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TAKAI DUALITY FOR CROSSED PRODUCTS BY HILBERT C^*-BIMODULES

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ABSTRACT. We discuss the crossed product by the dual action δ of the circle on the crossed product $A \rtimes X$ of a C^*-algebra A by a Hilbert C^*-bimodule X. When X is an A-A Morita equivalence bimodule, the double crossed product $A \rtimes X \rtimes_\delta S^1$ is shown to be Morita equivalent to the C^*-algebra A.

KEYWORDS: *Takai duality, crossed products by Hilbert C^*-bimodules.*

MSC (2000): 46L08, 46L55.

1. INTRODUCTION

The crossed product $A \rtimes X$ of a C^*-algebra A by a Hilbert C^*-bimodule X was introduced in [2] and shown to be a generalization of the crossed product by an automorphism. There is an obvious generalization of the dual action to this context, which raises the question of whether there is an analog of Takai duality [12]. We show in this work that when X is an A-A Morita equivalence bimodule, that is, when it is a full Hilbert C^*-module both on the left and the right, then the double crossed product $A \rtimes X \rtimes_\delta \mathbb{Z}$ is Morita equivalent to the C^*-algebra A. Namely, if E_X denotes the right Hilbert C^*-module over A defined by $E_X = \bigoplus\limits_{n\in\mathbb{Z}} X^{\otimes n}$, then we identify $A \rtimes X \rtimes_\delta \mathbb{Z}$ with $\mathcal{K}(E_X)$, the C^*-algebra of compact operators on E_X, and we describe the double dual action on $\mathcal{K}(E_X)$. Our proof heavily relies on the universal properties of the crossed products by an automorphism and by a Hilbert C^*-bimodule, much as in [10].

This work is organized as follows. After establishing some preliminary results and notation in Section 2, we introduce in Section 3 representations on the crossed product $A \rtimes X$ induced by representations on A. Section 4 is devoted to the discussion of certain actions of amenable locally compact groups on $A \rtimes X$ that leave A and X invariant. We show that the crossed product of $A \rtimes X$ by an action of this kind can be written as the crossed product of a C^*-algebra by a

Hilbert C^*-bimodule. These results enable us to represent, in Section 5, the double crossed product $A \rtimes X \rtimes_\delta \mathbb{Z}$ as adjointable operators on E_X. When X is a Morita equivalence bimodule this representation turns out to be an isomorphism onto $\mathcal{K}(E_X)$. This yields the Morita equivalence between $A \rtimes X \rtimes_\delta \mathbb{Z}$ and A.

2. PRELIMINARIES

We next establish our basic notation concerning Hilbert C^*-modules and bimodules. We refer the reader to [8] for further details.

Let X and Y be right Hilbert C^*-modules over a C^*-algebra A. We denote by $\mathcal{L}(X,Y)$ the space of adjointable maps from X to Y and by $\mathcal{K}(X,Y)$ the space of compact operators, that is, the closed subspace spanned by $\{\theta_{y,x} :\ x \in X,\ y \in Y\}$, where $\theta_{y,x} : X \to Y$ is given by $\theta_{y,x}(z) = y\langle x, z\rangle$. We will also use the notation above when X and Y are Hilbert C^*-bimodules, thus viewing them as right Hilbert modules. Undecorated inner products will always denote right inner products.

Throughout this work we consider Hilbert C^*-bimodules in the sense of [4]. That is, a Hilbert C^*-bimodule X over a C^*-algebra A consists of a vector space X which is both a right and a left Hilbert C^*-module over A and satisfies $\langle x,y\rangle_{\mathrm{L}} z = x\langle y,z\rangle_{\mathrm{R}}$ and $(ax)b = a(xb)$, for all $x,y,z \in X$ and $a,b \in A$. Note that then both the left and the right action of A on X are adjointable. In fact, for all $x,y,z \in X$ and $a \in A$ we have:

$$\langle xa, y\rangle_{\mathrm{L}} z = xa\langle y,z\rangle_{\mathrm{R}} = x\langle ya^*, z\rangle_{\mathrm{R}} = \langle x, ya^*\rangle_{\mathrm{L}} z,$$

and analogously for the action on the left. This shows that Hilbert C^*-bimodules are Hilbert bimodules as those discussed in [9] or [7].

Let X and Y be Hilbert C^*-bimodules over the C^*-algebras A and B, respectively. A morphism of Hilbert C^*-bimodules

$$(\phi_A, \phi_X) : (A, X) \to (B, Y)$$

consists of a $*$-homomorphism $\phi_A : A \to B$ and a linear map $\phi_X : X \to Y$ such that, for all $x, y \in X$ and $a \in A$,

$$\begin{aligned} &\phi_X(ax) = \phi_A(a)\phi_X(x), \quad \phi_X(xa) = \phi_X(x)\phi_A(a), \\ &\langle \phi_X(x), \phi_X(y)\rangle_{\mathrm{L}} = \phi_A(\langle x,y\rangle_{\mathrm{L}}), \quad \text{and} \quad \langle \phi_X(x), \phi_X(y)\rangle_{\mathrm{R}} = \phi_A(\langle x,y\rangle_{\mathrm{R}}). \end{aligned}$$

REMARK 2.1. The map ϕ_X is norm decreasing, and it is isometric when ϕ_A is, since

$$\|\phi_X(x)\|^2 = \|\langle \phi_X(x), \phi_X(x)\rangle\| = \|\phi_A(\langle x,x\rangle)\| \leqslant \|\langle x,x\rangle\| = \|x\|^2.$$

A representation of (A, X) on a C^*-algebra B consists of a morphism

$$(\phi_A, \phi_X) : (A, X) \longrightarrow (B, B)$$

where B is viewed as a Hilbert C^*-bimodule over itself in the usual way.

The crossed product $A \rtimes X$ is ([2]) the universal C^*-algebra carrying a representation (i_A, i_X) of (A, X). Besides, the maps i_A and i_X are isometric ([2], 2.10), and $A \rtimes X$ is generated as a C^*-algebra by the images of i_A and i_X. The crossed product $A \rtimes X$ carries an action of S^1, called the dual action, which is the identity on $i_A(A)$ and is given by $i_X(x) \mapsto \lambda i_X(x)$ for $x \in X$ and $\lambda \in S^1$. Moreover, $i_A(A)$ and $i_X(X)$ are the fixed-point subalgebra and the first spectral subspace of the dual action, respectively (see 3 of [2]). A discussion of the connections between $A \rtimes X$ and the Cuntz–Pimsner C^*-algebra $\mathcal{O}_X$ defined in [9] can be found in [3].

REMARK 2.2. Let $(\phi_A, \phi_X) : (A, X) \longrightarrow (B, Y)$ be a morphism of Hilbert C^*-bimodules. Then there is a unique $*$-homomorphism

$$\phi_A \rtimes \phi_X : A \rtimes X \longrightarrow B \rtimes Y$$

such that the diagram

$$\begin{array}{ccc} A \rtimes X & \xrightarrow{\phi_A \rtimes \phi_X} & B \rtimes Y \\ {\scriptstyle (i_A, i_X)}\uparrow & & \uparrow{\scriptstyle (i_B, i_Y)} \\ (A, X) & \xrightarrow{(\phi_A, \phi_X)} & (B, Y) \end{array}$$

commutes. The map $\phi_A \rtimes \phi_X$ is injective when ϕ_A is, and it is surjective if ϕ_A and ϕ_X are.

Besides, the correspondence $(A, X) \mapsto A \rtimes X$, $(\phi_A, \phi_X) \mapsto \phi_A \rtimes \phi_X$ is functorial.

Proof. Since $(i_B \circ \phi_A, i_Y \circ \phi_X)$ is a representation of (A, X) on $B \rtimes Y$, the existence and the uniqueness of $\phi_A \rtimes \phi_X : A \rtimes X \longrightarrow B \rtimes Y$ follow from the universal property of $A \rtimes X$.

The map $\phi_A \rtimes \phi_X$ is covariant for the dual actions of S^1 on $A \rtimes X$ and $B \rtimes Y$ respectively, so it is injective if and only if it is injective when restricted to the fixed-point algebra ([5], 2.9), that is, when ϕ_A is injective. When ϕ_A and ϕ_X are surjective, the image of $\phi_A \rtimes \phi_X$ contains B and Y, so it is all of $B \rtimes Y$. The last statement is apparent. ∎

When X and Y are Hilbert C^*-bimodules over A and $x \in X$, we denote by $T_x \in \mathcal{L}(Y, X \otimes Y)$ the creation operator defined by $T_x(y) = x \otimes y$, for $y \in Y$.

The following facts are well known and easy to check, for $a \in A$, $x, x_0, x_1 \in X$ and $y \in Y$:

$$T_{ax}(y) = aT_x(y), \quad T_{xa}(y) = T_x(ay); \tag{2.1}$$

$$\|T_x\| \leqslant \|x\|, \quad T^*_{x_0}(x \otimes y) = \langle x_0, x\rangle_{\mathrm{R}} y; \tag{2.2}$$

$$T_{x_0} T^*_{x_1}(x \otimes y) = \langle x_0, x_1\rangle_{\mathrm{L}} x \otimes y \quad \text{and} \quad T^*_{x_0} T_{x_1}(y) = \langle x_0, x_1\rangle_{\mathrm{R}} y. \tag{2.3}$$

DEFINITION 2.3. Let X be a Hilbert C^*-bimodule over the C^*-algebra A, $x \in X$, and let n be a non-negative integer. We denote by T^n_x the map $T^n_x \in \mathcal{L}(X^{\otimes n}, X^{\otimes n+1})$ described above, where X and $X \otimes A$ are identified in the usual way when $n = 0$.

When $n < 0$, $X^{\otimes n}$ denotes $(\widetilde{X})^{\otimes -n}$, $\widetilde{X}$ being the dual bimodule of X as defined in [11]. That is, $\widetilde{X}$ is the conjugate vector space of X and carries the A-Hilbert C^*-bimodule structure given by

$$a \cdot \widetilde{x} = \widetilde{xa^*}, \quad \widetilde{x} \cdot a = \widetilde{a^* x}, \quad \langle \widetilde{x}, \widetilde{y} \rangle_{\mathrm{L}} = \langle x, y \rangle_{\mathrm{R}}, \quad \langle \widetilde{x}, \widetilde{y} \rangle_{\mathrm{R}} = \langle x, y \rangle_{\mathrm{L}},$$

where $\widetilde{x}$ denotes the element $x \in X$ viewed as an element of the dual bimodule $\widetilde{X}$. For negative values of n we define

$$T_x^n := (T_{\widetilde{x}}^{-n-1})^*. \tag{2.4}$$

Thus, for $n < 0$,

$$T_x^n(\widetilde{x}_1 \otimes \widetilde{x}_2 \otimes \cdots \otimes \widetilde{x}_{-n}) = \langle x, x_1 \rangle_{\mathrm{L}}\, \widetilde{x}_2 \otimes \cdots \otimes \widetilde{x}_{-n}, \tag{2.5}$$

$$(T_x^n)^*(\widetilde{x}_1 \otimes \widetilde{x}_2 \otimes \cdots \otimes \widetilde{x}_{-n-1}) = \widetilde{x} \otimes \widetilde{x}_1 \otimes \widetilde{x}_2 \otimes \cdots \otimes \widetilde{x}_{-n-1}. \tag{2.6}$$

Let L^n denote the left action of A on $X^{\otimes n}$ for $n \in \mathbb{Z}$. It follows from equations (2.1)–(2.3) for $n \geqslant 0$, and from some straightforward maneuvering and equations (2.5) and (2.6) for $n < 0$, that, for all $n \in \mathbb{Z}$, $a \in A$, and $x, y \in X$,

$$T_{ax}^n = L_a^{n+1} T_x^n, \quad T_{xa}^n = T_x^n L_a^n; \tag{2.7}$$

$$(T_x^n)(T_y^n)^* = L_{\langle x,y \rangle_{\mathrm{L}}}^{n+1}, \quad (T_x^n)^*(T_y^n) = L_{\langle x,y \rangle_{\mathrm{R}}}^n. \tag{2.8}$$

Throughout this work we will denote by E_X the right A-Hilbert C^*-module defined by $E_X = \bigoplus_{-\infty}^{+\infty} X^{\otimes n}$.

REMARK 2.4. We will often view $\mathcal{L}(X^{\otimes n}, X^{\otimes m})$ as a closed subspace of $\mathcal{L}(E_X)$ by means of the isometric linear map $i_{n,m} : \mathcal{L}(X^{\otimes n}, X^{\otimes m}) \to \mathcal{L}(E_X)$ given by the following, for $T \in \mathcal{L}(X^{\otimes n}, X^{\otimes m})$, $\eta \in E_X$, and $k \in \mathbb{Z}$:

$$[(i_{n,m}T)(\eta)](k) = \delta_m(k) T(\eta(n)).$$

Note that

$$i_{n,m}(T)^* = i_{m,n}(T^*), \quad \text{and} \quad i_{m,p}(S) \circ i_{n,m}(T) = i_{n,p}(S \circ T).$$

Besides,

$$i_{n,m}(\theta_{u,v}) = \theta_{u\delta_m, v\delta_n},$$

so under this identification $\mathcal{K}(X^{\otimes n}, X^{\otimes m}) \subset \mathcal{K}(E_X)$.

3. INDUCED REPRESENTATIONS ON CROSSED PRODUCTS

We discuss in this section certain representations of $A \rtimes X$ that are induced from representations of A.

Let X be a Hilbert C^*-bimodule over A, and let E_X be the right Hilbert C^*-module defined at the end of Section 2.

We define $(\Lambda_A, \Lambda_X) : (A, X) \to \mathcal{L}(E_X)$ by the following, for $a \in A$, $x \in X$, $\eta \in E_X$, and $T_x^{(n)}$ as in Definition 2.3:

$$[\Lambda_A(a)(\eta)](n) = L_a^n(\eta(n)), \quad [\Lambda_X(x)(\eta)](n) = T_x^{(n-1)}(\eta(n-1)). \tag{3.1}$$

It is easily checked that $(\Lambda_X(x)^*(\eta))(n) = (T_x^n)^*(\eta(n+1))$. It follows from equations (2.7) and (2.8) that (Λ_A, Λ_X) is a representation on $\mathcal{L}(E_X)$ and induces the following $*$-homomorphism as in Remark 2.2:

$$\Lambda := \Lambda_A \times \Lambda_X : A \rtimes X \longrightarrow \mathcal{L}(E_X).$$

PROPOSITION 3.1. *Let A, X and $\Lambda : A \rtimes X \longrightarrow \mathcal{L}(E_X)$ be as above. For $\lambda \in S^1$, let U_λ be the unitary operator on E_X defined by*

$$(U_\lambda \eta)(n) = \lambda^n \eta(n),$$

for $\eta \in E_X$, $n \in \mathbb{Z}$. Then:

(i) *Conjugation by U_λ defines a strongly continuous action β of S^1 on the image of Λ.*

(ii) *The map Λ is injective and covariant for the dual action and β.*

Proof. It is easily checked that the map $\lambda \mapsto U_\lambda$ is a group homomorphism from S^1 to the unitary group of $\mathcal{L}(E_X)$ and that, for all $x \in X$ and $a \in A$,

$$U_\lambda \Lambda_A(a) U_\lambda^* = \Lambda_A(a), \quad U_\lambda \Lambda_X(x) U_\lambda^* = \lambda \Lambda_X(x). \tag{3.2}$$

On the other hand, the set

$$\{T \in \mathcal{L}(E_X) : \ \lambda \mapsto U_\lambda T U_\lambda^* \text{ is continuous on } S^1\}$$

is a C^*-algebra of $\mathcal{L}(E_X)$. Therefore, since $A \rtimes X$ is generated as a C^*-algebra by $i_A(A)$ and $i_X(X)$, conjugation by U_λ defines a strongly continuous action β of S^1 on the image of Λ. An analogous reasoning, together with equation (3.2), shows that Λ is covariant for β and the dual action on $A \rtimes X$. Finally, since the restriction Λ_A of Λ to the fixed-point subalgebra is injective, so is Λ by 2.9 of [5]. ∎

REMARK 3.2. If $A \rtimes X$ is viewed as the cross-sectional C^*-algebra of a Fell bundle as in 2.9 of [2], then Λ is, in the terminology of [6], the left regular representation and its injectivity follows from [6] and the amenability of $\mathbb{Z}$.

DEFINITION 3.3. Let X be a Hilbert C^*-bimodule over a C^*-algebra A. A non-degenerate representation π of A on a Hilbert space H gives rise to the representation $\Lambda \otimes \mathrm{id}_H$ of $A \rtimes X$ on the Hilbert space $E_X \otimes_\pi H$. We will refer to $\Lambda \otimes \mathrm{id}_H$ as the representation of $A \rtimes X$ induced by π.

REMARK 3.4. For $\alpha \in \mathrm{Aut}(A)$ let A_α be the A-Hilbert C^*-bimodule consisting of A as a vector space with structure defined by

$$a \cdot x = ax, \quad x \cdot a = x\alpha(a), \quad \langle x, y \rangle_{\mathrm{L}} = xy^* \quad \text{and} \quad \langle x, y \rangle_{\mathrm{R}} = \alpha^{-1}(x^* y),$$

for $a \in A$ and $x, y \in A_\alpha$. It was shown in 3.2 of [2] that there is an isomorphism $J : A \rtimes A_\alpha \to A \rtimes_\alpha \mathbb{Z}$ given by

$$J(i_A(a)) = a\delta_0 \in C_c(\mathbb{Z}, A) \quad \text{and} \quad J(i_X(x)) = x\delta_1 \in C_c(\mathbb{Z}, A), \tag{3.3}$$

for $a \in A$ and $x \in A_\alpha$. Denote by $I_n : (A_\alpha)^{\otimes n} \to A,\ n \in \mathbb{Z}$ the map defined by

$$\begin{cases} I_n(a_1 \otimes \cdots \otimes a_n) = \alpha^{-n}(a_1)\alpha^{-n+1}(a_2)\cdots\alpha^{-1}(a_n) \text{ for } n \geqslant 0; \\ I_n = \mathrm{id}_A, \text{ for } n = 0; \\ I_n(\widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_{-n}) = \alpha^{-n-1}(a_1^*)\alpha^{-n-2}(a_2^*)\cdots\alpha(a_{-n}^*), \text{ for } n \geqslant 0. \end{cases}$$

Straightforward computations show that I_n is a homomorphism of right Hilbert C^*-modules over A and that, for all $a \in A$, $x \in A_\alpha$ and $c \in (A_\alpha)^{\otimes n}$,

$$I_n(L^n(a)(c)) = \alpha^{-n}(a)I_n(c), \tag{3.4}$$

$$I_{n+1}(T_x^n(c)) = \alpha^{-(n+1)}(x)I_n(c). \tag{3.5}$$

Now, given a non-degenerate representation π of A on a Hilbert space H, we define $U : E_X \otimes_\pi H \to l^2(\mathbb{Z}, H)$ by

$$[U(\eta \otimes h)](n) = \pi[I_n(\eta(n))](h),$$

for $X = A_\alpha$, $\eta \in E_X$, $h \in H$, and $n \in \mathbb{Z}$. Note that U extends to a unitary operator because, for $\eta_i, \xi_j \in E_X$ and $h_i, k_j \in H$,

$$\begin{aligned}\Big\langle U\Big(\sum_{i=1}^{p}\eta_i\otimes h_i\Big), U\Big(\sum_{j=1}^{q}\xi_j\otimes k_j\Big)\Big\rangle &= \sum_{i,j,n}\langle \pi[I_n(\eta_i(n))](h_i), \pi[I_n(\xi_j(n))](k_j)\rangle \\ &= \sum_{i,j}\langle h_i, \pi(\langle \eta_i, \xi_j\rangle_{\mathrm{R}})(k_j)\rangle = \Big\langle \sum_i \eta_i\otimes h_i, \sum_j \xi_j\otimes k_j\Big\rangle.\end{aligned}$$

Let now $\pi_\alpha \times \lambda$ denote the representation of $A \rtimes_\alpha \mathbb{Z}$ on $l^2(\mathbb{Z}, H)$ induced by π. That is, $\pi_\alpha \times \lambda$ is the integrated form of the covariant pair (π_α, λ) defined by

$$[(\pi_\alpha(a))(\xi)](n) = [\pi(\alpha^{-n}(a))](\xi(n)), \quad (\lambda_k\xi)(n) = \xi(n-k),$$

for $\xi \in l^2(\mathbb{Z}, H)$. Then the following diagram commutes for all $\phi \in A \rtimes A_\alpha$ and J as in equation (3.3):

$$\begin{array}{ccc} E_X \otimes_\pi H & \xrightarrow{\ U\ } & l^2(\mathbb{Z}, H) \\ {\scriptstyle (\Lambda\otimes\mathrm{id}_H)(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle (\pi_\alpha\times\lambda)J(\phi)} \\ E_X \otimes_\pi H & \xrightarrow{\ U\ } & l^2(\mathbb{Z}, H)\,. \end{array}$$

It suffices to check this statement for $\phi = i_A(a)$ and $\phi = i_X(x)$, for $a \in A$ and $x \in A_\alpha$, which follows from equations (3.4) and (3.5) above.

REMARK 3.5. Let X be a Hilbert C^*-bimodule over A, and let π be a non-degenerate representation of A on a Hilbert space H. Denote by $V : A \otimes_\pi H \to H$ the unitary operator defined by $V(a \otimes h) = \pi(a)(h)$. Let i be the isometric embedding of $A \otimes_\pi H$ into $E_X \otimes_\pi H$ given by $i(a \otimes h) = a\delta_0 \otimes h$, and let $S : H \longrightarrow E_X \otimes_\pi H$ be the isometry defined by $S = i \circ V^*$.

Then the following diagram commutes for all $a \in A$:

$$\begin{array}{ccc} H & \xrightarrow{\ \ S\ \ } & E_X \otimes_\pi H \\ {\scriptstyle \pi(a)}\Big\downarrow & & \Big\downarrow{\scriptstyle (\Lambda \otimes \mathrm{id}_H)(i_A(a))} \\ H & \xrightarrow{\ \ S\ \ } & E_X \otimes_\pi H\,. \end{array}$$

Proof. Straightforward computations prove the statement. ∎

PROPOSITION 3.6. *Let X be a Hilbert C^*-bimodule over A, and let π be a faithful non-degenerate representation of A on a Hilbert space H. Then the induced representation $\Lambda \otimes \mathrm{id}_H$ on $E_X \otimes_\pi H$ is faithful.*

Proof. Let β denote the strongly continuous action of S^1 on the image of Λ defined in Proposition 3.1. Then $\beta \otimes \mathrm{id}$ is a strongly continuous action of S^1 on the image of $\Lambda \otimes \mathrm{id}_H$. Besides, by Proposition 3.1 (ii), $\Lambda \otimes \mathrm{id}_H$ is covariant for the dual action δ and $\beta \otimes \mathrm{id}$.

Thus, by 2.9 of [5], it suffices to show that $\Lambda \otimes \mathrm{id}$ is injective on the fixed-point subalgebra $i_A(A)$. This last fact follows from Remark 3.5 and the injectivity of π. ∎

4. COVARIANT ACTIONS ON CROSSED PRODUCTS

Throughout this section all integrals over groups are taken with respect to Haar measure.

DEFINITION 4.1. Let G be a locally compact group, and let X be a Hilbert C^*-bimodule over a C^*-algebra A. A strongly continuous covariant action (α_A, α_X) of G on (A, X) consists of a strongly continuous action α_A of G on A and a group homomorphism α_X from G to the group of invertible linear maps on X such that:

(i) The map $t \longrightarrow (\alpha_X)_t(x)$ is continuous on G for all $x \in X$.

(ii) The pair $((\alpha_A)_t, (\alpha_X)_t)$ is an isomorphism of Hilbert C^*-bimodules for all $t \in G$.

REMARK 4.2. For (α_A, α_X) as above, the maps $(\alpha_X)_t$ in Definition 4.1 are isometric for all $t \in G$, by Remark 2.1.

REMARK 4.3. Let X be a Hilbert C^*-bimodule over a C^*-algebra A and let (α_A, α_X) be a strongly continuous action of a locally compact group G on (A, X). Then $\alpha_t := (\alpha_A)_t \rtimes (\alpha_X)_t$ defines a strongly continuous action α of G on $A \rtimes X$.

Proof. It follows from Remark 2.2 that $t \mapsto \alpha_t$ is a group homomorphism from G to $\mathrm{Aut}(A \rtimes X)$. Since $\|\alpha_t\| = 1$ for all $t \in G$, the set $\{c \in A \rtimes X :\ t \mapsto \alpha_t(c) \text{ is continuous}\}$ is a C^*-subalgebra of $A \rtimes X$ containing $i_A(A)$ and $i_X(X)$, which shows that α is strongly continuous. ∎

REMARK 4.4. Let f be in $C_c(G,X)$ for a locally compact group G and a Hilbert C^*-bimodule X over a C^*-algebra A. By identifying X with its isometric copy $i_X(X)$ in $A \rtimes X$, we view f as an $A \rtimes X$-valued map. Thus the integral $\int_G f \mathrm{d}\mu$ has its usual meaning. Notice that this integral belongs to $i_X(X)$, since it can be approximated by sums $\sum_1^n c_i f(t_i)$, for $c_i \in \mathbb{C}$ and $t_i \in G$. This is the way we will view $\int_G f \mathrm{d}\mu$ as an element of X throughout this work.

The same procedure could be followed in order to view A-valued functions as being $A \rtimes X$-valued. The integral does not depend on the approach, since the restriction to A of a non-degenerate faithful representation of $A \rtimes X$ is again a non-degenerate faithful representation.

PROPOSITION 4.5. *Let X be a Hilbert C^*-bimodule over A, and let α be the strongly continuous action on $A \rtimes X$ induced by a covariant action (α_A, α_X) of an amenable locally compact group G on (A, X).*

Let i_A and i_X be the embeddings of A and X, respectively, in $A \rtimes X$. Then the map that sends $\phi \in C_c(G, i_A(A))$ to $i_A^{-1} \circ \phi$ and $f \in C_c(G, i_X(X))$ to $i_X^{-1} \circ f$ extends to an isomorphism from $(A \rtimes X) \rtimes_\alpha G$ to $(A \rtimes_{\alpha_A} G) \rtimes Y$, where Y is the Hilbert C^*-bimodule over $A \rtimes_{\alpha_A} G$ consisting of the completion of $C_c(G,X)$ with the structure defined by

$$(\phi f)(t) = \int_G \phi(u)(\alpha_X)_u[f(u^{-1}t)]\mathrm{d}u, \quad (f\phi)(t) = \int_G f(u)(\alpha_A)_u[\phi(u^{-1}t)]\mathrm{d}u,$$

$$\langle f, g\rangle_{\mathrm{L}}(t) = \int_G \Delta(t^{-1}u)\langle f(u), (\alpha_X)_t[g(t^{-1}u)]\rangle_{\mathrm{L}}\mathrm{d}u \quad \text{and}$$

$$\langle f, g\rangle_{\mathrm{R}}(t) = \int_G (\alpha_A)_{u^{-1}}\langle f(u), g(ut)\rangle_{\mathrm{R}}\mathrm{d}u,$$

where $f, g \in C_c(G,X)$, $\phi \in C_c(G,A)$, and Δ denotes the modular function on G.

Proof. Let σ denote the dual action of S^1 on $A \rtimes X$. Note that σ and α commute, since so do their restrictions to the images of i_A and i_X. Therefore ([1], 1.2) σ induces an action γ of S^1 on $(A \rtimes X) \rtimes_\alpha G$ given by the following, for $\phi \in C_c(G, A \rtimes X)$, $t \in G$, and $\lambda \in S^1$:

$$[\gamma_\lambda(\phi)](t) = \sigma_\lambda[\phi(t)].$$

Let B_0 and B_1 be the closures in $(A \rtimes X) \rtimes_\alpha G$ of the sets of functions in $C_c(G, A \rtimes X)$ whose image lies, respectively, in $i_A(A)$ and $i_X(X)$. We next show that B_0 and B_1 are the fixed-point subalgebra and the first spectral subspace, respectively, for the action γ. We will then prove that the action γ is semi-saturated and that (B_0, B_1) and $(A \rtimes_{\alpha_A} G, Y)$ are isomorphic as Hilbert C^*-bimodules. At this point the statement will follow from 3.1 of [2].

Let P_n denote the n^{th} spectral projection for the action γ, that is,

$$P_n(c) = \int_{S^1} \lambda^{-n}\gamma_\lambda(c)\mathrm{d}\lambda.$$

The n^{th} spectral subspace of $(A \rtimes X) \rtimes_\alpha G$ is the image of P_n, so we will show that $\operatorname{Im} P_i = B_i$, for $i = 0, 1$. It is clear that the restriction of P_i to B_i is the identity map, so $\operatorname{Im} P_i \supset B_i$, for $i = 0, 1$.

Let $\phi \in C_c(G, A \rtimes X)$ be such that for some $n > 1$

$$\phi(t) = i_X(x_1(t))i_X(x_2(t)) \cdots i_X(x_n(t)),$$

where $x_i(t) \in X$ for all $t \in \operatorname{supp} \phi$. Then $\gamma_\lambda(\phi) = \lambda^n\phi$, and $P_i(\phi) = 0$, for $i = 0, 1$. Analogously, $P_i(\phi) = 0$ for $i = 0, 1$ if $\phi \in C_c(G, A \rtimes X)$ is such that for some $n > 0$ we have the following (for all $t \in \operatorname{supp} \phi$):

$$\phi(t) = i_X(x_1(t))^* i_X(x_2(t))^* \cdots i_X(x_n(t))^*.$$

Let $\mathcal{C}$ denote the dense $*$-subalgebra of $A \rtimes X$ generated by $\{i_A(A),\ i_X(X)\}$. Given $\phi \in C_c(G, A \rtimes X)$ and $\varepsilon > 0$, let U be a precompact open set containing $\operatorname{supp} \phi$, and let $\varepsilon' = \varepsilon/\mu(U)$, μ being Haar measure.

For each $t \in \operatorname{supp} \phi$ choose $c_t \in \mathcal{C}$ and a neighborhood $N_t \subset U$ of t such that $\|\phi(s) - c_t\| < \varepsilon'$ for all $s \in N_t$. Let $\{N_{t_i}\}$ be a finite subcovering of $\operatorname{supp} \phi$ and $\{h_i\}$ a partition of unity subordinate to it. Then $\left\|\phi - \sum_i h_i c_{t_i}\right\|_{(A\rtimes X)\rtimes_\alpha G} \leqslant \left\|\phi - \sum_i h_i c_{t_i}\right\|_{L^1(G, A\rtimes X)} < \varepsilon.$

Therefore $P_i(\phi) \in B_i$, and (for $i = 0, 1$)

$$P_i((A \rtimes X) \rtimes_\alpha G) \subset \overline{P}_i(C_c(G, A \rtimes X)) \subset B_i.$$

We next show that γ is semi-saturated, i.e. that $(A \rtimes X) \rtimes_\alpha G$ is generated as a C^*-algebra by B_0 and B_1. As above, any function in $C_c(G, A \rtimes X)$ can be approximated by finite sums $\sum f_i$, where either f_i or f_i^* is of the form

$$f(t) = h(t)i_A(x_0)i_X(x_1)i_X(x_2) \cdots i_X(x_n),$$

for $h \in C_c(G)$, $n \geqslant 0$, $x_0 \in A$, and $x_1, x_2, \ldots, x_n \in X$. Therefore it suffices to show that these maps belong to $C^*(B_0, B_1)$. We show this by induction on n. Note that the result holds for $n = 0, 1$.

Given $\varepsilon > 0$ and f as above for $n > 1$, let V be a neighborhood of e in G such that $\|h(t)x_n - h(s^{-1}t)(\alpha_X)_s(x_n)\| < \varepsilon$ for all $t \in \operatorname{supp} h$ and $s \in V$. Let $\lambda \in C_c(G)$ be a positive function such that $\operatorname{supp} \lambda \subset V$ and $\int_G \lambda = 1$. We now set $k(t) = h(t)i_X(x_n)$ and $g(t) = \lambda(t)y$, where $y = i_A(x_0)i_X(x_1)i_X(x_2) \cdots i_X(x_{n-1})$, so that $g, k \in C^*(B_0, B_1)$.

Then

$$\|f(t) - (gk)(t)\| = \left\| \int_G \lambda(s)y[h(t)i_X(x_n) - h(s^{-1}t)i_X((\alpha_X)_s(x_n)]\mathrm{d}s \right\| < \|y\|\varepsilon.$$

Thus $C_c(G, A \rtimes X) \subset C^*(B_0, B_1)$, and γ is semi-saturated.

We now show that B_0 is isomorphic to $A \rtimes_{\alpha_A} G$. Let $j_{A\rtimes X}$ and j_G denote the canonical inclusions of $A \rtimes X$ and G in the multiplier algebra of $(A \rtimes X) \rtimes_\alpha G$, respectively. The non-degenerate $*$-homomorphism $j_{A\rtimes X} \circ i_A$ is non-degenerate and the pair

$$(j_{A\rtimes X} \circ i_A, j_G) : (A, G) \longrightarrow M((A \rtimes X) \rtimes_\alpha G)$$

is covariant for the system (A, G, α_A). Therefore it induces, as in Proposition 2 of [10], a $*$-homomorphism $J : A \rtimes_{\alpha_A} G \longrightarrow (A \rtimes X) \rtimes_\alpha G$ such that $(J\phi)(t) = i_A(\phi(t))$ for all $\phi \in C_c(G, A)$. This shows that the image of J is B_0, and it remains to prove that J is one-to-one.

Let π be a non-degenerate faithful representation of A on a Hilbert space H, and let $\widetilde{\pi} := \Lambda \otimes \mathrm{id}_H$ be the representation of $A \rtimes X$ on $E_X \otimes_\pi H$ induced by π as in Definition 3.3. Denote by θ the representation of $(A \rtimes X) \rtimes_\alpha G$ on $L^2(G, E_X \otimes_\pi H)$ induced by $\widetilde{\pi}$.

Let V be the unitary operator from $A \otimes_\pi H$ to H defined in Remark 3.5. Note that $A \otimes_\pi H \subset E_X \otimes_\pi H$ is invariant under $\widetilde{\pi}(i_A(a))$ for all $a \in A$ and that

$$V\widetilde{\pi}(i_A(a))V^* = \pi(a) \quad \text{for all } a \in A,$$

because for $a, b \in A$ and $h \in H$

$$V\widetilde{\pi}(i_A(a))V^*(\pi(b)h) = V(ab \otimes h) = \pi(a)(\pi(b)h).$$

Fix now $\xi \in L^2(G, H)$ and $\phi \in C_c(G, A) \subset A \rtimes_{\alpha_A} G$. Then $V^* \circ \xi \in L^2(G, A \otimes_\pi H) \subset L^2(G, E_X \otimes_\pi H)$, and

$$\begin{aligned}(\theta_{J(\phi)}(V^* \circ \xi))(r) &= \int_G \widetilde{\pi}[\alpha_{r^{-1}}(J(\phi)(t))]V^*[\xi(t^{-1}r)]\mathrm{d}t \\ &= V^*\Big(\int_G \pi[(\alpha_A)_{r^{-1}}(\phi(t))](\xi(t^{-1}r))\Big)\mathrm{d}t = [V^*\pi_0(\phi)(\xi)](r),\end{aligned}$$

for all $r \in G$, where π_0 denotes the representation of $A \rtimes_{\alpha_A} G$ on $L^2(G, H)$ induced by π. Thus,

$$\|J(\phi)\| = \|\theta_{J(\phi)}\| \geqslant \|\pi_0(\phi)\| = \|\phi\|,$$

which shows that J is isometric.

Now, B_1 carries a natural structure of Hilbert C^*-bimodule over B_0, both the left and the right action consisting of multiplication, and the B_0-valued inner products being given by $\langle b, c\rangle_{\mathrm{L}} = bc^*$, $\langle b, c\rangle_{\mathrm{R}} = b^*c$. On the other hand, the map $f \mapsto i_X \circ f$ and the isomorphism J above identify $C_c(G, X)$ with $C_c(G, i_X(X)) \subset B_1$ and B_0 with $A \rtimes_{\alpha_A} G$, respectively. It only remains to show that under these identifications (B_0, B_1) and $(A \rtimes_{\alpha_A} G, Y)$ agree as Hilbert C^*-bimodules. For

$f, g \in C_c(G, X)$ we have

$$\begin{aligned} i_A[\langle f,g\rangle_{\mathrm{L}}(t)] &= (i_X \circ f)(i_X \circ g)^*(t) = \int_G (i_X \circ f)(u)\alpha_u[(i_X \circ g)^*(u^{-1}t)]\mathrm{d}u \\ &= \int_G i_X(f(u))[i_X(\Delta(t^{-1}u)(\alpha_X)_t(g(t^{-1}u)))]^*\mathrm{d}u \\ &= i_A\Big(\int_G \Delta(t^{-1}u)\langle f(u),(\alpha_X)_t[g(t^{-1}u)]\rangle_{\mathrm{L}}\mathrm{d}u\Big). \end{aligned}$$

The remaining cases are shown by means of similar computations. ∎

5. THE DUALITY THEOREM

In this section we make use of the results in Proposition 4.5 in order to discuss the crossed product by the dual action on the crossed product by a Hilbert C^*-bimodule. We establish in Theorem 5.4 conditions that ensure the Morita equivalence between A and the double crossed product $A \rtimes X \rtimes_\delta S^1$.

PROPOSITION 5.1. *Let X be a Hilbert C^*-bimodule over A, and let $C_0(\mathbb{Z}, X)$ be the Hilbert C^*-bimodule over $C_0(\mathbb{Z}, A)$ defined by*

$$(\phi f)(n) = \phi(n)f(n), \quad (f\phi)(n) = f(n)\phi(n-1), \tag{5.1}$$

$$\langle f,g\rangle_{\mathrm{L}}(n) = \langle f(n),g(n)\rangle_{\mathrm{L}}, \quad \langle f,g\rangle_{\mathrm{R}}(n) = \langle f(n+1),g(n+1)\rangle_{\mathrm{R}}, \tag{5.2}$$

for $\phi \in C_0(\mathbb{Z}, A)$ and $f, g \in C_0(\mathbb{Z}, X)$.

Let δ denote the dual action of S^1 on $A \rtimes X$. Then there is an isomorphism

$$I : (A \rtimes X) \rtimes_\delta S^1 \longrightarrow C_0(\mathbb{Z}, A) \rtimes C_0(\mathbb{Z}, X)$$

such that

$$(I(\phi))(n) = \int_{S^1} \lambda^n i_A^{-1}(\phi(\lambda))\mathrm{d}\lambda \quad \textit{and} \quad (I(f))(n) = \int_{S^1} \lambda^{n-1} i_X^{-1}(f(\lambda))\mathrm{d}\lambda,$$

for $\phi \in C(S^1, i_A(A))$ and $f \in C(S^1, i_X(X))$.

Proof. In the notation of Remark 4.3, δ is the action induced by (δ_A, δ_X), where, for $\lambda \in S^1$, $(\delta_A)_\lambda$ is the identity and $(\delta_X)_\lambda$ is multiplication by λ.

It follows from Proposition 4.5 that

$$(A \rtimes X) \rtimes_\delta S^1 \simeq (A \rtimes_{\mathrm{id}} S^1) \rtimes Y,$$

Y being the completion of $C(S^1, X)$ with the norm coming from the $A \rtimes_{\mathrm{id}} S^1$-Hilbert C^*-bimodule structure given by:

$$(\phi f)(\mu) = \int_{S^1} \lambda\phi(\lambda)f(\lambda^{-1}\mu)\mathrm{d}\lambda, \quad (f\phi)(\mu) = \int_{S^1} f(\lambda)\phi(\lambda^{-1}\mu)\mathrm{d}\lambda; \tag{5.3}$$

$$\langle f,g\rangle_{\mathrm{L}}(\mu)=\int_{S^1}\langle f(\lambda),\mu g(\mu^{-1}\lambda)\rangle_{\mathrm{L}}\mathrm{d}\lambda,\quad \langle f,g\rangle_{\mathrm{R}}(\mu)=\int_{S^1}\langle f(\lambda),g(\lambda\mu)\rangle_{\mathrm{R}}\mathrm{d}\lambda. \tag{5.4}$$

Let J_A denote the isomorphism $J_A : A\rtimes_{\mathrm{id}} S^1 \longrightarrow C_0(\mathbb{Z},A)$ given by

$$(J_A\phi)(n)=\int_{S^1}\lambda^n\phi(\lambda)\mathrm{d}\lambda,$$

for $\phi\in C(S^1,A)$, and define J_Y on $C(S^1,X)$ by

$$(J_Y f)(n)=\int_{S^1}\lambda^{n-1}f(\lambda)\mathrm{d}\lambda. \tag{5.5}$$

Note that for $f,g\in C(S^1,X)$ we have

$$\begin{aligned}J_A(\langle f,g\rangle_{\mathrm{R}})(n)&=\int_{S^1\times S^1}\lambda^n\langle f(\mu),g(\mu\lambda)\rangle_{\mathrm{R}}\mathrm{d}\mu\,\mathrm{d}\lambda=\int_{S^1\times S^1}\mu^{-n}\lambda^n\langle f(\mu),g(\lambda)\rangle_{\mathrm{R}}\mathrm{d}\mu\,\mathrm{d}\lambda\\&=\langle (J_Y(f))(n+1),(J_Y(g))(n+1)\rangle_{\mathrm{R}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\lim_n\|J_Y(f)(n)\|^2=\lim_n\|J_A(\langle f,f\rangle_{\mathrm{R}})(n)\|=0,\quad\text{and}\\&\|f\|^2=\|\langle f,f\rangle_{\mathrm{R}}\|=\|J_A(\langle f,f\rangle_{\mathrm{R}})\|=\|\langle J_Y(f),J_Y(f)\rangle_{\mathrm{R}}\|=\|J_Y(f)\|^2,\end{aligned}$$

which shows that J_Y extends to $J_Y : Y\to C_0(\mathbb{Z},X)$. In fact, $J_Y(Y)$ is all of $C_0(\mathbb{Z},X)$, since the Hilbert C^*-bimodule norm in $C_0(\mathbb{Z},X)$ is the supremum norm, J_Y is linear and isometric, and $J_Y(f_{x,k})=x\delta_{1-k}$ for $x\in X$, $k\in\mathbb{Z}$, $f_{x,k}(\lambda)=\lambda^k x$.

Routine computations similar to those above show that $(J_A,\ J_Y)$ is an isomorphism of Hilbert C^*-bimodules between $(A\rtimes_{\mathrm{id}} S^1,Y)$, as defined by equations (5.3) and (5.4), and $(C_0(\mathbb{Z},A),C_0(\mathbb{Z},X)))$, as defined by equations (5.1) and (5.2). Finally, it follows from Remark 2.2 that $J_A\rtimes J_X$ is an isomorphism from $(A\rtimes_{\mathrm{id}} S^1)\rtimes Y$ to $C_0(\mathbb{Z},A)\rtimes C_0(\mathbb{Z},X)$; the isomorphism I is obtained by composing $J_A\rtimes J_X$ with the isomorphism in Proposition 4.5. ∎

PROPOSITION 5.2. *Let X be a Hilbert C^*-bimodule over A, and let*

$$(\pi_0,\pi_1):(C_0(\mathbb{Z},A),C_0(\mathbb{Z},X))\longrightarrow\mathcal{L}(E_X)$$

be defined by

$$[(\pi_0\phi)(\eta)](n)=\phi(n)\eta(n),\quad [(\pi_1 f)(\eta)](n)=T^{n-1}_{f(n)}(\eta(n-1)),$$

for $\phi\in C_0(\mathbb{Z},A)$ and $f\in C_0(\mathbb{Z},X)$, where $T^{n-1}_{f(n)}$ and E_X are as in Definition 2.3. *Then*

(i) *The pair (π_0,π_1) is a representation, so it induces a $*$-homomorphism*

$$\pi : C_0(\mathbb{Z},A)\rtimes C_0(\mathbb{Z},X)\longrightarrow\mathcal{L}(E_X),$$

and $\operatorname{Im}\pi$ *is the C^*-subalgebra of $\mathcal{L}(E_X)$ generated by $\{T^n_x, L^n_a : n\in\mathbb{Z}, x\in X, a\in A\}$, where L^n_a is as in Definition* 2.3, *and the set above is viewed as a subset of $\mathcal{L}(E_X)$ as in Remark* 2.4.

(ii) *The map π is injective if and only if so are the homomorphisms $A \mapsto \mathcal{L}(X_A)$ and $A \mapsto \mathcal{L}({}_AX)$ induced by the left and the right action of A on X, respectively.*

(iii) *The image of π contains $\mathcal{K}(E_X)$, and it is $\mathcal{K}(E_X)$ when X is full both as a left and a right Hilbert C^*-module over A.*

Proof. (i) In what follows $\phi \in C_0(\mathbb{Z}, A)$, $f, g \in C_0(\mathbb{Z}, X)$, $\xi, \eta \in E_X$, and $n \in \mathbb{Z}$. By virtue of equations (2.7), (2.8), (5.1), and (5.2), we have

$$\begin{aligned}[\pi_1(f\phi)(\eta)](n) &= T^{n-1}_{f(n)\phi(n-1)}(\eta(n-1)) = T^{n-1}_{f(n)}(\phi(n-1)\eta(n-1)) \\ &= [(\pi_1 f)(\pi_0\phi)(\eta)](n)].\end{aligned}$$

Note that, since

$$\langle(\pi_1 f)(\eta), \xi\rangle = \sum_n \langle T^{n-1}_{f(n)}(\eta(n-1)), \xi(n)\rangle_{\mathrm{R}} = \sum_n \langle \eta(n), (T^n_{f(n+1)})^*(\xi(n+1))\rangle_{\mathrm{R}},$$

we have

$$[(\pi_1 f)^*(\xi)](n) = (T^n_{f(n+1)})^*(\xi(n+1)). \tag{5.6}$$

Therefore

$$\begin{aligned}[\pi_0(\langle f, g\rangle_{\mathrm{R}})(\eta)](n) &= \langle f, g\rangle_{\mathrm{R}}(n)\eta(n) = \langle f(n+1), g(n+1)\rangle_{\mathrm{R}}\eta(n) \\ &= [(T^n_{f(n+1)})^* T^n_{g(n+1)}](\eta(n)) = [((\pi_1 f)^*(\pi_1 g))(\eta)](n).\end{aligned}$$

The remaining properties are checked in a similar fashion. The last statement follows from the fact that, for $a \in A,\ x \in X, n \in \mathbb{Z}$,

$$\pi_0(a\delta_n) = L^n_a, \quad \pi_1(x\delta_{n+1}) = T^n_x.$$

(ii) Let U_λ be the unitary operator defined in Proposition 3.1. Conjugation by U_λ yields, as in Proposition 3.1, a strongly continuous action β of S^1 on the image of π, and π is covariant for the dual action and β.

Therefore, by 2.9 of [5], π is injective if and only if so is π_0. Clearly the left and right actions of A on X must be faithful when π is injective because they correspond to π_0 on maps supported in $\{1\}$ and $\{-1\}$, respectively.

On the other hand, if the left and right actions of A on X are injective, then so are the left and right actions of A on $X^{\otimes n}$ for all $n \in \mathbb{Z}$. This is shown by induction on n for positive values of n, since $a(x_1 \otimes \cdots \otimes x_n) = 0$ for all $x_i \in X$ implies that

$$0 = \langle |ax_1|(x_2 \otimes \cdots \otimes x_n), |ax_1|(x_2 \otimes \cdots \otimes x_n)\rangle_{\mathrm{R}},$$

for all $x_i \in X$, where $|ax_1| = (\langle ax_1, ax_1\rangle_{\mathrm{R}})^{1/2}$. The equality above implies that $|ax_1| = 0$ for all $x_1 \in X$ and, consequently, that $a = 0$. A similar reasoning with the left inner product yields the proof for the right action of A. It is apparent that the statement always holds for $n = 0$. Finally, the case $n < 0$ is taken care of by applying the results above to the dual bimodule $\widetilde{X}$.

Assume now that the left and right actions of A on X are faithful, and let $\phi \in \ker \pi_0$. Then

$$0 = ((\pi_0\phi)\eta) = \phi(n)\eta(n),$$

for all $\eta \in E_X$ and $n \in \mathbb{Z}$. It follows from the remarks above that $\phi = 0$.

(iii) First note that $\mathcal{K}(E_X)$ is generated as a C^*-algebra by the set

$$\{\theta_{\eta\delta_k,\xi\delta_l} : k,l \in \mathbb{Z},\ k \geqslant l,\ \eta = x_1 \otimes \cdots \otimes x_k, \xi = y_1 \otimes \cdots \otimes y_l\}.$$

Notice also that for a positive integer n, $x \in X$, $u \in X^{\otimes n-1}$, and $\eta \in E_X$ we have

$$\theta_{(x\otimes u)\delta_n,\eta} = T_x^{n-1}\theta_{u\delta_{n-1},\eta} = \pi_1(x\delta_n)\theta_{u\delta_{n-1},\eta}.$$

Similarly, for a positive integer m, $\widetilde{y} \in \widetilde{X}$, $\eta \in E_X$, and $v \in \widetilde{X}^{\otimes m-1}$,

$$\theta_{\eta,(\widetilde{y}\otimes v)\delta_{-m}} = \theta_{\eta,v\delta_{-m+1}}\pi_1(\widetilde{y}\delta_{-m+1}).$$

Finally, since $\theta_{a,b} = \pi_0(ab^*\delta_0)$ for all $a,b \in A$, and $\theta_{\eta,\xi} = \theta^*_{\xi,\eta}$ for all $\eta, \xi \in E_X$, the inclusion $\mathcal{K}(E_X) \subset \operatorname{Im}\pi$ follows.

When X is full on the left and the right, so are the bimodules $X^{\otimes n}$ for all $n \in \mathbb{Z}$, and consequently A acts by compact operators on $X^{\otimes n}$ for all $n \in \mathbb{Z}$. Besides, T_x^n is compact for all $n \in \mathbb{Z}$ and $x \in X$: for $n \geqslant 0$ approximate x by

$$x' = x_0 \sum_{i=1}^{N} \langle u_i, v_i\rangle_{\mathrm{L}}^{A},$$

for appropriate $u_i, v_i \in X^{\otimes n}$. Then T_x^n gets approximated by

$$T_{x'}^n = \sum_{i=1}^{N} \theta_{x_0\otimes u_i,v_i} \in \mathcal{K}(X^{\otimes n}, X^{\otimes n+1}).$$

Therefore $(T_x^n)^*$ is compact when $n \geqslant 0$, which shows, by equation (2.4), that so is T_x^n for negative values of n. It follows from Remark 2.4 that the images of both π_0 and π_1, and consequently that of π, are contained in $\mathcal{K}(E_X)$. ∎

REMARK 5.3. Let A, X, and π be as in Proposition 5.2 and identify $(A \rtimes X) \rtimes_\delta S^1$ and $C_0(\mathbb{Z}, A) \rtimes C_0(\mathbb{Z}, X)$ through the isomorphism I in Proposition 5.1. Then, when π is injective, the bidual action of $\mathbb{Z}$ on $\operatorname{Im}\pi$ becomes the automorphism σ given by

$$\sigma(L_a^n) = L_a^{n-1}, \quad \sigma(T_x^n) = T_x^{n-1},$$

for all $a \in A$, $x \in X$, and $n \in \mathbb{Z}$. (Notice that by part (i) of Proposition 5.2 σ is determined by the equations above)

Proof. In the notation of Propositions 5.1 and 5.2 we have the following, for all $a \in A$ and $n \in \mathbb{Z}$:

$$L_a^n = \pi(a\delta_n) = \pi \circ I(f_{n,a}), \text{ where } f_{n,a}(\lambda) = \lambda^{-n}a.$$

Then

$$\sigma(L_a^n) = \pi \circ I(\widehat{\delta}(f_{n,a})) = \pi \circ I(f_{n-1,a}) = L_a^{n-1}.$$

Analogously, $T_x^n = \pi \circ I(g_{n,x})$, for $g_{n,x}(\lambda) = \lambda^{-n}x$, $x \in X$, and $n \in \mathbb{Z}$, and $\sigma(T_x^n) = T_x^{n-1}$. ∎

THEOREM 5.4. *Let X be an A-A Morita equivalence bimodule, that is, a Hilbert C^*-bimodule over A that is full both on the left and the right. Let δ denote the dual action of S^1 on $A \rtimes X$. Then the crossed product $(A \rtimes X) \rtimes_\delta S^1$ is Morita equivalent to A.*

Namely, there is an isomorphism between $(A \rtimes X) \rtimes_\delta S^1$ and $\mathcal{K}(E_X)$ through which the bidual action $\widehat{\delta}$ becomes the action induced by the automorphism σ given by

$$\sigma(L_a^n) = L_a^{n-1}, \quad \sigma(T_a^n) = T_x^{n-1},$$

for all $a \in A$, $x \in X$, and $n \in \mathbb{Z}$.

Proof. It suffices to show the second statement, since E_X is a Morita equivalence bimodule between $\mathcal{K}(E_X)$ and A. In view of Proposition 5.1, Proposition 5.2, and Remark 5.3, it only remains to notice that the left and right actions of A on X are faithful because X is full. ∎

COROLLARY 5.5. *If X is a Morita equivalence bimodule over A, then $A \rtimes X$ is Morita equivalent to the crossed product (by an automorphism) $\mathcal{K}(E_X) \rtimes_\sigma \mathbb{Z}$, where σ is the automorphism defined in Remark* 5.3.

Proof. By Takai duality ([12]), $A \rtimes X$ is Morita equivalent to $A \rtimes X \rtimes_\delta S^1 \rtimes_{\widehat{\delta}} \mathbb{Z}$, δ being the dual action on $A \rtimes X$. Now, by Theorem 5.4, $A \rtimes X \rtimes_\delta S^1 \rtimes_{\widehat{\delta}} \mathbb{Z}$ is isomorphic to $\mathcal{K}(E_X) \rtimes_\sigma \mathbb{Z}$. ∎

REMARK 5.6. Let $\alpha \in \mathrm{Aut}(A)$, so that $A \rtimes_\alpha \mathbb{Z}$ is isomorphic to $A \rtimes X$ for $X = A_\alpha$ as in Remark 3.4. By identifying $A^{\otimes n}$ and A through the isomorphism I_n in Remark 3.4, we get an isomorphism I of right Hilbert modules from E_X to $l^2(\mathbb{Z}) \otimes A$. Namely, $(I\eta)(n) = I_n(\eta(n))$.

This isomorphism yields, in turn, an identification between $\mathcal{K}(E_X)$ and $\mathcal{K} \otimes A$ which, by virtue of equations (3.4) and (3.5), maps L_a^n and T_x^n to $E_{nn} \otimes \alpha^{-n}(a)$ and $E_{n+1,n} \otimes \alpha^{-(n+1)}(x)$, respectively, where, as usual, E_{ij} denotes the matrix having a 1 at its ij-entry and all other entries 0. In this setting the automorphism σ in Theorem 5.4 becomes, as in [12], $\mathrm{Ad}\rho \otimes \alpha$, ρ being translation by 1.

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