

## STRICT ESSENTIAL EXTENSIONS OF $C^*$ -ALGEBRAS AND HILBERT $C^*$ -MODULES

MICHAEL FRANK and ALEXANDER A. PAVLOV

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**ABSTRACT.** In the present paper we develop both ideas of [1] and the categorical approach to multipliers from [9], [13], [14] for the introduction and study of left multipliers of Hilbert  $C^*$ -modules. Some properties and, in particular, the property of maximality among all strict essential extensions of a Hilbert  $C^*$ -module for left multipliers are proved. Also relations between left essential and left strict essential extensions in different contexts are obtained. Left essential and left strict essential extensions of matrix algebras are considered. In the final paragraph the topological approach to the left multiplier theory of Hilbert  $C^*$ -modules is worked out.

**KEYWORDS:** *Hilbert  $C^*$ -modules,  $C^*$ -algebras of operators, left multipliers, left strict essential extensions.*

**MSC (2000):** Primary 46L08; Secondary 46L05, 47L25.

### 1. INTRODUCTION

There are a lot of ways to include a non-unital  $C^*$ -algebra as an (essential) ideal into unital ones. Among those extensions there is a maximal extension, the algebra of multipliers. This object can be considered from different points of view. Historically the first definition of this algebra arose in the context of centralizers in [2]. There exists another definition of multipliers given via the universal representations of  $C^*$ -algebras. This approach may be generalized for arbitrary non-degenerated faithful representations of  $C^*$ -algebras on Hilbert  $C^*$ -modules, cf. [9]. Besides this we can understand algebras of multipliers as the set of all adjointable maps from a  $C^*$ -algebra to itself. Indeed, the latter approach is the most suitable for a generalization of these constructions to Hilbert  $C^*$ -modules. In [1] multipliers of Hilbert  $C^*$ -modules were introduced and their universal property was obtained. In [16] these notions were significantly used both for a generalization of the Kasparov stabilization theorem to the non-unital case and for an extension of the concept of module frames in Hilbert  $C^*$ -modules over non-unital  $C^*$ -algebras as a continuation of ideas on module frame concepts explained in [4], [5].

In the present paper we will continue both the ideas of [1] and the categorical approach to multipliers from [9], [13], [14] for an introduction of a notion of left multipliers of Hilbert  $C^*$ -modules. The text is organized in the following way. In Section 2 we include some remarks on Hilbert  $C^*$ -modules and on categorical constructions of (left) multipliers of  $C^*$ -algebras. In Section 3 some properties and, in particular, the property to be a left strictly essential extension (Theorem 3.8) and the property of maximality (Theorem 3.9) for left multipliers of Hilbert  $C^*$ -modules are obtained. In Section 4 we study differences between essential and strictly essential extensions both in  $C^*$ - and Banach situations. In Section 5 left essential and left strictly essential extensions of matrix algebras are studied and the property of their maximality is considered. Finally, Section 6 is dedicated to the approach to the left multiplier theory considering appropriate analogs of strict topologies.

## 2. PRELIMINARIES AND REMINDING

To begin with, let us remind that for a  $C^*$ -algebra  $A$  a *pre-Hilbert  $A$ -module* is a (right)  $A$ -module  $V$  equipped with a semi-linear map  $\langle \cdot, \cdot \rangle : V \times V \rightarrow A$  such that

- (i)  $\langle x, x \rangle \geq 0$  for all  $x \in V$ ;
- (ii)  $\langle x, x \rangle = 0$  if and only if  $x = 0$ ;
- (iii)  $\langle x, y \rangle^* = \langle y, x \rangle$  for all  $x, y \in V$ ;
- (iv)  $\langle x, ya \rangle = \langle x, y \rangle a$  for all  $x, y \in V, a \in A$ .

The map  $\langle \cdot, \cdot \rangle$  is called an  $A$ -valued inner product. A norm can be defined for any pre-Hilbert module  $V$  by the formula

$$\|x\| = \|\langle x, x \rangle\|^{1/2}, \quad x \in V.$$

A pre-Hilbert  $A$ -module is a *Hilbert  $A$ -module* if it is complete with respect to this norm.

Let  $V_1, V_2$  be Hilbert  $A$ -modules. Then by  $\text{Hom}_A(V_1, V_2)$  we will denote the set of all  $A$ -linear bounded operators from  $V_1$  to  $V_2$ . When  $V_1 = V_2 = V$  we will write  $\text{End}_A(V)$  instead of  $\text{Hom}_A(V, V)$ .

An operator  $T \in \text{Hom}_A(V_1, V_2)$  admits an adjoint operator if there exists an element  $T^* \in \text{Hom}_A(V_2, V_1)$  such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \text{for all } x \in V_1, y \in V_2.$$

By  $\text{End}_A^*(V)$  we denote the subset of  $\text{End}_A(V)$  consisting of operators which possess adjoint ones.

Any  $C^*$ -algebra  $A$  can be considered as a right Hilbert  $A$ -module over itself with the inner product  $\langle a, b \rangle = a^*b$ . Then the  $C^*$ -algebra  $M(A)$  of multipliers of  $A$  can be defined as  $M(A) = \text{End}_A^*(A)$  and the Banach algebra  $LM(A)$  of left multipliers of  $A$  can be defined as  $LM(A) = \text{End}_A(A)$ , cf. [6], [8], [10].

In [13], [14] this definition has been extended to rather more general situations. Let us briefly remind these notions and results, because we will need them later. Let  $\mathcal{B}$  be a Banach algebra and suppose the existence of a  $C^*$ -subalgebra  $A \subseteq \mathcal{B}$  which is a left ideal of  $\mathcal{B}$ .

DEFINITION 2.1.  $A$  is said to be a *left essential ideal* of  $\mathcal{B}$  if one of the following equivalent conditions holds:

- (i) any non-zero two-sided ideal of  $\mathcal{B}$  has a non-trivial intersection with  $A$ ;
- (ii) there does not exist any non-zero element  $b \in \mathcal{B}$  such that  $ba = 0$  for all  $a \in A$ , i.e. the two-sided ideal  $\mathcal{B}_0 := \{b \in \mathcal{B} : ba = 0 \text{ for all } a \in A\}$  of  $\mathcal{B}$  equals to zero.

DEFINITION 2.2.  $A$  is said to be a *left strictly essential ideal* of  $\mathcal{B}$  (and  $\mathcal{B}$  is said to be a *left strictly essential extension* of  $A$ ) if the following condition holds:

$$(2.1) \quad \|b\| = \sup\{\|ba\| : a \in A, \|a\| \leq 1\} \quad \text{for all } b \in \mathcal{B}.$$

Any left strictly essential ideal is a left essential one, because the equality (2.1) implies the second condition of Definition 2.1. But the inverse statement is not true, i.e. a left essential ideal of a Banach algebra might not be a left strictly essential one in contradistinction to the case when  $\mathcal{B}$  is a  $C^*$ -algebra, because in the  $C^*$ -case both these properties of ideals coincide (see Lemma 7 of [13]). To give an example of a Banach algebra  $\mathcal{B}$  with a  $C^*$ -algebra  $A$  as a left essential ideal such that  $A$  is not a left strictly essential ideal of  $\mathcal{B}$  consider any non-unital  $C^*$ -algebra  $A$  and its algebraic unitization  $\mathcal{B} = A + \mathbb{C}e$  setting the norm of the identity  $e$  to a fixed positive integer  $\alpha$  greater than one and the norm of composite elements of  $\mathcal{B}$  to the sum of the norms of their two components. Then

$$\|e\| = \alpha > 1 = \sup\{\|ea\| : a \in A, \|a\| \leq 1\}$$

because  $\|ea\| = \|a\|$  for any  $a \in A$ .

Let us remark that a Banach algebra  $\mathcal{B}$  may contain not only the fixed  $C^*$ -algebra  $A$ , but its isomorphic copy too as a left essential or strictly essential ideal. In such situations we also will call  $\mathcal{B}$  as either a *left essential* or a *left strictly essential extension* of  $A$ .

Let  $A$  be a  $C^*$ -algebra and  $\mathcal{B}$  be an algebra with an involution containing  $A$  as a left essential ideal. Then the map

$$\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$$

introduced by the formula (2.1) defines a norm on  $\mathcal{B}$  such that  $A$  is a left strictly essential ideal of  $\mathcal{B}$  with respect to it. But, in general, we do not require  $\mathcal{B}$  to be complete with respect to that norm. Let us discuss this point a bit more thoroughly.

THEOREM 2.3. *Let  $A$  be a  $C^*$ -algebra, let  $\mathcal{B}$  be an algebra with an involution containing  $A$  as a left essential ideal. Then the following conditions are equivalent:*

- (i)  $\mathcal{B}$  admits a structure of a  $C^*$ -algebra such that  $A \subset \mathcal{B}$  is a  $C^*$ -subalgebra;

(ii)  $\mathcal{B}$  is complete with respect to the norm (2.1).

*Proof.* To start with, let us verify that (i) implies (ii). Consider the map

$$\beta : \mathcal{B} \rightarrow \text{End}_A^*(A), \quad \beta(b) = ba \quad (b \in \mathcal{B}, a \in A).$$

Then under our assumptions  $\beta$  is injective. Consequently, the map  $\beta$  is an isometry, cf. Theorem 3.1.5 of [12]. Therefore, the norm(2.1) coincides with the  $C^*$ -norm on  $\mathcal{B}$ .

The inverse statement is clear. ■

LEMMA 2.4 ([13]). *Let  $A, C$  be  $C^*$ -algebras,  $\mathcal{B}$  be a Banach algebra,  $A \subset \mathcal{B}$  be a left ideal,  $E$  be a Hilbert  $C$ -module and  $\rho : A \rightarrow \text{End}_C^*(E)$  be a non-degenerate representation of  $A$  in  $E$ . Then there is a unique extension of  $\rho$  to a morphism  $\tilde{\rho} : \mathcal{B} \rightarrow \text{End}_C(E)$  of Banach algebras. If in addition  $A$  is a left strictly essential ideal and  $\rho$  is injective, then  $\tilde{\rho}$  is an isometry.*

DEFINITION 2.5. Let  $A, C$  be  $C^*$ -algebras,  $E$  be a Hilbert  $C$ -module and  $\rho : A \rightarrow \text{End}_C^*(E)$  be a faithful non-degenerate representation of  $A$  in  $E$ . Then  $(E, C, \rho)$  is an admissible for  $A$  triple.

DEFINITION 2.6. Let  $(E, C, \rho)$  be an admissible for  $A$  triple. Then the set of left  $(E, C, \rho)$ -multipliers of  $A$  is defined as

$$LM_{(E, C, \rho)}(A) = \{T \in \text{End}_C(E) : T\rho(A) \subset \rho(A)\}.$$

The standard definition of the left multipliers  $LM(A)$  of a  $C^*$ -algebra  $A$  is a special case of Definition 2.6 corresponding to the triple  $(A, A, \alpha)$ , where

$$\alpha : A \rightarrow \text{End}_A^*(A), \quad \alpha(a)b = ab \quad (a, b \in A).$$

DEFINITION 2.7. A left strictly essential extension  $\widehat{\mathcal{B}}$  of  $A$  is maximal if for any other left strictly essential extension  $\mathcal{B}$  of  $A$  there is an isometrical homomorphism from  $\mathcal{B}$  to  $\widehat{\mathcal{B}}$ , which acts identically on the two copies of  $A$ .

THEOREM 2.8 ([13]). *For any admissible for  $A$  triple  $(E, C, \rho)$  the algebra of the left  $(E, C, \rho)$ -multipliers is a maximal left strictly essential extension of  $A$ .*

THEOREM 2.9 ([13]). *The Banach algebras of left  $(E, C, \rho)$ -multipliers are isomorphic for all admissible for  $A$  triples  $(E, C, \rho)$ , and these isomorphisms act as the identity map on the embedded copies of  $A$ .*

### 3. LEFT MULTIPLIERS OF HILBERT $C^*$ -MODULES

In this section we introduce the notion of left multipliers of Hilbert  $C^*$ -modules as a particular form of a strict essential extension of the respective Hilbert  $C^*$ -module. We prove the generic maximality for the left multipliers of a Hilbert  $C^*$ -module among all of its strict essential extensions.

DEFINITION 3.1. Let  $A$  be a  $C^*$ -algebra, let  $V$  be a Hilbert  $A$ -module. A Banach extension of  $V$  is a triple  $(W, \mathcal{B}, \Phi)$ , where:

- (i)  $\mathcal{B}$  is a Banach algebra,  $A \subset \mathcal{B}$  is a left ideal;
- (ii)  $W$  is a Banach  $\mathcal{B}$ -module;
- (iii)  $\Phi : V \rightarrow W$  is an  $A$ -linear isometric map;
- (iv)  $\text{Im}\Phi = WA := \overline{\text{span}_A\{y : y \in W\}}$ .

REMARK 3.2. The fourth condition of Definition 3.1 is an analogue of the requirement to the representation  $\rho$  from Definition 2.6 to be non-degenerate.

DEFINITION 3.3. The Banach extension  $(W, \mathcal{B}, \Phi)$  of  $A$  is a *strictly essential* one if  $A \subset \mathcal{B}$  is a left strictly essential ideal and the following condition holds

$$(3.1) \quad \|y\| = \sup\{\|ya\| : a \in A, \|a\| \leq 1\} \quad \text{for all } y \in W.$$

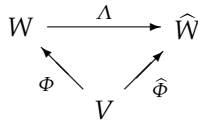
EXAMPLE 3.4. For any Hilbert  $A$ -module  $V$  the triple  $(V, A, \text{id}_V)$ , where  $\text{id}_V : V \rightarrow V$  is an identical map, is (*an identical*) strictly essential Banach extension of  $V$ , because for any approximative unit  $\{e_\alpha\}$  of  $A$  and for any  $x \in V$  the net  $xe_\alpha$  converges with respect to the norm to  $x$  (see Lemma 1.3.8 of [11]). In particular, the triple  $(A, A, \text{id}_A)$  is a strictly essential Banach extension of the Hilbert  $A$ -module  $A$ .

EXAMPLE 3.5. Let  $A \subset \mathcal{B}$  be a left strictly essential ideal and let us denote this embedding by  $i$ . Let us consider  $\mathcal{B}$  as a Banach module over itself and  $A$  as a Hilbert module over itself. Then the triple  $(\mathcal{B}, \mathcal{B}, i)$  is a strictly essential extension of  $A$ .

DEFINITION 3.6. A Banach strictly essential extension  $(\widehat{W}, \widehat{\mathcal{B}}, \widehat{\Phi})$  of a Hilbert  $A$ -module  $V$  is *maximal* if for any other Banach strictly essential extension  $(W, \mathcal{B}, \Phi)$  there are an isometrical homomorphism  $\lambda : \mathcal{B} \rightarrow \widehat{\mathcal{B}}$  which is identical on  $A$  and an isometrical linear map  $\Lambda : W \rightarrow \widehat{W}$  such that it is a  $\lambda$ -homomorphism, i.e.

$$\Lambda(yb) = \Lambda(y)\lambda(b) \quad \text{for all } y \in W, b \in \mathcal{B},$$

and the following diagram



is commutative.

As a consequence  $\Lambda$  maps  $\Phi(V)$  onto  $\widehat{\Phi}(V)$  identifying both the copies of  $V$ . If we consider only Banach algebras  $\mathcal{B}$  which are both modules over themselves and left strict essential extensions of  $A$ , i.e. only Banach strictly essential extensions of kind  $(\mathcal{B}, \mathcal{B}, i)$ , where  $i$  denotes the embedding of  $A$  into  $\mathcal{B}$ , then apparently Definition 3.6 coincides with Definition 2.7.

DEFINITION 3.7. Let  $V$  be a Hilbert module over a  $C^*$ -algebra  $A$ . Then the set of left multipliers  $LM(V)$  of  $V$  is defined by

$$LM(V) := \text{Hom}_A(A, V).$$

Let us define the action of  $LM(A)$  on  $LM(V)$  by the formula

$$(3.2) \quad (yb)(a) = y(b(a)), \quad y \in LM(V), b \in LM(A), a \in A,$$

which turns the set  $LM(V)$  into a right  $LM(A)$ -module. Furthermore, define the map  $\Gamma : V \rightarrow LM(V)$  in the following way

$$(3.3) \quad (\Gamma(x))(a) = xa, \quad x \in V, a \in A.$$

If  $V = A$ , then, obviously, Definition 3.7 coincides with the standard definition of the left multipliers of the  $C^*$ -algebra  $A$ . The map  $\Gamma$  is an isometric embedding of  $V$  into  $LM(V)$  as a Banach  $A$ -submodule.

THEOREM 3.8. Let  $V$  be a Hilbert module over a  $C^*$ -algebra  $A$ . For  $LM(V)$  being the set of left multipliers of  $V$  the triple  $(LM(V), LM(A), \Gamma)$  is a strictly essential Banach extension of a Hilbert  $A$ -module  $V$ .

*Proof.* It is clear that  $LM(V)$  is a Banach  $LM(A)$ -module with respect to the action (3.2). Beside this,  $A$  is a left strictly essential ideal in  $LM(A)$  by Theorem 2.8. Moreover, it is a straightforward verification that the map (3.3) is an  $A$ -module isometry and that the equality (3.1) holds. So it remains only to check the fourth condition of Definition 3.1.

Let us choose any approximative identity  $\{e_\alpha\}$  in  $A$ . Then for any  $x \in V$  we can write

$$\Gamma(x) = \lim_{\alpha} \Gamma(xe_\alpha) = \lim_{\alpha} \Gamma(x)e_\alpha$$

and, consequently,  $\text{Im}\Gamma \subset LM(V)A$ . To obtain the inverse inclusion let us take any  $T \in LM(V)$ ,  $a \in A$ , then for all  $b \in A$  we have

$$(Ta)(b) = T(ab) = T(a)b = \Gamma(T(a))b,$$

so  $Ta = \Gamma(T(a))$  and we have got the desired set coincidence  $\text{Im}\Gamma = LM(V)A$ . ■

THEOREM 3.9. Let  $V$  be a Hilbert module over a  $C^*$ -algebra  $A$ . The strictly essential Banach extension  $(LM(V), LM(A), \Gamma)$  of any Hilbert  $A$ -module  $V$  is maximal.

*Proof.* Consider any other strictly essential Banach extension  $(W, \mathcal{B}, \Phi)$  of  $V$ . An isometrical homomorphism  $\lambda : \mathcal{B} \rightarrow LM(A)$  which is identical on  $A$  exists by Theorem 2.8, and, moreover, the uniqueness of this homomorphism (cf. Lemma 2.4) is the reason why the equality  $\lambda(b)(a) = ba$  has to hold for all  $b \in \mathcal{B}, a \in A$ .

Now let us define the map  $\Lambda : W \rightarrow LM(V)$  by the formula

$$\Lambda(y)(a) := \Gamma(\Phi^{-1}(y))(a) = \Phi^{-1}(ya), \quad y \in W, a \in A.$$

This definition is correct because  $ya \in \text{Im}\Phi$  and  $\Phi$  is an isometry. Further, for any  $y \in W$  the following equalities hold due to (3.1):

$$\|\Lambda y\| = \sup\{\|\Lambda(y)(a)\| : a \in A, \|a\| \leq 1\} = \sup\{\|ya\| : a \in A, \|a\| \leq 1\} = \|y\|.$$

Consequently,  $\Lambda$  is an isometry. The properties of  $\Lambda$  to be a  $\lambda$ -homomorphism and to fulfil the equality  $\Lambda\Phi = \Gamma$  can be derived by obvious computations. ■

#### 4. ESSENTIAL AND STRICT ESSENTIAL EXTENSIONS OF HILBERT $C^*$ -MODULES

The left strict extensions of Hilbert  $C^*$ -modules are Banach module extensions over Banach algebras, in general. So a wide variety of them might occur in particular situations in difference to the quite canonical situations appearing in the case of multiplier modules and (two-sided) strict extensions, cf. [1]. We start the investigation of characteristic situations with the known definition of (two-sided) essential extensions of Hilbert  $C^*$ -modules for the situation of  $C^*$ -extensions of the  $C^*$ -algebra of coefficients.

DEFINITION 4.1 ([1]). Let  $V$  be a Hilbert  $A$ -module over a  $C^*$ -algebra  $A$ . An extension of  $V$  is a triple  $(W, B, \Phi)$  such that:

- (i)  $B$  is a  $C^*$ -algebra,  $A \subset B$  is an ideal;
- (ii)  $W$  is a Hilbert  $B$ -module;
- (iii)  $\Phi : V \rightarrow W$  is a map satisfying  $\langle \Phi x, \Phi y \rangle = \langle x, y \rangle$  for all  $x, y \in V$ ;
- (iv)  $\text{Im}\Phi = WA$ .

If in addition  $A$  is an essential ideal of  $B$ , then the extension  $(W, B, \Phi)$  is called *essential*.

THEOREM 4.2. Let  $(W, B, \Phi)$  be an essential extension of a Hilbert  $A$ -module  $V$ . Then the mentioned extension is automatically strictly essential, i.e. the conditions (2.1), (3.1) hold for it.

*Proof.* The following equalities can be established for any  $y \in W, a \in A$ :

$$\|ya\|^2 = \|\langle ya, ya \rangle\| = \|a^* \langle y, y \rangle a\| = \|\langle y, y \rangle^{1/2} a\|^2.$$

Any essential ideal is automatically strictly essential in the  $C^*$ -case, i.e. the property (2.1) holds (see Lemma 7 of [13]). Therefore,

$$\sup\{\|\langle y, y \rangle^{1/2} a\| : a \in A, \|a\| \leq 1\} = \|\langle y, y \rangle^{1/2}\| = \|y\|$$

and, consequently, the desired property is obtained. ■

At the contrary, the results in the situation of Banach extensions of Hilbert  $A$ -modules are completely different from the one mentioned above as the next statement shows.

**THEOREM 4.3.** *For any non-unital  $C^*$ -algebra  $A$  there exists a Hilbert  $A$ -module  $V$  and a Banach extension  $(W, \mathcal{B}, \Phi)$  of  $V$  such that  $A$  is a strictly essential ideal of  $\mathcal{B}$ , but the condition (3.1) does not hold.*

*Proof.* Let us take into consideration any non-unital  $C^*$ -algebra  $A$  and put  $V = A$ ,  $\mathcal{B} = A$ . Further let us choose  $W = \tilde{A}$ , where  $\tilde{A}$  is the  $C^*$ -algebra with an adjoint unit equipped with the (Banach, but not  $C^*$ -) norm

$$\|(a, \lambda)\| = \|a\| + |\lambda|, \quad a \in A, \lambda \in \mathbb{C}.$$

Then  $A$  will be a left essential, but not left strictly essential ideal of  $\tilde{A}$  (see Lemma 8 of [13]). Let us choose the map  $\Phi = i$  to be the canonical embedding of  $A$  into  $\tilde{A}$ . Then the condition (3.1) does not hold for the Banach extension  $(\tilde{A}, A, i)$ . ■

Let  $(W, \mathcal{B}, \Phi)$  be a Banach extension of a Hilbert  $A$ -module  $V$ . Let us define a closed  $\mathcal{B}$ -submodule  $W_0$  of  $W$  by the formula

$$(4.1) \quad W_0 = \{y \in W : ya = 0 \text{ for all } a \in A\}.$$

Then the assertion  $W_0 = \{0\}$  would be a reasonable analogue to the condition (ii) of Definition 2.1. Let us remark in addition that  $W_0 = \{0\}$  holds for any strict essential Banach extensions.

**REMARK 4.4.** We can a bit strengthen the result of Theorem 4.3. More precisely, the example  $(\tilde{A}, A, i)$  of a Banach extension with a non-unital  $C^*$ -algebra  $A$  from the proof of Theorem 4.3 shows that there are a Hilbert  $A$ -module  $V$  and its Banach extension  $(W, \mathcal{B}, \Phi)$  such that  $A$  is a strictly essential ideal of  $\mathcal{B}$  and  $W_0 = 0$ , but the condition (3.1) does not hold.

Finally, let us discuss one question, which was formulated by D. Bakić. In [1] the set of multipliers of a Hilbert  $A$ -module  $V$  was defined as the Hilbert  $M(A)$ -module  $V_d = M(V) := \text{Hom}^*(A, V)$ . Then the question was raised whether there exists an admissible situation, when Hilbert modules  $V_1$  and  $V_2$  over a non-unital  $C^*$ -algebra are not isomorphic, but their modules of multipliers are isomorphic. Let us demonstrate by example that the answer on this question is affirmative.

**EXAMPLE 4.5.** Let  $A$  be the  $C^*$ -algebra  $K(H)$  of all compact operators on a separable Hilbert space  $H$ , and let  $B$  be the  $C^*$ -algebra  $B(H)$  of all bounded linear operators on  $H$ . Consider the  $C^*$ -algebra  $C$  and two Hilbert  $C$ -modules  $V_1$  and  $V_2$  defined by

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad V_1 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad V_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then  $V_1$  and  $V_2$  are not isomorphic as Hilbert  $C$ -modules, because the first one is a full Hilbert  $C$ -module, but the second one is not. At the same time both their sets of (two-sided, left) multipliers can be described by the Hilbert  $M(C)$ -module

$$M(V_1) = M(V_2) = LM(V_1) = LM(V_2) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$



5. ESSENTIAL AND STRICT ESSENTIAL EXTENSIONS OF MATRIX ALGEBRAS

The aim of the present section is to check to which extent an essential extension of a  $C^*$ -algebra  $A$  to a  $*$ -algebra  $\mathbb{B}$  preserves the property of  $\mathbb{B}$  to be Banach with respect to the induced norm (2.1) for any of its finite matricial extensions  $M_n(\mathbb{B})$ ,  $n \geq 2$ , with  $M_n(A) \subseteq M_n(\mathbb{B})$  being an essential extension of  $M_n(A)$ , and vice versa. Note, that Definition 2.1 of a left essential ideal is formulated for algebraic representations of the  $*$ -algebra  $A$  in  $*$ -algebras  $\mathbb{B}$  without any reference to topologies on both these algebras.

LEMMA 5.1. *Let  $A$  be a  $C^*$ -algebra and  $\mathcal{B}$  be a Banach algebra. Then the following conditions are equivalent:*

- (i)  $A$  is a left essential ideal of  $\mathcal{B}$ ;
- (ii)  $M_n(A)$  is a left essential ideal of  $M_n(\mathcal{B})$  for any integer  $n \geq 2$ .

*Proof.* Observe that  $M_n(\mathcal{B}_0) = M_n(\mathcal{B})_0$  for any integer  $n \geq 2$  under the notations of Definition 2.1. Consequently, the assertion holds. ■

THEOREM 5.2. *Let  $A$  be a  $C^*$ -algebra and  $\mathcal{B}$  be a normed algebra containing  $A$  as a left essential ideal. Then the following conditions are equivalent:*

- (i) *The algebra  $\mathcal{B}$  is Banach with respect to the induced by the essential extension of  $A$  norm (2.1).*
- (ii) *The algebra  $M_n(\mathcal{B})$  is Banach with respect to the induced by the essential extension of  $M_n(A)$  norm (2.1) for any integer  $n \geq 2$ .*

*Proof.* We will denote by  $\|\cdot\|$  the norm (2.1) for elements either from  $\mathcal{B}$  or from  $M_n(\mathcal{B})$ . For the convenience of the reader let us remind the following well known inequalities (cf. Remark 3.4.1 of [12])

$$(5.1) \quad \|a_{i,j}\| \leq \|a\| \quad (i, j = 1, \dots, n), \quad \|a\| \leq \sum_{i,j=1}^n \|a_{i,j}\|,$$

which hold for any  $a = (a_{i,j}) \in M_n(A)$ .

To begin with, let us prove that (i) implies (ii). For  $b = (b_{i,j})$  from  $M_n(\mathcal{B})$  we have the following estimates:

$$(5.2) \quad \|b\| \geq \sup \left\{ \left\| \sum_{k=1}^n b_{i,k} a_{k,r} \right\| : \|(a_{i,j})\| \leq 1, (a_{i,j}) \in M_n(A) \right\} \\ \geq \sup \{ \|b_{i,j} a_{j,r}\| : \|a_{j,r}\| \leq 1, a_{j,r} \in A \} = \|b_{i,j}\|$$

for all  $1 \leq i, j \leq n$  by (5.1). Therefore, any Cauchy sequence  $\{b^{(N)} = (b_{i,j}^{(N)}) : N \in \mathbb{N}\}$  from  $M_n(\mathcal{B})$  generates Cauchy sequences  $\{b_{i,j}^{(N)}\}$  from  $\mathcal{B}$  for any pair  $(i, j)$  with  $1 \leq i, j \leq n$ . Let us denote the limits of the sequences  $\{b_{i,j}^{(N)} : N \in \mathbb{N}\}$

in  $\mathcal{B}$  by  $b_{i,j}$  for any pair  $(i, j)$ , i.e.

$$\lim_{N \rightarrow \infty} \sup \{ \| (b_{i,j}^{(N)} - b_{i,j})a \| : \|a\| \leq 1, a \in A \} = 0,$$

and let  $b = (b_{i,j})$  denote a corresponding matrix from  $M_n(\mathcal{B})$ . Then, taking into consideration (5.1), we deduce

$$\begin{aligned} \|b - b^{(N)}\| &\leq \sup \left\{ \left\| \sum_{i,j=1}^n \left\| \sum_{k=1}^n (b_{i,k} - b_{i,k}^{(N)})a_{k,j} \right\| : \| (a_{k,j}) \| \leq 1, (a_{k,j}) \in M_n(A) \right\} \right\} \\ (5.3) \quad &\leq \sum_{i,j=1}^n \sum_{k=1}^n \sup \{ \| (b_{i,k} - b_{i,k}^{(N)})a_{k,j} \| : \| (a_{k,j}) \| \leq 1, (a_{k,j}) \in M_n(A) \} \\ &= \sum_{i,j=1}^n \sum_{k=1}^n \sup \{ \| (b_{i,k} - b_{i,k}^{(N)})a_{k,j} \| : \| a_{k,j} \| \leq 1, a_{k,j} \in A \} \\ &= \sum_{i,k=1}^n n \| b_{i,k} - b_{i,k}^{(N)} \|. \end{aligned}$$

Therefore the sequence  $\{b^{(N)}\}$  converges to  $b$  with respect to the norm and, hence, the space  $M_n(\mathcal{B})$  is complete.

Now we have to verify that (ii) implies (i). Let us consider any Cauchy sequence  $\{b^{(N)} : N \in \mathbb{N}\}$  of  $\mathcal{B}$  and define a corresponding sequence of matrices  $\{\tilde{b}^{(N)}\}$  of  $M_n(\mathcal{B})$  where the element at position  $(1, 1)$  of the respectively derived matrix equals to  $b^{(N)}$  and all the other elements of the matrices are equal to zero. Then inequality (5.3) is the reason why

$$\|\tilde{b}^{(N)}\| \leq n \|b^{(N)}\|$$

for any  $N \in \mathbb{N}$ . Denote the limit of the sequence  $\{\tilde{b}^{(N)}\}$  in  $M_n(\mathcal{B})$  by  $\tilde{b} = (\tilde{b}_{i,j})$ . Immediately (5.2) implies that, firstly,  $\tilde{b}_{i,j} = 0$  if  $(i, j) \neq (1, 1)$  and, secondly, the sequence  $\{b^{(N)}\}$  converges to  $\tilde{b}_{1,1}$ . ■

Finally, considering the particular case of maximal left strict essential extensions of matrix algebras  $M_n(A)$  for  $C^*$ -algebras  $A$  one obtains the identification  $LM(M_n(A)) \simeq M_n(LM(A))$  for any integer  $n \geq 1$ . The equality may be verified using the strict topology approach to left multipliers of  $C^*$ -algebras (see [17] for details).

## 6. LEFT STRICT TOPOLOGY AND LEFT MULTIPLIERS

Essential left extensions of  $C^*$ -algebras are strongly interrelated with some kind of topological closures of the embedded copy of the extended  $C^*$ -algebra, where these left strict topologies are generated by certain sets of semi-norms. We

are going to look for analogous sets of semi-norms for essential extensions of  $C^*$ -algebras of bounded  $C^*$ -linear operators on Hilbert  $C^*$ -modules and for essential extensions of Hilbert  $C^*$ -modules.

Before we discuss an approach to the left multipliers of Hilbert  $C^*$ -modules which is connected with the notion of a certain left strict topology let us introduce the analogue of the one for the left  $(E, C, \rho)$ -multipliers of a  $C^*$ -algebra  $A$  (see Definition 2.6).

DEFINITION 6.1. Let  $(E, C, \rho)$  be an admissible for  $A$  triple. The left strict topology on  $\text{End}_C(E)$  is defined by the family of semi-norms

$$(6.1) \quad \{v_a\}_{a \in A}, \quad \text{where } v_a(T) = \|T\rho(a)\|, \quad T \in \text{End}_C(E).$$

We will denote this topology by *l.s.*

PROPOSITION 6.2. Let  $(E, C, \rho)$  be an admissible for a  $C^*$ -algebra  $A$  triple and  $\mathcal{B}$  be a Banach subalgebra of  $\text{End}_C(E)$  containing  $\rho(A)$  as a left ideal. The following conditions are equivalent:

- (i) the left strict topology on  $\mathcal{B}$  is Hausdorff;
- (ii)  $\rho(A)$  is an essential ideal of  $\mathcal{B}$ .

In particular, the set of left  $(E, C, \rho)$ -multipliers of  $A$  equipped with the left strict topology is a Hausdorff space.

*Proof.* The second condition of Definition 2.1 is, obviously, equivalent to the requirement that the system (6.1) of semi-norms separates points of  $\mathcal{B}$ . ■

PROPOSITION 6.3. The set of all left  $(E, C, \rho)$ -multipliers  $LM_{(E, C, \rho)}(A)$  of  $A$  is a closed space with respect to the left strict topology.

*Proof.* Consider any net  $\{T_\alpha\}$  from  $LM_{(E, C, \rho)}(A)$  converging to  $T \in \text{End}_C(E)$  with respect to the left strict topology. It means that the net  $\{T_\alpha\rho(a)\}$  from  $\rho(A)$  converges to  $T\rho(a)$  with respect to norm for any  $a \in A$ . Therefore  $T\rho(a)$  belongs to  $\rho(A)$  for any  $a \in A$  and, hence,  $T$  belongs to  $LM_{(E, C, \rho)}(A)$ . ■

PROPOSITION 6.4. The set of all left  $(E, C, \rho)$ -multipliers  $LM_{(E, C, \rho)}(A)$  of  $A$  coincides with the closure  $\overline{\rho(A)}^{l.s.}$  of  $\rho(A)$  inside  $\text{End}_C(E)$  with respect to the left strict topology.

*Proof.* Because of Proposition 6.3 it remains to check that  $LM_{(E, C, \rho)}(A) \subset \overline{\rho(A)}^{l.s.}$ . Take an approximative identity  $\{e_\alpha\}$  in  $\rho(A)$ . Then for any left multiplier  $T \in LM_{(E, C, \rho)}(A)$  the net  $\{Te_\alpha\} \in \rho(A)$  converges to  $T$  with respect to the left strict topology. Indeed

$$\lim_{\alpha} \|Te_\alpha\rho(a) - T\rho(a)\| \leq \lim_{\alpha} \|T\| \|e_\alpha\rho(a) - \rho(a)\| = 0$$

for any  $a \in A$ . ■

Now let us consider a Banach extension  $(W, \mathcal{B}, \Phi)$  of a Hilbert  $A$ -module  $V$ . Then  $V$  is isomorphic to  $\Phi(V)$  and  $\text{Hom}_A(A, \Phi(V))$  is a Banach  $LM(A)$ -submodule of  $\text{Hom}(A, W)$  with respect to the action (3.2).

DEFINITION 6.5. The left strict topology on  $\text{Hom}_A(A, W)$  is defined by the family of semi-norms

$$(6.2) \quad \{v_a\}_{a \in A}, \quad \text{where } v_a(S) = \|Sa\|, \quad S \in \text{Hom}_A(A, W).$$

Here we understand  $A$  canonically embedded into  $LM(A)$ , and  $Sa$  means the result of the action (3.2). We will denote this topology by *l.s.*

DEFINITION 6.6. Let  $(W, \mathcal{B}, \Phi)$  be a Banach extension of a Hilbert  $A$ -module  $V$ . The left  $(W, \mathcal{B}, \Phi)$ -multipliers of  $V$  are defined as

$$LM_{(W, \mathcal{B}, \Phi)}(V) = \text{Hom}_A(A, \Phi(V)).$$

It is clear that left  $(W, \mathcal{B}, \Phi)$ -multipliers of  $V$  are isomorphic for all Banach extensions  $(W, \mathcal{B}, \Phi)$  of  $V$ . Beside this, the previous Definition 3.7 of left multipliers  $LM(V)$  of  $V$  is a special case of Definition 6.6 corresponding to the identical Banach extension discussed as Example 3.4.

In the sequel we will need the generalization  $\Gamma_\Phi : \Phi(V) \rightarrow LM_{(W, \mathcal{B}, \Phi)}(V)$  of the map (3.3) which will be defined in the following way

$$(6.3) \quad (\Gamma_\Phi(y))(a) := ya, \quad y \in \Phi(V), a \in A.$$

This map is an isometric  $A$ -linear map.

PROPOSITION 6.7. Let  $(W, \mathcal{B}, \Phi)$  be a Banach extension of a Hilbert  $A$ -module  $V$ . The following conditions are equivalent:

- (i) the left strict topology on  $\text{Hom}_A(A, W)$  is Hausdorff;
- (ii) the submodule  $W_0$  of  $W$  introduced in (4.1) equals to zero.

In particular, the set of all left  $(W, \mathcal{B}, \Phi)$ -multipliers of  $V$  equipped with the left strict topology is a Hausdorff space.

*Proof.* Obviously,  $W_0$  equals zero if and only if the system (6.2) of semi-norms separates points of  $\text{Hom}_A(A, W)$ . ■

THEOREM 6.8. The set of all left  $(W, \mathcal{B}, \Phi)$ -multipliers  $LM_{(W, \mathcal{B}, \Phi)}(V)$  of  $V$  is a closed space with respect to the left strict topology.

*Proof.* Consider any net  $\{S_\alpha\} \in LM_{(W, \mathcal{B}, \Phi)}(V)$  converging to the element  $S \in \text{Hom}_A(A, W)$  with respect to the left strict topology. This means that the net  $\{S_\alpha a\}$  from  $\text{Hom}_A(A, \Phi(V))$  converges to  $Sa$  with respect to the norm for any  $a \in A$ . Therefore,  $Sa$  belongs to  $\text{Hom}_A(A, \Phi(V))$  for any  $a \in A$  and, consequently,  $(Sa)(b) = S(ab)$  belongs to  $\Phi(V)$  for any  $a, b \in A$ . The latter implies that the image of  $S$  belongs to  $\Phi(V)$ , i.e.  $S$  is an element of  $LM_{(W, \mathcal{B}, \Phi)}(V)$ . ■

**THEOREM 6.9.** *The set of all left  $(W, \mathcal{B}, \Phi)$ -multipliers  $LM_{(W, \mathcal{B}, \Phi)}(V)$  of  $V$  coincides with the closure  $\overline{\Gamma_\Phi(\Phi(V))}^{1.s.}$  of the image of the map (6.3) inside  $\text{Hom}_A(A, W)$  with respect to left strict topology.*

*Proof.* Because of Theorem 6.8 it remains to check that  $LM_{(W, \mathcal{B}, \Phi)}(V) \subseteq \overline{\Gamma_\Phi(\Phi(V))}^{1.s.}$ . Consider an approximative identity  $\{e_\alpha\}$  for  $A$ . Then for any  $S \in LM_{(W, \mathcal{B}, \Phi)}(V)$  the net  $\{\Gamma_\Phi(S(e_\alpha))\}$  converges with respect to the left strict topology to  $S$ . Indeed,

$$\begin{aligned} \lim_\alpha \|(\Gamma_\Phi(S(e_\alpha))a)(b) - (Sa)(b)\| &= \lim_\alpha \|(\Gamma_\Phi(S(e_\alpha)))(ab) - S(ab)\| \\ &= \lim_\alpha \|S(e_\alpha)ab - S(ab)\| \\ &= \lim_\alpha \|S(e_\alpha ab) - S(ab)\| \\ &\leq \lim_\alpha \|S\| \|b\| \|e_\alpha a - a\| = 0 \end{aligned}$$

for any  $a, b \in A$ . ■

As a summary, in the present paper we have extended the results by D. Bakić and B. Guljaš from [1] about multipliers of Hilbert  $C^*$ -modules to the case of left multipliers of Hilbert  $C^*$ -modules. The well known facts about the left multiplier algebra of a  $C^*$ -algebra are particular cases of our results in case a  $C^*$ -algebra is considered as a Hilbert  $C^*$ -module over itself. In forthcoming research it would be interesting to investigate an analogue of quasi-multipliers of  $C^*$ -algebras for Hilbert  $C^*$ -modules.

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MICHAEL FRANK, HOCHSCHULE FÜR TECHNIK, WIRTSCHAFT UND KULTUR (HTWK) LEIPZIG, FACHBEREICH IMN, PF 301166, D-04251 LEIPZIG, F.R.GERMANY  
*E-mail address:* mfrank@imn.htwk-leipzig.de

ALEXANDER A. PAVLOV, MOSCOW STATE UNIVERSITY, DEPARTMENT OF GEOGRAPHY AND DEPARTMENT OF MECHANICS AND MATHEMATICS, 119 992 MOSCOW, RUSSIA  
*E-mail address:* axpavlov@mail.ru

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