J. OPERATOR THEORY
64:1(2010), 131–147

ON HONESTY OF PERTURBED SUBSTOCHASTIC C_0-SEMIGROUPS IN L^1-SPACES

MUSTAPHA MOKHTAR-KHARROUBI and JÜRGEN VOIGT

Communicated by Nikolai K. Nikolski

ABSTRACT. Let T be the generator of a positive contraction semigroup on $L^1(\Omega,\mu)$, and let $B\colon D(T)\to L^1(\Omega,\mu)$ be a positive linear operator such that $\int(Tf+Bf)\leqslant 0$ for all $f\in D(T)_+$. It is known that there exists a minimal positive contraction semigroup generated by some operator $K\supseteq T+B$. This paper deals mainly with the total mass carried by trajectories $(\mathrm{e}^{tK}f;t\geqslant 0)$ with non-negative initial data f. In particular, our analysis covers the problem of whether $K=\overline{T+B}$ or $K\supsetneqq\overline{T+B}$ and the related problem of the stochasticity or lack of stochasticity of "formally conservative" perturbed positive semigroups in L^1-spaces.

KEYWORDS: C_0*-semigroup, perturbation, substochastic, honest trajectory.*

MSC (2000): 47D06, 47D07, 35F10.

INTRODUCTION

Let (Ω,μ) be a measure space and $X:=L^1(\Omega,\mu)$. A substochastic operator A on X is a positive contraction on X. The operator A is called stochastic if additionally A is norm-preserving on the positive cone X_+ (consisting of the non-negative functions of X). Let $(U(t);t\geqslant 0)$ be a substochastic C_0-semigroup on X (i.e., $U(t)$ is substochastic for all $t\geqslant 0$) with generator T, and let $B\colon D(T)\to X$ be such that

$$Bf\in X_+\quad\text{for all } f\in D(T)_+:=D(T)\cap X_+$$

and

$$\int_\Omega (Tf+Bf)\mathrm{d}\mu\leqslant 0\quad\text{for all } f\in D(T)_+.$$

Then there exists a minimal substochastic semigroup $(V(t);t\geqslant 0)$ generated by an extension K of $T+B$; in particular, $T+B$ is closable.

This result goes back to the seminal paper by T. Kato [16] on Kolmogorov's differential equations, revisited later ([23], [1], [2], [6]) in view of transport theory.

More recently, essentially from the beginning of the 2000's, this abstract framework found a new life in the context of kinetic theory of dilute gases, semiconductor theory, fragmentation equations, birth-and-death problems etc. ([3], [4], [5], [6], [7], [8], [10], [11], [15]); we refer to the monograph [9] for more information.

By the Miyadera perturbation theorem, for each $r \in [0,1)$, the operator $T + rB$ generates a substochastic semigroup $(U_r(t);\, t \geqslant 0)$ on X such that $U_r(t) \leqslant U_{r'}(t)$ (in the lattice sense) for $r < r' < 1$, and then $(V(t);\, t \geqslant 0)$ is obtained as a strong limit of $(U_r(t);\, t \geqslant 0)$ as $r \to 1-$. (We refer to [23] for this procedure, to [20] for more recent developments, and to [21] and Section III.3.b of [13] for the Miyadera perturbation theorem.)

The discussion whether $D(T)$ is a core for K is an important issue of the theory. In particular, in the case

$$\int_{\Omega} (Tf + Bf)\mathrm{d}\mu = 0 \quad \text{for all } f \in D(T) \tag{0.1}$$

which we refer to as the "conservative" case, $(V(t);\, t \geqslant 0)$ is norm-preserving on the positive cone if and only if the closure property $K = \overline{T+B}$ holds (cf. Remark 1.7(iii) in combination with Remark 2.3(iii)). Thus, an expected property of the model fails to hold if $K \supsetneq \overline{T+B}$, i.e., a mass loss occurs. The closure property was investigated by different techniques in [16], [23], [9]. More generally, intuition suggests that the functional

$$c\colon f \in D(T) \mapsto -\int (T+B)f \,\mathrm{d}\mu \tag{0.2}$$

should be responsible for the mass loss along trajectories. This is because, for $f \in D(K)_+$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)f\| = \int KV(t)f \,\mathrm{d}\mu \quad (t \geqslant 0),$$

and the latter is equal to $-c(V(t)f)$ if $V(t)f \in D(T)$. We will define a "minimal" extension of c to $D(K)$ (again denoted by c), and a trajectory will considered to be "honest" (i.e., "well-behaved") if $c(V(t)f) = -\int KV(t)f \,\mathrm{d}\mu$ holds for all $t \geqslant 0$; cf. Section 2 for the precise definition.

In this work, we improve and extend recent work on the subject ([3] and Chapter 6 of [9]). Our paper is an improved version of [18].

In Section 1 we analyse the functional c defined in (0.2) and two distinguished extensions $c \leqslant \widehat{c}$ of this functional to $D(K)$. The main result of this section is that an element $f \in D(K)_+$ belongs to $D(\overline{T+B})$ if and only if $c(f) = \widehat{c}(f)$ (Proposition 1.6). In [3] and Chapter 6 of [9], related results are shown for the case that c is of a special form; cf. Remark 1.2.

In Section 2 we introduce and investigate the concept of honesty of trajectories $(\mathrm{e}^{tK}f;\, t \geqslant 0)$ on subintervals of $[0,\infty)$, for $f \in L^1(\mu)_+$. This is in the spirit of Section 6.2 of [9]; cf. in particular Remark 6.16 of [9]. A fundamental fact observed in this section is the result that the set of initial values giving rise to an honest trajectory is the positive cone of a closed invariant ideal; cf. Remark 2.1(iii) and Proposition 2.4. In addition, this section contains characterisations and sufficient conditions for honesty.

In Section 3 we present some examples illustrating the notions introduced previously. In particular it becomes clear from these examples that, in general, not much can be said concerning the band disjoint to the functions giving rise to honest trajectories.

1. ON SOME FUNCTIONALS

In the remainder of the paper, we assume that the measure space (Ω, μ), the operators T, B, K, and the C_0-semigroups $(U(t); t \geqslant 0)$ and $(V(t); t \geqslant 0)$ are as in the introduction. Unless otherwise stated, the various integrals appearing in this paper refer to the measure μ.

The substochasticity of $(V(t);\ t \geqslant 0)$ implies

$$\int Kf \leqslant 0 \quad (f \in D(K)_+ = D(K) \cap X_+). \tag{1.1}$$

Define the functional

$$\widehat{c}\colon f \in D(K) \mapsto -\int Kf \in \mathbb{R}.$$

Due to (1.1), this functional is positive, i.e., $\widehat{c}(f) \geqslant 0$ for all $f \in D(K)_+$. We note that the definition of $\widehat{c}$ implies that $\widehat{c}\colon D_K \to \mathbb{R}$ is a continuous linear functional, where we denote by D_K the domain of K, equipped with the graph norm. We denote by c the restriction of $\widehat{c}$ to $D(T)$,

$$c\colon f \in D(T) \mapsto \widehat{c}(f) = -\int (T+B)f.$$

Let $\lambda > 0$. The strong and monotone convergence of the C_0-semigroups U_r to V, as $r \to 1-$, implies the strong and monotone convergence of the resolvents, $(\lambda - T - rB)^{-1} \to (\lambda - K)^{-1}$ $(r \to 1-)$, which in turn implies the existence of

$$c_\lambda((\lambda - K)^{-1}f) := \lim_{r \to 1-} c((\lambda - T - rB)^{-1}f)$$

for all $f \in X_+$, and therefore also for all $f \in X = X_+ - X_+$.

In order to get another expression for c_λ we note the identity

$$(\lambda - T - rB)^{-1} = \sum_{n=0}^{\infty} r^n (\lambda - T)^{-1} (B(\lambda - T)^{-1})^n, \tag{1.2}$$

valid for $r \in [0, 1)$ because of $\|B(\lambda - T)^{-1}\| \leqslant 1$. The series is convergent in D_T, and because of $(\lambda - T)^{-1} \leqslant (\lambda - K)^{-1}$ the embedding $D_T \hookrightarrow D_K$ is continuous.

This implies

$$c((\lambda - T - rB)^{-1}f) = \sum_{n=0}^{\infty} r^n c((\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f)$$

for all $f \in X$. For $r \to 1-$ we obtain

$$c_\lambda((\lambda - K)^{-1}f) = \sum_{n=0}^{\infty} c((\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f),$$

first for $f \in X_+$, but again this carries over to all $f \in X = X_+ - X_+$.

By construction

$$0 \leqslant c_\lambda((\lambda - K)^{-1}f) \leqslant \widehat{c}((\lambda - K)^{-1}f) \quad (f \in X_+). \tag{1.3}$$

PROPOSITION 1.1. *Let* $0 < \lambda < \mu$. *Then:*

(i) $c_\lambda\big|_{D(T)} = c$.

(ii) $c_\mu = c_\lambda$.

(iii) *The extension of* c *to* $D(K)$, *defined by* $c := c_\lambda$ *is continuous with respect to the graph norm of* K, $c\big|_{D(\overline{T+B})} = \widehat{c}\big|_{D(\overline{T+B})}$, *and* $0 \leqslant c(f) \leqslant \widehat{c}(f)$ *for all* $f \in D(K)_+$.

Proof. (i) Let $f \in X$, $g := (\lambda - K)^{-1}f \in D(T)$. Then

$$\begin{aligned}
c_\lambda(g) &= \sum_{n=0}^{\infty} c((\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f) \\
&= \sum_{n=0}^{\infty} c(((\lambda - T)^{-1}B)^n(\lambda - T)^{-1}(\lambda - T - B)g) \\
&= \sum_{n=0}^{\infty} c(((\lambda - T)^{-1}B)^n g) - c_\lambda((\lambda - K)^{-1}Bg) \\
&= c(g) + c_\lambda((\lambda - K)^{-1}Bg) - c_\lambda((\lambda - K)^{-1}Bg) = c(g).
\end{aligned}$$

(ii) Let $f, g \in X$, $f \geqslant 0$, $(\lambda - K)^{-1}f = (\mu - K)^{-1}g$. Then

$$g = (\mu - K)(\lambda - K)^{-1}f = f + (\mu - \lambda)(\lambda - K)^{-1}f \geqslant f \geqslant 0.$$

We obtain (using the resolvent equation in the second equality)

$$\begin{aligned}
&c_\mu((\mu - K)^{-1}g) - c_\lambda((\lambda - K)^{-1}f) \\
&= \lim_{r\to 1-} c((\mu - T - rB)^{-1}g - (\lambda - T - rB)^{-1}f) \\
&= \lim_{r\to 1-} c((\mu-T-rB)^{-1}((f+(\mu-\lambda)(\lambda-K)^{-1}f)-(I+(\mu-\lambda)(\lambda-T-rB)^{-1})f)) \\
&= (\mu - \lambda) \lim_{r\to 1-} c((\mu - T - rB)^{-1}((\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f)).
\end{aligned}$$

In the last expression we know that $(\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f \geqslant 0$ for all $r \in [0, 1)$, and therefore $c_\mu((\mu - K)^{-1}g) - c_\lambda((\lambda - K)^{-1}f) \geqslant 0$. In order to estimate from above we note that

$$c((\mu - T - rB)^{-1}((\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f)) \\ \leqslant \widehat{c}((\mu - K)^{-1}((\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f))$$

for all $r \in [0,1)$. Further $(\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f \to 0$, and therefore $(\mu - K)^{-1}((\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f) \to 0$ in D_K, which implies

$$\lim_{r\to 1-} \widehat{c}((\mu - K)^{-1}((\lambda - K)^{-1}f - (\lambda - T - rB)^{-1}f)) = 0.$$

(iii) For $f \in D(K)$ we estimate

$$\begin{aligned}|c(f)| &\leqslant c_\lambda((\lambda - K)^{-1}|(\lambda - K)f|) \\ &\leqslant \widehat{c}((\lambda - K)^{-1}|(\lambda - K)f|) = -\int K(\lambda - K)^{-1}|(\lambda - K)f| \\ &= \|(\lambda - K)f\| - \lambda\|(\lambda - K)^{-1}|(\lambda - K)f|\| \leqslant \lambda\|f\| + \|Kf\| - \lambda\|f\| = \|Kf\|,\end{aligned}$$

which shows the continuity of c with respect to the graph norm of K.

The functionals c and $\widehat{c}$ coincide on $D(T)$, are continuous with respect to the graph norm of K, and $D(\overline{T+B})$ is the closure of $D(T)$ in D_K. Therefore c and $\widehat{c}$ coincide on $D(\overline{T+B})$.

Let $f \in D(K)_+$. Inequality (1.3) implies that $0 \leqslant \overline{c}(\lambda(\lambda - K)^{-1}f) \leqslant \widehat{c}(\lambda(\lambda - K)^{-1}f)$ for all $\lambda > 0$. Since $\lim_{\lambda\to\infty} \lambda(\lambda - K)^{-1}f = f$ in D_K we obtain $0 \leqslant c(f) \leqslant \widehat{c}(f)$. ∎

In the following remark we establish the connection between our definitions and the construction proposed in Sections 6.2 and 10.4 of [9].

REMARK 1.2. Assume that there exist a measure space (Λ, ν) and a positive linear operator $Z\colon D(T) \to L^1(\nu)$ such that

$$-\int_\Omega (T + B)f \,\mathrm{d}\mu = \int_\Lambda Zf \,\mathrm{d}\nu \quad (f \in D(T)).$$

(This assumption should be regarded as an interpretation of the somewhat imprecisely formulated hypothesis in relation (6.8) of [9].) Then the definition

$$Z(((\lambda - K)^{-1})f) := \lim_{r\to 1-} Z((\lambda - T - rB)^{-1}f) := \sum_{n=0}^{\infty} Z((\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f) \tag{1.4}$$

for $f \in X$, $\lambda > 0$, yields a (well-defined) positive K-bounded extension of Z to $D(K)$. The existence of the limits in (1.4), for $f \in X_+$, is a consequence of the monotone convergence theorem. The proof of the other properties of the extension Z is analogous to the proof of Proposition 1.1. It then follows that

$$c(f) = \int_\Lambda Zf \,\mathrm{d}\nu \quad (f \in D(K)).$$

Assume now that $\widetilde{Z}\colon D(K) \to L^1(\nu)$ is an extension of $Z\big|_{D(T)}$ "respecting monotone convergence", i.e., whenever $(f_n) \subseteq D(K)$ is a monotone sequence converging in X to $f \in D(K)$, then $\widetilde{Z}f_n \to \widetilde{Z}f$ in $L^1(\nu)$. Then (1.4) implies $\widetilde{Z} = Z$. (The requirement of such a property for $\widetilde{Z}$ appears in Remark 6.2 of [9] and is used in the proof of Theorem 6.8 of [9].)

We note that there may be other K-bounded positive extensions of $Z\big|_{D(T)}$; cf. Example 3.1.

REMARK 1.3. Letting $r \to 1-$ in (1.2) one obtains

$$(\lambda - K)^{-1} = \sum_{n=0}^{\infty} (\lambda - T)^{-1}(B(\lambda - T)^{-1})^n,$$

with strong and monotone convergence of the series (cf. [2], [6]).

For $\lambda > 0$ we define the functional $\beta_\lambda \in X'_+$ by

$$\beta_\lambda(f) := \widehat{c}((\lambda - K)^{-1}f) - c((\lambda - K)^{-1}f) \quad (f \in X).$$

The following lemma can be considered as a variant of Theorem 6.8, p. 163 of [9]. The existence and importance of the first expression in the following lemma was already recognised and exploited in Kato's paper [16].

LEMMA 1.4. *Let $\lambda > 0$, $f \in X$. Then*

$$\beta_\lambda(f) = \lim_{n\to\infty} \int (B(\lambda - T)^{-1})^n f = \lim_{r\to 1-} (1 - r) \int B(\lambda - T - rB)^{-1} f.$$

Proof. We compute

$$\begin{aligned} c((\lambda - K)^{-1}f) &= \sum_{n=0}^{\infty} c((\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f) \\ &= \sum_{n=0}^{\infty} \int (\lambda - T - B - \lambda)(\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f \\ &= \sum_{n=0}^{\infty} \Big(\int (B(\lambda - T)^{-1})^n f - \int (B(\lambda - T)^{-1})^{n+1} f \Big) \\ &\qquad - \lambda \int \sum_{n=0}^{\infty} (\lambda - T)^{-1}(B(\lambda - T)^{-1})^n f \\ &= \int f - \lim_{n\to\infty} \int (B(\lambda - T)^{-1})^n f - \lambda \int (\lambda - K)^{-1} f \\ &= \widehat{c}((\lambda - K)^{-1}f) - \lim_{n\to\infty} \int (B(\lambda - T)^{-1})^n f, \end{aligned}$$

and

$$\begin{aligned} c((\lambda - K)^{-1}f) &= \lim_{r\to 1-} \int (\lambda - T - rB - \lambda - (1 - r)B)(\lambda - T - rB)^{-1} f \\ &= \int f - \lambda \int (\lambda - K)^{-1} f - \lim_{r\to 1-} (1 - r) \int B(\lambda - T - rB)^{-1} f \end{aligned}$$

$$= \widehat{c}((\lambda - K)^{-1} f) - \lim_{r \to 1-} (1 - r) \int B(\lambda - T - rB)^{-1} f. \quad \blacksquare$$

It is known ([15]) that β_λ is an eigenvector of $(B(\lambda - T)^{-1})'$ associated with the eigenvalue 1,

$$(B(\lambda - T)^{-1})' \beta_\lambda = \beta_\lambda.$$

We give a proof of this property which provides us with the additional information that β_λ is the maximal nonnegative eigenvector $\leqslant 1$ associated with the eigenvalue 1.

COROLLARY 1.5. *If $\beta_\lambda \neq 0$, then 1 is an eigenvalue of $(B(\lambda - T)^{-1})'$, and β_λ is the maximal eigenvector $\leqslant 1$.*

Proof. Lemma 1.4 implies that, for all $f \in X$,

$$\langle [(B(\lambda - T)^{-1})']^n 1, f \rangle = \int (B(\lambda - T)^{-1})^n f \to \beta_\lambda(f)$$

$(n \to \infty)$, i.e., the sequence $([(B(\lambda - T)^{-1})']^n 1)_n$ is weak*-convergent to β_λ in $L^1(\mu)'_+$. The equation

$$(B(\lambda - T)^{-1})' [(B(\lambda - T)^{-1})']^n 1 = [(B(\lambda - T)^{-1})']^{n+1} 1$$

together with the weak*-continuity of $(B(\lambda - T)^{-1})'$ implies that β_λ is an eigenvector of $(B(\lambda - T)^{-1})'$ for the eigenvalue 1.

Let $\psi \in L^1(\mu)'_+$, $\psi \leqslant 1$, be such that $(B(\lambda - T)^{-1})' \psi = \psi$. Then

$$\psi = [(B(\lambda - T)^{-1})']^n \psi \leqslant [(B(\lambda - T)^{-1})']^n 1$$

$(n \in \mathbb{N})$, and thus $\psi \leqslant \beta_\lambda$. $\blacksquare$

The following result is an improvement of one of the assertions in Proposition 1.1(iii).

PROPOSITION 1.6. *Let $f \in D(K)_+$. Then one has $c(f) = \widehat{c}(f)$ if and only if $f \in D(\overline{T + B})$.*

Proof. In view of Proposition 1.1(iii), we only have to show necessity. Thus, let $f \in D(K)_+$, $(\widehat{c} - \overline{c})(f) = 0$. Then $f_\lambda := \lambda(\lambda - K)^{-1} f \to f$ $(\lambda \to \infty)$ in D_K, and this implies that $\lambda \beta_\lambda(f) = (\widehat{c} - \overline{c})(f_\lambda) \to (\widehat{c} - c)(f) = 0$.

Let $\varepsilon > 0$. There exists $\lambda > 0$ such that $\|f_\lambda - f\| \leqslant \varepsilon$ and $\|K f_\lambda - K f\| \leqslant \varepsilon$, i.e., $(f_\lambda, K f_\lambda)$ is ε-close to (f, Kf) in $X \times X$, and such that $\lambda \beta_\lambda(f) < \varepsilon$.

We define

$$u_n := \sum_{j=0}^{n-1} (\lambda - T)^{-1} (B(\lambda - T)^{-1})^j (\lambda f).$$

Then $u_n \in D(T) = D(T + B)$ $(n \in \mathbb{N})$,

$$u_n \to (\lambda - K)^{-1}(\lambda f) = f_\lambda \quad (n \to \infty),$$

and therefore $(\lambda \vee 1)\|u_n - f_\lambda\| \leqslant \varepsilon$ for large n. Moreover

$$(\lambda - T - B)u_n = \lambda f - (B(\lambda - T)^{-1})^n(\lambda f) = (\lambda - K)f_\lambda - \lambda(B(\lambda - T)^{-1})^n f,$$

and Lemma 1.4 implies that

$$\|(\lambda - T - B)u_n - (\lambda - K)f_\lambda\| = \lambda\|(B(\lambda - T)^{-1})^n f\| \leqslant \varepsilon$$

and hence $\|(T + B)u_n - Kf_\lambda\| \leqslant 2\varepsilon$, for large n. This shows that (f_λ, Kf_λ) is 2ε-close to the graph of $T + B$, and hence (f, Kf) is 3ε-close to the graph of $T + B$.

Thus we obtain that (f, Kf) belongs to the closure of the graph of $T + B$, and therefore $f \in D(\overline{T + B})$. ∎

REMARKS 1.7. (i) If $f \in D(\overline{T + B})_+$, $g \in D(K)_+$, $g \leqslant f$, then $g \in D(\overline{T + B})$. This "ideal property" of $D(\overline{T + B})$ is a consequence of Proposition 1.6 and the positivity of $\widehat{c} - c\colon D(K) \to \mathbb{R}$.

(ii) Let $\lambda > 0$, $f \in X_+$. Proposition 1.6 implies that $\beta_\lambda(f) = 0$ (i.e., $(\widehat{c} - \overline{c})((\lambda - K)^{-1}f) = 0$) if and only if $(\lambda - K)^{-1}f \in D(\overline{T + B})$.

(iii) Proposition 1.6 implies that $K = \overline{T + B}$ if and only if $c = \widehat{c}$, and by part (ii) above this holds if and only if $\beta_\lambda = 0$ for some (or equivalently all) $\lambda > 0$. Part of this result is contained in Theorem 6.11 combined with Theorem 6.13 of [9], for the case considered there.

2. HONESTY

Let $f \in X_+$. Recall that then $\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r \in D(K)$, for all $t \geqslant s \geqslant 0$, and

$$\mathrm{e}^{tK} f - \mathrm{e}^{sK} f = K \int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r.$$

By integration we obtain

$$\|\mathrm{e}^{tK} f\| = \|\mathrm{e}^{sK} f\| - \widehat{c}\Big(\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r\Big). \tag{2.1}$$

Let $J \subseteq [0, \infty)$ be an interval. We will call the trajectory $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ *honest on J* if

$$\|\mathrm{e}^{tK} f\| = \|\mathrm{e}^{sK} f\| - c\Big(\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r\Big)$$

(or equivalently, $c\Big(\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r\Big) = \widehat{c}\Big(\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r\Big)$) for all $s, t \in J$, $s \leqslant t$. The trajectory will be called *honest* if it is honest on $[0, \infty)$. We will use the notation

$$H_J := \{f \in X_+;\ (\mathrm{e}^{tK} f;\, t \geqslant 0) \text{ honest on } J\},$$

and $H := H_{[0,\infty)}$ will denote the set of $f \in X_+$ with honest trajectories. The semigroup $(\mathrm{e}^{tK};\, t \geqslant 0)$ will be called *honest* if all trajectories are honest, i.e., if $H = X_+$.

For the comparison of our definition of honesty with the definition given in [9] we refer to Definition 6.4, Proposition 6.9, Remark 6.16 of [9]; see also Remark 1.2.

REMARKS 2.1. (i) Let $f \in X_+$. It is obvious from the definition that $f \in H$ if and only if

$$\|\mathrm{e}^{tK} f\| = \|f\| - c\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s\Big),$$

for all $t \geqslant 0$. Also, if $0 \leqslant a < b < \infty$ and

$$\|\mathrm{e}^{bK} f\| = \|\mathrm{e}^{aK} f\| - c\Big(\int_a^b \mathrm{e}^{rK} f \,\mathrm{d}r\Big),$$

then Proposition 1.1(iii) together with (2.1) implies that $f \in H_{[a,b]}$.

(ii) Let $f \in X_+$, let $J_1, J_2 \subseteq \mathbb{R}$ be intervals, $J_1 \cap J_2 \neq \varnothing$, and assume that $f \in H_{J_1} \cap H_{J_2}$. It is immediate that then $f \in H_{J_1 \cup J_2}$.

(iii) Let $f \in X_+$, $0 \leqslant a < b \leqslant \infty$, and assume that $f \in H_{[a,b)}$. If $s \geqslant 0$, then it is clear from the definition that the trajectory starting from $\mathrm{e}^{sK} f$ is honest on $[(a-s) \vee 0, (b-s) \vee 0)$.

If $b = \infty$, then we conclude that $H_{[a,\infty)}$ is invariant under $(\mathrm{e}^{tK};\, t \geqslant 0)$. In particular, the set $H = H_{[0,\infty)}$ is invariant under $(\mathrm{e}^{tK};\, t \geqslant 0)$.

(iv) Let $f \in X_+$, $0 \leqslant a < b \leqslant \infty$, $s \geqslant 0$, and assume that $\mathrm{e}^{sK} f \in H_{[a,b)}$. Then obviously $f \in H_{[a+s,b+s)}$.

For the special case described in Remark 1.2, the following result is contained in Theorem 6.11, p. 166 of [9].

THEOREM 2.2. *Let $f \in X_+$. Then $f \in H = H_{[0,\infty)}$ if and only if $\beta_\lambda(f) = 0$ for some (or equivalently all) $\lambda > 0$.*

Proof. We recall the formula

$$(\lambda - K)^{-1} f = \int_0^\infty \mathrm{e}^{-\lambda t} \mathrm{e}^{tK} f \,\mathrm{d}t = \lambda \int_0^\infty \mathrm{e}^{-\lambda t} \Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s\Big) \,\mathrm{d}t. \tag{2.2}$$

The function $t \mapsto \int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s$ is continuous and linearly bounded as a D_K-valued function, and therefore the outer integral in the last expression in (2.2) can be considered as an integral in D_K. This implies that

$$\widehat{c}((\lambda - K)^{-1}f) = \lambda \int_0^\infty \mathrm{e}^{-\lambda t}\, \widehat{c}\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s \Big)\,\mathrm{d}t$$

and

$$c((\lambda - K)^{-1}f) = \lambda \int_0^\infty \mathrm{e}^{-\lambda t}\, c\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s \Big)\,\mathrm{d}t.$$

Recalling Proposition 1.1(iii) we obtain that $f \in H$, i.e.,

$$c\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s \Big) = \widehat{c}\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s \Big)$$

for all $t \geqslant 0$, if and only if $\overline{c}((\lambda - K)^{-1}f) = \widehat{c}((\lambda - K)^{-1}f)$. ∎

REMARKS 2.3. (i) Combining Remark 1.7(ii) with Theorem 2.2 we obtain that $f \in H$ if and only if $f \in X_+$ and $(\lambda - K)^{-1}f \in D(\overline{T+B})$ for some (or equivalently all) $\lambda > 0$.

(ii) Let $f \in H$ be quasi-interior, i.e., the lattice ideal generated by f is dense in X. Then $H = X_+$.

Indeed, Theorem 2.2 implies that $\beta_1(f) = 0$. The positivity of β_1 implies that β_1 vanishes on the closed ideal generated by f, i.e. $\beta_1 = 0$. Now Theorem 2.2 shows $H = X_+$.

In Corollary 2.9 we will obtain a stronger version of the result stated above.

(iii) Assuming the "conservativity condition" (0.1) we have $c = 0$, and for $f \in X_+$ we obtain that

$$\|\mathrm{e}^{tK} f\| = \|f\| \quad (t \geqslant 0)$$

if and only if $\beta_\lambda(f) = 0$ for some (or equivalently all) $\lambda > 0$. This equivalence is already contained in Corollary of Theorem 2 of [16].

PROPOSITION 2.4. *Let $J \subseteq [0, \infty)$ be a subinterval. Then $\widehat{H}_J := \operatorname{lin} H_J = H_J - H_J$ is a closed lattice ideal (i.e., a band, since $X = L^1$), and $H_J = (\widehat{H}_J)_+$.*

Proof. It is clear that H_J is stable under formation of linear combinations with positive scalars, which immediately implies $\operatorname{lin} H_J = H_J - H_J$.

Let $f \in H_J$, and let $0 \leqslant g \leqslant f$. Let $s, t \in J$, $s < t$. Then $(\widehat{c} - c)\Big(\int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r \Big) = 0$. From the positivity of $\widehat{c} - c$ on $D(K)$ (see Proposition 1.1(iii)) and the inequality $0 \leqslant \int_s^t \mathrm{e}^{rK} g \,\mathrm{d}r \leqslant \int_s^t \mathrm{e}^{rK} f \,\mathrm{d}r$ we conclude that $(\widehat{c} - c)\Big(\int_s^t \mathrm{e}^{rK} g \,\mathrm{d}r \Big) = 0$. This shows $g \in H_J$.

Let $s, t \in J$, $s < t$. Then the mapping $X \ni f \mapsto \int_s^t e^{rK} f \, dr \in D_K$ is continuous, and therefore the functional $X \ni f \mapsto (\widehat{c} - c)\Big(\int_s^t e^{rK} f \, dr\Big)$ is continuous. This shows that H_J is closed.

These properties show that $\widehat{H}_J$ is a closed ideal and that $H_J = (\widehat{H}_J)_+$. ∎

REMARKS 2.5. (i) Combining the statements of Remark 2.1(iii) and Proposition 2.4 we conclude that the set $\widehat{H} := \operatorname{lin} H$ is a closed lattice ideal in $X = L^1$, and therefore a projection band, which is invariant under the semigroup $(e^{tK}; t \geqslant 0)$. The positive cone of the disjoint complement $\widehat{H}^{\mathrm{d}}$ of $\widehat{H}$ does not contain non-trivial elements with an honest trajectory. In Section 3 we illustrate that it is not clear whether more structure can be obtained for the band disjoint to $\widehat{H}$.

More generally, for $a \geqslant 0$ the set $H_{[a,\infty)}$ is a projection band in X which is invariant under $(e^{tK}; t \geqslant 0)$.

(ii) If the measure space (Ω, μ) is localisable, then there exist locally measurable subsets $\Omega_h, \Omega_d \subseteq \Omega$, $\Omega_h \cap \Omega_d = \emptyset$, $\Omega_h \cup \Omega_d = \Omega$, such that $\widehat{H} = L^1(\Omega_h, \mu)$, $\widehat{H}^{\mathrm{d}} = L^1(\Omega_d, \mu)$. For the functional $\beta_\lambda \in X'_+ = L^\infty(\mu)_+$, Theorem 2.2 implies that $\beta_\lambda = 0$ on Ω_h, $\beta_\lambda(x) > 0$ on Ω_d locally a.e., for all $\lambda > 0$.

We refer to 14M, pp. 128, 129 of [17] and Kap. IV.3 of [12] for the definition of localisable measure spaces as well as for pertinent properties. (In particular, σ-finite measure spaces are localisable.)

(iii) Let $f \in H$, $\lambda > 0$. Using that $\widehat{H}$ is a closed, $(e^{tK}; t \geqslant 0)$-invariant ideal we obtain that $(\lambda - K)^{-1} f = \int e^{-\lambda t} e^{tK} f \, dt \in H$ for all $\lambda > 0$.

In the following proposition we present sufficient conditions for honesty.

PROPOSITION 2.6. *Let $\lambda > 0$.*

(i) *Let $f \in X_+$, $B(\lambda - T)^{-1} f \leqslant f$. Then $f \in H$.*

(ii) *Let $g \in D(T)_+$, $(T + B)g \leqslant \lambda g$. Then $g \in H$.*

Proof. (i) We start by noting that $h \in X_+$, $B(\lambda - T)^{-1} h = h$ implies that $0 \leqslant \lambda \int (\lambda - T)^{-1} h \leqslant \int (\lambda - T - B)(\lambda - T)^{-1} h = \int (h - B(\lambda - T)^{-1} h) = 0$, and thus $h = 0$.

Because of the positivity of $B(\lambda - T)^{-1}$ the hypothesis implies that the sequence $((B(\lambda - T)^{-1})^n f)_n$ in X_+ is decreasing, and therefore

$$h := \lim_{n \to \infty} (B(\lambda - T)^{-1})^n f$$

exists. As a consequence, $B(\lambda - T)^{-1} h = h$, and the initial remark implies $h = 0$. Now Lemma 1.4 implies $\beta_\lambda(f) = \int h = 0$, and Theorem 2.2 shows $f \in H$.

(ii) The function $f := (\lambda - T)g \geqslant Bg \geqslant 0$ satisfies the hypotheses of (i), and therefore $f \in H$. Since H is the positive cone of a closed ideal, and $g = (\lambda - T)^{-1} f \leqslant (\lambda - K)^{-1} f$, we obtain $g \in H$ using Remark 2.5(iii). ∎

REMARK 2.7. The condition used in Proposition 2.6(ii) was used in Remark 2.2(b) of [23] in connection with sufficient conditions for $K = \overline{T + B}$.

PROPOSITION 2.8. *Assume that there exists $\varepsilon > 0$ such that $H_{[0,\varepsilon)} = X_+$. Then $H = X_+$ (i.e., the semigroup $(\mathrm{e}^{tK};\, t \geqslant 0)$ is honest).*

Proof. Let $f \in X_+$. For $s \geqslant 0$, the hypothesis implies that the trajectory $(\mathrm{e}^{tK}(\mathrm{e}^{sK} f);\, t \geqslant 0)$ is honest on $[0, \varepsilon)$, i.e., the trajectory $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ is honest on $[s, s + \varepsilon)$. Now Remark 2.1(ii) implies that $f \in H$. ∎

The following two corollaries express the so-called "universality of honesty" which has been proved in the concrete situation of birth-and-death problems or fragmentation equations in [11] (see also Chapters 7 and 9 of [9]). We mention that, if (Ω, μ) is localisable, then an element $f \in X_+$ is quasi-interior if and only if the set where f vanishes is a local null set.

COROLLARY 2.9. *Let $\varepsilon > 0$, and let $f \in H_{[0,\varepsilon)}$ be a quasi-interior element of X. Then $H = X_+$.*

Proof. Since $\widehat{H}_{[0,\varepsilon)}$ is a band containing f, and X is the smallest band containing f, we conclude that $\widehat{H}_{[0,\varepsilon)} \supseteq X$, and therefore $H_{[0,\varepsilon)} = X_+$. Now Proposition 2.8 implies the assertion. ∎

COROLLARY 2.10. *Let $H \neq \{0\}$, and assume that the C_0-semigroup $(\mathrm{e}^{tK};\, t \geqslant 0)$ is irreducible (i.e., there exists no non-trivial closed ideal which is invariant under $(\mathrm{e}^{tK};\, t \geqslant 0)$). Then $H = X_+$.*

Proof. Let $f \in H \setminus \{0\}$. Then $g := (1 - K)^{-1} f \in H$, by Remark 2.5(iii), and the irreducibility of $(\mathrm{e}^{tK};\, t \geqslant 0)$ implies that g is quasi-interior; cf. C-III, Definition 3.1 of [19]. Therefore Corollary 2.9 implies the assertion. ∎

In Corollary 6.12 of [9] it is shown that in the case that $(\mathrm{e}^{tK};\, t \geqslant 0)$ is not honest, there are always elements $f \in X_+$ such that the trajectory $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ is *immediately dishonest*, i.e.,

$$\|\mathrm{e}^{tK} f\| < \|f\| - c\Big(\int_0^t \mathrm{e}^{sK} f \,\mathrm{d}s\Big),$$

for all $t > 0$. The following consequence of Proposition 2.8 is another result concerning immediately dishonest trajectories.

COROLLARY 2.11. *Assume that the C_0-semigroup $(\mathrm{e}^{tK};\, t \geqslant 0)$ is not honest. Let $f \in X_+$ be such that $f_d := (I - P_H) f$ is a quasi-interior element of $\widehat{H}^{\mathrm{d}}$, where P_H denotes the band projection onto $\widehat{H}$. Then the trajectory $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ is immediately dishonest.*

Proof. Assume that $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ is not immediately dishonest. This implies that there exists $\varepsilon > 0$ such that $(\mathrm{e}^{tK} f;\, t \geqslant 0)$ is honest on $[0, \varepsilon)$, i.e. $f \in H_{[0,\varepsilon)}$. The ideal generated by f contains the ideal generated by f_d ($\leqslant f$), and by hypothesis, the latter is dense in $\widehat{H}^{\mathrm{d}}$. As $H_{[0,\varepsilon)}$ is the positive cone of a closed ideal we conclude that $\widehat{H}^{\mathrm{d}}_+ \subseteq H_{[0,\varepsilon)}$. We trivially have $H \subseteq H_{[0,\varepsilon)}$, and so we conclude that $X_+ = H + \widehat{H}^{\mathrm{d}}_+ \subseteq H_{[0,\varepsilon)}$. Now Proposition 2.8 leads to a contradiction. ∎

3. EXAMPLES

The examples we present will be of the following kind. We start with the generator S of a stochastic C_0-semigroup. We define a measurable function $V\colon \Omega \to [0,\infty)$ for which $\operatorname{s-lim}_{n\to\infty} \mathrm{e}^{t(S-V\wedge n)}$ ($t \geqslant 0$) defines a C_0-semigroup, and we define T as the generator of this absorption semigroup, i.e., V is admissible and $T = S_V$, with the notions introduced in Section 2 of [22]. We note that then $D(T) \subseteq D(V)$ and $\int (T+V)f \leqslant 0$ ($f \in D(T)_+$), by Lemma 4.1 of [22]. With $B := V$ (maximal multiplication operator) we will describe situations where $\int (T+B)f = 0$ ($f \in D(T)$) and where the semigroup generated by K is not stochastic. In the terminology of Section 4 of [22], we have $K = S_{0,V}$. The semigroup generated by K is dominated by the semigroup generated by S, and in our examples we will have $K \neq S$ (i.e., V is not regular in the sense of Definition 2.12 of [22]).

EXAMPLE 3.1. Let S be the generator of the C_0-semigroup of left translation on $L_1(\mathbb{R})$, i.e., $D(S) = W_1^1(\mathbb{R})$, $Sf = f'$, and $V(x) := \frac{1}{x}\mathbf{1}_{(0,\infty)}(x)$. For $0 \leqslant r < 1$ we obtain that $T + rB$ generates the C_0-semigroup corresponding to left translation with absorption by $(1-r)V$ (cf. Proposition 4.2 of [22]), i.e.,

$$\mathrm{e}^{t(T+rB)} f(x) = \exp\Big(-(1-r) \int\limits_{x\vee 0}^{(x+t)\vee 0} y^{-1}\,\mathrm{d}y\Big) f(x+t).$$

Letting $r \to 1-$ we see that the C_0-semigroup $(\mathrm{e}^{tK};\, t \geqslant 0)$ decomposes into left translation on $L^1(-\infty,0)$ (with zero coming in from the right), which is stochastic, and left translation on $L^1(0,\infty)$.

From this description we immediately obtain that K is given by

$$D(K) = \{f \in W_1^1(\mathbb{R}\setminus\{0\});\ f(0-) = 0\}, \quad Kf = f'.$$

It then follows from Corollary 4.3(a) of [22] that $T = K - V$, i.e.,

$$D(T) = D(K) \cap D(V) = \{f \in W_1^1(\mathbb{R});\ Vf \in L^1(\mathbb{R})\}, \quad Tf = f' - Vf.$$

Thus we finally obtain that $(T+B)f = f'$ for all $f \in D(T)$, and therefore

$$c(f) = -\int (T+B)f = -\int\limits_{\mathbb{R}} f' = 0.$$

(In order to preclude irritations we note that the derivative in the formula $Kf = f'$ has to be understood as distributional derivative on $\mathbb{R} \setminus \{0\}$, and that

$$\widehat{c}(f) = -\int Kf = f(0+) \quad (f \in D(K)).)$$

We thus obtain that the set $H = H_{[0,\infty)}$ (recall the notation from Section 2) of initial values for an honest trajectory is $L^1(-\infty,0)_+$. More general, for $0 \leqslant a < b \leqslant \infty$, the set $H_{[a,b)}$ is given by $\{f \in L^1(\mathbb{R})_+;\ \operatorname{spt} f \cap (a,b) = \varnothing\}$.

We define the measure space (Λ,ν) by $\Lambda := \{0\}$, ν the Dirac measure, and $Z\colon D(T) \to L^1(\nu)$ $(= \mathbb{R})$ by $Zf := f(0+)$ $(= 0)$. Then, evidently, $Z = 0$ is the extension $Z\colon D(K) \to L^1(\nu)$ described in Remark 1.2. We further note that $\widehat{Z}\colon D(K) \to \mathbb{R}$, defined by $\widehat{Z}(f) := f(0+)$ $(f \in D(K))$ yields a different positive K-bounded extension of $Z|_{D(T)}$. This extension does not "respect monotone convergence" (in the sense expressed in Remark 1.2). The operators Z, $\widehat{Z}$ correspond to the functionals c, $\widehat{c}$, respectively.

In the above example the semigroup $(e^{tK};\ t \geqslant 0)$ is the direct sum of an honest semigroup and a semigroup without honest trajectories. Our next example shows that the band disjoint to $H_{[0,\infty)}$ is not necessarily invariant under the semigroup $(e^{tK};\ t \geqslant 0)$.

EXAMPLE 3.2. The operator S is intended to generate left translation on $\mathbb{R}$ with half of the particles arriving at 1 jumping to 0. This means

$$D(S) = \{f \in W_1^1(\mathbb{R} \setminus \{0,1\});\ f(1-) = f(1+)/2, f(0-) = f(0+) + f(1+)/2\},$$
$$Sf = f' \quad (f \in D(S)).$$

(Note that the stochasticity of the generated semigroup corresponds to $\int f' = f(0-) + (f(1-) - f(0+)) - f(1+) = 0$ $(f \in D(S))$.) As in Example 3.1 we define $V(x) := \frac{1}{x}\mathbf{1}_{(0,\infty)}(x)$. Without carrying out details (which are similar to Example 3.1) we state that the semigroup $(e^{tK};\ t \geqslant 0)$ is left translation on $L^1(-\infty,0)$, $L^1(0,1)$, $L^1(0,\infty)$ separately, with the additional requirement that functions arriving at 1 are to equal parts transported to 1 (for the interval $(0,1)$) and 0 (for $(-\infty,0)$), and that functions arriving at 0 die. We further state that

$$D(K) = \{f \in W_1^1(\mathbb{R} \setminus \{0,1\}); f(1-) = f(0-) = f(1+)/2\},$$
$$Kf = f' \quad (f \in D(K)),$$
$$D(T) = D(K) \cap D(V) = \{f \in W_1^1(\mathbb{R}\setminus\{0,1\}); f(1-) = f(0-) = f(1+)/2, Vf \in L^1(\mathbb{R})\},$$
$$Tf = f' - Vf \quad (f \in D(T)).$$

As in Example 3.1 we obtain

$$c(f) = -\int (T+B)f = -\int f' = f(1+) - f(1-) + f(0+) - f(0-) = 0.$$

(Note that $Vf \in L^1(\mathbb{R})$ forces $f(0+) = 0$.)

Again we obtain that the set $H = H_{[0,\infty)}$ of initial values for an honest trajectory is $L^1(-\infty,0)_+$, and that, for $0 \leqslant a < b \leqslant \infty$, the set $H_{[a,b)}$ is given by $\{f \in L^1(\mathbb{R})_+;\ \operatorname{spt} f \cap (a,b) = \emptyset\}$. However, in contrast to Example 3.1, the disjoint band to H, i.e. $L^1(0,\infty)$, is no longer invariant under $(\mathrm{e}^{tK};\ t \geqslant 0)$. (Even stronger, there exists no decomposition of $L^1(\mathbb{R})$ into two complementary bands which are invariant under $(\mathrm{e}^{tK};\ t \geqslant 0)$.)

In both of the above examples, the set $\bigcup_{a\geqslant 0} H_{[a,\infty)}$ (of elements whose trajectories are *eventually honest*) is dense in X_+. The third example will make it clear that this is not always the case.

Modifying the above examples by starting with left translation semigroups on $(-\infty,a)$, where $a > 0$, and otherwise proceeding as above we obtain situations where even $H_{[a,\infty)} = X_+$.

EXAMPLE 3.3. As the starting stochastic C_0-semigroup we use the semigroup associated with the heat equation on $L^1(0,\infty)$ with Neumann boundary condition, i.e.,

$$D(S) := \{f \in W_1^2(0,\infty);\ f'(0) = 0\}, \quad Sf := f'',$$

and we define $V(x) := x^{-2}$. As before, we note that for $0 \leqslant r < 1$, the semigroup generated by $T + rB$ is the semigroup obtained as the absorption semigroup with the absorption rate $(1-r)V$.

In order to determine T and K we use that the above semigroup is also a C_0-semigroup on $L^2(0,\infty)$, generated by a symmetric operator, and that forming absorption semigroups is consistent in different L^p-spaces; cf. Section 3 of [22].

From Proposition 5.8 and its proof of [22] we obtain that in $L^2(0,\infty)$, the semigroup associated with the absorption $(1-r)V$ is generated by the negative of the operator associated with the form

$$s_r(u,v) := \int uv + (1-r)\int Vuv, \quad D(s_r) := \{u \in W_2^1(0,\infty);\ Vu^2 \in L^1\}.$$

It is easy to see that $D(s_r) = \{u \in W_{2,0}^1(0,\infty);\ Vu^2 \in L^1\}$. Since $C_c^\infty(0,\infty)$ is dense in $W_{2,0}^1(0,\infty)$, a form convergence theorem (together with the consistency mentioned above) implies that K generates the heat semigroup on $L^1(0,\infty)$ with Dirichlet boundary conditions, i.e.,

$$D(K) = W_{1,0}^1(0,\infty) \cap W_1^2(0,\infty), \quad Kf = f''.$$

And then, as above,

$$\begin{aligned} &D(T) = D(K) \cap D(V) = \{f \in W_1^2(0,\infty);\ f(0) = 0,\ Vf \in L_1\}, \\ &Tf = f'' - Vf. \end{aligned}$$

The inverse quadratic singularity of V at zero forces $f'(0) = 0$ for $f \in D(T)$, and therefore

$$\widehat{c}(f) = -\int Kf = -\int f'' = f'(0) \quad (f \in D(K)),$$
$$c(f) = 0 \quad (f \in D(T)).$$

Making use of the explicit representation of e^{tK} (in terms of the heat kernel on $\mathbb{R}$) one obtains that $H_J = \{0\}$ for any nonempty subinterval $J \subseteq [0, \infty)$.

Acknowledgements. The first-named author thanks J. Banasiak and B. Lods for helpful remarks and suggestions. The second-named author is grateful to H. Vogt for useful discussions.

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MUSTAPHA MOKHTAR-KHARROUBI, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, F-25030 BESANÇON, FRANCE
E-mail address: mmokhtar@univ-fcomte.fr

JÜRGEN VOIGT, FACHRICHTUNG MATHEMATIK, TECHNISCHE UNIVERSITÄT DRESDEN, D-01062 DRESDEN, GERMANY
E-mail address: juergen.voigt@tu-dresden.de

Received October 30, 2007.