MORITA TYPE EQUIVALENCES AND REFLEXIVE ALGEBRAS

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ABSTRACT. Two unital dual operator algebras $\mathcal{A}, \mathcal{B}$ are called Δ-equivalent if there exists an equivalence functor $\mathcal{F} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ which "extends" to a $*$-functor implementing an equivalence between the categories ${}_{\mathcal{A}}\mathfrak{DM}$ and ${}_{\mathcal{B}}\mathfrak{DM}$. Here ${}_{\mathcal{A}}\mathfrak{M}$ denotes the category of normal representations of $\mathcal{A}$ and ${}_{\mathcal{A}}\mathfrak{DM}$ denotes the category with the same objects as ${}_{\mathcal{A}}\mathfrak{M}$ and $\Delta(\mathcal{A})$-module maps as morphisms ($\Delta(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^*$). We prove that any such functor maps completely isometric representations to completely isometric representations, "respects" the lattices of the algebras and maps reflexive algebras to reflexive algebras. We present applications to the class of CSL algebras.

KEYWORDS: *Operator algebras, dual operator algebras, Morita equivalence, TRO, reflexive algebras, CSL.*

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INTRODUCTION

The Morita equivalence of rings has been extended to many settings. In particular, Rieffel developed the appropriate notion and theory for C^*-algebras and w*-algebras ([10], [11]). Blecher, Muhly and Paulsen generalized Rieffel's C^*-algebraic version to nonselfadjoint operator algebras [3]. In [5], [6] we obtained a generalization of Rieffel's concept of Morita equivalence of w*-algebras to the class of (not necessarily selfadjoint) unital dual operator algebras. The purpose of the present work is to apply this theory to the case of reflexive algebras, especially to CSL algebras. As we know this is the first connection between Morita theory and nonselfadjoint reflexive algebras in literature.

We say that two unital dual operator algebras are Δ-equivalent if there is an equivalence functor between their categories of normal representations which not only preserves intertwiners of representations of the algebras, but also preserves intertwiners of their restrictions to the diagonals (see Definition 1.6). This reduces to Rieffel's definition in the selfadjoint case. In Section 2 we study properties of

Δ-equivalence functors. We show that every such functor maps completely isometric representations to completely isometric ones. Furthermore, every equivalence functor sends the invariant projection lattice of a representation onto the invariant projection lattice of the corresponding representation, and maps reflexive algebras to reflexive algebras.

In Section 3 we give examples of Δ-equivalent algebras and of Δ-inequivalent algebras. We show that two CSL algebras $\mathcal{A}, \mathcal{B}$ are Δ-equivalent if and only if there exists a ternary ring of operators $\mathcal{M}$ such that $\mathcal{A} = [\mathcal{M}^* \mathcal{B} \mathcal{M}]^{-\mathrm{w}^*}$ and $\mathcal{B} = [\mathcal{M} \mathcal{A} \mathcal{M}^*]^{-\mathrm{w}^*}$ (see Definition 1.2.) It follows from [5] that two separably acting CSL algebras with either continuous or totally atomic invariant projection lattices are Δ-equivalent if and only if they have isomorphic lattices. We show however that isomorphism of the invariant projection lattices does not always imply Δ-equivalence of the algebras, even in the case of multiplicity free nests and isomorphic diagonals. Nevertheless, if there exists an order isomorphism between two nests, preserving the dimensions of intervals (equivalently, if the nests are *similar*), there always exists an equivalence functor between the categories of normal, completely contractive representations of the nest algebras, which is normal and completely isometric.

We inform the reader that in another paper [7], written after the research reported in this paper was completed, we proved jointly with Vern Paulsen that two unital dual operator algebras $\mathcal{A}, \mathcal{B}$ are Δ-equivalent if and only if they are stably isomorphic, i.e. if there exists a Hilbert space H such that the algebras $\mathcal{A} \overline{\otimes} B(H)$ and $\mathcal{B} \overline{\otimes} B(H)$ (where $\overline{\otimes}$ denotes the normal spatial tensor product [2]) are isomorphic as dual operator algebras.

1. PRELIMINARIES

We present some symbols used below. If $\mathcal{A}$ is an operator algebra ([2], [9]), we denote its diagonal $\mathcal{A} \cap \mathcal{A}^*$ by $\Delta(\mathcal{A})$. The symbol $[\mathcal{S}]$ denotes the linear span of $\mathcal{S}$. The commutant of a set $\mathcal{L}$ of bounded operators on a Hilbert space H is denoted $\mathcal{L}'$. If $\mathcal{U}$ is a linear space and $n, m \in \mathbb{N}$ we denote by $M_{n,m}(\mathcal{U})$ the space of $n \times m$ matrices with entries from $\mathcal{U}$ and by $M_n(\mathcal{U})$ the space $M_{n,n}(\mathcal{U})$. If $\mathcal{U}, \mathcal{V}$ are linear spaces, α is a linear map from $\mathcal{U}$ to $\mathcal{V}$ and $n, m \in \mathbb{N}$ we denote the linear map

$$M_{n,m}(\mathcal{U}) \to M_{n,m}(\mathcal{V}) : (A_{ij})_{i,j} \to (\alpha(A_{ij}))_{i,j}$$

again by α. If $\mathcal{U}$ is a subspace of $B(H, K)$ for H, K Hilbert spaces we equip $M_{n,m}(\mathcal{U})$, $n, m \in \mathbb{N}$ with the norm inherited from the embedding $M_{n,m}(\mathcal{U}) \subset B(H^m, K^n)$. If $(\mathcal{X}, \|\cdot\|)$ is a normed space we denote by $\mathrm{Ball}(\mathcal{X})$ the unit ball of $\mathcal{X} : \{X \in \mathcal{X} : \|X\| \leqslant 1\}$. If $x_1, \ldots, x_n$ are in a vector space $\mathcal{V}$, we write $(x_1, \ldots, x_n)^{\mathrm{t}}$ for the column vector in $M_{n,1}(\mathcal{V})$.

A C^*-algebra which is a dual Banach space is called a w*-*algebra*. A *dual operator algebra* is an abstract operator algebra which is the operator dual of an

operator space. Every w*-algebra is a dual operator algebra. For every dual operator algebra $\mathcal{A}$ there exists a Hilbert space H_0 and an algebraic homomorphism $\alpha_0 : \mathcal{A} \to B(H_0)$ which is a complete isometry and a w*-continuous map [2].

A set of projections on a Hilbert space will be called a *lattice* if contains the zero and identity operators and is closed under arbitrary suprema and infima. If $\mathcal{A}$ is a subalgebra of $B(H)$ for some Hilbert space H, the set

$$\mathrm{Lat}(\mathcal{A}) = \{L \in pr(B(H)) : L^{\perp}\mathcal{A}L = 0\}$$

is a lattice. Dually if $\mathcal{L}$ is a lattice the space

$$\mathrm{Alg}(\mathcal{L}) = \{A \in B(H) : L^{\perp}AL = 0 \,\forall\, L \in \mathcal{L}\}$$

is an algebra. The *reflexive hull* of a unital algebra $\mathcal{A}$ is the algebra

$$\mathrm{Ref}(\mathcal{A}) \equiv \mathrm{Alg}(\mathrm{Lat}(\mathcal{A})).$$

Whenever $\mathcal{A} = \mathrm{Ref}(\mathcal{A})$ we call $\mathcal{A}$ a *reflexive algebra*.

A subspace $\mathcal{M}$ of $B(H_1, H_2)$ where H_1, H_2 are Hilbert spaces, is called a *ternary ring of operators (TRO)* if

$$\mathcal{M}\mathcal{M}^*\mathcal{M} \subset \mathcal{M}.$$

In this case the spaces $[\mathcal{M}^*\mathcal{M}], [\mathcal{M}\mathcal{M}^*]$ are selfadjoint algebras. We call $\mathcal{M}$ *essential* if the algebras $[\mathcal{M}^*\mathcal{M}]^{-w^*}, [\mathcal{M}\mathcal{M}^*]^{-w^*}$ contain the identity operators. The following lemma is known. See for example 8.5.32 of [2] or Lemma 3.1 of [5].

LEMMA 1.1. *Let $\mathcal{C}, \mathcal{E}$ be von Neumann algebras acting on Hilbert spaces H_1 and H_2 respectively, $\theta : \mathcal{C} \to \mathcal{E}$ be a $*$-isomorphism and*

$$\mathcal{M} = \{T \in B(H_1, H_2) : TA = \theta(A)T \text{ for all } A \in \mathcal{C}\}.$$

The space $\mathcal{M}$ is an essential TRO.

DEFINITION 1.2 ([5]). Let $\mathcal{A}, \mathcal{B}$ be w*-closed algebras acting on Hilbert spaces H_1 and H_2 respectively. If there exists a TRO $\mathcal{M} \subset B(H_1, H_2)$ such that $\mathcal{A} = [\mathcal{M}^*\mathcal{B}\mathcal{M}]^{-w^*}$ and $\mathcal{B} = [\mathcal{M}\mathcal{A}\mathcal{M}^*]^{-w^*}$ we write $\mathcal{A} \overset{\mathcal{M}}{\sim} \mathcal{B}$. We say that the algebras $\mathcal{A}, \mathcal{B}$ are *TRO equivalent* if there exists a TRO $\mathcal{M}$ such that $\mathcal{A} \overset{\mathcal{M}}{\sim} \mathcal{B}$.

We also need the following main result of [5].

THEOREM 1.3. *Two unital reflexive algebras $\mathcal{A}, \mathcal{B}$ are* TRO *equivalent if and only if there exists a $*$-isomorphism $\theta : \Delta(\mathcal{A})' \to \Delta(\mathcal{B})'$ such that $\theta(\mathrm{Lat}(\mathcal{A})) = \mathrm{Lat}(\mathcal{B})$. If θ is as above and $\mathcal{M} = \{T : TA = \theta(A)T \text{ for all } A \in \Delta(\mathcal{A})'\}$ then $\mathcal{A} \overset{\mathcal{M}}{\sim} \mathcal{B}$.*

We now define the category $_{\mathcal{A}}\mathfrak{M}$ for a unital dual operator algebra $\mathcal{A}$ [2]. An object of $_{\mathcal{A}}\mathfrak{M}$ is a Hilbert space H for which there exists a unital algebraic homomorphism $\alpha : \mathcal{A} \to B(H)$ which is completely contractive and w*-continuous. We shall call such a map a *normal representation* of $\mathcal{A}$. Throughout this work we

denote this object by H or (H, α). If $(H_i, \alpha_i), i = 1, 2$ are objects of $_A\mathfrak{M}$ the space of homomorphisms $\text{Hom}_A(H_1, H_2)$ is the following:

$$\text{Hom}_A(H_1, H_2) = \{T \in B(H_1, H_2) : T\alpha_1(A) = \alpha_2(A)T \text{ for all } A \in \mathcal{A}\}.$$

Observe that the map $\alpha_i|_{\Delta(A)}$ is a $*$-homomorphism since α_i is a contraction, [2]. We also define the category $_A\mathfrak{DM}$ which has the same objects as $_A\mathfrak{M}$ but for every pair of objects $(H_i, \alpha_i), i = 1, 2$ the space of homomorphisms $\text{Hom}_A^{\mathfrak{D}}(H_1, H_2)$ is the following:

$$\text{Hom}_A^{\mathfrak{D}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\alpha_1(A) = \alpha_2(A)T \text{ for all } A \in \Delta(A)\}.$$

If the algebra $\mathcal{A}$ is a w^*-algebra the categories $_A\mathfrak{M}$ and $_A\mathfrak{DM}$ are the same. Also observe that $\text{Hom}_A(H_1, H_2) \subset \text{Hom}_A^{\mathfrak{D}}(H_1, H_2)$.

DEFINITION 1.4 ([6]). Let $\mathcal{A}, \mathcal{B}$ be unital dual operator algebras and $\mathcal{F} : {}_A\mathfrak{M} \rightarrow {}_B\mathfrak{M}$ be a functor. We say that the functor $\mathcal{F}$ has a Δ-extension if there exists a functor $\mathcal{G} : {}_A\mathfrak{DM} \rightarrow {}_B\mathfrak{DM}$ such that the following diagram is commutative:

$$\begin{array}{ccc} _A\mathfrak{M} & \hookrightarrow & _A\mathfrak{DM} \\ \mathcal{F} \downarrow & & \mathcal{G} \downarrow \\ _B\mathfrak{M} & \hookrightarrow & _B\mathfrak{DM} \end{array}.$$

The following extends Rieffel's definition [10]:

DEFINITION 1.5 ([6]). Let $\mathcal{A}, \mathcal{B}$ be unital dual operator algebras and $\mathcal{F} : {}_A\mathfrak{DM} \rightarrow {}_B\mathfrak{DM}$ be a functor. We say that $\mathcal{F}$ is a $*$-*functor* if for every pair of objects H_1, H_2 of $_A\mathfrak{DM}$ every operator $F \in \text{Hom}_A^{\mathfrak{D}}(H_1, H_2)$ satisfies

$$\mathcal{F}(F^*) = \mathcal{F}(F)^* \in \text{Hom}_B^{\mathfrak{D}}(\mathcal{F}(H_2), \mathcal{F}(H_1)).$$

DEFINITION 1.6 ([6]). Let $\mathcal{A}, \mathcal{B}$ be unital dual operator algebras. If there exists an equivalence functor $\mathcal{F} : {}_A\mathfrak{M} \rightarrow {}_B\mathfrak{M}$ which has a Δ-extension as a $*$-functor implementing an equivalence between the categories $_A\mathfrak{DM}, {}_B\mathfrak{DM}$, then $\mathcal{A}, \mathcal{B}$ are called Δ-*equivalent* algebras.

The main theorem in [6] which is a generalization of the main theorem of Rieffel [10], is the following:

THEOREM 1.7. *Two unital dual operator algebras $\mathcal{A}, \mathcal{B}$ are Δ-equivalent if and only if they have completely isometric normal representations α, β on Hilbert spaces such that the algebras $\alpha(\mathcal{A}), \beta(\mathcal{B})$ are* TRO *equivalent.*

2. PROPERTIES OF THE EQUIVALENCE FUNCTORS

In this section we fix unital dual operator algebras $\mathcal{A}, \mathcal{B}$ and a functor $\mathcal{F}$ implementing the equivalence of Theorem 1.7. We are going to investigate properties of the functor $\mathcal{F}$, especially in case the algebras are reflexive.

In [6], Section 1, we proved that there exists $(H_0, \alpha_0) \in {}_{\mathcal{A}}\mathfrak{M}$ with corresponding object $(\mathcal{F}(H_0), \beta_0) \in {}_{\mathcal{B}}\mathfrak{M}$ such that α_0, β_0 are complete isometries and there exists an essential w^*-closed TRO $\mathcal{M} \subset B(H_0, \mathcal{F}(H_0))$ such that $\alpha_0(\mathcal{A}) \overset{\mathcal{M}}{\sim} \beta_0(\mathcal{B})$. In what follows we identify $\mathcal{A}$ with $\alpha_0(\mathcal{A})$ and $\mathcal{B}$ with $\beta_0(\mathcal{B})$. We denote by $\mathcal{U}$ and $\mathcal{V}$ the spaces $\mathcal{U} = [\mathcal{M}\mathcal{A}]^{-w^*}, \mathcal{V} = [\mathcal{A}\mathcal{M}^*]^{-w^*}$ which satisfy the following relations:

$$\mathcal{B}\mathcal{U}\mathcal{A} \subset \mathcal{U}, \quad \mathcal{A}\mathcal{V}\mathcal{B} \subset \mathcal{V}, \quad [\mathcal{V}\mathcal{U}]^{-w^*} = \mathcal{A}, \quad [\mathcal{U}\mathcal{V}]^{-w^*} = \mathcal{B}.$$

Let us briefly recall from [6], Section 2, the definition of a functor $\mathcal{F}_{\mathcal{U}} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ and its Δ-extension to a $*$-functor $\mathcal{F}_{\mathcal{U}} : {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{D}\mathfrak{M}$ denoted by the same symbol.

For every $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ the Hilbert space $\mathcal{F}_{\mathcal{U}}(H)$ is the Hausdorff completion of the algebraic tensor product $\mathcal{U} \otimes H$ with respect to the following seminorm:

$$\left\| \sum_{j=1}^m T_j \otimes x_j \right\|_{\mathcal{F}_{\mathcal{U}}(H)} = \sup_{S \in \text{Ball}(M_{n,1}(\mathcal{V})), n \in \mathbb{N}} \left\| \sum_{j=1}^m \alpha(ST_j)(x_j) \right\|_{H^n}.$$

The representation $\mathcal{F}(\alpha)$ is defined by the formula

$$\beta = \mathcal{F}(\alpha) : \mathcal{B} \to B(\mathcal{F}_{\mathcal{U}}(H)) : \beta(B)(T \otimes x) = BT \otimes x$$

where $B \in \mathcal{B}, T \in \mathcal{U}, x \in H$ and $T \otimes x$ is identified with its image in the quotient.

Also for every $H_1, H_2 \in {}_{\mathcal{A}}\mathfrak{M}$ we define a map

$$\mathcal{F}_{\mathcal{U}} : \text{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H_1, H_2) \to \text{Hom}_{\mathcal{B}}^{\mathfrak{D}}(\mathcal{F}_{\mathcal{U}}(H_1), \mathcal{F}_{\mathcal{U}}(H_2))$$

by the formula

$$\mathcal{F}_{\mathcal{U}}(F)(M \otimes x) = M \otimes F(x) \quad \text{for all } F \in \text{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H_1, H_2), M \in \mathcal{M}, x \in H_1.$$

The map $\mathcal{F}_{\mathcal{U}}(F)$ is well defined by this formula because $\mathcal{M} \otimes H_1$ is dense in $\mathcal{F}_{\mathcal{U}}(H_1)$, ([6], Corollary 2.5).

We will need the following theorem in [6]:

THEOREM 2.1. *The functors $\mathcal{F}, \mathcal{F}_{\mathcal{U}}$ are equivalent as functors between the categories ${}_{\mathcal{A}}\mathfrak{M}$, ${}_{\mathcal{B}}\mathfrak{M}$ and their Δ-extensions are equivalent as $*$-functors between the categories ${}_{\mathcal{A}}\mathfrak{D}\mathfrak{M}$, ${}_{\mathcal{B}}\mathfrak{D}\mathfrak{M}$.*

We now come to the concepts which will occupy us in this work.

DEFINITION 2.2. Let $\mathcal{A}_1, \mathcal{B}_1$ be unital dual operator algebras.

(i) A functor $\mathcal{G} : {}_{\mathcal{A}_1}\mathfrak{M} \to {}_{\mathcal{B}_1}\mathfrak{M}$ is called *completely isometric (respectively normal)* if for every pair of objects H_1, H_2 the map

$$\mathcal{G} : \text{Hom}_{\mathcal{A}_1}(H_1, H_2) \to \text{Hom}_{\mathcal{B}_1}(\mathcal{G}(H_1), \mathcal{G}(H_2))$$

is a complete isometry (respectively w^*-continuous).

(ii) We say that the functor $\mathcal{G} : {}_{\mathcal{A}_1}\mathfrak{M} \to {}_{\mathcal{B}_1}\mathfrak{M}$ *respects isometries* if whenever $(H, \alpha) \in {}_{\mathcal{A}_1}\mathfrak{M}$ is such that the map $\alpha : \mathcal{A}_1 \to B(H)$ is a complete isometry the corresponding map $\mathcal{G}(\alpha) : \mathcal{B}_1 \to B(\mathcal{G}(H))$ is a complete isometry too.

(iii) We say that the functor $\mathcal{G} : {}_{\mathcal{A}_1}\mathfrak{M} \to {}_{\mathcal{B}_1}\mathfrak{M}$ *respects reflexivity* if whenever $(H, \alpha) \in {}_{\mathcal{A}_1}\mathfrak{M}$ is such that the map $\alpha : \mathcal{A}_1 \to B(H)$ is a complete isometry and the algebra $\alpha(\mathcal{A}_1)$ is reflexive, the map $\beta : \mathcal{B}_1 \to B(\mathcal{G}(H))$ is a complete isometry and the algebra $\beta(\mathcal{B}_1)$ is reflexive, where $(\mathcal{G}(H), \beta)$ is the corresponding object of ${}_{\mathcal{B}_1}\mathfrak{M}$.

(iv) A functor $\mathcal{G} : {}_{\mathcal{A}_1}\mathfrak{DM} \to {}_{\mathcal{B}_1}\mathfrak{DM}$ is called a *lattice respecting* functor if for every object (H, α) of ${}_{\mathcal{A}_1}\mathfrak{M}$

$$\mathcal{G}(\mathrm{Lat}(\alpha(\mathcal{A}_1))) = \mathrm{Lat}(\beta(\mathcal{B}_1))$$

where $(\mathcal{G}(H), \beta)$ is the corresponding object of the category ${}_{\mathcal{B}_1}\mathfrak{M}$. (Observe that $\mathrm{Lat}(\alpha(\mathcal{A}_1)) \subset \mathrm{Hom}_{\mathcal{A}_1}^{\mathfrak{D}}(H, H)$ and $\mathrm{Lat}(\beta(\mathcal{B}_1)) \subset \mathrm{Hom}_{\mathcal{B}_1}^{\mathfrak{D}}(\mathcal{G}(H), \mathcal{G}(H))$.)

The following lemma is essentially due to Paschke; see for example 8.5.23 of [2].

LEMMA 2.3. *There exist partial isometries* $\{W_k, k \in J\} \subset \mathcal{M}(\{V_k, k \in I\} \subset \mathcal{M})$ *such that* $W_k^* W_k \perp W_m^* W_m (V_k V_k^* \perp V_m V_m^*)$ *for* $k \neq m$ *and* $I_{H_0} = \sum_k \bigoplus W_k^* W_k$ $(I_{\mathcal{F}(H_0)} = \sum_k \bigoplus V_k V_k^*)$.

LEMMA 2.4. *The functor* $\mathcal{F}_{\mathcal{U}} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ *respects isometries.*

Proof. Let $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ be such that the representation α is a complete isometry. Suppose that $(\mathcal{F}_{\mathcal{U}}(H), \beta)$ is the corresponding object. We shall prove that the representation β is a complete isometry too.

Let $n \in \mathbb{N}$. Fix a vector $y = (y_1, \ldots, y_n)^{\mathrm{t}} \in \mathrm{Ball}(\mathcal{F}(H_0)^n)$ and a matrix $(B_{ij}) \in M_n(\mathcal{B})$. We recall from Lemma 2.3 the partial isometries $\{V_k, k \in I\} \subset \mathcal{M}$. For $\varepsilon > 0$ there exists a subset $\{i_1, \ldots, i_N\} \subset I$ such that

$$\|(B_{ij})(y)\|^2 - \varepsilon = \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} B_{ik}(y_k) \right\|^2 - \varepsilon \leqslant \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} B_{ik} \sum_{l=1}^{N} V_{i_l} V_{i_l}^*(y_k) \right\|^2 - \frac{\varepsilon}{2}.$$

Using again Lemma 2.3 there exists a subset $\{j_1, \ldots, j_m\} \subset I$ such that

$$\sum_{i=1}^{n} \left\| \sum_{k=1}^{n} B_{ik} \sum_{l=1}^{N} V_{i_l} V_{i_l}^*(y_k) \right\|^2 - \frac{\varepsilon}{2} \leqslant \sum_{i=1}^{n} \left\| \sum_{t=1}^{m} V_{j_t} V_{j_t}^* \Big(\sum_{k=1}^{n} B_{ik} \sum_{l=1}^{N} V_{i_l} V_{i_l}^*(y_k) \Big) \right\|^2 - \frac{\varepsilon}{4}.$$

Since the projections $(V_{j_t} V_{j_t}^*)$ are mutually orthogonal it follows that

$$\|(B_{ij})(y)\|^2 - \varepsilon \leqslant \sum_{i=1}^{n} \sum_{t=1}^{m} \left\| V_{j_t}^* \Big(\sum_{k=1}^{n} B_{ik} \sum_{l=1}^{N} V_{i_l} V_{i_l}^*(y_k) \Big) \right\|^2 - \frac{\varepsilon}{4}$$

$$= \|(V^* \oplus \cdots \oplus V^*)(B_{ij})(U \oplus \cdots \oplus U)(z)\|^2 - \frac{\varepsilon}{4},$$

where

$$z = (V_{i_1}^*(y_1), \ldots, V_{i_N}^*(y_1), V_{i_1}^*(y_2), \ldots, V_{i_N}^*(y_2), \ldots, V_{i_N}^*(y_n))^{\mathrm{t}},$$
$$V = (V_{j_1}, \ldots, V_{j_m}), \quad U = (V_{i_1}, \ldots, V_{i_N}).$$

Observe that

$$\|z\|^2 = \sum_{l=1}^n \sum_{k=1}^N \|V_{i_k}^*(y_l)\|^2 = \sum_{l=1}^n \sum_{k=1}^N \|V_{i_k} V_{i_k}^*(y_l)\|^2$$

$$= \sum_{l=1}^n \Big\| \sum_{k=1}^N V_{i_k} V_{i_k}^*(y_l) \Big\|^2 \leqslant \sum_{l=1}^n \|y_l\|^2 = \|y\|^2 \leqslant 1.$$

It follows that

$$\|(B_{ij})(y)\|^2 - \varepsilon \leqslant \|(V^* \oplus \cdots \oplus V^*)(B_{ij})(U \oplus \cdots \oplus U)\|^2 - \frac{\varepsilon}{4}$$

$$= \|\alpha((V^* \oplus \cdots \oplus V^*)(B_{ij})(U \oplus \cdots \oplus U))\|^2 - \frac{\varepsilon}{4}$$

(the last equality holds because the map α is a complete isometry). We can find vectors $x_{lk} \in H$ such that the vector

$$x = (x_{11}, \ldots, x_{N1}, x_{12}, \ldots, x_{N2}, \ldots, x_{Nn})^{\mathrm{t}} \in H^{Nn}$$

has norm one and

$$\|(B_{ij})(y)\|^2 - \varepsilon \leqslant \|\alpha((V^* \oplus \cdots \oplus V^*)(B_{ij})(U \oplus \cdots \oplus U))(x)\|^2.$$

Thus

(2.1) $$\|(B_{ij})(y)\|^2 - \varepsilon \leqslant \sum_{s=1}^n \sum_{r=1}^m \Big\| \sum_{k=1}^n \sum_{l=1}^N \alpha(V_{j_r}^* B_{sk} V_{i_l})(x_{lk}) \Big\|^2.$$

Let

$$\omega_k = \sum_{l=1}^N V_{i_l} \otimes x_{lk} \in \mathcal{F}_{\mathcal{U}}(H), \quad k = 1, \ldots, n.$$

We have

$$\sum_{k=1}^n \|\omega_k\|_{\mathcal{F}_{\mathcal{U}}(H)}^2 = \sum_{k=1}^n \sup_{S \in \mathrm{Ball}(M_{p,1}(\mathcal{V})), p \in \mathbb{N}} \Big\| \sum_{l=1}^N \alpha(SV_{i_l})(x_{lk}) \Big\|^2$$

$$= \sum_{k=1}^n \sup_{S \in \mathrm{Ball}(M_{p,1}(\mathcal{V})), p \in \mathbb{N}} \Big\| \alpha(S(V_{i_1} \cdots V_{i_N})) \begin{pmatrix} x_{1k} \\ \vdots \\ x_{Nk} \end{pmatrix} \Big\|^2$$

$$\leqslant \sum_{k=1}^n \Big\| \begin{pmatrix} x_{1k} \\ \vdots \\ x_{Nk} \end{pmatrix} \Big\|^2 \leqslant 1.$$

The inequality is a consequence of the fact that $\|(V_{i_1} \cdots V_{i_N})\| = 1$ and the map α is a complete isometry.

It follows that

$$\|\beta((B_{ij}))\|^2 \geqslant \|\beta((B_{ij}))(\omega_1 \cdots \omega_n)^t\|^2$$

$$= \sum_{s=1}^{n} \Big\| \sum_{k=1}^{n} \beta(B_{sk})(\omega_k) \Big\|^2_{\mathcal{F}_{\mathcal{U}}(H)} = \sum_{s=1}^{n} \Big\| \sum_{k=1}^{n} \sum_{l=1}^{N} B_{sk} V_{i_l} \otimes x_{lk} \Big\|^2_{\mathcal{F}_{\mathcal{U}}(H)}$$

$$= \sum_{s=1}^{n} \sup_{S \in \mathrm{Ball}(M_{t,1}(\mathcal{V})), t \in \mathbb{N}} \Big\| \sum_{k=1}^{n} \sum_{l=1}^{N} \alpha(S B_{sk} V_{i_l})(x_{lk}) \Big\|^2.$$

Observe that

$$S = (V_{j_1}^* \cdots V_{j_m}^*)^t \in \mathrm{Ball}(M_{m,1}(\mathcal{V}))$$

so we have

$$\|\beta((B_{ij}))\|^2 \geqslant \sum_{s=1}^{n} \Big\| \sum_{k=1}^{n} \sum_{l=1}^{N} \alpha((V_{j_1}^* \cdots V_{j_m}^*)^t B_{sk} V_{i_l})(x_{lk}) \Big\|^2$$

$$= \sum_{s=1}^{n} \sum_{r=1}^{m} \Big\| \sum_{k=1}^{n} \sum_{l=1}^{N} \alpha(V_{j_r}^* B_{sk} V_{i_l})(x_{lk}) \Big\|^2.$$

From inequality (2.1) it follows that $\|(B_{ij})(y)\|^2 - \varepsilon \leqslant \|\beta((B_{ij}))\|^2$, hence $\|(B_{ij})\| \leqslant \|\beta((B_{ij}))\|$. Since β is a complete contraction we have equality:

$$\|(B_{ij})\| = \|\beta((B_{ij}))\|. \qquad \blacksquare$$

Combining this lemma and Theorem 2.1 we obtain the next theorem.

THEOREM 2.5. *Every functor implementing the equivalence of Theorem* 1.7 *respects isometries.*

LEMMA 2.6. *Let* $(H, \alpha) \in {}_\mathcal{A}\mathfrak{M}$ *and* $(\mathcal{F}_{\mathcal{U}}(H), \beta)$ *be the corresponding object. Then*

$$\mathcal{F}_{\mathcal{U}}(\mathrm{Lat}(\alpha(\mathcal{A}))) \subset \mathrm{Lat}(\beta(\mathcal{B})).$$

Proof. Suppose that L is a projection of the lattice of $\alpha(\mathcal{A})$. We shall prove that $\mathcal{F}_{\mathcal{U}}(L) \in \mathrm{Lat}(\beta(\mathcal{B}))$. The operator $\mathcal{F}_{\mathcal{U}}(L)$ is a projection because $\mathcal{F}_{\mathcal{U}}$ is a $*$-functor ([6], Theorem 2.10). If $B \in \mathcal{B}, M \in \mathcal{M}$ and $x \in H$ then

$$\beta(B)\mathcal{F}_{\mathcal{U}}(L)(M \otimes x) = \beta(B)(M \otimes L(x)) = BM \otimes L(x).$$

By Lemma 2.4 of [6] we have

$$BM \otimes L(x) \in [N \otimes L(y) : N \in \mathcal{M}, y \in H]^- = [\mathcal{F}_{\mathcal{U}}(L)(N \otimes y) : N \in \mathcal{M}, y \in H]^-.$$

It follows that

$$\beta(B)\mathcal{F}_{\mathcal{U}}(L)(M \otimes x) \in \mathcal{F}_{\mathcal{U}}(L)(\mathcal{F}_{\mathcal{U}}(H)).$$

Since the space $\mathcal{M} \otimes H$ is dense in $\mathcal{F}_{\mathcal{U}}(H)$, ([6], Corollary 2.5), we obtain

$$\beta(B)\mathcal{F}_{\mathcal{U}}(L)(z) \in \mathcal{F}_{\mathcal{U}}(L)(\mathcal{F}_{\mathcal{U}}(H)) \quad \text{for all } z \in \mathcal{F}_{\mathcal{U}}(H), B \in \mathcal{B}.$$

This shows that $\mathcal{F}_{\mathcal{U}}(L) \in \mathrm{Lat}(\beta(\mathcal{B}))$. We proved that

$$\mathcal{F}_{\mathcal{U}}(\mathrm{Lat}(\alpha(\mathcal{A}))) \subset \mathrm{Lat}(\beta(\mathcal{B})). \qquad \blacksquare$$

THEOREM 2.7. *Every functor implementing the equivalence of Theorem* 1.7 *is a lattice respecting functor.*

Proof. Let $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ and $(\mathcal{F}(H), \mathcal{F}(\alpha)) \in {}_{\mathcal{B}}\mathfrak{M}$ be the corresponding object. Since the functor $\mathcal{F}$ is equivalent to $\mathcal{F}_{\mathcal{U}}$ it follows from the last lemma that

$$\mathcal{F}(\mathrm{Lat}(\alpha(\mathcal{A}))) \subset \mathrm{Lat}(\mathcal{F}(\alpha)(\mathcal{B})).$$

Suppose that $\mathcal{G}$ is the inverse functor of $\mathcal{F}$ which maps $(\mathcal{F}(H), \mathcal{F}(\alpha)) \in {}_{\mathcal{B}}\mathfrak{M}$ to $(\mathcal{G}\mathcal{F}(H), \mathcal{G}\mathcal{F}(\alpha)) \in {}_{\mathcal{A}}\mathfrak{M}$. By the same argument

$$\mathcal{G}(\mathrm{Lat}(\mathcal{F}(\alpha)(\mathcal{B}))) \subset \mathrm{Lat}(\mathcal{G}\mathcal{F}(\alpha)(\mathcal{A})).$$

If $\mathcal{F}(\mathrm{Lat}(\alpha(\mathcal{A})))$ is strictly contained in $\mathrm{Lat}(\mathcal{F}(\alpha)(\mathcal{B}))$ then $\mathcal{G}\mathcal{F}(\mathrm{Lat}(\alpha(\mathcal{A})))$ is strictly contained in $\mathrm{Lat}(\mathcal{G}\mathcal{F}(\alpha)(\mathcal{A}))$. The functor $\mathcal{G}\mathcal{F}$ is equivalent to the identity functor of the category ${}_{\mathcal{A}}\mathfrak{M}$. So there exists a unitary $U \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}\mathcal{F}(H), H)$ satisfying $U^*FU = \mathcal{G}\mathcal{F}(F)$ for all $F \in \mathrm{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H, H)$. So

$$\mathcal{G}\mathcal{F}(\mathrm{Lat}(\alpha(\mathcal{A}))) = U^*\mathrm{Lat}(\alpha(\mathcal{A}))U = \mathrm{Lat}(\mathcal{G}\mathcal{F}(\alpha)(\mathcal{A})).$$

This is a contradiction, and hence we have the equality

$$\mathcal{F}(\mathrm{Lat}(\alpha(\mathcal{A}))) = \mathrm{Lat}(\mathcal{F}(\alpha)(\mathcal{B})). \quad \blacksquare$$

THEOREM 2.8. *Let $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ be such that α is a complete isometry. If $(\mathcal{F}(H), \beta) \in {}_{\mathcal{B}}\mathfrak{M}$ is the corresponding object then β is a complete isometry and the algebras $\alpha(\mathcal{A}), \beta(\mathcal{B})$ are* TRO *equivalent.*

Proof. By Theorem 2.5, β is a complete isometry. We denote by σ the map

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H, H) = \alpha(\Delta(\mathcal{A}))' \to \beta(\Delta(\mathcal{B}))' = \mathrm{Hom}_{\mathcal{B}}^{\mathfrak{D}}(\mathcal{F}(H), \mathcal{F}(H))$$

which is a $*$-isomorphism. By Lemma 1.1 the space

$$\mathcal{Y} = \{N \in B(H, \mathcal{F}(H)) : NA = \sigma(A)N \text{ for all } A \in \alpha(\Delta(\mathcal{A}))'\}$$

is an essential TRO. In the sequel, if K is a Hilbert space, T is an operator on K and $\mathcal{C} \subset B(K)$ we denote by K^{∞} the countably infinite direct sum $K \oplus K \oplus \cdots$, by $T^{\infty} \in B(K^{\infty})$ the operator $T \oplus T \oplus \cdots$, and by $\mathcal{C}^{\infty}$ the set $\{C^{\infty} : C \in \mathcal{C}\}$.

The map $\alpha^{\infty} : \mathcal{A} \to B(H^{\infty})$ given by $\alpha^{\infty}(A) = \alpha(A)^{\infty}$ is a normal representation so $(H^{\infty}, \alpha^{\infty}) \in {}_{\mathcal{A}}\mathfrak{M}$ hence $(\mathcal{F}(H^{\infty}), \mathcal{F}(\alpha^{\infty})) \in {}_{\mathcal{B}}\mathfrak{M}$.

Let $U \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(H^{\infty}), \mathcal{F}(H)^{\infty})$ be a unitary (see Lemma 3.2 of [6]). This defines a unitary equivalence between the algebras

$$\mathrm{Hom}_{\mathcal{B}}^{\mathfrak{D}}(\mathcal{F}(H^{\infty}), \mathcal{F}(H^{\infty})) = (\mathcal{F}(\alpha^{\infty})(\Delta(\mathcal{B})))'$$

and

$$\mathrm{Hom}_{\mathcal{B}}^{\mathfrak{D}}(\mathcal{F}(H)^{\infty}, \mathcal{F}(H)^{\infty}) = (\beta(\Delta(\mathcal{B}))^{\infty})'.$$

This equivalence maps the invariant projection lattice of the algebra $\mathcal{F}(\alpha^{\infty})(\mathcal{B})$ onto the lattice of $\beta(\mathcal{B})^{\infty}$. The functor $\mathcal{F}$ defines a $*$-isomorphism between the spaces

$$\mathrm{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H^{\infty}, H^{\infty}) = (\alpha(\Delta(\mathcal{A}))^{\infty})' \quad \text{and} \quad \mathrm{Hom}_{\mathcal{B}}^{\mathfrak{D}}(\mathcal{F}(H^{\infty}), \mathcal{F}(H^{\infty}))$$

which by Theorem 2.7 maps the lattice of the algebra $\alpha(\mathcal{A})^\infty$ onto the lattice of the algebra $\mathcal{F}(\alpha^\infty)(\mathcal{B})$. Composing with the unitary U we obtain a $*$-isomorphism

$$\theta : (\alpha(\Delta(\mathcal{A}))^\infty)' \to (\beta(\Delta(\mathcal{B}))^\infty)'$$

such that $\theta(\mathrm{Lat}(\alpha(\mathcal{A})^\infty)) = \mathrm{Lat}(\beta(\mathcal{B})^\infty)$ and which satisfies

$$\theta((F_{ij})_{i,j}) = (\sigma(F_{ij}))_{i,j} \quad \text{for all } (F_{ij})_{i,j} \in (\alpha(\Delta(\mathcal{A}))^\infty)'.$$

From this we conclude that the space

$$\mathcal{X} = \{(T_{ij}) \in B(H^\infty, \mathcal{F}(H)^\infty) : (T_{ij})(F_{ij}) = \theta((F_{ij}))(T_{ij}) \forall (F_{ij}) \in (\alpha(\Delta(\mathcal{A}))^\infty)'\},$$

equals $\mathcal{Y}^\infty$. Since the algebras $\alpha(\mathcal{A})^\infty, \beta(\mathcal{B})^\infty$ are reflexive (see for Example A.1.5 of [2]) by Theorem 1.3 we have

$$\alpha(\mathcal{A})^\infty \overset{\mathcal{X}}{\sim} \beta(\mathcal{B})^\infty \Rightarrow \alpha(\mathcal{A}) \overset{\mathcal{Y}}{\sim} \beta(\mathcal{B}). \quad \blacksquare$$

THEOREM 2.9. *Every functor implementing the equivalence of Theorem 1.7 respects reflexivity.*

Proof. Let $(H, \alpha) \in {}_\mathcal{A}\mathfrak{M}$ be such that α is a complete isometry and the algebra $\alpha(\mathcal{A})$ is reflexive. Suppose that $(\mathcal{F}(H), \beta) \in {}_\mathcal{B}\mathfrak{M}$ is the corresponding object. By Theorem 2.5 β is a complete isometry. By the above theorem the algebras $\alpha(\mathcal{A}), \beta(\mathcal{B})$ are TRO equivalent. Since $\alpha(\mathcal{A})$ is reflexive so is $\beta(\mathcal{B})$ ([5], Remark 2.7). $\blacksquare$

3. APPLICATIONS AND EXAMPLES

We present some definitions and concepts used in this section. A *commutative subspace lattice (CSL)* is a projection lattice $\mathcal{L}$ whose elements commute; the algebra $\mathrm{Alg}(\mathcal{L})$ is called a *CSL algebra*. In the special case where $\mathcal{L}$ is totally ordered we call $\mathcal{L}$ a *nest* and the algebra $\mathrm{Alg}(\mathcal{L})$ a *nest algebra*. CSL algebras are of course reflexive. When $\mathcal{A}$ is a CSL algebra there exists a smallest w^*-closed algebra contained in $\mathcal{A}$, which contains the diagonal $\Delta(\mathcal{A})$ and whose reflexive hull is $\mathcal{A}$ [1], [12]. We denote this algebra by $\mathcal{A}_{\min}$. Whenever $\mathcal{A} = \mathcal{A}_{\min}$ we call $\mathcal{A}$ *synthetic*. The first example of a nonsynthetic CSL algebra was given in [1].

PROPOSITION 3.1. *If $\mathcal{A}$ is a CSL algebra which is Δ-equivalent to a unital dual operator algebra $\mathcal{B}$ then there exists a completely isometric normal representation β of $\mathcal{B}$ such that the algebras $\mathcal{A}$ and $\beta(\mathcal{B})$ are TRO equivalent. It follows that the algebra $\beta(\mathcal{B})$ is a CSL algebra too.*

Proof. Suppose that $\mathcal{F} : {}_\mathcal{A}\mathfrak{M} \to {}_\mathcal{B}\mathfrak{M}$ is an equivalence functor which has a Δ-extension to an equivalence $*$-functor between the categories ${}_\mathcal{A}\mathfrak{D}\mathfrak{M}$ and ${}_\mathcal{B}\mathfrak{D}\mathfrak{M}$. Also suppose that $\mathcal{A} \subset B(H)$ and let $(\mathcal{F}(H), \beta)$ be the object corresponding to identity representation of $\mathcal{A}$. By Theorem 2.9 β is a complete isometry and the

algebra $\beta(\mathcal{B})$ is reflexive. Also by Theorem 2.8 the algebras $\mathcal{A}, \beta(\mathcal{B})$ are TRO equivalent. Now Theorem 1.3 shows that the lattice $\text{Lat}(\beta(\mathcal{B}))$ is a CSL. ∎

Although Δ-equivalent algebras are not necessarily TRO-equivalent, even when they are reflexive, in CSL algebras this is indeed the case:

THEOREM 3.2. *Two* CSL *algebras are Δ-equivalent if and only if they are TRO-equivalent.*

Proof. TRO-equivalent algebras are Δ-equivalent (Theorem 1.7). For the converse, suppose that the CSL algebras $\mathcal{A}$ and $\mathcal{B}$ are Δ-equivalent. By the previous proposition there exists a completely isometric normal representation β of $\mathcal{B}$ such that the algebras $\mathcal{A}$ and $\beta(\mathcal{B})$ are TRO equivalent. Since $\beta(\mathcal{B})$ is a CSL algebra, as just shown, it is easily checked that $\beta(\text{Lat}(\mathcal{B})) = \text{Lat}(\beta(\mathcal{B}))$ and $\beta(\Delta(\mathcal{B})') = (\Delta(\beta(\mathcal{B})))'$. It follows (Theorem 1.3) that the algebras $\mathcal{B}$ and $\beta(\mathcal{B})$ are TRO equivalent. The conclusion is a consequence of the fact that TRO equivalence is an equivalence relation (Theorem 2.3 in [5]). ∎

REMARK 3.3. (i) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are separably acting CSL algebras with totally atomic lattices. By the previous theorem and Theorem 5.3 of [5] they are Δ-equivalent if and only if they have isomorphic lattices. Two separably acting nest algebras are Δ-equivalent if and only if their lattices are isomorphic through a lattice isomorphism which respects the continuous parts of the nests. This can be concluded from Theorem 5.6 of [5].

(ii) If two nests are isomorphic, their nest algebras are not always Δ-equivalent, even if they have isomorphic diagonals (see Example 3.7).

PROPOSITION 3.4. *If $\mathcal{A}$ is a nonsynthetic* CSL *algebra there exists no isometric normal representation $\alpha : \mathcal{A}_{\min} \to B(H)$ such that $\alpha(\mathcal{A}_{\min})$ is a* CSL *algebra. It follows from Proposition 3.1 that the algebra $\mathcal{A}_{\min}$ cannot be Δ-equivalent to any* CSL *algebra.*

Proof. Let $\mathcal{A}$ be a nonsynthetic CSL algebra, H a Hilbert space and $\alpha : \mathcal{A}_{\min} \to B(H)$ be a w*-continuous isometric homomorphism such that $\mathcal{B} \equiv \alpha(\mathcal{A}_{\min})$ is a CSL algebra. Since $\mathcal{A}$ equals $\text{Ref}(\mathcal{A}_{\min})$ and $\text{Lat}(\mathcal{A}) \subset \mathcal{A}_{\min}$ we can check that $\alpha(\text{Lat}(\mathcal{A})) = \text{Lat}(\mathcal{B})$. From Theorem 4.7 of [5] the algebra $\mathcal{B}$ is not synthetic, so the algebra $\mathcal{B}_{\min}$ is strictly contained in $\mathcal{B}$. Thus the algebra $\alpha^{-1}(\mathcal{B}_{\min})$ is strictly contained in $\mathcal{A}_{\min}$. This is a contradiction because $\Delta(\mathcal{A}) \subset \alpha^{-1}(\mathcal{B}_{\min}), \text{Lat}(\mathcal{A}) = \text{Lat}(\alpha^{-1}(\mathcal{B}_{\min}))$ and the algebra $\mathcal{A}_{\min}$ is the smallest w*-closed subalgebra of $\mathcal{A}$ with these properties ([1], [12]). ∎

We will prove that similar nest algebras [4] have equivalent categories. So we fix nests $\mathcal{N}_1, \mathcal{N}_2$ acting on the separable Hilbert spaces H_1, H_2 respectively and an order isomorphism $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ which preserves dimension of intervals. We say that an invertible operator $T \in B(H_1, H_2)$ *implements* θ if $\theta(N)$ is the projection onto $TN(H_1)$ for all $N \in \mathcal{N}_1$. Define the spaces

$$\mathcal{U} = \{T \in B(H_1, H_2) : \theta(N)^{\perp} TN = 0 \text{ for all } N \in \mathcal{N}_1\},$$

$$\mathcal{V} = \{ S \in B(H_2, H_1) : N^\perp S\theta(N) = 0 \text{ for all } N \in \mathcal{N}_1 \}.$$

If $\mathcal{A} = \mathrm{Alg}(\mathcal{N}_1)$ and $\mathcal{B} = \mathrm{Alg}(\mathcal{N}_2)$, one verify easily that

$$\mathcal{VU} \subset \mathcal{A}, \quad \mathcal{UV} \subset \mathcal{B}, \quad \mathcal{BUA} \subset \mathcal{U}, \quad \mathcal{AVB} \subset \mathcal{V}.$$

We will need the Similarity Theorem ([4], Theorem 13.20).

THEOREM 3.5 (Davidson). *For every $\varepsilon > 0$ there exists an invertible operator T which implements θ such that $\|T\| < 1 + \varepsilon, \|T^{-1}\| < 1 + \varepsilon$.*

PROPOSITION 3.6. *There exists an equivalence functor $\mathcal{F}_{\mathcal{U}} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ which is normal and completely isometric.*

Proof. Let $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$. We define the following seminorm on the algebraic tensor product $\mathcal{U} \otimes H$:

$$\Big\| \sum_{i=1}^n T_i \otimes x_i \Big\| = \sup_{S \in \mathrm{Ball}(\mathcal{V})} \Big\| \sum_{i=1}^n \alpha(ST_i)(x_i) \Big\|.$$

This seminorm satisfies the parallelogram identity. Let $\mathcal{F}_{\mathcal{U}}(H)$ be the corresponding Hausdorff completion of $\mathcal{U} \otimes H$ and identify every $T \otimes x$ with its image in $\mathcal{F}_{\mathcal{U}}(H)$. If $B \in \mathcal{B}, T_1, \ldots, T_m \in \mathcal{U}, x_1, \ldots, x_m \in H$ we can check that

$$\Big\| \sum_{j=1}^m BT_j \otimes x_j \Big\|_{\mathcal{F}_{\mathcal{U}}(H)} \leqslant \|B\| \Big\| \sum_{j=1}^m T_j \otimes x_j \Big\|_{\mathcal{F}_{\mathcal{U}}(H)}.$$

The map $T \otimes x \to BT \otimes x, B \in \mathcal{B}, T \in \mathcal{U}, x \in H$ extends to a map $\beta(B) \in B(\mathcal{F}_{\mathcal{U}}(H))$ and clearly β is a unital algebraic homomorphism. As in Proposition 2.7 of [6] we can prove that the map $\beta : \mathcal{B} \to B(\mathcal{F}_{\mathcal{U}}(H))$ is w*-continuous. It follows from Corollary 20.17 of [4] that β is a complete contraction and hence $(\mathcal{F}_{\mathcal{U}}(H), \beta) \in {}_{\mathcal{B}}\mathfrak{M}$. In order to define a functor $\mathcal{F}_{\mathcal{U}} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ we have to define the map

$$\mathcal{F}_{\mathcal{U}} : \mathrm{Hom}_{\mathcal{A}}(H_1, H_2) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}_{\mathcal{U}}(H_1), \mathcal{F}_{\mathcal{U}}(H_2))$$

for every pair of objects $(H_j, \alpha_j), j = 1, 2$. If $F \in \mathrm{Hom}_{\mathcal{A}}(H_1, H_2)$, then

$$\begin{aligned} \Big\| \sum_i T_i \otimes F(x_i) \Big\|_{\mathcal{F}_{\mathcal{U}}(H_2)} &= \sup_{S \in \mathrm{Ball}(\mathcal{V})} \Big\| \sum_i \alpha_2(ST_i)F(x_i) \Big\| \\ &= \sup_{S \in \mathrm{Ball}(\mathcal{V})} \Big\| F \sum_i \alpha_1(ST_i)(x_i) \Big\| \leqslant \|F\| \Big\| \sum_i T_i \otimes x_i \Big\|_{\mathcal{F}_{\mathcal{U}}(H_1)}. \end{aligned}$$

So we can define a map $\mathcal{F}_{\mathcal{U}}(F) \in B(\mathcal{F}_{\mathcal{U}}(H_1), \mathcal{F}_{\mathcal{U}}(H_2))$ by the formula

$$\mathcal{F}_{\mathcal{U}}(F)(T \otimes x) = T \otimes F(x), \quad T \in \mathcal{U}, x \in H_1.$$

It is easy to check that $\mathcal{F}_{\mathcal{U}}(F) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}_{\mathcal{U}}(H_1), \mathcal{F}_{\mathcal{U}}(H_2))$. The definition of the functor $\mathcal{F}_{\mathcal{U}}$ is complete. Symmetrically we can define a functor $\mathcal{F}_{\mathcal{V}} : {}_{\mathcal{B}}\mathfrak{M} \to {}_{\mathcal{A}}\mathfrak{M}$.

Now fix $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ with corresponding object $(\mathcal{F}_{\mathcal{U}}, \beta)$. If $S_i \in \mathcal{V}, T_i \in \mathcal{U}, x_i \in H, i = 1, \ldots, r$, then

$$\left\| \sum_{i=1}^{r} S_i \otimes (T_i \otimes x_i) \right\|_{\mathcal{F}_{\mathcal{V}}\mathcal{F}_{\mathcal{U}}(H)} = \sup_{U \in \mathrm{Ball}(\mathcal{U})} \left\| \sum_{i=1}^{r} \beta(US_i)(T_i \otimes x_i) \right\|_{\mathcal{F}_{\mathcal{U}}(H)}$$

$$= \sup_{U \in \mathrm{Ball}(\mathcal{U})} \left\| \sum_{i=1}^{r} US_i T_i \otimes x_i \right\|_{\mathcal{F}_{\mathcal{U}}(H)}$$

$$= \sup_{U \in \mathrm{Ball}(\mathcal{U})} \sup_{V \in \mathrm{Ball}(\mathcal{V})} \left\| \sum_{i=1}^{r} \alpha(VU)\alpha(S_i T_i)(x_i) \right\|_{H}$$

$$\leqslant \left\| \sum_{i=1}^{r} \alpha(S_i T_i)(x_i) \right\|_{H}.$$

By Theorem 3.5 for arbitrary $\varepsilon > 0$ we can choose $T \in \mathcal{U}$ such that $T^{-1} \in \mathcal{V}$ and $\|T\| < 1 + \varepsilon, \|T^{-1}\| < 1 + \varepsilon$. By the definition of the norm $\| \cdot \|_{\mathcal{F}_{\mathcal{V}}\mathcal{F}_{\mathcal{U}}(H)}$ we have

$$\left\| \sum_{i=1}^{r} S_i \otimes (T_i \otimes x_i) \right\|_{\mathcal{F}_{\mathcal{V}}\mathcal{F}_{\mathcal{U}}(H)} \geqslant \left\| \sum_{i=1}^{r} \alpha\left(\frac{T^{-1}}{\|T^{-1}\|} \frac{T}{\|T\|} S_i T_i \right)(x_i) \right\|$$

$$\geqslant \frac{1}{(1+\varepsilon)^2} \left\| \sum_{i=1}^{r} \alpha(S_i T_i)(x_i) \right\|.$$

Letting $\varepsilon \to 0$ we obtain

$$\left\| \sum_{i=1}^{r} S_i \otimes (T_i \otimes x_i) \right\|_{\mathcal{F}_{\mathcal{V}}\mathcal{F}_{\mathcal{U}}(H)} \geqslant \left\| \sum_{i=1}^{r} \alpha(S_i T_i)(x_i) \right\|$$

and hence equality holds. It follows that we can define a unitary

$$U_H : \mathcal{F}_{\mathcal{V}}\mathcal{F}_{\mathcal{U}}(H) \to H : U_H(S \otimes (T \otimes x)) = \alpha(ST)(x), \quad S \in \mathcal{V}, T \in \mathcal{U}, x \in H.$$

One can now easily check that the family of unitary operators $\{U_H : H \in {}_{\mathcal{A}}\mathfrak{M}\}$ implements the required equivalence. The proofs of the facts that the functor $\mathcal{F}_{\mathcal{U}}$ is normal and completely isometric are similar to Lemmas 3.4, 3.5 of [6] so we omit them. ∎

In spite of the last proposition, we show in the following example that the similarity of nest algebras does not imply Δ-equivalence even in the case of isomorphic diagonals.

EXAMPLE 3.7. In Examples 13.25, 13.22 of [4] there exist similar nests $\mathcal{N}_1, \mathcal{N}_2$ acting on separable Hilbert spaces H_1, H_2 respectively, such that the algebra $\mathcal{N}_1''$ is a totally atomic maximal abelian selfadjoint algebra (masa) and the algebra $\mathcal{N}_2''$ is a masa with the property that the algebra $\mathcal{N}_2''|_{N(H_2)}$ has nontrivial continuous part for every nonzero projection $N \in \mathcal{N}_2$. We define the nests

$$\mathcal{M}_1 = \{0 \oplus N : N \in \mathcal{N}_1\} \cup \{N \oplus H_1 : N \in \mathcal{N}_2\} \subset B(H_2 \oplus H_1),$$
$$\mathcal{M}_2 = \{0 \oplus N : N \in \mathcal{N}_2\} \cup \{N \oplus H_2 : N \in \mathcal{N}_1\} \subset B(H_1 \oplus H_2).$$

These nests are similar too.

So by the previous proposition if $\mathcal{A} = \mathrm{Alg}(\mathcal{M}_1), \mathcal{B} = \mathrm{Alg}(\mathcal{M}_2)$ the categories $_{\mathcal{A}}\mathfrak{M}$, $_{\mathcal{B}}\mathfrak{M}$ are equivalent. Observe that the diagonals of these algebras are isomorphic because $\Delta(\mathcal{A}) = \mathcal{N}_2'' \oplus \mathcal{N}_1''$ and $\Delta(\mathcal{B}) = \mathcal{N}_1'' \oplus \mathcal{N}_2''$. The algebras $\mathcal{A}$ and $\mathcal{B}$ are not Δ-equivalent because if they were by Theorem 3.2 they would be TRO equivalent. So by Theorem 1.3 there would exist a $*$-isomorphism

$$\pi : \Delta(\mathcal{A}) \to \Delta(\mathcal{B}) \quad \text{such that } \pi(\mathcal{M}_1) = \mathcal{M}_2.$$

Since the diagonals are masas this map is unitarily implemented ([8], Theorem 9.3.1). Now there are two possibilities: either $\pi(0 \oplus I_{H_1}) = 0 \oplus N$ for some $N \in \mathcal{N}_2$ in which case the algebras $\mathcal{N}_1''$ and $\mathcal{N}_2''|_{N(H_2)}$ are unitarily equivalent, or $\pi(0 \oplus I_{H_1}) = M \oplus I_{H_2}$ for some $M \in \mathcal{N}_1$ in which case the algebras $\mathcal{N}_1''$ and $\mathcal{N}_1''|_{M(H_1)} \oplus \mathcal{N}_2''$ are unitarily equivalent. This is a contradiction because in both cases the first algebra is totally atomic but the second one is not.

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