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CONTINUITY OF CP-SEMIGROUPS IN THE POINT-STRONG OPERATOR TOPOLOGY

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ABSTRACT. We prove that if $\{\phi_t\}_{t\geqslant 0}$ is a CP-semigroup acting on a von Neumann algebra $M \subseteq B(H)$, then for every $A \in M$ and $\xi \in H$, the map $t \mapsto \phi_t(A)\xi$ is norm-continuous. We discuss the implications of this fact to the existence of dilations of CP-semigroups to semigroups of endomorphisms.

KEYWORDS: *CP-semigroup, E_0-semigroup, strong operator continuity, Bhat's dilation theorem, dilations.*

MSC (2000): 46L55, 46L57.

1. INTRODUCTION

Let H be a Hilbert space, not necessarily separable, and let $M \subseteq B(H)$ be a von Neumann algebra. A *CP-semigroup* on M is a family $\phi = \{\phi_t : M \to M\}_{t\geqslant 0}$ of contractive normal completely positive maps which satisfies the following properties:

(i) $\phi_0(A) = A, \forall A \in M$;
(ii) $\phi_{s+t} = \phi_s \circ \phi_t$, $s, t \geqslant 0$;
(iii) for all $A \in M$ and $\omega \in M_*$, $\lim\limits_{t\to t_0} \omega(\phi_t(A)) = \omega(\phi_{t_0}(A))$;

where M_* denotes the predual of M. We shall refer to continuity condition (iii) as *continuity in the point-σ-weak topology*. It is equivalent to *continuity in the point-weak operator topology*, i.e.

$$\lim_{t\to t_0} \langle \phi_t(A)\xi, \eta\rangle = \langle \phi_{t_0}(A)\xi, \eta\rangle, \quad A \in M, \xi, \eta \in H.$$

A CP-semigroup ϕ is called an *E-semigroup* if ϕ_t is a $*$-endomorphism for all $t \geqslant 0$.

In this note we prove that CP-semigroups satisfy a seemingly stronger continuity condition, namely

$$\lim_{t\to t_0} \|\phi_t(A)\xi - \phi_{t_0}(A)\xi\| = 0, \tag{1.1}$$

for all $A \in M, \xi \in H$. A semigroup satisfying (1.1) will be said to be *continuous in the point-strong operator topology*. The proposition that CP-semigroups are continuous in the point-strong operator topology has appeared in the literature earlier, but the proofs that are available seem to be incomplete. In the proofs of which we are aware, only continuity *from the right* in the point-strong operator topology is established. By this we mean that (1.1) holds for limits taken with $t \searrow t_0$.

We consider the continuity of CP-semigroups in the point-strong operator topology to be an important property, because it plays a crucial role in the existence of dilations of CP-semigroups to E-semigroups. We are aware of five different proofs for the fact that every CP-semigroup has a dilation to an E-semigroup: Bhat [2], Selegue [7], Bhat–Skeide [4], Muhly–Solel [6] and Arveson [1] (some of the authors require some additional conditions, notably that the CP-semigroup be unital or that the Hilbert space be separable). In order to show that the minimal dilation of a CP-semigroup to an E-semigroup is continuous in the point-weak operator topology, all authors make use of continuity of the CP-semigroup in the point-strong operator topology, either implicitly or explicitly.

2. PRELIMINARIES

Let M be a von Neumann algebra acting on a Hilbert space, which is not assumed to be separable. Let $\phi = \{\phi_t : M \to M\}_{t\geqslant 0}$ be a CP-semigroup acting on M. We denote by M_* the set of σ-weakly continuous linear functionals on M. We shall denote by $\sigma(M_*, M)$ the pointwise convergence topology of M_* as a subset of the dual space of M.

Let δ be the generator of ϕ, and let $D(\delta)$ denote its domain:

$$D(\delta) = \Big\{ A \in M : \exists \delta(A) \in M \, \forall \omega \in M_* \lim_{t\to 0+} t^{-1}\omega(\phi_t(A) - A) = \omega(\delta(A)) \Big\}.$$

LEMMA 2.1. *For every $A \in M$ and $\xi \in H$, the map $t \mapsto \phi_t(A)\xi$ is continuous from the right (in norm).*

The proof of this result can be found in the literature, for example as Lemma A.1 of [3] or Proposition 4.1 item 1 in [6]. For completeness, let us present the argument from [3]. Let $A \in M$, $\xi \in H$ and $t \geqslant 0$. For all $h > 0$, we have, using the Schwartz inequality for completely positive maps,

$$\begin{aligned} &\|\phi_{t+h}(A)\xi - \phi_t(A)\xi\| \\ &= \langle \phi_{t+h}(A)^*\phi_{t+h}(A)\xi, \xi\rangle - 2\,\mathrm{Re}\langle \phi_{t+h}(A)\xi, \phi_t(A)\xi\rangle + \|\phi_t(A)\xi\|^2 \\ &\leqslant \langle \phi_h(\phi_t(A)^*\phi_t(A))\xi, \xi\rangle - 2\,\mathrm{Re}\langle \phi_{t+h}(A)\xi, \phi_t(A)\xi\rangle + \|\phi_t(A)\xi\|^2 \xrightarrow[h\to 0]{} 0. \end{aligned}$$

We remark, however, that two-sided continuity does not follow directly from continuity from the right. This is in contrast with the situation of the classical theory of C_0-semigroups on Banach spaces (see for example [5]). If $T =$

$\{T_t\}_{t \geqslant 0}$ is a semigroup of contractions on a Banach space X such that the maps

$$t \mapsto T_t(x)$$

are continuous from the right in norm for all $x \in X$, then it is easy to show that these maps are also continuous from the left in norm (for given $x \in X$, $0 \leqslant t \leqslant a$, $\|T_{a-t}(x) - T_a(x)\| = \|T_{a-t}(x - T_t(x))\| \leqslant \|x - T_t(x)\|$) . In fact, when X is separable, for instance, it can be proven by measurability and integrability techniques that if the maps $t \mapsto f(T_t(x))$ are measurable for all $x \in X$ and $f \in X^*$, then the maps $t \mapsto T_t(x)$ are continuous in norm for $t > 0$. In the case of CP-semigroups on von Neumann algebras, however, these techniques seem to require considerable modification. We provide here an alternative approach to the problem.

Recall that a function $g : [0,1] \to H$ is *weakly measurable* if for all $\eta \in H$, the complex-valued function $g_\eta(t) = \langle \eta, g(t) \rangle$ is measurable. We will say that the function g is *strongly measurable* if there exists a family of countably-valued functions (i.e. assuming a set of values which is at most countable) converging Lebesgue almost everywhere to g. (For more details, see Definition 3.5.4, p. 72, and the surrounding discussion in [5]).

LEMMA 2.2. *For all $\xi \in H, A \in M$, the function $f : [0,1] \to H$ given by $f(t) = \phi_t(A)\xi$ is strongly measurable and Bochner integrable on the interval $[0,1]$.*

Proof. The function f is weakly continuous, since ϕ is continuous in the point-weak operator topology. In particular, it is weakly measurable. Furthermore, by Lemma 2.1, the function f is continuous from the right in norm, hence it is separably valued (i.e., its range is contained in a separable subspace of H). By Theorem 3.5.3 of [5], the function f is strongly measurable because it is is weakly measurable and separably valued.

Thanks to Theorem 3.7.4, p. 80 of [5], in order to show that f is Bochner integrable it is enough to show that f is strongly measurable and that

$$\int_0^1 \|f(t)\| \mathrm{d}t < \infty.$$

The latter condition is easy to verify, as $t \mapsto \|f(t)\|$ is a right-continuous, bounded function on $[0,1]$. ∎

We thank Michael Skeide for the idea to use the continuity of f from the right in order to avoid making the assumption that H is separable.

LEMMA 2.3. *Let $A \in B(H)$ be positive. Then there exists a sequence $A_n \in D(\delta)$ of positive operators such that $A_n \to A$ in the σ-strong* topology.*

Proof. Recall that the sequence

$$A_n = n \int_0^{1/n} \phi_t(A)\mathrm{d}t$$

(integral taken in the σ-weak sense) converges in the σ-weak topology to A. Furthermore $A_n \in D(\delta)$ and it is a positive operator for each $n > 0$ since ϕ_t is a CP map for all t. It is easy to check that $\|A_n\| \leqslant \|A\|$ for all n since ϕ_t is contractive.

Now observe that for each $\xi \in H$, the map $t \mapsto \phi_t(A)\xi$ is Bochner integrable on $[0,1]$ (see Lemma 2.2), hence in fact we have

$$A_n\xi = n \int_0^{1/n} \phi_t(A)\xi\mathrm{d}t$$

where the integral is taken in the Bochner sense. The identity holds because for all $\eta \in H$, $n \in \mathbb{N}$ we have:

$$\langle A_n\xi, \eta\rangle = n \int_0^{1/n} \langle\phi_t(A)\xi, \eta\rangle\mathrm{d}t = \Big\langle n \int_0^{1/n} \phi_t(A)\xi\mathrm{d}t, \eta\Big\rangle.$$

We now show that $A_n \to A$ strongly. Let $\xi \in H$ be fixed.

$$\|A\xi - A_n\xi\| = \Big\|n \int_0^{1/n} A\xi\mathrm{d}t - n \int_0^{1/n} \phi_t(A)\xi\mathrm{d}t\Big\| \leqslant n \int_0^{1/n} \|A\xi - \phi_t(A)\xi\|\mathrm{d}t.$$

The latter goes to zero by continuity from the right (Lemma 2.1). Since A_n, A are positive operators, by considering adjoints we obtain that $A_n \to A$ in the strong* topology. Finally, since the sequence is bounded, we have convergence in the σ-strong* topology. ∎

LEMMA 2.4. *Let A_n be a bounded sequence of operators in M converging to A in the σ-strong* topology and let $t_0 \geqslant 0$. Then for every sequence $t_k \to t_0$, $\xi \in H$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$,*

$$\|\phi_{t_k}(A_n - A)\xi\| < \varepsilon, \quad \textit{for all } k.$$

Proof. Let $B_n = (A_n - A)^*(A_n - A)$, $\omega_k(X) = \langle\phi_{t_k}(X)\xi, \xi\rangle$ and $\omega(X) = \langle\phi_{t_0}(X)\xi, \xi\rangle$. Then we have that

$$\|\phi_{t_k}(A_n - A)\xi\|^2 = \langle\phi_{t_k}(A_n - A)^*\phi_{t_k}(A_n - A)\xi, \xi\rangle \leqslant \omega_k(B_n)$$

since ϕ_t is a CP map for all t. Since ϕ is a point-σ-weakly continuous semigroup, we have that (ω_k) is a sequence of σ-weakly continuous linear functionals such that $\omega_k(X) \to \omega(X)$ for all $X \in M$. Furthermore, B_n is a bounded sequence converging in the σ-strong* topology to 0. The latter holds because A_n is a bounded sequence converging to A in the σ-strong* topology and multiplication is jointly continuous with respect to this topology in bounded sets (of course * is also continuous). Finally, we obtain the desired conclusion by applying Lemma III.5.5, p. 151 of [8], which states the following. Let M be a von Neumann algebra and let ρ_k be a sequence in M_* converging to $\rho_0 \in M_*$ in the $\sigma(M_*, M)$ topology. If a bounded sequence (a_n) converges σ-strongly* to 0, then $\lim_{n\to\infty} \rho_k(a_n) = 0$ uniformly in k. ∎

3. THE MAIN RESULT

THEOREM 3.1. *Let ϕ be a* CP*-semigroup acting on a von Neumann algebra $M \subseteq B(H)$. Then for all $\xi \in H$, $A \in M$ and $t_0 \geqslant 0$,*

$$\lim_{t\to t_0} \|\phi_t(A)\xi - \phi_{t_0}(A)\xi\| = 0.$$

Proof. Let $\varepsilon > 0$ be given, and let (t_k) be a sequence converging to t_0. By Lemma 2.3, there is a bounded sequence (A_n) of operators $A_n \in D(\delta)$ such that $A_n \to A$ in the σ-strong* topology. By Lemma 2.4, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$,

$$\|\phi_{t_k}(A_n - A)\xi\| < \frac{\varepsilon}{3}, \quad \text{for all } k \geqslant 0.$$

By an application of the Principle of Uniform Boundedness, if $X \in D(\delta)$ there exists $C_X > 0$ such that

$$\sup_{s>0} \frac{1}{s}\|\phi_s(X) - X\| \leqslant C_X < \infty.$$

Now notice that $A_n \in D(\delta)$ for all n, and in particular $\exists C > 0$ such that

$$\sup_{s>0} \frac{1}{s}\|\phi_s(A_N) - A_N\| \leqslant C.$$

Because ϕ_s is a contraction for all s, we obtain that for all k,

$$\begin{aligned}\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| &\leqslant \|\phi_{t_k}(A_N) - \phi_{t_0}(A_N)\|\,\|\xi\| \\ &\leqslant \|\phi_{|t_k-t_0|}(A_N) - A_N\|\,\|\xi\| \leqslant C\|\xi\|\,|t_k - t_0|.\end{aligned}$$

In particular, we must have that $\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| \to 0$ as $k \to \infty$. Thus there is $K \in \mathbb{N}$ such that for $k \geqslant K$,

$$\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| < \frac{\varepsilon}{3}.$$

We conclude that for $k \geqslant K$,

$$\begin{aligned}&\|\phi_{t_k}(A)\xi - \phi_{t_0}(A)\xi\| \\ &\quad\leqslant \|\phi_{t_k}(A - A_N)\xi\| + \|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| + \|\phi_{t_0}(A_N - A)\xi\| < \varepsilon.\end{aligned}$$ ∎

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