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ADDITIVE MAPS PRESERVING STRONGLY GENERALIZED INVERSES

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Dedicated to Professor Ángel Rodríguez Palacios on the occasion of his 60 birthday

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ABSTRACT. Let $\mathcal{A}, \mathcal{B}$ be unital complex Banach algebras. An additive map ϕ from $\mathcal{A}$ to $\mathcal{B}$ is said to preserve strongly generalized inverses if $\phi(a)$ is a generalized inverse of $\phi(b)$ whenever a is a generalized inverse of b. In this paper, we characterize additive maps preserving strongly generalized inverses.

KEYWORDS: *Generalized inverses, Jordan homomorphism, linear preserver problems.*

MSC (2000): 47B48, 47A10, 46H05.

1. INTRODUCTION

An additive map $\phi : \mathcal{R} \to \mathcal{R}'$ between two rings is said to be a *Jordan homomorphism* if $\phi(a^2) = \phi(a)^2$ for every a in $\mathcal{R}$. The class of Jordan homomorphisms contains obviously every homomorphism or anti-homomorphism (recall that an additive map ϕ is an anti-homomorphism if $\phi(ab) = \phi(b)\phi(a)$ for all a, b in $\mathcal{R}$). Note that if ϕ is an additive map defined between two real or complex vector spaces, then it is easy to see that ϕ is rational linear and that for rational linear maps between Banach spaces, continuity is equivalent to boundedness as for linear maps. It is well known that there is a myriad of discontinuous additive automorphisms on the complex field [17]. Also, in any finite dimensional algebra, one can easily construct discontinuous automorphisms. But in infinite dimensional algebras, interesting results become possible. Indeed Arnold [1] proved that every automorphism of the algebra of all bounded linear operators on an infinite dimensional complex Banach space X is automatically real-linear (and hence, linear or conjugate linear). This result was generalized by Kaplansky [20] as follows: If ϕ is an isomorphism from one semisimple Banach algebra $\mathcal{A}$ onto another, then $\mathcal{A}$ is a direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$ with $\mathcal{A}_3$ finite-dimensional, ϕ linear on $\mathcal{A}_1$ and conjugate linear on $\mathcal{A}_2$.

Let $\mathcal{A}$ be an algebra. An element $b \in \mathcal{A}$ is called a *generalized inverse* of $a \in \mathcal{A}$ if b satisfies the following two identities

$$aba = a \quad \text{and} \quad bab = b. \tag{1.1}$$

Let $\mathcal{A}^\wedge$ denote the set of all the elements of $\mathcal{A}$ having a generalized inverse. Then by a theorem of Kaplansky [19], if $\mathcal{A} = \mathcal{A}^\wedge$ then $\mathcal{A}$ is finite dimensional. Furthermore, if $\mathcal{A}$ is semi-simple, then $\mathcal{A} = \mathcal{A}^\wedge \iff \dim(A) < \infty$. For more details on the generalized inverse we refer the reader to [12], [13], [23], [30] and the references therein. Now let $\mathcal{A}, \mathcal{B}$ be unital algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. We say that ϕ is unital if $\phi(1) = 1$, where 1 stands for the unit of both $\mathcal{A}$ and $\mathcal{B}$. We denote by $\mathcal{A}^{-1}$ the set of invertible elements of $\mathcal{A}$. Following [25], we shall say that an additive map $\phi : \mathcal{A} \to \mathcal{B}$ *preserves strongly invertibility* if $\phi(x^{-1}) = \phi(x)^{-1}$ for every $x \in \mathcal{A}^{-1}$. Similarly, we shall say that ϕ *preserves strongly generalized invertibility* if $\phi(y)$ is a generalized inverse of $\phi(x)$ whenever y is a generalized inverse of x.

One easily checks that a Jordan homomorphism preserves strongly generalized inverses. In the present paper we deal with additive maps preserving strongly generalized inverses. The motivation for this problem is Hua's theorem which states that every unital additive map ϕ between two fields such that $\phi(x^{-1}) = \phi(x)^{-1}$ is an isomorphism or an anti-isomorphism (see [16], [2]). This result has been later extended to the algebra of matrices over some fields (see [9], [7]) and recently to Banach algebras (see [25]). The case of a nonunital mapping preserving strongly generalized inverses seems to be challenging. However, we show that if $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is an additive map preserving strongly generalized inverses such that $\phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1}$ is nonvoid, then $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ. Other partial results are given.

Our study is closely connected with questions concerning linear (additive) preserver problems. One of the most famous problems in this direction is Kaplansky's conjecture [21] : Let ϕ be a unital surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$ which preserves invertibility (i.e., $\phi(x) \in \mathcal{B}^{-1}$ whenever $x \in \mathcal{A}^{-1}$), then ϕ is a Jordan homomorphism. In the commutative case the well-known Gleason–Kahane–Żelazko theorem provides an affirmative answer. In the non-commutative case the best known results so far are due to Aupetit and Sourour. They showed that the answer to Kaplansky's conjecture is in affirmative for von Neumann algebras [4] and for bijective unital linear invertibility preserving maps acting on the algebra of all bounded operators on a Banach space [34]. For more details on this subject, we refer the reader to some survey articles [4], [3], [6], [33], [34], [15], [24], [28] and the references therein. Recently, characterization of maps preserving generalized inverses in infinite dimensional spaces was attacked by Mbekhta, Rodman and Šemrl in [26] and [27].

It should be pointed out that our proofs are rather technical and crucially based on Hua's identity [2], [16]: If a, b and $a - b^{-1}$ are invertible, then $a^{-1} - (a -$

$b^{-1})^{-1}$ is invertible and

$$[a^{-1} - (a - b^{-1})^{-1}]^{-1} = a - aba. \tag{1.2}$$

We mention that in this work we present an algebraic approach in contrast to the more analytic approach in [25] where only unital continuous maps are considered.

We now fix some notation. Let $\mathcal{A}$ be a complex unital Banach algebra. For each $a \in \mathcal{A}$ let $\sigma(a)$ denote the spectrum of a, and rad$A$ denotes the Jacobson radical of $\mathcal{A}$. If $\mathcal{S}$ is any subset of $\mathcal{A}$ define the commutant of $\mathcal{S}$ as follows: $\{\mathcal{S}\}' = \{x \in \mathcal{A} : xs = sx \text{ for all } s \in \mathcal{S}\}$. Let X be a Banach space. As usual, $\mathcal{B}(X)$ will denote the Banach algebra of all bounded linear operators from X into itself and $\mathcal{F}(X)$ the ideal of finite rank operators. Finally, let ϕ be any map, Imϕ will denote the range of ϕ.

2. A GENERALIZATION OF HUA'S THEOREM

In this section we give a generalization of Hua's theorem to the case of Banach algebras. More precisely, we shall examine the special situation when the map is not unital.

Let us start with a simple observation.

REMARK 2.1. The property "preserving strongly invertibility" is invariant by similarity. That is ϕ preserves strongly invertibility if and only if there is $b \in \mathcal{B}^{-1}$ such that the map $x \mapsto b\phi(x)b^{-1}$ preserves strongly invertibility if and only if for all $b \in \mathcal{B}^{-1}$ the map $x \mapsto b\phi(x)b^{-1}$ preserves strongly invertibility.

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. Then ϕ preserves strongly invertibility if and only if $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

Proof. The "if" part is obvious. We prove the "only if" part. Since $1^{-1} = 1$ and ϕ preserves strongly invertibility, we have $\phi(1) = (\phi(1))^{-1}$. That is $(\phi(1))^2 = 1$. Now, let x be an arbitrary element in $\mathcal{A}$ and let λ, μ be two rational numbers such that $\lambda, \mu \notin \sigma(x)$ and $\lambda \neq \mu$. Take $a = x - \lambda$ and $b = (\mu - \lambda)^{-1}$. Then $a - b^{-1} = x - \mu$ is invertible. On the other hand, $\phi(a) = \phi(x - \lambda), \phi(b) = (\mu - \lambda)^{-1}\phi(1)$ and $\phi(a) - (\phi(b))^{-1} = \phi(x - \mu)$ are invertible. It follows from (1.2) and the assumption on ϕ that

$$\phi((x - \lambda)^2) = (\phi(x) - \lambda\phi(1))\phi(1)(\phi(x) - \lambda\phi(1)),$$

thus

$$\phi(x^2) = \phi(x)\phi(1)\phi(x).$$

Therefore, for all $x \in \mathcal{A}$, $\phi(1)\phi(x^2) = (\phi(1)\phi(x))^2$. That is $\phi(1)\phi$ is a Jordan homomorphism.

Next, we will show that $\phi(1)$ commutes with the range of ϕ. Let $x \in \mathcal{A}$ and let $\lambda, \mu \in \mathbb{Q}$ such that $\mu \neq 0$ and $\lambda, \lambda + \mu \notin \sigma(x)$. Then $a = x - (\lambda + \mu)$, $b = (x - \lambda)^{-1}$, $a - b^{-1} = -\mu$ are invertible. On the other hand, $\phi(a), \phi(b), \phi(a) - \phi(b)^{-1} = -\mu\phi(1)$ are also invertible. Using (1.2) and the assumption on ϕ, we get that $\phi(aba) = \phi(a)\phi(b)\phi(a)$. It follows that $\phi((x - \lambda))^{-1} = \phi(1)\phi((x - \lambda))^{-1}\phi(1)$. Since $(\phi(1))^2 = 1$, it is easy to see that $\phi(1)\phi(x) = \phi(x)\phi(1)$ for all $x \in \mathcal{A}$. This completes the proof. ∎

For the special case of the complex matrix algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, we derive the following corollary that provides a more explicit form of linear maps preserving strongly invertibility.

COROLLARY 2.3. *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, be a linear map. Then ϕ preserves strongly invertibility if and only if there is $\lambda \in \{-1, 1\}$ such that ϕ takes one of the following forms:*

$$\phi(x) = \lambda axa^{-1} \quad or \quad \phi(x) = \lambda ax^{\mathrm{tr}}a^{-1},$$

for some invertible element $a \in \mathcal{M}_n(\mathbb{C})$. Here, x^{tr} denotes the transpose of a matrix x.

Proof. The "if" part is obvious. We prove the "only if" part. By Theorem 2.2, $\psi = \phi(1)\phi$ is a unital Jordan homomorphism. Now, by Lemma 1 in [29], ψ is injective. Hence ψ is a bijective Jordan homomorphism. Thus, ψ is an automorphism or an anti-automorphism. Clearly, $(\phi(1))^2 = 1$. Since $\phi(1) \in \{\mathrm{Im}(\phi)\}'$ and ϕ is surjective, it is easy to see that $\phi(1) \in \{-1, 1\}$. The conclusion follows from the classical result that every automorphism (respectively, anti-automorphism) of $\mathcal{M}_n(\mathbb{C})$ is inner. ∎

3. ADDITIVE MAPS PRESERVING STRONGLY GENERALIZED INVERSES

We start with the following remarks.

REMARK 3.1. The property "preserving strongly generalized invertibility" is invariant by similarity. That is ϕ preserves strongly generalized invertibility if and only if there is $b \in \mathcal{B}^{-1}$ such that the map $x \mapsto b\phi(x)b^{-1}$ preserves strongly generalized invertibility, and if and only if for all $b \in \mathcal{B}^{-1}$ the map $x \mapsto b\phi(x)b^{-1}$ preserves strongly generalized invertibility.

REMARK 3.2. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a map and let $x \in \mathcal{A}^{-1}$. Suppose that $\phi(x) \in \mathcal{B}^{-1}$. If ϕ preserves strongly generalized invertibility then

$$\phi(x^{-1}) = \phi(x)^{-1}. \tag{3.1}$$

The main results of this section are the following.

THEOREM 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital additive map. Then ϕ preserves strongly generalized inverses if and only if ϕ is a Jordan homomorphism.*

THEOREM 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map such that $1 \in \mathrm{Im}(\phi)$. Then the following conditions are equivalent:*
(i) *ϕ preserves strongly generalized invertibility;*
(ii) *$\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

THEOREM 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map such that $\phi(1)$ is invertible. Then the following conditions are equivalent:*
(i) *ϕ preserves strongly generalized invertibility;*
(ii) *$\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

For the proofs we need the following lemmas.

LEMMA 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. If ϕ preserves strongly generalized invertibility then*

$$\mathrm{Im}(\phi) \cap \mathcal{B}^{-1} \neq \emptyset \Rightarrow \phi(1) \neq 0.$$

Proof. Pick $u \in \mathcal{A}$ such that $\phi(u)$ is invertible and suppose that $\phi(1) = 0$. We distinguish two cases.

Step 1. Suppose that u is invertible. Take $a = u + \lambda$ and $b = \lambda^{-1}u$ where $\lambda \in \mathbb{Q}$ and $\lambda \neq 0$. Then $\phi(a) = \phi(u)$ and $\phi(b) = \lambda^{-1}\phi(u)$. Since u and $\phi(u)$ are invertible, for $|\lambda|$ sufficiently small, we have $a = u + \lambda$, b, $a - b^{-1} = u + \lambda(1 - u^{-1})$, $\phi(a) = \phi(u)$, $\phi(b)$ and $\phi(a) - \phi(b)^{-1} = \phi(u) - \lambda\phi(u)^{-1}$ are invertible. Therefore, by (1.2), $\phi(a - aba) = \phi(a) - \phi(a)\phi(b)\phi(a)$. It follows that $\phi(u) = 0$, a contradiction. Thus $\phi(1) \neq 0$.

Step 2. If u is not invertible, choose $\mu \in \mathbb{Q}$ such that $x = u + \mu$ is invertible. Then x and $\phi(x) = \phi(u)$ are invertible. This is Step 1. ∎

LEMMA 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. If ϕ preserves strongly generalized invertibility then*

$$\phi(1)^2 = 1 \iff \phi(1) \text{ is invertible } \iff \phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1} \neq \emptyset.$$

Moreover, if $\phi(x) \in \phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1}$ (i.e. x and $\phi(x)$ are invertible) then:

$$\phi(x) = \phi(1)\phi(x)\phi(1); \tag{3.2}$$

$$2\phi(x^2) = \phi(1)(\phi(x))^2 + (\phi(x))^2\phi(1). \tag{3.3}$$

Proof. It follows from our assumption that $(\phi(1))^3 = \phi(1)$. Thus, the first equivalence is obvious. We only need to show that $\phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1} \neq \emptyset$ implies that $\phi(1)$ is invertible. By Lemma 3.6 $\phi(1) \neq 0$. Now, let $u \in \mathcal{A}^{-1}$ such that $\phi(u)$ is invertible. Take $a = u + \lambda$ and $b = \lambda^{-1}u$, where $\lambda \in \mathbb{Q}$ and $\lambda \neq 0$. Then $a - b^{-1} = u + \lambda(1 - u^{-1}), \phi(a) = \phi(u) + \lambda\phi(1), \phi(b) = \lambda^{-1}\phi(u)$ and $\phi(a) - \phi(b)^{-1} = \phi(u) + \lambda(\phi(1) - \phi(u)^{-1})$. For $|\lambda|$ sufficiently small, $a, b, a - b^{-1}$, $\phi(a)$, $\phi(b)$ and $\phi(a) - \phi(b)^{-1}$ are invertible. Therefore, by (1.2), $\phi(a - aba) = \phi(a) -$

$\phi(a)\phi(b)\phi(a)$. It follows that

$$\phi(u) = \phi(1)\phi(u)\phi(1) \quad \text{and} \quad 2\phi(u^2) = \phi(1)(\phi(u))^2 + (\phi(u))^2\phi(1).$$

Thus

$$(\phi(u))^{-1}\phi(1)\phi(u)\phi(1) = 1 = \phi(1)\phi(u)\phi(1)(\phi(u))^{-1}.$$

Hence $\phi(1)$ is invertible. This completes the proof. ∎

With these results at hand, we are ready to prove our main results in this section.

Proof of Theorem 3.5. Suppose that (i) holds. Pick $x \in \mathcal{A}^{-1}$ such that $\phi(x) \in \mathcal{B}^{-1}$. Then by (3.2) and (3.3), $\phi(x) = \phi(1)\phi(x)\phi(1)$ and $2\phi(x^2) = \phi(1)(\phi(x))^2 + (\phi(x))^2\phi(1)$. Since $\phi(1)^2 = 1$, we have

$$\phi(1)\phi(x) = \phi(x)\phi(1) \quad \text{and} \quad \phi(1)\phi(x^2) = (\phi(1)\phi(x))^2. \tag{3.4}$$

Now, for any $x \in \mathcal{A}$ and $\lambda \in \mathbb{Q}$ such that $x + \lambda$ and $\phi(x+\lambda) = \phi(x) + \lambda\phi(1)$ are invertible, it follows from (3.4) that $\phi(1)\phi(x+\lambda) = \phi(x+\lambda)\phi(1)$. Thus, $\phi(1)\phi(x) = \phi(x)\phi(1)$. Consequently, $\phi(1)$ commutes with the range of ϕ. On the other hand, if we put $\psi(x) = \phi(1)\phi(x)$, then again from (3.4), we have $\psi((x+\lambda)^2) = (\psi(x+\lambda))^2$. It follows that $\psi(x^2) = (\psi(x))^2$ for every $x \in \mathcal{A}$ (i.e. $\phi(1)\phi$ is a Jordan homomorphism), and (ii) is proved.

The converse can be checked straightforwardly. ∎

Proof of Theorem 3.4. Suppose that $1 = \phi(u) \in \mathrm{Im}(\phi)$ and let λ be a non-zero rational number such that $x = u - \lambda$ is invertible and $\lambda^{-1} \notin \sigma(\phi(1)) \subseteq \{-1, 0, 1\}$ (since $(\phi(1))^3 = \phi(1)$). Then x and $\phi(x) = 1 - \lambda\phi(1) = -\lambda(\phi(1) - \lambda^{-1})$ are invertible . Hence, $\phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1} \neq \emptyset$. By Lemma 3.7, $\phi(1)$ is invertible. Now, the proof is completed by using Theorem 3.5. ∎

The proof of Theorem 3.3 is a direct consequence of Theorem 3.4 or Theorem 3.5.

We note in passing that Theorem 3.3 provides a positive answer to the question raised by the second author in Remark 2.3 of [25]. Moreover, by Lemma 3.7, the assumption in Theorem 3.5 can be weakened to $\phi(\mathcal{A}^{-1}) \cap \mathcal{B}^{-1} \neq \emptyset$. It would be a major advance if the conclusion of Theorem 3.5 remains true if the requirement $\phi(1) \in \mathcal{B}^{-1}$ is relaxed to Im ϕ contains an invertible element. Note that our arguments yield the following.

LEMMA 3.8. *Let $\mathcal{A}, \mathcal{B}$ be unital complex Banach algebras. Suppose that $\mathcal{A}$ (or $\mathcal{B}$) is finite-dimensional modulo the Jacobson radical and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map preserving strongly generalized inverses. Then*

$$\phi(1)^2 = 1 \iff \phi(1) \textit{ is invertible } \iff \mathrm{Im}\,\phi \cap \mathcal{B}^{-1} \neq \emptyset.$$

Next we treat our problems in the case of the algebra of bounded linear operators on a Hilbert space H.

THEOREM 3.9. *Let H be a complex Hilbert space and let $\mathcal{A}$ be a unital Banach algebra. If $\phi : \mathcal{B}(H) \to A$ is an additive map that preserves strongly generalized invertibility, then $\phi(1)\phi$ is a Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

For the proof we need the following two propositions.

PROPOSITION 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let ϕ be an additive map such that $\phi(1)$ commutes with the range of ϕ. If ϕ preserves strongly generalized invertibility then $\phi(1)\phi$ is a Jordan homomorphism.*

Proof. Consider the unital Banach algebra $B' = (\phi(1))^2 A(\phi(1))^2$ and let $\psi : A \to B'$ be the additive map defined by $\psi(a) = (\phi(1))^2\phi(a)$ for every $a \in A$. Clearly, ψ preserves strongly generalized invertibility and $\psi(1) = \phi(1)$ lies in $(B')^{-1}$. By Theorem 3.5, $\psi(1)\psi = \phi(1)\phi$ is a Jordan homomorphism. ∎

PROPOSITION 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. Suppose that ϕ preserves strongly generalized invertibility. Then for every idempotent $e \in A$,*

$$\phi(1)\phi(e) = \phi(e)\phi(1) = (\phi(e))^2 \quad \textit{and} \quad (\phi(1))^2\phi(e) = \phi(e).$$

In particular, $\phi(1)\phi$ preserves idempotents.

Proof. Let $e \in \mathcal{A}$ be an idempotent. The relations $(\phi(1-e))^3 = \phi(1-e)$ and $\phi(e)^3 = \phi(e)$ yield that

$$(\phi(e))^2\phi(1) + \phi(1)(\phi(e))^2 - \phi(1)\phi(e)\phi(1) + \phi(e)\phi(1)\phi(e) = (\phi(1))^2\phi(e) + \phi(e)(\phi(1))^2. \tag{3.5}$$

Next let $\lambda \in \mathbb{Q}$ such that $\lambda \notin \{0, -1\}$. Then $e + \lambda$ is invertible and $(e + \lambda)^{-1} = \frac{-1}{\lambda^2+\lambda}e + \frac{1}{\lambda}$. By our assumption on ϕ we infer that

$$(\phi(e) + \lambda\phi(1))\Big(\frac{-1}{\lambda^2+\lambda}\phi(e) + \frac{1}{\lambda}\phi(1)\Big)(\phi(e) + \lambda\phi(1)) = \phi(e) + \lambda\phi(1).$$

This implies that

$$\begin{aligned}-\lambda(\phi(1)(\phi(e))^2 + (\phi(e))^2\phi(1)) + (\lambda+1)\phi(e)\phi(1)\phi(e)\\ - \lambda^2\phi(1)\phi(e)\phi(1) + (\lambda^2+\lambda)((\phi(1))^2\phi(e) + \phi(e)(\phi(1))^2)\\ = (\lambda^2+\lambda+1)\phi(e).\end{aligned}$$

Applying (3.5), we deduce

$$\begin{aligned}\lambda^2(\phi(1)(\phi(e))^2 + (\phi(e))^2\phi(1)) + (\lambda^2+2\lambda+1)\phi(e)\phi(1)\phi(e) - (2\lambda^2+\lambda)\phi(1)\phi(e)\phi(1)\\ = (\lambda^2+\lambda+1)\phi(e).\end{aligned}$$

It follows that

$$\phi(e) = \phi(e)\phi(1)\phi(e), \quad \phi(1)\phi(e)\phi(1) = \phi(e),$$

and

$$\phi(1)(\phi(e))^2 + (\phi(e))^2\phi(1) = 2\phi(e).$$

Then

$$\phi(e)\phi(1)(\phi(e))^2 + \phi(e)\phi(1) = 2(\phi(e))^2.$$

Consequently,

$$(\phi(e))^2 + \phi(e)\phi(1) = 2(\phi(e))^2$$

and so $\phi(e)\phi(1) = (\phi(e))^2$. Analogously, $\phi(1)\phi(e) = (\phi(e))^2$. Thus

$$\phi(1)\phi(e) = (\phi(e))^2 = \phi(e)\phi(1). \qquad \blacksquare$$

Proof of Theorem 3.9. By Proposition 3.11, $\phi(1)\phi(e)=\phi(e)\phi(1)$ for each idempotent e. Since every bounded operator on H can be expressed as finite linear combination of idempotents [32], we infer that $\phi(1)$ commutes with the range of ϕ. Now the result follows from Proposition 3.10 $\blacksquare$

REMARK 3.12. With the same hypotheses as in Theorem 3.9, if the Hilbert space $\mathcal{H}$ is separable, then $\mathrm{Im}(\phi) \cap \mathcal{A}^{-1} \neq \emptyset$ implies that $\phi(1)$ is invertible. Indeed, by Lemma 3.7 and Theorem 3.5, we may assume that for every invertible operator T, $\phi(T)$ is not invertible. Pick $T \in B(H)$ such that $\phi(T) \in \mathcal{A}^{-1}$. If T is not semi-Fredholm, then by [8], there exist two quasi-nilpotent operators T_1, T_2 such that $T = T_1T_2$. If $\lambda, \beta \in \mathbb{Q}$ are sufficiently small and nonzero, the operator $U = T + \lambda T_1 + \beta T_2 + \lambda\beta$ is invertible and $\phi(U)$ is invertible, contradicting our assumption. Now if T is semi-Fredholm, without loss of generality, we may suppose that T is left invertible. Since Hua's identity still holds for left invertible elements, we proceed analogously to the proof of Lemma 3.7 to get the desired conclusion.

For the special case of linear maps over the complex matrix algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, we derive the following corollary that provides a more explicit form of the linear maps preserving strongly generalized invertibility.

COROLLARY 3.13. *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, be a linear map. Then ϕ preserves strongly generalized inverses if and only if either $\phi = 0$ or there is $\lambda \in \{-1, 1\}$ such that ϕ takes one of the following forms:*

$$\phi(x) = \lambda axa^{-1} \quad \text{or} \quad \phi(x) = \lambda ax^{\mathrm{tr}}a^{-1},$$

for some invertible element $a \in \mathcal{M}_n(\mathbb{C})$. Here, x^{tr} denotes the transpose of a matrix x.

Proof. The "if" part is obvious. We prove the "only if" part. By Proposition 3.11, $\psi = \phi(1)\phi$ preserves idempotents. According to Theorem 2.1 in [5] ψ is a Jordan homomorphism. Now, by Lemma 1 in [29], $\psi = 0$ or ψ is injective. Assume that $\psi \neq 0$, then ψ is a bijective Jordan homomorphism. Thus, ψ is an automorphism or an anti-automorphism. Therefore $\psi(1) = 1$ and thus, $(\phi(1))^2 = 1$. Now it is easy to see that $\phi(1) \in \{-1, 1\}$. The conclusion follows from the classical result that every automorphism (respectively, anti-automorphism) of $\mathcal{M}_n(\mathbb{C})$ is inner. $\blacksquare$

As a consequence of the above corollary, we have the following result.

COROLLARY 3.14. *Let $\mathcal{A}$ be a complex unital Banach algebra and suppose that $\mathcal{A}/\mathrm{rad}\mathcal{A}$ is simple and finite-dimensional. If $\phi : \mathcal{A} \to \mathcal{A}$ is a linear map preserving strongly generalized inverses, then either $\phi(1) \in \{1, -1\}$ and $\phi(1)\phi$ is a Jordan homomorphism, or $\phi(1) = 0$, and ϕ has finite range. Moreover, if A is finite-dimensional, then $\phi = 0$ or $\phi(1)$ is invertible.*

Proof. By [10], $\mathcal{A}$ is the (vector space) topological direct sum of its radical and a closed subalgebra $\mathcal{S}$ of $\mathcal{A}$, isomorphic to $\mathcal{A}/\mathrm{rad}\mathcal{A}$. Thus $\mathcal{S}$ is isomorphic to a matrix algebra $\mathcal{M}_n(\mathbb{C})$, for some nonzero integer n. It is easy to check that $1 \in \mathcal{S}$. Let $\pi : \mathcal{A} \to \mathcal{S}$ be the natural surjection, and denote by ϕ' the restriction of $\pi \circ \phi$ to $\mathcal{S}$. Clearly, ϕ' preserves strongly generalized inverses. Assume first that $\phi(1)$ is invertible. It follows from Theorem 3.5 that $\phi(1)^2 = 1$ and $\phi(1)\phi$ is a Jordan homomorphism. Since $\phi'(1)$ is also invertible, Corollary 3.13 implies that $\phi'(1) \in \{1, -1\}$. Now we check easily that $\phi(1) \in \{1, -1\}$. Next suppose that $\phi(1)$ is not invertible. Then $\phi'(1)$ is not invertible. Corollary 3.13 yields that $\phi' = 0$. Pick $r \in \mathrm{rad}\mathcal{A}$. Then $\phi(r) = \phi(1 + r) \in \mathcal{A}^{\wedge}$, thus $\phi(r) \notin \mathrm{rad}\mathcal{A}$. Since $\mathcal{S}$ has finite-dimension, $\mathrm{rad}\mathcal{A}$ can be seen as a vector space direct sum of two vector spaces H_1 and H_2, where H_1 has finite dimension and H_2 is contained in the kernel of ϕ. As a result, ϕ has finite range.

Now suppose that $\mathrm{rad}\mathcal{A}$ has finite dimension and let $r \in \mathrm{rad}\mathcal{A}$. Then r is nilpotent. Applying the fact that $1 - \lambda r$ is invertible with inverse $\Sigma\lambda^n r^n$, for sufficiently small numbers λ, we deduce that $\phi(\Sigma\lambda^n r^n)$ is a generalized inverse of $-\lambda\phi(r)$. Thus $\phi(r) = 0$. ∎

REMARK 3.15. Recall that every surjective Jordan homomorphism of a prime algebra is a homomorphism or an anti-homomorphism [14]. Since algebras of bounded operators on complex Banach spaces are prime and central, we infer that if X and Y are two Banach spaces and $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective additive map, then ϕ preserves strongly generalized invertibility if and only if there is $\lambda \in \{-1, 1\}$ such that $\lambda\phi$ is a unital homomorphism or anti-homomorphism. Moreover, in the context of separable Hilbert spaces, using Theorem 6.1 in [34] and Theorem 3.9, we can get the following result.

COROLLARY 3.16. *Let H be a separable Hilbert space and Y a Banach space. Let $\phi : \mathcal{B}(H) \to \mathcal{B}(Y)$ be a surjective linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*

(ii) *there is an isomorphism A from H onto Y and $\lambda \in \{-1, 1\}$ such that ϕ takes one of the following forms:*

$$\phi(T) = \lambda ATA^{-1} \quad \text{or} \quad \phi(T) = \lambda AT^{\mathrm{tr}}A^{-1},$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of H.

In the context of Banach spaces, it turns out that the surjectivity assumption is not enough to conclude that ϕ is bijective, as shown by the following example. We would like to thank M. Gonzalez for helpful discussions concerning the following example.

EXAMPLE 3.17. Let $\mathcal{J}$ be the quasi-reflexive James space over the complex field (see [18]). It is well known that the weakly compact operators on $\mathcal{B}(\mathcal{J})$ form a closed one-codimensional ideal in $\mathcal{B}(\mathcal{J})$. Thus every operator T in $\mathcal{B}(\mathcal{J})$ admits a decomposition $T = \lambda_T I + S_T$, where $\lambda_T \in \mathbb{C}$ and S_T is weakly compact. Now consider the map $\phi : B(\mathcal{J}) \to \mathbb{C}$, given by $\phi(T) = \lambda_T$ then ϕ is a continuous surjective homomorphism which is not injective.

REMARK 3.18. Let us point out that, in addition, this seems to be the first example where ϕ annihilates the finite rank operators (see Remark 4 after Theorem 3.4 in [34]).

If we suppose in addition that ϕ does not annihilate all finite rank operators then using Theorem 3.4 and Theorem 3.4 in [34], we obtain the following.

COROLLARY 3.19. *Let X and Y be Banach spaces and let $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective linear map. Suppose that $\phi(\mathcal{F}(X)) \neq \{0\}$. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*

(ii) *there is $\lambda \in \{-1, 1\}$ such that either*

(a) *there is an isomorphism A from X onto Y such that*

$$\phi(T) = \lambda A T A^{-1} \quad \text{for all } T \in \mathcal{B}(X), \text{ or}$$

(b) *there is an isomorphism B from Y onto X^* such that*

$$\phi(T) = \lambda B^{-1} T^* B \quad \text{for all } T \in \mathcal{B}(X).$$

In this case X and Y must be reflexive.

4. FINAL REMARKS

Let $\mathcal{A}$ be a unital algebra. An element $b \in \mathcal{A}$ is called the *Drazin inverse* of $a \in \mathcal{A}$, if b is a solution of the following equations

$$ab = ba, \quad bab = b \quad \text{and} \quad a^k ba = a^k \quad \text{for some positive integer } k. \tag{4.1}$$

When $k = 1$, then we say that b is the *group inverse* of a. Notice that contrary to the generalized inverse, the Drazin inverse, and consequently the group inverse, is unique whenever it exists. Let $\mathcal{A}^{\mathrm{D}}$ (respectively $\mathcal{A}^{\mathrm{G}}$) denote the set of all the elements of $\mathcal{A}$ having a Drazin inverse (respectively group inverse) and let a^{D} (respectively a^{G}) denote the Drazin inverse (respectively group inverse) of a for any $a \in \mathcal{A}^{\mathrm{D}}$ (respectively $\mathcal{A}^{\mathrm{G}}$). We have the following inclusions

$$\mathcal{A}^{-1} \subseteq \mathcal{A}^{\mathrm{G}} \subseteq \mathcal{A}^{\wedge} \cap \mathcal{A}^{\mathrm{D}}.$$

We will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$
- *preserves strongly group invertibility* if $\phi(x^{\mathrm{G}}) = \phi(x)^{\mathrm{G}}$ for all $x \in \mathcal{A}^{\mathrm{G}}$;
- *preserves strongly Drazin invertibility* if $\phi(x^{\mathrm{D}}) = \phi(x)^{\mathrm{D}}$ for all $x \in \mathcal{A}^{\mathrm{D}}$.

REMARK 4.1. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a map and let $x \in \mathcal{A}^{-1}$ such that $\phi(x) \in \mathcal{B}^{-1}$. If ϕ preserves strongly group invertibility (respectively Drazin invertibility) then

$$\phi(x^{-1}) = \phi(x)^{-1}. \tag{4.2}$$

In [25], it is proved that every Jordan homomorphism preserves strongly the Drazin (respectively group) invertibility. Conversely, according to (4.2), the same arguments used in the proof of Theorems 3.3, 3.4, 3.5, yield the following result.

THEOREM 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. If ϕ is unital (respectively $\phi(1)$ is invertible, $1 \in Im(\phi)$), then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*
(ii) *ϕ preserves strongly group invertibility;*
(iii) *ϕ preserves strongly Drazin invertibility;*
(iv) *ϕ (respectively $\phi(1)\phi$) is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

As an application in the context of the Banach algebra of bounded linear operator on a complex Banach space, we derive the following result.

COROLLARY 4.3. *Let X and Y be Banach spaces and let $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a bijective linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*
(ii) *ϕ preserves strongly group invertibility;*
(iii) *ϕ preserves strongly Drazin invertibility;*
(iv) *ϕ preserves strongly invertibility;*
(v) *there is $\lambda \in \{-1, 1\}$ such that either*
  (a) *there is an isomorphism A from X onto Y such that*

$$\phi(T) = \lambda A T A^{-1} \quad \text{for all } T \in \mathcal{B}(X), \text{ or}$$

  (b) *there is an isomorphism B from Y onto X^* such that*

$$\phi(T) = \lambda B^{-1} T^* B \quad \text{for all } T \in \mathcal{B}(X).$$

In this case X and Y must be reflexive.

REMARK 4.4. We thank the referee for drawing our attention to spectral algebras (of Palmer [31]), or equivalently, m-convex Q-algebras. As noted by the referee, a careful reading of the proofs of Theorems 2.2, 3.3, 3.4, 3.5 and 4.2 show that our methods still work for these algebras. Recall that a unital topological algebra $\mathcal{A}$ is said to be locally multiplicatively convex (briefly, m-convex) if there is a basis of neighborhoods of the origin consisting of sets which are absolutely convex and multiplicative. If moreover the set of invertible elements of $\mathcal{A}$ is open, then we say that $\mathcal{A}$ is a Q-algebra (see for instance [11], [22]). Recall also that if $\mathcal{A}$ is an m-convex Q-algebra, then the spectrum $\sigma(a)$ is non-empty and compact for each $a \in \mathcal{A}$. As shown in Proposition 6.20 and Theorem 6.18 in [11], for every m-convex Q-algebra $\mathcal{A}$ there exists a multiplicative semi-norm p (belonging to the family of semi-norms defining the topology of $\mathcal{A}$) such that $r(a) \leqslant p(a)$ for every $a \in \mathcal{A}$ (here, $r(a)$ denotes the spectral radius of a).

To emphasize how different the study of maps preserving strongly inverses can be from that of maps preserving strongly generalized inverses, we quote the following simple examples.

EXAMPLES 4.5. Let $\mathcal{A} = \mathcal{B}(H)$, and $\mathcal{B} = \mathcal{B}(H \oplus H)$ where H is a Hilbert space. Let $\phi_i : \mathcal{A} \to \mathcal{B} i = 1, 2$, be defined by

$$\phi_1(T) = 0 \oplus T \quad \text{and} \quad \phi_2(T) = 0 \oplus (-T), \quad T \in \mathcal{A}.$$

Then $\phi_i, i = 1, 2$, are linear maps preserving strongly generalized invertibility. It is easily verified that $\mathrm{Im}(\phi_i) \cap \mathcal{B}^{-1} = \emptyset$ and $\phi_i(1) \neq 0$. Further, ϕ_1 is a Jordan homomorphism, but ϕ_2 is not a Jordan homomorphism.

We conclude this paper by stating a conjecture which arises in a natural way from our results.

CONJECTURE 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. If ϕ preserves strongly generalized invertibility (respectively group invertibility, Drazin invertibility) then $\phi(1)\phi$ is a Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

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