J. OPERATOR THEORY
64:1(2010), 155–169

THE CUNTZ SEMIGROUP OF IDEALS AND QUOTIENTS AND A GENERALIZED KASPAROV STABILIZATION THEOREM

ALIN CIUPERCA, LEONEL ROBERT and LUIS SANTIAGO

Communicated by Kenneth R. Davidson

ABSTRACT. Let A be a C^*-algebra and I a closed two-sided ideal of A. We use the Hilbert C^*-modules picture of the Cuntz semigroup to investigate the relations between the Cuntz semigroups of I, A and A/I. We obtain a relation on two elements of the Cuntz semigroup of A that characterizes when they are equal in the Cuntz semigroup of A/I. As a corollary, we show that the Cuntz semigroup functor is exact. Replacing the Cuntz equivalence relation of Hilbert modules by their isomorphism, we obtain a generalization of Kasparov's Stabilization theorem.

KEYWORDS: *Cuntz semigroup, Kasparov stabilization theorem, Multiplier algebras.*

MSC (2000): Primary 46L08; Secondary 46L35.

1. INTRODUCTION

In recent years the Cuntz semigroup has emerged as a powerful invariant in the classification of C^*-algebras, simple and nonsimple (e.g., [1], [2], [13], [15]). In [15] Andrew Toms provides examples of simple AH C^*-algebras that cannot be distinguished by their standard Elliott invariant (K-theory and traces) but that have different Cuntz semigroups. The first author and G.A. Elliott show in [2] that in the nonsimple case, the Cuntz semigroup is a classifying invariant for all AI C^*-algebras (their approach relies on Thomsen's classification of AI C^*-algebras; see [14]).

Here we define the Cuntz semigroup in terms of countably generated Hilbert C^*-modules over the algebra, following the approach introduced by K. Coward, G.A. Elliott and C. Ivanescu in [3]. This construction of the Cuntz semigroup is analogous to the description of K_0 in terms of finitely generated projective modules, and is based on an appropriate translation of the notion of Cuntz equivalence of positive elements to the context of Hilbert C^*-modules. Our investigation is initially motivated by the following question: is the Cuntz semigroup of a quotient of a C^*-algebra implicitly determined by the Cuntz semigroup of the algebra? We deduce a satisfactory answer from the inequality in Theorem 1.1 below, of interest in its own right.

Given a countably generated right Hilbert C^*-module M over A, let us denote by $[M]$ the element that it defines in $Cu(A)$, the Cuntz semigroup of A. We denote by $W(A)$ the subsemigroup of $Cu(A)$ consisting of the elements $[M]$ that satisfy $M \subseteq A^n$ for some n. The semigroup $W(A)$ can also be described in terms of positive elements of A (and $M_n(A)$) and it is often referred to as the Cuntz semigroup of A. However, in this paper we use this name for the semigroup $Cu(A)$. Let I be a σ-unital ideal of A. Then MI is a countably generated right Hilbert C^*-module over I. We will see that $[MI]$ only depends on the equivalence class of M. Therefore we write $[M]I := [MI]$.

THEOREM 1.1. *Let I be a σ-unital, closed, two-sided ideal of the C^*-algebra A and let $\pi\colon A \to A/I$ denote the quotient homomorphism. Let M and N be countably generated right Hilbert C^*-modules over A. Then $Cu(\pi)([M]) \leqslant Cu(\pi)([N])$ if and only if*

$$[M] + [N]I \leqslant [N] + [M]I.$$

It follows from Theorem 1.1 that $Cu(\pi)([M]) = Cu(\pi)([N])$ if and only if $[M] + [N]I = [N] + [M]I$. Adding $[l_2(I)]$ on both sides and using Kasparov's stabilization theorem we get that

$$Cu(\pi)([M]) = Cu(\pi)([N]) \Longleftrightarrow [M] + [l_2(I)] = [N] + [l_2(I)]. \tag{1.1}$$

We will show that the map $Cu(\pi)\colon Cu(A) \to Cu(A/I)$ is surjective. We conclude that the restriction of $Cu(\pi)$ to $Cu(A) + [l_2(I)]$ is an isomorphism onto $Cu(A/I)$.

In a similar way, the semigroup $W(A/I)$ is obtained as the quotient of $W(A)$ by the equivalence relation: $[M] \sim_I [N]$ if $[M] \leqslant [N] + [C_1]$ and $[N] \leqslant [M] + [C_2]$ for some C_1 and C_2, Hilbert C^*-modules over I. Here the assumption that the ideal I is σ-unital is not needed. This result, which we prove in this paper, was first obtained by Francesc Perera in an unpublished work. It can also be deduced from Lemma 4.12 of [7].

A suitable notion of exactness of sequences of ordered semigroups can be defined such that the isomorphism between $Cu(A) + [l_2(I)]$ and $Cu(A/I)$, implemented by $Cu(\pi)$, implies the exactness in the middle of the sequence

$$0 \longrightarrow Cu(I) \xrightarrow{Cu(\iota)} Cu(A) \xrightarrow{Cu(\pi)} Cu(A/I) \longrightarrow 0.$$

In Theorem 4.1 we will show that this is a short exact sequence of ordered semigroups, with splittings of the maps $Cu(\iota)$ and $Cu(\pi)$.

We can express (1.1) more directly as follows: M/MI and N/NI are Cuntz equivalent as A/I-Hilbert C^*-modules if and only if $M \oplus l_2(I)$ and $N \oplus l_2(I)$ are also Cuntz equivalent. In Section 5 we obtain an improvement of this result, with isomorphism of Hilbert C^*-modules instead of Cuntz equivalence. We prove the following theorem.

THEOREM 1.2. *Let A be a C^*-algebra and I a σ-unital, closed, two-sided ideal of A. Let M and N be countably generated right Hilbert C^*-modules over A and suppose that $\phi\colon M/MI \to N/NI$ is an isomorphism of A/I-Hilbert C^*-modules. Then there is an isomorphism of Hilbert C^*-modules $\Phi\colon M \oplus l_2(I) \to N \oplus l_2(I)$ that induces ϕ after passing to the quotient.*

Taking $I = A$ we get Kasparov's Stabilization Theorem ([6], Theorem 2.1). The module $M \oplus l_2(I)/MI \oplus l_2(I)$ is canonically isomorphic to M/MI. It is using this identification, applied also to N, that Φ induces ϕ. Theorem 1.2 is proved by an adaptation of the proof given by Mingo and Phillips in [11] of Kasparov's Stabilization Theorem.

In the last two sections we apply Theorem 1.2 in the context of multiplier algebras and we prove an equivariant version of Theorem 1.2 assuming that the group is compact.

2. PRELIMINARIES ON HILBERT C^*-MODULES

Let M and N be right Hilbert C^*-modules over a C^*-algebra A. We shall denote by $K(M,N)$ the norm closure of the space spanned by the A-module maps $\theta_{u,v}\colon M \to N$, given by $\theta_{u,v}(x) := v\langle u, x\rangle$, $u \in M$, $v \in N$. We shall denote by $B(M,N)$ the space of adjointable operators from M to N. If $T \in B(M,N)$, $\ker T$ and $\operatorname{im} T$ will denote the kernel and the image of T respectively. When $M = N$ the spaces $K(M,N)$ and $B(M,N)$ are C^*-algebras that we shall denote by $K(M)$ and $B(M)$ respectively. The elements of $B(M,N)$ will often be referred to simply as operators, while the elements of $K(M,N)$ will be called compact operators. Sometimes we will drop the prefix C^* and refer to Hilbert C^*-modules as Hilbert modules. The C^*-algebra will always act on the right of the Hilbert C^*-modules.

Given a Hilbert C^*-module M, the Hilbert C^*-module $l_2(M)$ is defined as the sequences $(m_i)_{i\in\mathbb{N}}$, $m_i \in M$, with the property that $\sum\limits_i \langle m_i, m_i\rangle$ converges in norm. This module is endowed with the inner product $\langle (m_i^1), (m_i^2)\rangle := \sum\limits_i \langle m_i^1, m_i^2\rangle$.

Let I be a closed two-sided ideal of A. By MI we denote the span of the elements of the form $m \cdot i$, with $m \in M$, $i \in I$. By Cohen's Theorem (see Lemma 4.4 of [9]), this set is a closed submodule of M consisting of all the vectors z of M for which $\langle z, z\rangle \in I$. The quotient M/MI is a right A/I-Hilbert C^*-module module with inner product $\langle x + MI, y + MI\rangle := \langle x, y\rangle + I$.

The submodule MI is invariant by any operator $T \in B(M)$. More generally, if $T \in B(M,N)$, then $T(MI) \subseteq NI$. In this way every operator T induces an operator $\widetilde{\pi}(T) \in B(M/MI, N/NI)$.

We say that a Hilbert C^*-module M is countably generated if there is a countable set $\{v_i\}_{i=1}^{\infty} \subset M$ with dense span in M. We will make use of the following two theorems on countably generated Hilbert modules.

THEOREM 2.1 (Noncommutative Tietze extension Theorem for Hilbert C^*-modules). *Let M and N be countably generated Hilbert C^*-modules and $\phi \in B(M/MI, N/NI)$. Then there is $\Phi \in B(M,N)$ that induces ϕ in the quotient.*

Proof. Let $H = M \oplus N$. We have $H/HI \simeq M/MI \oplus N/NI$. Using this isomorphism, we define $\psi\colon H/HI \to H/HI$, adjointable operator, by the matrix

$$\psi := \begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix}.$$

The homomorphism $\widetilde{\pi}\colon B(H) \to B(H/HI)$ maps $\theta_{u,v}$ to $\theta_{\pi(u),\pi(v)}$ (here $\pi\colon H \to H/HI$ is the quotient map). Thus, $K(H)$ is mapped surjectively onto $K(H/HI)$ by $\widetilde{\pi}$. Since H is countably generated, $K(H)$ is σ-unital. Thus, by the noncommutative Tiezte extension Theorem ([16], Theorem 2.3.9), $\widetilde{\pi}$ is also surjective. Let $\Psi \in B(H)$ be a selfadjoint preimage of ψ given by the matrix

$$\Psi = \begin{pmatrix} A & \Phi^* \\ \Phi & B \end{pmatrix}.$$

Then the operator Φ is a lift of ϕ. ∎

The following theorem is due to Michael Frank ([5], Theorem 4.1).

THEOREM 2.2. *Let M and N be Hilbert C^*-modules, M countably generated. Let $T\colon M \to N$ be a module morphism that is bounded and bounded from below (but not necessarily adjointable). Then M is isomorphic to* $\operatorname{im} T$ *as Hilbert C^*-modules.*

3. CUNTZ SEMIGROUPS

Let A be a C^*-algebra. Let us briefly review the construction of $Cu(A)$ and $W(A)$ in terms of countably generated Hilbert C^*-modules over A. We refer to [3] for further details.

Let M be a Hilbert C^*-module over A. A submodule F of M is said to be compactly contained in M if there is $T \in K(M)^+$ such that T restricted to F is the identity of F. In this case we write $F \subseteq\subseteq M$. Given two Hilbert C^*-modules M and N we say that M is Cuntz smaller than N, denoted by $M \preceq N$, if for all F, $F \subseteq\subseteq M$, there is F', $F' \subseteq\subseteq N$, isomorphic to F. This relation defines a preorder relation on the isomorphism classes of Hilbert modules over A. Let us say that M is Cuntz equivalent to N if $M \preceq N$ and $N \preceq M$. Let $[M]$ denote the equivalence class of all the modules Cuntz equivalent to a given module M. Then the relation $[M] \leqslant [N]$ if $M \preceq N$ defines an order on the Cuntz equivalence classes of right Hilbert modules over A.

Following [3], the Cuntz semigroup is defined as the ordered set of Cuntz equivalence classes of countably generated Hilbert modules over A endowed with the addition opperation $[M] + [N] := [M \oplus N]$. We denote this ordered semigroup by $Cu(A)$. It is shown in the appendix of [3] that the subsemigroup of $Cu(A)$ formed by the Cuntz equivalence classes $[M]$ of A-Hilbert modules M such that $M \subseteq A^n$ for some $n \geqslant 1$ coincides with the ordered semigroup defined by Cuntz in [4]. The latter semigroup, denoted by $W(A)$ in [13], is defined in terms of positive elements of $\bigcup_{n=1}^{\infty} M_n(A)$. It was also shown in [3] that $Cu(A) = W(A \otimes K)$. Furthermore, we can define functors $Cu(\cdot)$ and $W(\cdot)$ from the category of C^*-algebras to the category of ordered semigroups. By choosing a suitable subcategory of the category of ordered semigroups, Coward, Elliott and Ivanescu were able to show in [3] that the functor $Cu(\cdot)$ is continuous with respect to inductive limits.

Let I be a σ-unital closed two-sided ideal of A. If M is a countably generated Hilbert module over A then MI is also countably generated. Let us see that if $[M] \leqslant [N]$ then $[MI] \leqslant [NI]$. Suppose that $F \subseteq\subseteq MI$. Then there is $F' \subseteq\subseteq N$ isomorphic to it. Since F and F' are isomorphic and $FI = F$, we must have $F'I = F'$. So $F' \subseteq NI$. Hence $[F] = [F'] \leqslant [NI]$. Taking supremum over $F \subseteq\subseteq MI$ we get that $[MI] \leqslant [NI]$. In particular, if M and M' are Cuntz equivalent then MI and $M'I$ are also Cuntz equivalent. This justifies writing $[MI] := [M]I$. We have seen already that the map $[M] \mapsto [M]I$ is order preserving. Since $(M \oplus N)I = MI \oplus NI$, it is also additive. Notice that $M = MI$ (i.e., M is a Hilbert I-module) if and only if $[M]I = [M]$. If $M \subseteq A^n$ then $MI \subseteq A^n$, so the map $[M] \mapsto [M]I$ sends elements in $W(A)$ to elements in $W(A)$.

Let $\iota\colon I \to A$ and $\pi\colon A \to A/I$ denote the inclusion and quotient homomorphisms. The morphisms of ordered semigroups $Cu(\iota)$ and $Cu(\pi)$ are given by

$$\begin{aligned} Cu(\iota)([H]) &:= [H \otimes_\iota A] = [H], \\ Cu(\pi)([M]) &:= [M \otimes_\pi A/I] = [M/MI]. \end{aligned}$$

The restrictions of $Cu(\iota)$ and $Cu(\pi)$ to $W(I)$ and $W(A)$ respectively, give $W(\iota)$ and $W(\pi)$.

Proof of Theorem 1.1. The hypothesis $Cu(\pi)([M]) \leqslant Cu(\pi)([N])$ says that M/MI is Cuntz smaller than N/NI as A/I-Hilbert C^*-modules. We will first show that if M/MI is isomorphic to a submodule of N/NI then we have $[M] + [N]I \leqslant [N] + [M]I$.

Let $\phi\colon M/MI \to N'/N'I$ be an isomorphism of M/MI into $N'/N'I$, a submodule of N/NI. Let $C\colon M/MI \to M/MI$ be an arbitrary positive compact operator with dense range. This operator exists because M is countably generated. Then $\phi' = \phi \circ C$ is compact and satisfies that $\operatorname{im} \phi'^* \phi'$ is dense in M/MI. Since ϕ' is compact, it is also a compact operator after composing it with the inclusion of $N'/N'I$ into N/NI. Let us consider ϕ' as a compact operator having codomain N/NI. Let $T\colon M \to N$ be a compact operator that lifts ϕ'. We have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d] & N \arrow[d] \\ M/MI \arrow[r, "\phi'"] & N/NI. \end{tikzcd}$$

Since $\operatorname{im}\phi'^*\phi'$ is dense in M/MI, we have that $\operatorname{im}(T^*T) + MI$ is dense in M. Let $D_1\colon M \to M$ be positive and with $\operatorname{im} D_1$ dense in MI. The operator D_1 exists because MI is countably generated (here we use that I is σ-unital). Then $T^*T + D_1$ has dense range in M, that is, it is strictly positive. Let $\{F_n\}_{n=1}^{\infty}$ be an increasing sequence of submodules of M such that $T^*T + D_1$ is bounded from below on F_n and $\bigcup_n F_n$ is dense in M (e.g., $F_n = \operatorname{im}\phi_n(T^*T + D_1)$, where $\phi_n \in C_0(\mathbb{R}^+)$ has compact support and $\phi_n(t) \uparrow 1$). Let G be compactly contained in NI. We claim that $F_n \oplus G$ is isomorphic to a submodule on $N \oplus MI$. By Theorem 2.2, in order to prove this it is enough to find an operator (not necessarily adjointable) $\Phi\colon M \oplus NI \to N \oplus MI$ that is bounded from below when restricted to $F_n \oplus G$. Let us take

$$\Phi := \begin{pmatrix} T & -\iota_{NI,N} \\ D_1 & T^* \end{pmatrix},$$

where $\iota_{NI,I}$ is the inclusion map of NI in N. In order to show that Φ is bounded from below it is enough to show that $\Phi'\Phi$ is bounded from below, where Φ' is some bounded, possibly nonadjointable, operator. Let us choose $\Phi'\colon N \oplus MI \to M \oplus NI$ as follows:

$$\Phi' := \begin{pmatrix} T^* & \iota_{MI,M} \\ -D_2 & T \end{pmatrix},$$

where $D_2\colon N \to N$ has image in NI and is bounded from below on G. Then $\Phi'\Phi$ has the form

$$\Phi'\Phi = \begin{pmatrix} T^*T + D_1 & 0 \\ * & TT^* + D_2 \end{pmatrix}.$$

To show that the restriction of $\Phi'\Phi$ to $F_n \oplus G$ is bounded from below it is enough to show that the operators on the main diagonal are bounded from below (because the upper right corner is 0). This is true by our choice of F_n and D_2. So $F_n \oplus G$ is isomorphic to a submodule of $N \oplus MI$. Taking supremum over F_n and G we get that $[M] + [NI] \leqslant [N] + [MI]$.

Now suppose that $M/MI \preceq N/NI$. Let $F \subseteq\subseteq M$. Then $F/FI \subseteq\subseteq M/MI$, so F/FI is isomorphic to a submodule of N/NI. It follows that $[F] + [N]I \leqslant [N] + [F]I$. Taking supremum over all F, $F \subseteq\subseteq M$, we get $[M] + [NI] \leqslant [N] + [MI]$. ∎

COROLLARY 3.1. *Let I and J be σ-unital ideals. Suppose that $[M/M(I \cap J)] = [N/N(I \cap J)]$. Then*

$$[M]I + [N]J = [M](I + J) + [N](I \cap J) = [M](I \cap J) + [N](I + J).$$

Proof. We have $[M]I+[N]J=[M(I+J)]I+[NJ]=[M(I+J)]+[NJ]I=[M](I+J)+[N](I\cap J)$. ∎

COROLLARY 3.2. *The map $Cu(\pi)$ restricted to $Cu(A)+[l_2(I)]$ is an isomorphism onto $Cu(A/I)$.*

Proof. As remarked in the introduction, it follows from Theorem 1.1 and Kasparov's Stabilization Theorem that the map $Cu(\pi)$ is injective on $Cu(A)+[l_2(I)]$. The map $Cu(\pi)$ is surjective, since every A/I-Hilbert module can be embedded in $l_2(A/I)$, and then have its preimage taken by the quotient map $l_2(A)\to l_2(A/I)$. $Cu(\pi)$ is also surjective restricted to $Cu(A)+[l_2(I)]$, since adding $[l_2(I)]$ does not change the image in $Cu(A/I)$. Hence, $Cu(\pi)$ sends $Cu(A)$ $+[l_2(I)]$ isomorphically onto $Cu(A/I)$. ∎

The description of $Cu(A/I)$ obtained in Corollary 3.2 assumes that the ideal I is σ-unital. It is possible to obtain $W(A/I)$ as a quotient of $W(A)$ by a suitable equivalence relation without assuming that I is σ-unital. Since $Cu(A)\simeq W(A\otimes K)$, this result can also be applied to the Cuntz semigroup.

Recall that $W(A)$ can be defined as equivalence classes of positive elements on $\bigcup_n M_n(A)$ (see [13]). Given $[a],[b]\in W(A)$ let us say that $[a]\leqslant_I[b]$ if there is $c\in M_n(I)^+$ for some n such that $[a]\leqslant[b]+[c]$. We say that $[a]\sim_I[b]$ if $[a]\leqslant_I[b]$ and $[b]\leqslant_I[a]$.

PROPOSITION 3.3. *The semigroups $W(A)/\sim_I$ and $W(A/I)$ are isomorphic.*

Proof. Let $\pi\colon A\to A/I$ be, as before, the quotient homomorphism. Let us show that the map $W(\pi)([a])=\pi([a])$ induces an isomorphism after passing to the quotient by $\sim_I$. Since π is surjective, $W(\pi)$ is also surjective. It only rests to show that $W(\pi)([a])\leqslant W(\pi)([b])$ if and only if $[a]\leqslant_I[b]$.

Let a and b be positive elements in $M_n(A)$, such that $\pi(a)\preceq\pi(b)$. For all k, there is $d_k\in M_n(A/I)$ such that $\|\pi(a)-d_k\pi(b)d_k^*\|\leqslant 1/k$. By Lemma 2.2 of [8], there is $d_k'\in M_n(A/I)$ such that $(\pi(a)-1/k))_+=d_k'\pi(b)d_k'^*$. Let $f_k\in M_n(A)$ be such that $\pi(f_k)=d_k'$. We have

$$(a-1/k)_+=f_kbf_k^*+i_k\leqslant f_kbf_k^*+i_k^+,$$

for some $i_k^+\in M_n(I)^+$. We get that $[(a-1/k)_+]\leqslant[b]+[i_k^+]$. Let $i\in M_n(I)^+$ be an element such that $[i]$ majorizes the sequence $[i_k^+]$ (e.g., $i=\sum i_k^+/(2^k\|i_k^+\|)$). Taking supremum over k in $[(a-1/k)_+]\leqslant[b]+[i]$ we get $[a]\leqslant[b]+[i]$. Hence $[a]\leqslant_I[b]$. ∎

4. EXACTNESS OF THE CUNTZ SEMIGROUP FUNCTOR

Given S and T ordered, abelian semigroups, and $\phi\colon S \to T$ an order preserving semigroup map, let us define $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ as follows:

$$\begin{aligned}\mathrm{Ker}(\phi) &:= \{\, (s_1, s_2) \in S \times S : \phi(s_1) \leqslant \phi(s_2) \,\},\\ \mathrm{Im}(\phi) &:= \{\, (t_1, t_2) \in T \times T : \exists\, s \in S,\ t_1 \leqslant \phi(s) + t_2 \,\}.\end{aligned}$$

We denote by $\mathrm{im}\,\phi$ and $\ker\phi$ the image and the kernel of ϕ (i.e., the elements mapped to 0), in the standard sense.

By a short exact sequence of ordered semigroups we mean one which is exact with respect to the two notions of image and kernel mentioned above.

THEOREM 4.1. *Let I be a σ-unital ideal of A. The short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

induces split, short exact sequences of ordered abelian semigroups

$$0 \longrightarrow Cu(I) \overset{r}{\leftrightarrows} Cu(A) \overset{q}{\leftrightarrows} Cu(A/I) \longrightarrow 0, \tag{4.1}$$

$$0 \longrightarrow W(I) \overset{r}{\leftrightarrows} W(A) \longrightarrow W(A/I) \longrightarrow 0. \tag{4.2}$$

These sequences are also exact in the standard sense.

The maps r and q are defined as follows: $r([H]) := [HI]$ and $q([M]) := [M'] + [l_2(I)]$, where $[M']$ is such that $Cu(\pi)([M']) = [M]$.

Proof. We have already shown in Corollary 3.2 that the maps $Cu(\pi)$ and $W(\pi)$ are surjective. The maps $Cu(\iota)$ and $W(\iota)$ are injective, since if M is Cuntz smaller than N as I-modules, then the same holds when they are regarded as A-modules.

Let us prove the exactness of the sequence (4.1) and note that the same proof works also for the sequence (4.2). Exactness at $Cu(I)$ and $Cu(A/I)$ is easily verified. To check the exactness in the middle of the sequence (4.1) it suffices to prove that $\mathrm{Ker}(Cu(\pi)) \subseteq \mathrm{Im}(Cu(\iota))$, the other inclusion being obvious. The pair $([M],[N])$ belongs to $\mathrm{Ker}(Cu(\pi))$ precisely when $Cu(\pi)([M]) \leqslant Cu(\pi)([N])$, and this is equivalent by Theorem 1.1 with the fact that $[M] + [N]I \leqslant [N] + [M]I$. This shows that $([M],[N]) \in \mathrm{Im}(Cu(\iota))$, and hence $\mathrm{Ker}(Cu(\pi)) \subseteq \mathrm{Im}(Cu(\iota))$.

It only remains to show that the maps q and r define splittings of $Cu(\pi)$ and $Cu(\iota)$, respectively. We have already observed that $Cu(\pi)$ restricted to $Cu(A) + [l_2(I)]$ is an isomorphism of ordered semigroups. Its inverse is q. We have also noted that $M = MI$ (i.e., M is a Hilbert I-module) if and only if $[M]I = [M]$, which shows that r is a splitting of $Cu(\iota)$.

The restriction of r to $W(A)$ is a splitting of $W(\iota)$. ∎

REMARK 4.2. The maps r and q are not necessarily morphisms in the category defined by Coward, Elliott and Ivanescu in [3] since they might not preserve the relation of compact containment of elements in $Cu(A)$ (see [3] for the definition of this relation). To see this for r, let I be a σ-unital ideal in a C^*-algebra A such that $[A]$ is compactly contained in $[A]$, but $[I]$ is not compactly contained in $[I]$ (e.g., $A = C([0,1])$ and $I = C_0((0,1]))$. Then $r([A]) = [I]$, which shows that the map r does not preserve the compact containment relation. The map q does not preserve the compact containment relation either. One can see this by taking the zero element of $Cu(A/I)$, which is always compactly contained in itself, and noticing that q maps it into $[l_2(I)]$, which is not necessarily compactly contained in itself (e.g., I equal to the C^*-algebra of compact operators, and A the unitization of I).

Let us show that the map r preserves directed suprema.

PROPOSITION 4.3. *Let $\{[H_i]\}_{i=1}^{\infty}$ be an increasing sequence in $Cu(A)$ with supremum $[H]$. Then $[H]I = \sup_i([H_i]I)$.*

Proof. It will be enough to show that $[H]I \leqslant \sup_i([H_i]I)$, the other inequality being obvious. Let F be a compactly contained submodule of HI. Then F is compactly contained in H, hence we conclude that $[F] \leqslant [H_i]$ for some i (see Theorem 1 of [3]). This implies that $[F] \leqslant [H_i]I$, so $[F] \leqslant \sup_i([H_i]I)$. Taking supremum over F, we get that $[H]I \leqslant \sup_i([H_i]I)$. ∎

5. PROOF OF THEOREM 1.2.

Proof. By Theorem 2.1, there is an operator $T \in B(M,N)$ that lifts ϕ. The following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d] & N \arrow[d] \\ M/MI \arrow[r, "\phi"] & N/NI. \end{tikzcd}$$

The operator T in general will not be an isomorphism. However, by the commutativity of this diagram, and the fact that $\phi^* = \phi^{-1}$, we do have that

$$N = \operatorname{im} T + NI \quad \text{and} \quad M = \operatorname{im} T^* + MI.$$

We now follow the ideas of Mingo and Phillips's proof of the Stabilization Theorem ([11], Theorem 1.4) to find $\widetilde{T}\colon M \oplus l_2(I) \to N \oplus l_2(I)$ such that $\widetilde{T}$ and $\widetilde{T}^*$ have dense range. The desired isomorphism Φ will be obtained by the polar decomposition of $\widetilde{T}$.

Since I is σ-unital, the modules MI and NI are countably generated. Let $\{\eta_k\}$ and $\{\zeta_k\}$ be infinite sequences of generators of MI and NI respectively, such that each generator appears infinitely often. Let us define operators $\phi_1\colon l_2(I) \to N$, $\phi_2\colon l_2(I) \to l_2(I)$, and $\phi_3\colon l_2(I) \to M$ by the formulas

$$\phi_1((x_k)) := \sum_k \frac{1}{2^k}\zeta_k x_k, \quad \phi_2((x_k)) := (\frac{1}{4^k}x_k), \quad \phi_3((x_k)) := \sum_k \frac{1}{2^k}\eta_k x_k. \tag{5.1}$$

Let $\widetilde{T}\colon M \oplus l_2(I) \to N \oplus l_2(I)$ be defined by the matrix

$$\widetilde{T} := \begin{pmatrix} T & \phi_1 \\ \phi_3^* & \phi_2 \end{pmatrix}.$$

Notice that $\widetilde{T}$ is still a lift of ϕ. We have $\widetilde{T}(0,y) = (\phi_1 y, \phi_2 y)$, for $y \in l_2(I)$. It is argued in the proof of Theorem 1.4 of [11], that this set is dense in $NI \oplus l_2(I)$. Thus $NI \oplus l_2(I) \subseteq \overline{\operatorname{im}\widetilde{T}}$. Also, $\widetilde{T}(x,0) = (Tx,0) + (0,\phi_3^* x)$. So, $\operatorname{im} T \oplus 0 \subseteq \overline{\operatorname{im}\widetilde{T}}$. We conclude that $\operatorname{im}\widetilde{T}$ is dense in $N \oplus l_2(I)$. In the same way we show that $\widetilde{T}^*$ has dense range. Thus, the operator $\widetilde{T}$ admits a polar decomposition of the form $\widetilde{T} = \Phi(\widetilde{T}^*\widetilde{T})^{1/2}$, with Φ an isomorphism (see Proposition 15.3.7 of [16]). Passing to the quotients M/MI and N/NI, the operator $\widetilde{T}^*\widetilde{T}$ induces the identity. So Φ lifts ϕ. ∎

6. MULTIPLIER ALGEBRAS

Let A be a σ-unital algebra and I a σ-unital closed two-sided ideal of A. In this section we use Theorem 1.2 to explore the relationship between the multiplier algebras $M(A)$ and $M(A/I)$.

We shall consider A and I as countably generated right Hilbert modules over A. We shall identify the algebra $K(A)$ with A, and the algebra $B(A)$ with $M(A)$. All throughout this section we make the following two assumptions:

(1) the ideal I is stable;

(2) $A \simeq A \oplus I$ as A-Hilbert modules.

Let us denote by $s\colon M(A) \to M(I)$ the map given by restriction of the multipliers of A to the invariant submodule I. Let $\widetilde{\pi}\colon M(A) \to M(A/I)$ be the extension of the quotient map $\pi\colon A \to A/I$ by strict continuity. Recall that, by the noncommutative Tietze extension theorem, $\widetilde{\pi}$ is surjective.

Recall the fact that for $p, q \in M(A) \otimes M_n(\mathbb{C})$ projections, the modules pA^n and qA^n are isomorphic if and only if p and q are Murray–von Neumann equivalent.

LEMMA 6.1. *The following statements are equivalent:*

(i) *The ideal I is a direct summand of A as a right A-Hilbert module.*

(ii) *There is a projection $P_I \in M(A)$ such that $P_I A \subseteq I$ and $s(P_I)$ is Murray–von Neumann equivalent to the unit of $M(I)$.*

Any two projections of $M(A)$ that satisfy (ii) *are Murray–von Neumann equivalent in $M(A)$.*

Proof. Suppose we have (i). Let $A = I_1 \oplus N_1$, with $I_1 \simeq I$ as right Hilbert A-modules. Let $P_I \in M(A)$ be the projection onto I_1. Since I_1 is an I-module $I_1 I = I_1$. Hence $P_I A = (P_I A)I = P_I I \subseteq I$. Since the I-module $P_I I$ is isomorphic to I, it follows that P_I, as an I multiplier, is Murray–von Neumann equivalent to the identity of $M(I)$.

Suppose we have (ii). The I-modules I and $P_I I$ are isomorphic. Hence, they are isomorphic as A-modules. Since $P_I A \subseteq I$, we have $P_I I = P_I A$. Hence, $P_I A$ is a direct summand of A isomorphic to I.

If $P_I^{(1)}$ and $P_I^{(2)}$ satisfy (ii) then $P_I^{(1)} A = P_I^{(1)} I \simeq I \simeq P_I^{(2)} I = P_I^{(2)} A$. Thus, $P_I^{(1)}$ and $P_I^{(2)}$ are Murray–von Neumann equivalent. ∎

PROPOSITION 6.2. *Suppose A and I satisfy conditions* (i) *and* (ii) *above. The following propositions are true:*

(i) *Every unitary of $M(A/I)$ lifts to a unitary of $M(A)$.*

(ii) *If p and q are projections in $M(A)$ such that $\widetilde{\pi}(p)$ and $\widetilde{\pi}(q)$ are Murray–von Neumann equivalent in $M(A/I)$, then $p \oplus P_I$ and $q \oplus P_I$ are Murray–von Neumann equivalent in $M_2(M(A))$.*

(iii) *For every projection $p_0 \in M(A/I)$ there is $p \in M(A)$ such that $\widetilde{\pi}(p)$ is Murray–von Neumann equivalent to p_0.*

Proof. (i) Let $\Phi\colon A \to A \oplus I$ be an A-module isomorphism. This map induces an isomorphism $\phi\colon A/I \to (A \oplus I)/(A \oplus I)I$, and composing with the canonical identification of $(A \oplus I)/(A \oplus I)I$ and A/I, we get a unitary $\phi'\colon A/I \to A/I$. By Theorem 1.2, we can lift this unitary to a unitary $\Phi'\colon A \oplus I \to A \oplus I$. Now the map $\Phi_0 = (\Phi')^{-1}\Phi$ is an isomorphism of the Hilbert modules A and $A \oplus I$ that induces the identity in the quotient.

Let $u \in M(A/I)$ be unitary. By Theorem 1.2, there is a unitary $U\colon A \oplus I \to A \oplus I$ that lifts u. Then $\Phi_0^* U \Phi_0 \in M(A)$ is a unitary that lifts u.

(ii) Since the A/I-modules $\pi(p)A/I$ and $\pi(q)A/I$ are isomorphic, we get $pA \oplus I \simeq qA \oplus I$. We have $P_I A \simeq I$. Hence, $pA \oplus P_I A \simeq qA \oplus P_I A$. So $p \oplus P_I$ is Murray–von Neumann equivalent to $q \oplus P_I$.

(iii) The A-modules $\pi^{-1}(p_0 A/I) \oplus \pi^{-1}((1-p_0)A/I)$ and A are isomorphic in the quotient (to A/I). Thus $\pi^{-1}(p_0 A/I) \oplus \pi^{-1}((1-p_0)A/I) \oplus I \simeq A \oplus I \simeq A$. So $\pi^{-1}(p_0 A/I)$ is a direct summand of A. Let $p \in M(A)$ be such that $pA \simeq \pi^{-1}(p_0 A/I)$. Then $\pi(p)A/I \simeq p_0 A/I$, so $\pi(p)$ is Murray–von Neumann equivalent to p_0. ∎

REMARK 6.3. If A and I satisfy conditions (i) and (ii), then $M_n(A)$ and $M_n(I)$ satisfy them as well. So Proposition 6.2 applies to the pair $(M_n(A), M_n(I))$. If A is stable then $A \oplus I \simeq A$ (by the Stabilization Theorem), and I is stable. So (i) and (ii) are verified in this case too. More generally, suppose there is B stable such that $I \subseteq A \subseteq B$, and I is an ideal of B. Then there is $P_I \in M(B)$ that satisfies (ii) of

Lemma 6.1. The restriction of P_I to A is in $M(A)$ and satisfies (ii) of Lemma 6.1. Hence, in this case the pair A, I satisfies (i) and (ii).

Proposition 6.2 has some implications for the K-theory of the multiplier algebras $M(A)$ and $M(A/I)$. Part (i), applied to the algebras $M_n(A)$, implies that the map $K_1(M(A)) \to K_1(M(A/I))$ is surjective. Parts (ii) and (iii) imply that the map $K_0(M(A)) \to K_0(M(A/I))$ is an isomorphism. We can improve these results as follows.

Let B be a unital C^*-algebra. Let $A \otimes B$ be the minimal tensor product of A and B. Given H and E, Hilbert modules over A and B respectively, let us denote by $H \otimes E$ the external tensor product of H and E (see [9]). This is an $A \otimes B$ Hilbert module. Given A-Hilbert modules H_1 and H_2, let $B(H_1, H_2) \otimes B$ denote the norm closed subspace of $B(H_1 \otimes B, H_2 \otimes B)$ generated by operators of the form $T \otimes b$, with $T \in B(H_1, H_2)$ and $b \in B$. Note that the composition of operators in $B(H_1 \otimes B, H_2 \otimes B)$ with operators in $B(H_2 \otimes B, H_3 \otimes B)$ results in operators in $B(H_1 \otimes B, H_3 \otimes B)$.

Let $M(A, I)$ be the kernel of $\widetilde{\pi}\colon M(A) \to M(A/I)$. We have $M(A, I) = \{ x \in M(A) : xa, ax \in I \text{ for all } a \in A \}$.

PROPOSITION 6.4. *Let B be a unital C^*-algebra and A and I as before. Let $p \in M(A, I) \otimes B$ be a projection and $P_I' = P_I \otimes 1$, with P_I as in Lemma* 6.1(ii). *Then $p \oplus P_I'$ is Murray–von Neumann equivalent to* $0 \oplus P_I'$.

Proof. The multiplier projection p is an operator from $A \otimes B$ to $A \otimes B$ with range contained in $I \otimes B$. Let $\widetilde{p} \in B(A, I) \otimes B$ denote the adjointable operator obtained by simply restricting the codomain of p to $I \otimes B$. Let $\widetilde{P}_I' \in B(A, I) \otimes B$ be the corresponding operator for P_I'. Notice that $\widetilde{p}^*\widetilde{p} = p$, $\widetilde{p}\widetilde{p}^* = s(p) \in M(I) \otimes B$, and similarly for $\widetilde{P}_I'$. By Lemma 16.2 of [16], there is $V \in M_2(M(I) \otimes B)$, partial isometry, such that $V^*V = s(p) \oplus s(P_I')$ and $VV^* = 0 \oplus s(P_I')$. Let W be defined as

$$W := \begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{P}_I')^* \end{pmatrix} V \begin{pmatrix} \widetilde{p} & 0 \\ 0 & \widetilde{P}_I' \end{pmatrix}.$$

Then $W^*W = p \oplus P_I'$, $WW^* = 0 \oplus P_I'$, and $W \in M(A, I) \otimes B$. ∎

COROLLARY 6.5. *We have, for $i = 0, 1$,*

$$K_i(M(A, I)) = 0, \quad K_i(M(A)) \simeq K_i(M(A/I)), \quad K_i(M(A, I)/I) \simeq K_{1-i}(I).$$

Proof. From Proposition 6.4 we deduce that $K_0(M(A, I)) = 0$. Taking $B = C(\mathbb{T})$, we get $K_1(M(A, I)) = 0$. Now by the six term exact sequence associated to the extension $M(A, I) \to M(A) \to M(A/I)$, we have $K_i(M(A)) \simeq K_i(M(A/I))$, $i = 0, 1$. Looking at the extension $I \to M(A, I) \to M(A, I)/I$, we get that $K_i(I) = K_{1-i}(M(A, I)/I)$ for $i = 0, 1$. ∎

QUESTION 6.6. Suppose that A and I satisfy the conditions (i) and (ii) above. Is the unitary group of $M(A,I)^{\sim}$contractible in the norm or strict topologies?

If $I = A$ then A is stable, so the unitary group of $M(A)$ is contractible by the Kuiper–Mingo Theorem (see Theorem 16.8 of [16]).

7. EQUIVARIANT VERSION OF THEOREM 1.2

Let G be a locally compact Hausdorff group acting on the C^*-algebra A. A G-A Hilbert C^*-module, or simply a G-A-module, is a right Hilbert C^*-module endowed with a continuous action of G such that

$$\langle g\cdot v, g\cdot w\rangle = g(\langle v,w\rangle),\quad g\cdot(va) = (g\cdot v)(g(a)),\quad \text{for all } g\in G,\ v\in M, \text{ and } a\in A.$$

An operator between G-A-modules is equivariant if $T(g\cdot v) = gTv$. The action of g on T is defined as $(g\cdot T)(v) = gT(g^{-1}v)$. The operator T is G-continuous if the map $g \mapsto g\cdot T$ is continuous in the norm of operators.

Given a G-A-module M we denote by $L_2(G,M)$ the Hilbert C^*-module $L_2(G)\otimes M$, where $L_2(G)$ is the left regular representation of G. The action of G on $L_2(G,M)$ is defined as $g\cdot(\lambda\otimes m) = (g\cdot\lambda\otimes g\cdot m)$. The G-A-module $L_2(G,M)$ can also be viewed as the completion of $C_c(G,M)$, the M-valued continuous functions on G with compact support, with respect to the A-valued inner product $\langle h_1,h_2\rangle = \int\langle h_1(g),h_2(g)\rangle\,dg$.

Let I be a σ-unital, closed, two-sided ideal of A that is invariant by the action of G. Then we can define a quotient action of G on A/I. More generally, given a G-A-module M, we can define a natural (quotient) structure of G-A/I-module on M/MI.

We now state an equivariant version of Theorem 1.2 for compact groups (Theorem 2.1 of [6] and Theorem 2.5 of [11] in the case $I = A$).

THEOREM 7.1. *Suppose that the group G is compact. Let I be a σ-unital, invariant, closed, two-sided ideal of A. Let M and N be countably generated G-A-modules. Let $\phi\colon M/MI \to N/NI$ be an equivariant isomorphism. Then there is an equivariant isomorphism $\Phi\colon M\oplus L_2(G,l_2(I)) \to N\oplus L_2(G,l_2(I))$ that induces ϕ in the quotient.*

Proof. The proof is an adaptation of the proof of Theorem 1.2. The equivariant isomorphism $\phi\colon M/MI \to N/NI$ can be lifted to an equivariant operator $T\colon M\to N$ by first lifting it to an arbitrary operator T', and then averaging over the group: $Tx = \int(g\cdot T')x\,dg$. (This integration is possible because for all $x\in M$, the function $(g\cdot T')x$ is continuous in G.)

Next we construct the operator $\widetilde{T}$, this time making sure that it is equivariant. For this we need to replace the sequences of vectors $\{\eta_k\}$, $\{\zeta_k\}$, generators of MI and NI, by equivariant operators $\eta_k\colon L_2(G,I)\to M$, $\zeta_k\colon L_2(G,I)\to N$, such that $\sum \operatorname{im} \eta_k$ is dense in MI and $\sum \operatorname{im} \zeta_k$ is dense in NI. This is guaranteed by the following lemma.

LEMMA 7.2. *Suppose that H is a countably generated G-A-module. Let I be as before. Then there is a sequence $\eta_k \colon L_2(G, I) \to H$ of G-continuous maps such that $\sum \operatorname{im}(\eta_k)$ is dense in HI. If G is compact then these maps can be chosen equivariant.*

Before proving the lemma, let us proceed with the proof of the theorem. We define the maps ϕ_1 and ϕ_3 replacing the vectors η_k and ζ_k for the operators obtained using the lemma. The definition of the map ϕ_2 is unchanged. The resulting operator $\widetilde{T}$ is equivariant. Following the same argument of Mingo and Phillips, $\widetilde{T}$ and $\widetilde{T}^*$ have dense range. Since the unitary part of an equivariant operator is also equivariant, we get the equivariant isomorphism Φ by polar decomposition of $\widetilde{T}$.

Let us prove the lemma. First suppose that G is only locally compact. It is enough to find a G-continuous operator from $l_2(L_2(G, I))$ to H with range dense in HI. Let $C_1 \colon l_2(L_2(G, I)) \to HI$ be a G-continuous, surjective operator. Its existence is guaranteed by the Stabilization Theorem. Let $C_2 \colon HI \to HI$ be a compact operator with dense range. Then $C_2C_1 \colon l_2(L_2(G, I)) \to HI$ has dense range, and since it is compact, it is still an adjointable operator after composing it with the inclusion of HI in H. If G is compact we need to choose C_1 and C_2 equivariant. The equivariant C_1 exists by the Stabilization Theorem. We take $C_2 = \int (g \cdot C_2')\mathrm{d}g$, with $C_2' \in K(HI)^+$ strictly positive. In this way C_2 is strictly positive, thus it has dense range. ∎

REMARK 7.3. In the case that G is locally compact, Kasparov [6], and Mingo and Phillips [11], obtain a G-continuous isomorphism of $M \oplus L_2(G, l_2(A))$ and $L_2(G, l_2(A))$. Thus, it would be desirable to have a G-continuous version of Theorem 1.2. It is possible to obtain a G-continuous lift T of ϕ. Furthermore, the construction of the operator $\widetilde{T}$ can be carried through. However, the proof breaks down at the last step, since the unitary part of a G-continuous operator need not be G-continuous.

Acknowledgements. This work was completed during the 2007 Thematic Program on Operator Algebras at the Fields Institute for Research in Mathematical Sciences. Leonel Robert is supported by an NSERC PDF.

REFERENCES

[1] N.P. BROWN, F. PERERA, A. TOMS, The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras, *J. Reine Angew. Math.* **621**(2008), 191–211.

[2] A. CIUPERCA, G.A. ELLIOTT, A remark on invariants for C^*-algebras of stable rank one, *Internat. Math. Res. Notices* **5**(2008).

[3] K.T. COWARD, G.A. ELLIOTT, C. IVANESCU, The Cuntz semigroup as an invariant for C^*-algebras, *J. Reine Angew. Math.* **623**(2008), 161–193.

[4] J. CUNTZ, Dimension functions on simple C^*-algebras, *Math. Ann.* **233**(1978), 145–153.

[5] M. FRANK, Geometrical aspects of Hilbert C^*-modules, *Positivity* **3**(1999), 215–243.

[6] G.G. KASPAROV, Hilbert C^*-modules: theorems of Stinespring and Voiculescu, *J. Operator Theory* **4**(1980), 133–150.

[7] E. KIRCHBERG, M. RØRDAM, Non-simple purely infinite C^*-algebras, *Amer. J. Math.* **122**(2000), 637–666.

[8] E. KIRCHBERG, M. RØRDAM, Infinite non-simple C^*-algebras: absorbing the Cuntz algebra O_∞, *Adv. in Math.* **167**(2002), 195–264.

[9] E.C. LANCE, *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*, Cambridge Univ. Press, Cambridge 1995.

[10] J.A. MINGO, K-theory and multipliers of stable C^*-algebras, *Trans. Amer. Math. Soc.* **299**(1987), 397–411.

[11] J.A. MINGO, W.J. PHILLIPS, Equivariant triviality theorems for Hilbert C^*-modules, *Proc. Amer. Math. Soc.* **91**(1984), 225–230.

[12] G.K. PEDERSEN, SAW*-algebras and corona C^*-algebras, contributions to noncommutative topology, *J. Operator Theory* **15**(1986), 15–32.

[13] M. RØRDAM, The stable and the real rank of Z-absorbing C^*-algebras, *Internat. J. Math.* **15**(2004), 1065–1084.

[14] K. THOMSEN, Inductive limits of interval algebras: unitary orbits of positive elements, *Math. Ann.* **293**(1992), 47–63.

[15] A. TOMS, On the classification problem for nuclear C^*-algebras, *Ann. of Math. (2)* **167**(2008), 1029–1044.

[16] N.E. WEGGE-OLSEN, *K-Theory and C^*-Algebras*, Oxford Univ. Press, Oxford 1993.

ALIN CIUPERCA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON, M5S 2E4, CANADA
E-mail address: ciuperca@math.toronto.edu

LEONEL ROBERT, FIELDS INSTITUTE, TORONTO, ON, M5T 3J1, CANADA
E-mail address: lrobert@fields.toronto.edu

LUIS SANTIAGO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON, M5S 2E4, CANADA
E-mail address: santiago@math.toronto.edu

Received November 8, 2007.