(A)} : t \in \mathbb{R}\}$, provides the bilinear counterpart for $\mathcal{H}^{(A)}$ (and, in contrast, the transferred bilinear Hilbert transform $\mathbb{H}_{\mathcal{U}^{(A)}}(\cdot,\cdot)$ is bounded even in the instances where the index p_3 of the target space $L^{p_3}(\mathfrak{m}_K)$ satisfies $\frac{2}{3} < p \leqslant 1$).

Acknowledgements. The second author was supported by NSF Grant DMS-0556389.

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EARL BERKSON, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN ST., URBANA, IL 61801, U.S.A.

E-mail address: berkson@math.uiuc.edu

CIPRIAN DEMETER, DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, 831 EAST 3RD ST., BLOOMINGTON, IN 47405, U.S.A.

E-mail address: demeterc@indiana.edu

Received September 4, 2007; revised February 1, 2008.