C*-ALGEBRAS OF TILINGS WITH INFINITE ROTATIONAL SYMMETRY

MICHAEL F. WHITTAKER

Communicated by Kenneth R. Davidson

ABSTRACT. A tiling with infinite rotational symmetry, such as the Conway–Radin Pinwheel Tiling, gives rise to a topological dynamical system to which an étale equivalence relation is associated. A groupoid C*-algebra for a tiling is produced and a separating dense set is exhibited in the C*-algebra which encodes the structure of the topological dynamical system. In the case of a substitution tiling, natural subsets of this separating dense set are used to define an $\mathcal{A}\mathbb{T}$-subalgebra of the C*-algebra. Finally our results are applied to the Pinwheel Tiling.

KEYWORDS: *C*-algebras, dynamical systems, noncommutative geometry, operator algebras, substitution tilings, tilings.*

MSC (2000): 46L55, 47L40.

INTRODUCTION

A tiling of the plane refers to the covering of the 2-dimensional real vector space with polygons such that the polygons only intersect on their borders. A tiling of the plane gives rise to a compact metric space endowed with a continuous action of $\mathbb{R}^2$ which is a topological dynamical system. The connections between topological dynamical systems and operator algebras are well established [8]. On one hand, dynamical systems are used to produce interesting C*-algebras and on the other hand, invariants from operator algebras can be used to classify dynamical systems.

Associating a C*-algebra with a tiling began with a paper of Bellissard [2]. In 1995, Johannes Kellendonk [10] exhibited a groupoid structure on the dynamical system associated with a tiling and defined a C*-algebra using Renault's groupoid C*-algebras. The beauty of Kellendonk's construction was manifested in his reduction of the complexity, while preserving the structure, of the dynamics on the system which provided for the construction of a geometric separating subset of the C*-algebra as well as the formulation of natural C*-subalgebras.

Jared Anderson and Ian Putnam [1] later used topological methods to compute invariants of the dynamical system of a tiling, to which they associated a different C^*-algebra, and described these invariants in the framework of K-theory for C^*-algebras. Finally, we note the article [12] which gives a somewhat expository treatment of the articles [1] and [10], and serves as an excellent starting point for the definition of C^*-algebras from a tiling. In [12] the C^*-algebras produced in [1] and [10] were shown to be strongly Morita equivalent.

In the present article, Kellendonk's construction of a C^*-algebra from a tiling [10], [12] is extended to include tilings with infinite rotational symmetry. By now there are many examples of tilings with infinite rotational symmetry [7], but the most studied being the Conway–Radin Pinwheel Tiling [19]. The Pinwheel Tiling is built from a substitution on a finite base set of tiles called proto-tiles. Two iterations of the substitution on a proto-tile produces 25 tiles including a copy of the original proto-tile in the same orientation and a copy of the proto-tile which has been rotated at an irrational angle with respect to the original proto-tile. This fact implies that in a Pinwheel Tiling there are tiles which appear in an infinite number of distinct orientations. The reader who is unfamiliar with tiling theory may want to read the first three paragraphs of Section 5 in order to have an example at hand for the constructions in the first four sections.

We begin by presenting many of the standard definitions relating to tilings with sight modifications to admit tilings with infinite rotational symmetry. Of course, these modifications appear elsewhere in the literature in various guises [3], [15], [20], [19], [25], but our modifications anticipate the construction in the sequel. The dynamical system associated with a tiling is produced and the topological structure is discussed.

The construction of a C^*-algebra begins by producing a compact metric space based on a tiling which is a closed transversal, in the sense of [14], to the action of $\mathbb{R}^2$. Considering translational equivalence leads to an étale equivalence relation, viewed as a groupoid, and allows us to apply Renault's groupoid C^*-algebras to produce a C^*-algebra. A separating dense set is contained in the C^*-algebra and elements of this set are related by a commutation relation.

The C^*-algebra of a tiling with infinite rotational symmetry admits a particularly nice subalgebra when the tiling is produced from a substitution. We briefly discuss substitution tilings and exploit the substitution to define an $\mathcal{AT}$ C^*-subalgebra of our C^*-algebra. An $\mathcal{AT}$-algebra is the inductive limit of matrix algebras with entries that are continuous functions on the circle. Such algebras have been shown to be classifiable whenever they are simple [24].

In the final section, we apply our results to the Pinwheel Tiling and explicitly define the maps appearing in the construction of the $\mathcal{AT}$-algebra. We then show that the $\mathcal{AT}$-algebra for the Pinwheel Tiling is simple and comment on the K-theory of the inductive limit. We conclude with some open questions.

1. THE GEOMETRY OF TILINGS

Let $\mathbb{R}^2$ denote the 2-dimensional Euclidean vector space and $B(x, r)$ an open ball with radius r centered at x in $\mathbb{R}^2$. Given a subset X in $\mathbb{R}^2$, the translate of X by a vector y in $\mathbb{R}^2$ is given by $X + y = \{x + y : x \in X\}$.

DEFINITION 1.1. A *tiling* T of $\mathbb{R}^2$ is a countable collection of closed polygonal subsets $\{t_1, t_2, \ldots\}$ of $\mathbb{R}^2$, called tiles, whose union covers $\mathbb{R}^2$ and the pairwise intersection of the interiors of any two nonequal tiles is empty.

A tiling is a collection of subsets of $\mathbb{R}^2$, so the translate of a tiling T by a vector x in $\mathbb{R}^2$ is defined as $T + x = \{t + x : t \in T\}$. Notice that any translate of a tiling is also a tiling. To create a dynamical system we consider the set of all translates of a tiling T which is denoted $T + \mathbb{R}^2 = \{T + x : x \in \mathbb{R}^2\}$. We aim to complete this set in a suitable metric on tilings.

The construction of a metric on $T + \mathbb{R}^2$ requires some preliminary notions. To begin, define a group Γ as a closed subgroup of the group of orientation preserving isometries on $\mathbb{R}^2$ containing all translations, which we write as a semidirect product
$$\mathbb{R}^2 \subseteq \Gamma \subseteq \mathbb{R}^2 \rtimes \mathcal{S}^1$$
where $\mathcal{S}^1$ is identified with the group $SO(2, \mathbb{R})$. Elements in $\mathcal{S}^1$ are rotations about the origin of $\mathbb{R}^2$ and in the sequel these elements will be denoted by R_θ where
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
For (x, R_θ) and (y, R_ϕ) in Γ we have group operations
$$(x, R_\theta) \cdot (y, R_\theta) = (x + R_\theta(y), R_\theta \cdot R_\phi) \quad \text{and} \quad (x, R_\theta)^{-1} = (R_\theta^{-1}(-x), R_\theta^{-1})$$
and metric
$$d((x, R_\theta), (y, R_\phi)) = |x - y| + \left(\sum_{i,j=1}^{2} (a_{ij} - b_{ij})^2 \right)^{1/2}$$
where $R_\theta = (a_{ij})$ and $R_\phi = (b_{ij})$ in the matrix form given above. Furthermore, we allow an element (x, R_θ) in Γ to act on a tiling T via $(x, R_\theta) \cdot T = R_\theta(T + x)$. We shall often abuse notation and use $T + x$ for the tiling $(x, R_0) \cdot T$ and $R_\theta(T)$ for the tiling $(0, R_\theta) \cdot T$.

DEFINITION 1.2. A *patch* P in a tiling T is a finite subset of tiles in T. The notation $T \cap B(x, R)$ will denote the patch of tiles completely contained in the ball of radius R about the point x in $\mathbb{R}^2$.

DEFINITION 1.3. Suppose Λ is any collection of tilings and Γ is a closed subgroup of the orientation preserving isometries of $\mathbb{R}^2$. If T and T' are tilings in Λ, we define the *tiling metric*, d_Γ, on the Λ as follows. For $0 < \varepsilon < 1$ we say the distance between T and T' is less than ε if we can find elements (x, R_θ)

and $(x', R_{\theta'})$ in Γ such that $d((x, R_\theta), (0, R_0)) < \varepsilon$, $d((x', R_{\theta'}), (0, R_0)) < \varepsilon$, and $R_\theta(T + x) \cap B(0, \varepsilon^{-1}) = R_{\theta'}(T' + x') \cap B(0, \varepsilon^{-1})$. Now $d_\Gamma(T, T')$ is the infimum over the set consisting of each such ε satisfying the above hypothesis. If no such ε exists we define $d_\Gamma(T, T')$ to be 1.

The distance between two tilings is small when the tilings have the same pattern on a large ball about the origin up to a small orientation preserving isometry in Γ. Of course, the collection of tilings we are concerned about is $T + \mathbb{R}^2$.

DEFINITION 1.4. Given a tiling T, the *continuous hull* of T, denoted by Ω_T, is the completion of $T + \mathbb{R}^2$ in the tiling metric d_Γ.

We note that every element in the continuous hull is a tiling of $\mathbb{R}^2$ and the metric d_Γ extends to Ω_T. Moreover, for every tiling T in Ω_T and every x in $\mathbb{R}^2$, $T + x$ is also an element of the continuous hull. From this it follows that $\mathbb{R}^2$ is a continuous group action on Ω_T and we are interested in the dynamics of the system $(\Omega_T, \mathbb{R}^2)$. For further details see [12].

The local structure of a single tiling, more accurately the patterns in each tiling, play a large role in the global structure of the continuous hull. We briefly present the notion of finite local complexity of a tiling and its consequences.

DEFINITION 1.5. Suppose Λ is a collection of tilings, we say Λ has *finite local complexity* with respect to the group Γ if, for all $R > 0$, the set $\{T \cap B(x, R) : T \in \Lambda \text{ and } x \in \mathbb{R}^2\} / \Gamma$ is finite.

LEMMA 1.6. *The continuous hull of a tiling Ω_T has finite local complexity with respect to Γ if for all $R > 0$, the set $\{T' \cap B(0, R) : T' \in \Omega_T\} / \Gamma$ is finite.*

THEOREM 1.7 ([21]). *If T is a tiling satisfying finite local complexity with respect to Γ, then the metric space (Ω_T, d_Γ) is compact.*

The choice of the group Γ is contingent on the original tiling T in the sense that we would like to choose Γ to be the smallest subgroup of the orientation preserving isometries of $\mathbb{R}^2$ such that Ω_T has finite local complexity. For example, the Penrose Tiling has finite local complexity for $\Gamma = \mathbb{R}^2$, see [1]. However, the predominant example of a tiling with infinite rotational symmetry, the Pinwheel Tiling, has finite local complexity only when Γ is the full group of orientation preserving isometries of $\mathbb{R}^2$. We shall say more about this in Section 5. Finite local complexity also allows for the description of a base set of tiles from which a tiling is created, called proto-tiles.

DEFINITION 1.8. Given $\{p_1, p_2, \ldots, p_n\}$, a finite and non-empty collection of closed polygonal subsets of $\mathbb{R}^2$, we say that $\{p_1, p_2, \ldots, p_n\}$ is a set of proto-tiles for a collection of tilings Λ if for each tile t in T such that T is in Λ we have $t = \gamma(p_i)$ for some γ in the group Γ, $i \in \{1, 2, \ldots, n\}$.

DEFINITION 1.9. We say that a tiling T is *aperiodic* if $T + x \neq T$ for every non-zero x in $\mathbb{R}^2$. Furthermore, we say that the continuous hull Ω_T is *strongly aperiodic* if Ω_T contains no periodic tilings.

Our interest lies in tilings where the continuous hull is strongly aperiodic. In the sequel, we will construct a groupoid C^*-algebra from a subset of the continuous hull. Strongly aperiodic tiling systems allow us to consider translational equivalence classes which are viewed as the leaves of a foliation with a closed transversal that is homeomorphic to the product of a Cantor set and a circle in the infinite rotation case. The following theorem gives conditions for when the continuous hull is strongly aperiodic.

THEOREM 1.10 ([1]). *If the action of $\mathbb{R}^2$ on Ω_T is minimal and T is aperiodic, then Ω_T is strongly aperiodic.*

2. THE ÉTALE EQUIVALENCE RELATION OF A TILING

In this section we generalize the construction of Kellendonk [10], [12] to include tilings with infinite rotational symmetry such as the Pinwheel tiling. The basic idea is to use Kellendonk's punctured hull to define an étale equivalence relation which encodes the dynamics of the system $(\Omega_T, \mathbb{R}^2)$. Once we have shown that we have an étale equivalence relation, we construct, using Renault's groupoid C^*-algebras [22], a C^*-algebra on the dynamical system associated with a tiling.

Suppose X is a set and R is an equivalence relation on $X \times X$. The equivalence relation R is endowed with a natural groupoid structure as follows. Multiplication is partially defined on R with $(x, y) \cdot (w, z) = (x, z)$ if and only if $y = w$ and inverses are defined by $(x, y)^{-1} = (y, x)$. The range and source map $r : R \to X$ and $s : R \to X$ are defined via $r(x, y) = x$ and $s(x, y) = y$. The unit space is the diagonal and is denoted $\Delta = \{(x, x) : x \in X\}$. Furthermore, a triple $(X, R, \mathcal{T})$ is said to be an *étale equivalence relation* when X is a compact Hausdorff space and R has been endowed with an étale topology $\mathcal{T}$. For our purposes $\mathcal{T}$ is an *étale topology* on R if the following conditions are satisfied:

  (i) $(R, \mathcal{T})$ is σ-compact;
 (ii) $\Delta = \{(x, x) : x \in X\}$ is open in $(R, \mathcal{T})$;
(iii) every point (x, y) in R has an open neighborhood U in $(R, \mathcal{T})$ such that $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms;
 (iv) if U and V are open sets in $(R, \mathcal{T})$, then UV is open in $(R, \mathcal{T})$;
  (v) if U is an open set in $(R, \mathcal{T})$, then U^{-1} is open in $(R, \mathcal{T})$.

The running assumptions for the remainder of this paper are:

  • the group Γ is equal to the full group of orientation preserving isometries on $\mathbb{R}^2$,

- the continuous hull Ω_T satisfies finite local complexity with respect to Γ,
- the continuous hull Ω_T is strongly aperiodic.

We note that restriction from the case of $\Gamma = \mathbb{R}^2 \rtimes \mathcal{S}^1$ to any closed subgroup Γ is done in the obvious way.

DEFINITION 2.1 ([10]). For a tiling T define a point in the interior of each proto-tile which maximally breaks the symmetry of the proto-tile. Such a point is called a *puncture*. For each proto-tile p_i in $\{p_1, p_2, \ldots, p_n\}$, denote the puncture by $x(p_i)$. Since each tile in T is the image of a proto-tile under Γ, we may extend punctures to each tile $t \in T$ via Γ; i.e. if Γ relates two tiles ($t_1 = (x, R_\theta)t_2$ for some $(x, R_\theta) \in \Gamma$) then Γ relates their punctures in the same way $(x(t_1) = (x, R_\theta)(x(t_2)))$.

We remark that a puncture in a proto-tile is said to maximally break the symmetry of the proto-tile if the pointed polygon $(p, x(p))$ has trivial symmetry group.

DEFINITION 2.2 ([10]). Given the continuous hull of a tiling Ω_T, the *discrete hull* of Ω_T, denoted by Ω_{punc}, is defined to be all punctured tilings T' in Ω_T such that the origin of $\mathbb{R}^2$ is a puncture of some tile t in T'. Introducing the notation that $T'(0)$ is the tile in T' that contains the origin, 0 in $\mathbb{R}^2$, we have

$$\Omega_{\text{punc}} = \{T' \in \Omega_T : x(t) = 0 \text{ for } t = T'(0)\}.$$

Since punctures are in the interior of tiles, the tile t such that $x(t) = 0$ is unique.

Some notation will be necessary for the remainder of this note. From this point forward we shall consider the proto-tiles $\{p_1, \ldots, p_n\}$ to have their puncture on the origin and to be fixed in a standard orientation. We can also rotate the proto-tiles around the origin and keep track of the rotation. For a tiling T with tile $T(0)$ having puncture on the origin we define $\angle T(0) = \theta$ such that $R_\theta(p_i) = T(0)$ for some $i \in \{1, \ldots, n\}$. A natural subset of Ω_{punc} is the collection of tilings with a tile sitting on the origin in the same orientation as the proto-tiles; i.e. define

$$\Omega_{\text{punc}}^0 = \{T \in \Omega_{\text{punc}} : T(0) \in \{p_1, \ldots, p_n\}\}.$$

Now each tiling T in Ω_{punc} may be written as $R_\theta(T_0)$ for some tiling T_0 in Ω_{punc}^0. Furthermore, we define a homeomorphism $\psi : \Omega_{\text{punc}} \to \Omega_{\text{punc}}^0 \times \mathcal{S}^1$ via $\psi(T) = (T_0, R_\theta)$, where $\mathcal{S}^1$ is the group of rotations on $\mathbb{R}^2$. After stating some properties of the relationship between Ω_T, Ω_{punc}, and Ω_{punc}^0 we shall prove that ψ is a homeomorphism.

LEMMA 2.3. *Given the continuous hull of a tiling Ω_T, we have:*
(i) *If T' is in Ω_T, then $R_\theta(T' + x)$ is in Ω_{punc}^0 for some (x, R_θ) in Γ.*
(ii) *Ω_{punc} is closed in Ω_T.*
(iii) *There is an $\varepsilon > 0$ such that for any tiling T in Ω_{punc}, $T + x$ is not in Ω_{punc} for any $0 < |x| < \varepsilon$.*

(iv) *For every nontrivial rotation R_θ in $\mathcal{S}^1$ and every T_0 in Ω^0_{punc}, $R_\theta(T_0)$ is not in* Ω^0_{punc}.

(v) *There is an $\varepsilon > 0$ such that for any tiling T_0 in Ω^0_{punc}, $R_\theta(T_0 + x)$ is not in Ω^0_{punc} for any (x, R_θ) in Γ such that $0 < d((x, R_\theta), (0, R_0)) < \varepsilon$.*

Proof. Statement (i) is obvious. For (ii) take $T' \in \Omega_T$ a point of closure of Ω_{punc}. There is a sequence $\{T_n\}_{n=1}^\infty \in \Omega_{\text{punc}}$ such that $\lim_{n\to\infty} T_n = T'$. Let $T_n(0) = t_n$ and $T'(0) = t'$ where $x(t_n) = 0, \forall n \in \mathbb{N}$. Now $\lim_{n\to\infty} x(t_n) = x(t')$ which implies $0 = x(t')$. Whence, $T' \in \Omega_{\text{punc}}$. For (iii), if $\mathcal{P}$ is the set of proto-tiles for Ω_{punc}, let $\varepsilon > 0$ be smaller than the minimum distance from every proto-tile's puncture to the proto-tile's boundary. Statement (iv) is obvious and (v) follows from (iii). ∎

LEMMA 2.4. *The map $\psi : \Omega_{\text{punc}} \to \Omega^0_{\text{punc}} \times \mathcal{S}^1$ via $\psi(T) = (T_0, R_\theta)$ is a homeomorphism.*

Proof. Suppose $\psi(T) = (T_0, R_\theta) = (T'_0, R_{\theta'}) = \psi(T')$. Then $T_0 = T'_0$ and since punctures maximally break the symmetry of tiles, $\mathcal{S}^1$ acts freely on Ω^0_{punc} so that $R_\theta = R_{\theta'}$ and hence ψ is injective. To show that ψ is surjective, let (T_0, R_θ) be an element of $\Omega^0_{\text{punc}} \times \mathcal{S}^1$. Since $\Omega^0_{\text{punc}} \subset \Omega_{\text{punc}}$ we have T_0 in Ω_{punc} and Γ is the full group of orientation preserving isometries so that $R_\theta(T_0)$ is in Ω_{punc} as well. Continuity is automatic when Ω_{punc} has the tiling metric and $\Omega^0_{\text{punc}} \times \mathcal{S}^1$ has the product metric, so ψ is a homeomorphism since Ω_{punc} is compact. ∎

The topology of Ω_{punc}, given by the tiling metric, can now be described using the product structure coming from the homeomorphism ψ. Indeed, suppose P is a finite patch in some T in Ω_{punc} and t is a tile in P; then we define the set $U(P, t)$ to be all tilings T' in Ω_{punc} such that the image of the patch $P - x(t)$ under an element R_θ of $\mathcal{S}^1$ is a patch in T' with $R_\theta(P - x(t))$ at the origin. In symbols:

$$U(P, t) = \{T' \in \Omega_{\text{punc}} : R_\theta(P - x(t)) \subset T' \text{ for some } R_\theta \in \mathcal{S}^1\}.$$

Let $\mathcal{U}$ be the collection of all such sets for a given discrete hull Ω_{punc}.

Let us define the collection $\mathcal{U}_0$ to be the restriction of $\mathcal{U}$ to Ω^0_{punc}. Using the homeomorphism ψ it follows that $\mathcal{U}$ can be identified with $\mathcal{U}_0 \times \mathcal{S}^1$. It is easily verified that $\mathcal{U}_0$ generates the metric topology of Ω^0_{punc} which is the tiling metric restricted to $\mathbb{R}^2$ in Γ.

LEMMA 2.5. *The punctured hull of a tiling, Ω_{punc}, is the product of a Cantor set and a circle. Moreover, Ω^0_{punc} is a Cantor set in the relative topology of $\mathcal{U}_0$.*

Proof. We begin by showing that elements of $\mathcal{U}_0$ are closed. Let $U_0(P, t)$ be in $\mathcal{U}_0$ and pick $\varepsilon > 0$ so small that $P - x(t) \subset B(0, \varepsilon^{-1})$ and ε satisfies condition (v) of Lemma 2.3. Suppose T is a point of closure of $U_0(P, t)$ and $\{T_n\}$ is a sequence in $U_0(P, t)$ converging to T. For some N we have that $d_\Gamma(T, T_n) < \varepsilon$ for all $n \geqslant N$.

By (v) in Lemma 2.3 and finite local complexity it follows that $T \cap B(0, \varepsilon^{-1}) = T_n \cap B(0, \varepsilon^{-1})$ for all $n \geqslant N$. Since $P - x(t) \subset B(0, \varepsilon^{-1})$ it follows that T is in $U_0(P, t)$. So Ω_{punc}^0 is generated by closed and open sets which are invariant under translation via strong aperiodicity. Whence, Ω_{punc}^0 is totally disconnected and has no isolated points. Furthermore, Ω_{punc}^0 is compact and hence a Cantor set. ∎

The next step is to define translational equivalence on Ω_{punc}. We define the equivalence relation $R_{\text{punc}} \subset \Omega_{\text{punc}} \times \Omega_{\text{punc}}$ as follows:

$$R_{\text{punc}} = \{(T, T') : T \in \Omega_{\text{punc}} \text{ and } T' = T - x(t) \text{ for some } t \in T\}.$$

We want to define a suitable topology on R_{punc} which encodes both the topology on Ω_{punc} and the translation. The natural topology, which we denote $\mathcal{T}_R$, on R_{punc} is the metric topology defined as follows:

$$d_R((T, T - x(t)), (T', T' - x(t'))) = d_\Gamma(T, T') + |x(t) - x(t')|.$$

In analogy with the notation presented for Ω_{punc} we may view R_{punc} as $R_{\text{punc}}^0 \times \mathcal{S}^1$ where

$$R_{\text{punc}}^0 = \{(T_0, T_0 - x(t)) : T_0 \in \Omega_{\text{punc}}^0 \text{ and } t \in T_0\}.$$

If P is a finite patch in some T in Ω_{punc} and t, t' are tiles in P then we define the set $V(P, t, t')$ to be all elements (T, T') in R_{punc} such that $R_\theta(P - x(t)) \subset T$ for some R_θ in $\mathcal{S}^1$ and $T' = T - R_\theta(x(t') - x(t))$. In symbols:

$$V(P, t, t') = \{(T, T - R_\theta(x(t') - x(t))) : R_\theta(P - x(t)) \subset T \text{ for some } R_\theta \in \mathcal{S}^1\}.$$

Let $\mathcal{V}$ be the collection of all such sets for a given discrete hull Ω_{punc}. Moreover, let $\mathcal{V}_0$ be the restriction of $\mathcal{V}$ to R_{punc}^0.

The following lemma is verified directly, and implies that condition (iii) of the étale topology holds for R_{punc}.

LEMMA 2.6. *The range and source maps $r, s : R_{\text{punc}} \to \Omega_{\text{punc}}$ are local homeomorphisms such that, for $V(P, t, t')$ in $\mathcal{V}$,*

$$r(V(P, t, t')) = U(P, t) \quad and \quad s(V(P, t, t')) = U(P, t')$$

where $U(P, t)$ and $U(P, t')$ are in $\mathcal{U}$.

Since the range and source maps are homeomorphisms from $\mathcal{V}$ to $\mathcal{U}$ it follows that elements of $\mathcal{V}$ are compact and open via Lemma 2.5.

PROPOSITION 2.7. *The triple $(\Omega_{\text{punc}}, R_{\text{punc}}, \mathcal{T}_R)$ is an étale equivalence relation.*

Proof. We show that $\mathcal{T}_R$ is an étale topology on R_{punc}. Indeed, R_{punc} is locally compact from Lemma 2.6 since $R_{\text{punc}} = \bigcup \mathcal{V}$. The diagonal of R_{punc} may be written as:

$$\Delta = \bigcup \{V(P, t, t) : P \text{ is a finite patch and } t \in P\},$$

which is homeomorphic to Ω_{punc} and hence is compact and open. The fact that the range and source maps are local homeomorphisms is the content of

Lemma 2.6. The computation that the product and inverse of open sets in R_{punc} is open is left to the reader, however, we note that both follow easily by noting that open sets in R_{punc} may be written as unions of sets of the form $V_0 \times E$ where V_0 is in $\mathcal{V}_0$ and E is an open set in $\mathcal{S}^1$. ∎

3. A C^*-ALGEBRA ASSOCIATED WITH A TILING

In the previous section we defined an equivalence relation with topology coming from the punctured hull and translations in $\mathbb{R}^2$. In a sense we have discretized the action of $\mathbb{R}^2$ on the continuous hull and we aim to define a groupoid C^*-algebra based on this structure using Renault's groupoid C^*-algebras [22]. We note here that a construction using a crossed product $C(\Omega_T) \rtimes \mathbb{R}^2$ is also possible in this case, see [12] for details. The key point is that the algebra created here is Morita equivalent to the crossed product and has a much more tractable form. In the final two sections we describe a natural subalgebra in the case of a substitution tiling and apply the construction to the Pinwheel Tiling.

The construction presented here is the standard construction of the reduced C^*-algebra from an étale equivalence relation. The C^*-algebra comes with a generating set of functions consisting of partial isometries which have commutation relations very reminiscent of the irrational rotation algebra.

The reader is reminded of the assumptions presented at the beginning of Section 2. Furthermore, we shall denote equivalence classes of R_{punc} by $[T] = \{T' : (T, T') \in R_{\text{punc}}\}$ where T is a representative punctured tiling from the class.

We begin by considering the continuous functions of compact support on R_{punc}, denoted $C_c(R_{\text{punc}})$, which is a complex linear space. A product and adjoint are defined on continuous compactly supported functions f and g via

$$f \cdot g(T, T') = \sum_{T'' \in [T]} f(T, T'')g(T'', T'), \quad f^*(T, T') = \overline{f(T', T)},$$

which endows $C_c(R_{\text{punc}})$ with a $*$-algebra structure. A compactness argument paired with the fact that the range and source maps are local homeomorphisms implies that the sum in the product above is non-zero on only a finite number of terms. The inductive limit topology on $C_c(R_{\text{punc}})$, as a $*$-algebra, makes $C_c(R_{\text{punc}})$ into a topological $*$-algebra which reflects the groupoid structure. The inductive limit topology is described by: a sequence $\{f_n\}$ converges to f in $C_c(R_{\text{punc}})$ if and only if there exists a compact subset K in R_{punc} such that the support of f is contained in K, the support of $\{f_n\}$ is eventually in K, and $\{f_n\}$ converges uniformly to f on K.

The construction of the reduced C^*-algebra of R_{punc} is defined by representing $C_c(R_{\text{punc}})$, with the above topological $*$-algebra structure, as bounded operators on the Hilbert space $\ell^2([T])$ for some T in Ω_{punc}. In particular, for T in Ω_{punc}, ξ in $\ell^2([T])$, and T' in $[T]$ the induced representation from the unit space

$\pi_T : C_c(R_{\text{punc}}) \to \mathcal{B}(\ell^2([T]))$ is defined by

$$(\pi_T(f)\xi)(T') = \sum_{T'' \in [T]} f(T', T'')\xi(T'').$$

It is verified in [22] that the induced representations from the unit space are non-degenerate representations. The reduced C^*-algebra norm is defined, for f in $C_c(R_{\text{punc}})$, via

$$\|f\|_{\text{red}} = \sup\{\|\pi(f)\| : \pi \text{ is an induced representation}\}.$$

Renault [22] goes on to show that this defines a C^*-norm, called the reduced norm.

DEFINITION 3.1. The completion of $C_c(R_{\text{punc}})$ in the reduced C^*-norm is called the *reduced C^*-algebra of R_{punc}* and is denoted $C^*(R_{\text{punc}})$.

We have defined only the reduced C^*-algebra norm because the idea is to get our hands on a C^*-algebra rather than compare the full and reduced C^*-algebras. Furthermore, for a large class of tilings, namely tilings arising from a substitution, the étale equivalence relation is an amenable groupoid making the comparison trivial.

The remaining goal of this section is to define a collection of functions in $C_c(R_{\text{punc}})$ whose closed linear span is $C^*(R_{\text{punc}})$. The collection will consist of partial isometries and allows for easy manipulation of the reduced C^*-algebra. Moreover, these partial isometries come equipped with a commutation relation similar to the commutation relation in the irrational rotation algebra.

The generating set for our dense subalgebra consists of combinations of partial isometries and a unitary. Viewing Ω_{punc} as the product of a Cantor set and a circle, the partial isometries determine, up to some $\varepsilon > 0$, the location in the Cantor set with a translation and the unitary determines the location in the circle. Define for each $V(P, t, t')$ in $\mathcal{V}$ the characteristic function $e(P, t, t') = \mathcal{X}_{V(P,t,t')}$ and z to be the function with support on the diagonal of R_{punc} taking a tiling (T, T) to the character of the angle of the tile sitting at the origin of T with respect to the proto-tiles in fixed standard orientation. The explicit descriptions are as follows:

$$e(P,t,t')(T,T') = \begin{cases} 1 & \text{if } R_\theta(P{-}x(t)) \subset T \text{ for some } R_\theta \in \mathcal{S}^1 \text{ and } T' = T{-}R_\theta(x(t'){-}x(t)), \\ 0 & \text{otherwise;} \end{cases}$$

$$z(T,T') = \begin{cases} e^{\angle T(0)\,i} & \text{if } T = T', \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{E}$ be the set of functions

$$\mathcal{E} = \{z^k \cdot e(P, t, t') : P \text{ is a patch in } T \text{ and } t, t' \in P, k \in \mathbb{Z}\}.$$

LEMMA 3.2. *Let P be a patch in a tiling T in Ω_{punc} such that t, t' are tiles in P, both $e(P, t, t')$ and z are in $C_c(R_{\text{punc}})$. Moreover, functions in the set $\mathcal{E}$ have the following properties:*

(i) $e(P, t, t')^* = e(P, t', t)$;

(ii) $z \cdot e(P, t, t') = e^{(\angle t - \angle t') i}(e(P, t, t') \cdot z)$;

(iii) $e(P_1, t_1, t_1') \cdot e(P_2, t_2, t_2') = e(P_1 \cup P_2, t_1, t_2')$ if $t_1' = t_2$ and P_1, P_2 agree where they intersect; and

(iv) $(z^k \cdot e(P_1, t_1, t_1')) \cdot (z^l \cdot e(P_2, t_2, t_2')) = e^{l(\angle t_2 - \angle t_1) i}(z^{k+l} \cdot e(P_1 \cup P_2, t_1, t_2'))$ if $t_1' = t_2$ and P_1, P_2 agree where they intersect.

Proof. The facts that $e(P, t, t')$ and z are in $C_c(R_{\text{punc}})$ follow from $V(P, t, t') \in \mathcal{V}$ being compact and open and the diagonal of R_{punc} being compact and open respectively, see Lemma 2.5 and Proposition 2.7. For the properties; part (i) is obvious, we prove parts (ii) and (iii), and part (iv) follows by combining (ii) and (iii).

For (ii) we take the product of the two functions in each order and produce the commutation relation. Indeed, on one hand

$$z \cdot e(P, t, t')(T, T') = \sum_{T'' \in [T]} z(T, T'')e(P, t, t')(T'', T') = z(T, T)e(P, t, t')(T, T')$$

$$= e^{(\angle T(0)) i}e(P, t, t')(T, T')$$

$$= \begin{cases} e^{(\angle t + \theta) i} & \text{if } R_\theta(P - x(t)) \subset T \text{ and } T' = T - R_\theta(x(t') - x(t)) \\ & \text{for some } R_\theta \in \mathcal{S}^1; \\ 0 & \text{otherwise}; \end{cases}$$

$$= e^{(\angle t + \theta) i}e(P, t, t')(T, T').$$

On the other hand

$$e(P, t, t') \cdot z(T, T') = \sum_{T'' \in [T]} e(P, t, t')(T, T'')z(T'', T') = e(P, t, t')(T, T')z(T', T')$$

$$= e^{(\angle T'(0)) i}e(P, t, t')(T, T')$$

$$= \begin{cases} e^{(\angle t' + \theta) i} & \text{if } R_\theta(P - x(t)) \subset T \text{ and } T' = T - R_\theta(x(t') - x(t)) \\ & \text{for some } R_\theta \in \mathcal{S}^1; \\ 0 & \text{otherwise}; \end{cases}$$

$$= e^{(\angle t' + \theta) i}e(P, t, t')(T, T').$$

For (iii) we have

$$e(P_1, t_1, t_1') \cdot e(P_2, t_2, t_2')(T, T')$$

$$= \sum_{T'' \in [T]} e(P_1, t_1, t_1')(T, T'')e(P_2, t_2, t_2')(T'', T')$$

$$= \sum_{T'' \in [T]} \begin{cases} 1 & \text{if } R_\theta(P_1 - x(t_1)) \subset T \text{ and } R_\phi(P_2 - x(t_2)) \subset T'' \\ & \text{for some } R_\theta, R_\phi \in \mathcal{S}^1 \text{ such that} \\ & \quad T'' = T - R_\theta(x(t_1') - x(t_1)), \ T' = T'' - R_\phi(x(t_2') - x(t_2)); \\ 0 & \text{otherwise}; \end{cases}$$

$$
= \begin{cases} 1 & \text{if } R_\theta(P_1 - x(t_1)) \subset T \text{ and } R_\phi(P_2 - x(t_2)) \subset T - R_\theta(x(t_1') - x(t_1)) \\ & \text{such that } T' = T - R_\theta(x(t_1') - x(t_1)) - R_\phi(x(t_2') - x(t_2)) \text{ for some } R_\theta, R_\phi; \\ 0 & \text{otherwise}; \end{cases}
$$

$$
= \begin{cases} 1 & \text{if } R_\theta(P_1 \cup P_2 - x(t_1)) \subset T \text{ such that } R_\theta = R_\phi \in \mathcal{S}^1, \\ & t_1' = t_2, \text{ and } T' = T - R_\theta(x(t_2') - x(t_1)); \\ 0 & \text{otherwise}; \end{cases}
$$

$$
= e(P_1 \cup P_2, t_1, t_2')(T, T') \text{ if } t_1' = t_2. \quad \blacksquare
$$

Some remarks are in order. First we note that using the commutation relations in Lemma 3.2 we obtain

$$
(z^k \cdot e(P, t, t')) \cdot (z^k * e(P, t, t'))^* = e(P, t, t),
$$
$$
(z^k \cdot e(P, t, t'))^* \cdot (z^k \cdot e(P, t, t')) = e(P, t', t'),
$$

and both $e(P, t, t)$ and $e(P, t', t')$ are projections in $C^*(R_{\text{punc}})$. Whence, each of the functions in $\mathcal{E}$ is a partial isometry. Suppose $\{p_1, \ldots, p_n\}$ are the proto-tiles of Ω_{punc}, then the function $I = \sum_{i=1}^{n} e(\{p_i\}, p_i, p_i)$ is the identity of the C^*-algebra $C^*(R_{\text{punc}})$. Finally, through the induced representation, the function $z^k \cdot e(P, t, t')$ acts on $\ell^2([T])$ as follows: if $R_\theta(P - x(t)) \subset T'$, T' is in $[T]$, and ξ is in $\ell^2([T])$, then

$$
(\pi_T(z^k \cdot e(P, t, t'))\xi)(T') = e^{k(\angle t + \theta) i} \xi(T' - R_\theta(x(t') - x(t))).
$$

PROPOSITION 3.3. *The complex linear span of the functions in $\mathcal{E}$ is a uniformly dense $*$-subalgebra of $C^*(R_{\text{punc}})$.*

Proof. To begin we note that elements in the set $\mathcal{E}$ form an algebra under pointwise multiplication. The inductive limit topology ensures that convergent sequences in $C_c(R_{\text{punc}})$ are uniformly convergent. It follows from Lemma 3.2 that $\mathcal{E}$ separates points of R_{punc} and does not identically vanish on any point of R_{punc}. Whence, by the Stone–Weierstrass Theorem, $\mathcal{E}$ is uniformly dense in $C_c(R_{\text{punc}})$ and therefore also dense in $C^*(R_{\text{punc}})$. $\quad \blacksquare$

4. SUBSTITUTION TILINGS AND $\mathcal{AT}$-ALGEBRAS

In this section we begin by briefly giving some of the background required for substitution tilings. Given a substitution tiling with the group Γ as the full group of orientation preserving isometries whose continuous hull is strongly aperiodic and has finite local complexity, we can form an $\mathcal{AT}$-algebra which is a C^*-subalgebra of $C^*(R_{\text{punc}})$. We remind the reader that $\mathcal{AT}$-algebras are inductive limits of matrix algebras whose entries are continuous functions on the circle. In the setting where $\Gamma = \mathbb{R}^2$, Kellendonk and Putnam [12] show that an AF-algebra is produced, and the construction presented here extends their construction. An

example of a tiling with the above properties is the Pinwheel Tiling, and in the next section we apply the construction presented here to the Pinwheel Tiling.

The reader is again reminded of the assumptions presented at the beginning of Section 2. Also recall the definition of the proto-tiles of a tiling, Definition 1.8, and in particular the relationship of the proto-tiles and the group Γ.

A substitution rule consists of a scaling factor $\lambda > 1$ and a substitution ω, such that for each p_i in $\{p_1, p_2, \ldots, p_n\}$, $\omega(p_i)$ is a finite collection of tiles $\{t_1, \ldots, t_l\}$ such that:

$$t_j = \gamma(p_j) \text{ for some } \gamma \in \Gamma, p_j \in \{p_1, p_2, \ldots, p_n\}; \quad \text{Int}(t_j) \cap \text{Int}(t_k) = \varnothing; \quad \bigcup_{j=1}^{l} t_j = \lambda p_i.$$

Basically each proto-tile is being divided up into smaller versions of proto-tiles that have been moved by the group Γ and then expanded by the scaling factor λ so as to be the same size as the original proto-tiles. So for each proto-tile p_i we apply ω and receive a patch of moved proto-tiles. Now extend the definition of ω to the image of a proto-tile under Γ. We write this tile as $t = \gamma(p_i)$ for some γ in Γ, p_i in $\{p_1, p_2, \ldots, p_n\}$ and define $\omega(t) = \gamma(\omega(p_i))$. We may therefore define $\omega(T) = \{\omega(t) : t \in T\}$ for a tiling T.

A substitution rule ω on a set of proto-tiles $\{p_1, \ldots, p_n\}$ is said to be *primitive* if, for some $k \geqslant 1$, an image under Γ of each proto-tile p_i appears inside the patch $\omega^k(p_j)$ for each pair i, j in $\{1, 2, \ldots, n\}$.

Let us state some facts about substitutions and give references to where these facts are located in the literature. First, Kellendonk and Putnam [12] show that given a primitive substitution system one can produce a tiling T by iterating the substitution. So the definition of Ω_T and subsequent definitions are not vacuous. Now if the continuous hull of a tiling has been created from a primitive substitution tiling system, then the substitution is a continuous map from Ω_T onto Ω_T [1]. Finally, we add the assumption of "recognizability" [1]; that is, the substitution $\omega : \Omega_T \to \Omega_T$ is assumed to be injective. If a tiling is not strongly aperiodic then "recognizability" fails and in the classical case Solomyak [26] has shown these two notions to be equivalent when the subsitution is primitive.

We now begin the construction of an $\mathcal{A}\mathbb{T}$-algebra. To begin, recall that $U(P, t)$ consists of all tilings, up to rotation, with the the tile t at the origin and containing the patch P, see Section 2. In this section we will be interested in patches arising from the inflation of proto-tiles under the substitution. Indeed, for a proto-tile p we define $U(p, p)$ to consist of all tilings with some rotation of the proto-tile p having puncture on the origin. Notice that the definition of $U(p, p)$ should technically read $U(\{p\}, p)$ but we will abuse this notation for the remainder of this note. To extend to inflations of proto-tiles we define, for fixed N in $\mathbb{N}$,

$$U_N(p, t) = \{\omega^N(T) - R_\theta(x(t)) : R_\theta(p) \subset T \text{ for some } R_\theta \in \mathcal{S}^1, t \in \omega^N(p)\}.$$

At this point we remark that each $U_N(p, t)$ is an element of $\mathcal{U}$ for every N in $\mathbb{N}$, p a proto-tile, and t in $\omega^N(p)$.

Let $\{p_1, p_2, \ldots, p_n\}$ be the set of proto-tiles for a given discrete hull Ω_{punc} with punctures on the origin. The sets $U(p_1, p_1), U(p_2, p_2), \ldots, U(p_n, p_n)$ are pairwise disjoint and

$$U(p_1, p_1) \cup U(p_2, p_2) \cup \cdots \cup U(p_n, p_n) = \Omega_{\text{punc}}.$$

Since ω is injective on Ω_{punc} we have, for fixed N in $\mathbb{N}$, sets of the form $U_N(p_i, t)$ are pairwise disjoint for each p_i in $\{p_1, p_2, \ldots, p_n\}$ and t in $\omega^N(p_i)$ and the disjoint union of such sets is Ω_{punc}. This leads us to define, for each N in $\mathbb{N}$, the collection

$$\mathcal{U}_N = \{U_N(p_i, t) \in \mathcal{U} : p_i \in \{p_1, \ldots, p_n\} \text{ and } t \in \omega^N(p_i)\}$$

consisting of pairwise disjoint sets for each N in $\mathbb{N}$ whose union is Ω_{punc}. Following a similar procedure to that of Section 2 we can extend the collection $\mathcal{U}_N$ to a subcollection $\mathcal{V}_N$ of $\mathcal{V}$. Fix N in $\mathbb{N}$, for each pair of tiles t and t' in the patch $\omega^N(p_i)$ we define $V_N(p_i, t, t')$ as the pairs of tilings $(\omega^N(T) - R_\theta(x(t)), \omega^N(T) - R_\theta(x(t')))$ such that $R_\theta(p_i) \subset T$ for some R_θ in $\mathcal{S}^1$. Let

$$\mathcal{V}_N = \{V_N(p_i, t, t') : p_i \in \{p_1, \ldots, p_n\} \text{ and } t, t' \in \omega^N(p_i)\},$$

and notice that $\mathcal{V}_N$ is contained in $\mathcal{V}$ defined in Section 2 for each N in $\mathbb{N}$.

Let $e_N(p_i, t, t')$ be the characteristic function of $V_N(p_i, t, t')$ and let $\mathcal{E}_N$ denote the set of functions

$$\mathcal{E}_N = \{z^k \cdot e_N(p_i, t, t') : p_i \in \{p_1, \ldots, p_n\} \text{ and } t, t' \in \omega^N(p_i)\}.$$

Observe that $\mathcal{E}_N$ is a proper subset of $\mathcal{E}$ presented in Section 3. We may therefore complete the span of the functions $\mathcal{E}_N$ in the reduced C^*-norm used to construct $C^*(R_{\text{punc}})$. Let

$$\mathcal{A}_N = \overline{\text{span}}\{z^k \cdot e_N(p_i, t, t') : p_i \in \{p_1, \ldots, p_n\}, t, t' \in \omega^N(p_i) \text{ and } k \in \mathbb{Z}\}.$$

We aim to show that $\mathcal{A}_N$ is a C^*-algebra which is isomorphic to a matrix algebra with entries that are continuous functions on the circle. We shall require a lemma whose proof is obtained by restricting Lemma 3.2 to $\mathcal{E}_N$.

LEMMA 4.1. *Fix N in $\mathbb{N}$, for $p_i, p_j \in \{p_1, \ldots, p_n\}$, $t_1, t'_1 \in \omega^N(p_i)$, $t_2, t'_2 \in \omega^N(p_j)$, and $k_1, k_2 \in \mathbb{N}$ we have the following relations:*

(i) $[z^{k_1} \cdot e_N(p_i, t_1, t'_1)] \cdot [z^{k_2} \cdot e_N(p_j, t_2, t'_2)] = 0$ *if $p_i \neq p_j$;*

(ii) $[z^{k_1} \cdot e_N(p_i, t_1, t'_1)] \cdot [z^{k_2} \cdot e_N(p_j, t_2, t'_2)] = 0$ *if $p_i = p_j$ and $t'_1 \neq t_2$;*

(iii) $[z^{k_1} \cdot e_N(p_i, t_1, t'_1)] \cdot [z^{k_2} \cdot e_N(p_j, t_2, t'_2)] = e^{k_2(\angle t'_1 - \angle t_1)i} z^{k_1+k_2} \cdot e_N(p_i, t_1, t'_2)$ *if $p_i = p_j$ and $t'_1 = t_2$.*

These relations imply that the closure of the span of $\mathcal{E}_N$ in the reduced norm is a C^*-algebra for each N in $\mathbb{N}$. Moreover, from these relations one may deduce a great deal about the structure of the C^*-algebra $\mathcal{A}_N$. Our first observation is that for each p_i in $\{p_1, \ldots, p_n\}$ there are a finite number of tiles in $\omega^N(p_i)$. Thus, for

fixed N and p_i the closed span of $\{e_N(p_i, t, t') : t, t' \in \omega^N(p_i)\}$ is finite dimensional. Moreover, relations (ii) and (iii) imply that if we fix both N and p_i and define

$$\mathcal{A}_{(N,p_i)} = \overline{\text{span}}\{z^k \cdot e_N(p_i, t, t') : t, t' \in \omega^N(p_i) \text{ and } k \in \mathbb{N}\}$$

then we obtain a homomorphism from $\mathcal{A}_{(N,p_i)}$ into the algebra of $m \times m$ matrices with entries in $C(\mathbb{T})$, where m is the number of tiles in $\omega^N(p_i)$ and $\mathbb{T}$ denotes the circle $\{z \in \mathbb{C} : |z| = 1\}$. This follows from the fact that the convolution product and adjoint on $\mathcal{E}_N$ mimic the matrix operations on $C(\mathbb{T}) \otimes \mathbb{M}_m$. Furthermore, the first relation implies that $\mathcal{A}_{(N,p_i)}$ and $\mathcal{A}_{(N,p_j)}$ are orthogonal when $i \neq j$. We may therefore define

$$\mathcal{A}_N = \overline{\text{span}}\{z^k \cdot e_N(p_i, t, t') : p_i \in \{p_1, \ldots, p_n\}, t, t' \in \omega^N(p_i), \text{ and } k \in \mathbb{N}\}$$

$$= \bigoplus_{i=1}^{n} \mathcal{A}_{(N,p_i)}.$$

The number of summands in the direct sum is the number of proto-tiles and is therefore independent of N. However, the sizes of the algebras will depend on the inflation of each proto-tile and will increase as N increases. Our observations lead to the following result. The proof is quite standard and appears in [27].

PROPOSITION 4.2. *Fix N in $\mathbb{N}$. Given the discrete hull of a tiling, Ω_{punc}, there is a C^*-algebra isomorphism*

$$\psi_N : \mathcal{A}_N \to \bigoplus_{i=1}^{n}(C(\mathbb{T}) \otimes \mathbb{M}_{N,p_i})$$

defined on the dense set $\mathcal{E}_N$ as follows. Let $x \in [0,1)$ and $E_{t,t'}^{p_i}$ be a standard matrix unit in the row t and column t' of $\mathbb{M}_{N,p_i}$. Then,

$$\psi_N(z^k)(x) = \sum_{p_i, t \in \omega^N(p_i)} e^{(\angle t + 2\pi k x)i} \otimes E_{t,t}^{p_i}, \quad \psi_N(e_N(p_i, t, t'))(x) = 1 \otimes E_{t,t'}^{p_i}.$$

The next step is to show that these matrix algebras form an inductive system, for which the inductive limit is an $\mathcal{AT}$-algebra.

PROPOSITION 4.3. *For each N in $\mathbb{N}$, $\mathcal{A}_N \subseteq \mathcal{A}_{N+1}$ and the inclusion map, denoted by ϕ_N, is defined on the set $\mathcal{E}_N$ by*

$$\phi_N(z^k \cdot e_N(p_i, t', t''))$$
$$= \sum_{(p_j, t, R_\theta) \in I(p_i)} z^k \cdot e_{N+1}(p_j, R_\theta(p_i - \lambda^N x(t)) + x(t'), R_\theta(p_i - \lambda^N x(t)) + x(t''))$$

where $z^k \cdot e_N(p_i, t', t'')$ is in $\mathcal{E}_N$ and $I(p_i)$ denotes the set of triples (p_j, t, R_θ), where p_j is a proto-tile and t is a tile in $\omega(p_j)$ such that $R_\theta(p_i + x(t)) \in \omega(p_j)$ for some R_θ in $\mathcal{S}^1$.

Proof. We begin by showing that

$$(4.1) \qquad U_0(p_i, p_i) = \bigcup_{(p_j, t, R_\theta) \in I(p_i)} U_1(p_j, t)$$

where $U_0(p_i, p_i)$ is a clopen set in $\mathcal{U}_0$ and $U_1(p_j, t)$ is a clopen set in $\mathcal{U}_1$. Note that the sets in the union are pairwise disjoint since ω is injective. On one hand, if $T \in U_0(p_i, p_i)$ then $R_\theta(p_i) \subset T$ for some $R_\theta \in \mathcal{S}^1$. So $T(0) = R_\theta(p_i)$. Since substitution tilings are created with iterations of ω, $T(0)$ is a tile in $R_\theta(\omega(p_j) - x(t))$ for some $(p_j, t, R_\theta) \in I(p_i)$. Thus, $T \in U_1(p_j, t)$ and left containment follows. On the other hand, if $T \in U_1(p_j, t)$ for some $(p_j, t, R_\theta) \in I(p_i)$ then $R_\theta(\omega(p_j) - x(t)) \subset T$ and $T(0) = R_\theta(t - x(t))$. Since $t = R_\theta(p_i + x(t))$ we have $T \in U_0(p_i, p_i)$.

From (4.1) it follows that

$$e_0(p_i, p_i, p_i) = \sum_{(p_j, t, R_\theta) \in I(p_i)} e_1(p_j, t, t).$$

Applying ω^N to the equality of sets above we obtain for t' and t'' in $\omega^N(p_i)$,

$$e_N(p_i, t', t''))$$
$$= \sum_{(p_j, t, R_\theta) \in I(p_i)} e_{N+1}(p_j, R_\theta(p_i - \lambda^N x(t)) + x(t'), R_\theta(p_i - \lambda^N x(t)) + x(t'')).$$

Finally, we include the function z^k and obtain

$$z^k \cdot e_N(p_i, t', t'')$$
$$= \sum_{(p_j, t, R_\theta) \in I(p_i)} z^k \cdot e_{N+1}(p_j, R_\theta(p_i - \lambda^N x(t)) + x(t'), R_\theta(p_i - \lambda^N x(t)) + x(t'')).$$

Therefore, the span of functions in $\mathcal{E}_N$ is contained in the span of functions in $\mathcal{E}_{N+1}$, and the equality of sets above shows the required map. This extends to $\mathcal{A}_N$ by taking the span and closing both sides of the equality in the reduced C^*-norm. ∎

Combining Propositions 4.2 and 4.3, we have proved the following.

THEOREM 4.4. *The inductive limit*

$$\mathcal{AT}_T = \overline{\bigcup_{N=0}^{\infty} \mathcal{A}_N}$$

is a $\mathcal{AT}$-algebra; that is, an "approximately finite dimensional circle algebra".

5. PINWHEEL TILING

The primary example of a tiling with infinite rotational symmetry is the Pinwheel Tiling. In this section we define the Pinwheel Tiling and consider the

$A\mathbb{T}$-algebra contained in $C^*(R_{\mathrm{punc}})$ created from the Pinwheel Tiling, which we denote $A\mathbb{T}_{\mathrm{pin}}$ and A_{pin} respectively.

The substitution for the pinwheel tiling begins with a $(1, 2, \sqrt{5})$-right triangle and it is mirror image. We will denote these two tiles by p_0 and p_1. An important observation for the pinwheel tiling is that the smallest angle in these two triangles is equal to $\arctan(1/2)$, which is an irrational multiple of 2π [19]. As usual we consider these two tiles to have puncture on the origin and to be in fixed orientation. The substitution ω for the Pinwheel Tiling is illustrated, for the proto-tile p_0, in Figure 1. For the proto-tile p_1 we horizontally flip Figure 1. A Pinwheel Tiling is created from the substitution by iterating the substitution, see [12] for a construction of a tiling from a substitution.

Notice that the tile in the center of the patch $\omega(p_0)$ is the tile $R_\theta(p_0)$ modulo translation where $\theta = \arctan(1/2)$. One may now observe that the tile in the center of the patch $\omega^2(p_0)$ is the tile $R_{2\theta}(p_0)$ modulo translation and the tile in the center of the patch $\omega^n(p_0)$ is the tile $R_{n\theta}(p_0)$ modulo translation (see Figure 1 for the first two iterations). Since θ is irrational and the irrational rotations are dense in the circle we will have tiles appearing in an infinite number of orientations. Moreover, the continuous hull for the pinwheel tiling has tiles appearing in all orientations of $\mathcal{S}^1$. This implies that the group Γ consists of all orientation preserving isometries of $\mathbb{R}^2$. It is well known that the Pinwheel Tiling has finite local complexity and the continuous hull is strongly aperiodic in this case [19], [27].

To construct $A\mathbb{T}_{\mathrm{pin}}$ we begin by defining a labelling scheme. Observe that each of the patches $\omega^N(p_0)$ and $\omega^N(p_1)$ contains 5^N tiles. For N in $\mathbb{N}$, A_N is now defined to be the closed span of the functions in $\{z^k \cdot e_N(p_i, t, t')\}$. With iteration of ω we may label each tile in the consecutive patches with a sequence of length N. For example $\omega^1(p_0)$ has tiles labeled $\{t_1, t_2, t_3, t_4, t_5\}$, $\omega^2(p_0)$ has tiles labeled $\{t_{11}, \ldots, t_{15}, t_{21}, \ldots, t_{25}, t_{31}, \ldots, t_{35}, t_{41}, \ldots, t_{45}, t_{51}, \ldots, t_{55}\}$, and so on. In general, for N in $\mathbb{N}$ a function in $\mathcal{E}_N$ will have the form $z^k \cdot e_N(p_i, t_l, t_m)$ where $i = 1$ or 2, k is in $\mathbb{N}$, and $l = l_1 \cdots l_N$, $m = m_1 \cdots m_N$ where l_j and m_j take values in $\{1, 2, 3, 4, 5\}$. We aim to use the labelling scheme to define the inclusion map from A_N into A_{N+1} and the isomorphism of A_N onto $\mathbb{M}_{N,p_0}(C(\mathbb{T})) \oplus \mathbb{M}_{N,p_1}(C(\mathbb{T}))$ for functions in the generating set $\mathcal{E}_N$.

Let us begin with the inclusion map A_N into A_{N+1}: Beginning with the proto-tile p_i, let t_l and t_m be tiles in $\omega^N(p_i)$. Then, where $i + 1$ is taken mod2,

$$z^k \cdot e_N(p_i, t_l, t_m) \mapsto \sum_{j=3,4} z^k \cdot e_{N+1}(p_i, t_{jl}, t_{jm}) + \sum_{j=1,2,5} z^k \cdot e_{N+1}(p_{i+1}, t_{jl}, t_{jm}).$$

To describe the isomorphism of A_N onto $\mathbb{M}_{N,p_0}(C(\mathbb{T})) \oplus \mathbb{M}_{N,p_1}(C(\mathbb{T}))$ we begin with some notation. For labeled tiles t_l, t_m in $\omega^N(p_i)$ such that $l = l_1 \cdots l_N$
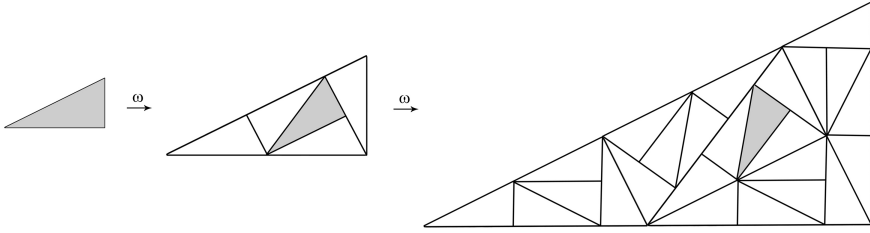
FIGURE 1. The first two substitutions of p_0

and $m = m_1 \cdots m_N$ we define row l and column m as:

$$\text{row } l = \text{ row } 1 + \sum_{j=1}^{N} 5^{N-j}(l_j - 1) \text{ in } M_{(N,p_i)}(C(\mathbb{T})),$$

$$\text{column } m = \text{ column } 1 + \sum_{j=1}^{N} 5^{N-j}(m_j - 1) \text{ in } M_{(N,p_i)}(C(\mathbb{T})).$$

Using this notation, the function $z^k \cdot e_N(p_i, t_l, t_m)$ goes to the direct sum of two matrices with zeros everywhere except in row l and column m of matrix p_i where the entry is $\mathrm{e}^{(\angle t_{l_1 \cdots l_N} + 2k\pi x)\mathrm{i}}$ which is a continuous function on the circle and $x \in [0, 1)$.

Recall that for an inductive limit C^*-algebra the map $\phi_{N,M}$ denotes the inclusion of $\mathcal{A}_N$ into $\mathcal{A}_M$ for $M > N$ and the map $\phi_{N,\infty}$ denotes the inclusion of $\mathcal{A}_N$ into the inductive limit. Using a result of Dadarlat, Nagy, Nemethi, and Pasnicu [5] the C^*-algebra $\mathcal{A}\mathbb{T}_{\text{pin}}$ is simple if we establish that for every N in $\mathbb{N}$ and for every f in A_N such that $\phi_{N,\infty}(f)$ is non-zero there is an $M > N$ such that $\phi_{N,M}(f)(x) \neq 0$ for every x in $[0, 1)$.

THEOREM 5.1. $\mathcal{A}\mathbb{T}_{\text{pin}}$ is simple.

Proof. Take N in $\mathbb{N}$ and f in $\mathcal{A}\mathbb{T}_{\text{pin}}$ such that $\phi_{\infty,N}(f) \neq 0$. Now there is an open set U about x in $[0, 1)$ so that f restricted to some matrix entry in $\mathcal{A}_N$ is non-zero on all of U, i.e. for labels l and m and p_i in $\{p_0, p_1\}$, $f_{p_i,l,m} \neq 0$ on all of U. Without loss of generality assume $i = 0$ so that $f_{p_0,l,m}$ is in row l and column m in the matrix $\mathbb{M}_{(N,p_0)}(C(\mathbb{T}))$. Now applying ϕ_N to the function yields

$$\phi_N(f_{p_0,l,m}) = \left\{ \begin{array}{lll} f_{p_1,1l,1m} \neq 0 & \text{on} & R_{-\theta}(U) \\ f_{p_1,2l,2m} \neq 0 & \text{on} & R_{-\theta}(U) \\ f_{p_0,3l,3m} \neq 0 & \text{on} & R_{\theta}(U) \\ f_{p_0,4l,4m} \neq 0 & \text{on} & R_{\theta+\pi}(U) \\ f_{p_1,5l,5m} \neq 0 & \text{on} & R_{-(\theta+\pi/2)}(U) \end{array} \right\} \in \bigoplus_{j=0}^{1} \mathbb{M}_{(N+1,p_j)}(C(\mathbb{T})).$$

Now for $M \geqslant 2$ even, if we apply $\phi_{N+M,N}$ to the function $f_{p_0,l,m}$ we receive 5^M functions in distinct matrix entries in $\mathbb{M}_{(N+M,p_0)}(C(\mathbb{T})) \oplus \mathbb{M}_{(N+M,p_1)}(C(\mathbb{T}))$. In particular, we receive the functions

$$
\begin{array}{llll}
f_{p_0,3^M l,3^M m} & \neq 0 & \text{on} & R_{M\theta}(U) \\
f_{p_0,2^2 3^{M-2} l,2^2 3^{M-2} m} & \neq 0 & \text{on} & R_{(M-2)\theta}(U) \\
\vdots & & \vdots & \quad\quad\quad\quad \in \mathbb{M}_{(N+M,p_0)}(C(\mathbb{T})) \\
f_{p_0,2^{M-2} 3^2 l,2^{M-2} 3^2 m} & \neq 0 & \text{on} & R_{2\theta}(U) \\
f_{p_0,2^M l,2^M m} & \neq 0 & \text{on} & U
\end{array}
$$

and the functions

$$
\begin{array}{llll}
f_{p_1,3^{M-2} 31 l,3^{M-2} 31 m} & \neq 0 & \text{on} & R_{(M-2)\theta}(U) \\
f_{p_1,2^2 3^{M-4} 31 l,2^2 3^{M-4} 31 m} & \neq 0 & \text{on} & R_{(M-4)\theta}(U) \\
\vdots & & \vdots & \quad\quad\quad\quad \in \mathbb{M}_{(N+M,p_1)}(C(\mathbb{T})) \\
f_{p_1,2^{M-4} 3^2 31 l,2^{M-4} 3^2 31 m} & \neq 0 & \text{on} & R_{2\theta}(U) \\
f_{p_1,2^{M-2} 31 l,2^{M-2} 31 m} & \neq 0 & \text{on} & U
\end{array}
$$

using the notation that row $3^4 2^2 l$ is the labeled row $333322l$. This shows that $\phi_{N,M}(f_{p_0,l,m})$ is nonzero on $U, R_{2\theta}(U), \ldots, R_{(M-4)\theta}(U), R_{(M-2)\theta}(U)$ in each of $\mathbb{M}_{(N+M,p_0)}(C(\mathbb{T}))$ and $\mathbb{M}_{(N+M,p_1)}(C(\mathbb{T}))$. Since irrational rotations of the circle are dense, we may find an even M so that $\phi_{N,M}(f_{p_0,l,m})$ is nonzero on all of $[0, 1)$. ∎

Finally, we comment on the K-theory of $\mathcal{A}\mathbb{T}_{\text{pin}}$. Both of the abelian groups $K_0(\mathcal{A}\mathbb{T}_{\text{pin}})$ and $K_1(\mathcal{A}\mathbb{T}_{\text{pin}})$ are inductive limits of the system:

$$
\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}} \cdots .
$$

We note that the connecting matrix is not invertible over the integers. Calculations reveal that $K_0(\mathcal{A}\mathbb{T}_{\text{pin}}) = K_1(\mathcal{A}\mathbb{T}_{\text{pin}})$ sit in the following exact sequence which does not split:

$$
0 \to \mathbb{Z} \to K_0(\mathcal{A}\mathbb{T}_{\text{pin}}) \to \mathbb{Z}[1/5] \to 0,
$$

that is, $K_0(\mathcal{A}\mathbb{T}_{\text{pin}})$ has $\mathbb{Z}$ as a normal subgroup and $\mathbb{Z}[1/5]$ as a quotient group. We refer the reader to Exercise 7.5.2 in [13] for a similar computation.

We conclude with two open questions regarding the Pinwheel Tiling. What is the K-theory of the C^*-algebra $\mathcal{A}_{\text{pin}}$ and, in particular, can the K-theory be computed in terms of cohomology in the sense of [1]? What is the exact relationship between $\mathcal{A}\mathbb{T}_{\text{pin}}$ and $\mathcal{A}_{\text{pin}}$?

paper was written while visiting the Fields Institute in Mathematical Sciences during the Thematic Program in Operator Algebras.

REFERENCES

[1] J.E. ANDERSON, I.F. PUTNAM, Topological invariants for substitution tilings and their associated C^*-algebras, *Ergodic Theory Dynamical Systems* **18**(1998), 509–537.

[2] J. BELLISSARD, K-theory of C^*-algebras in solid state physics, *Lecture Notes in Phys.* **257**(1986), 99–156.

[3] R. BENEDETTI, J.M. GAMBAUDO, On the dynamics of G-solenoids: application to Delone sets, *Ergodic Theory Dynamical Systems* **23**(2003), 673–691.

[4] B. BLACKADAR, *K-Theory for Operator Algebras*, Springer-Verlag, New York 1986.

[5] M. DADARLAT, G. NAGY, A. NEMETHI, C. PASNICU, Reduction of topological stable rank in inductive limits of C^*-algebras, *Pacific J. Math* **153**(1992), 323–341.

[6] K.R. DAVIDSON, C^*-*Algebras by Example*, Amer. Math. Soc., Providence, RI 1996.

[7] D. FRETTLÖH, Substitution tilings with statistical circular symmetry, preprint, 2007.

[8] T. GIORDANO, I.F. PUTNAM, C.F. SKAU, Topological orbit equivalence and C^*-crossed products, *J. Reine Angew. Math.* **469**(1995), 51–111.

[9] D. GONÇALVES, New C^*-algebras from substitution tilings, *J. Operator Theory* **57**(2007), 391–407.

[10] J. KELLENDONK, Noncommutative geometry of tilings and gap labelling, *Rev. Math. Phys.* **7**(1995), 1133–1180.

[11] J. KELLENDONK, The local structure of tilings and their integer group of coinvariants, *Comm. Math. Phys.* **187**(1997), 115–157.

[12] J. KELLENDONK, I.F. PUTNAM, Tilings, C^*-algebras, and K-theory, in *Directions in Mathematical Quasicrystals*, pp. 177–206, CRM Monogr. Ser., 13, Amer. Math. Soc., Providence, RI, 2000.

[13] D. LIND, B. MARCUS, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge 1995.

[14] P.S. MUHLY, J.N. RENAULT, D.P. WILLIAMS, Equivalence and isomorphism for groupoid C^*-algebras, *J. Operator Theory* **17**(1987), 3–22.

[15] N. ORMES, C. RADIN, L. SADUN, A homeomorphism invariant for substitution tiling spaces, *Geom. Dedicata* **90**(2002), 153–182.

[16] G.K. PEDERSEN, C^*-*Algebras and their Automorphism Groups*, Academic Press, London 1979.

[17] I.F. PUTNAM, C^*-algebras from smale spaces, *Canad. J. Math* **48**(1996), 175–195.

[18] I.F. PUTNAM, On the K-theory of C^*-algebras of principal groupoids, *Rocky Mountain J. Math.* **28**(1998), 1483–1518.

[19] C. RADIN, The pinwheel tilings of the plane, *Ann. of Math.* **139**(1994), 661–702.

[20] C. RADIN, L. SADUN, An algebraic invariant for substitution tiling systems, *Geom. Dedicata* **73**(1998), 21–37.

[21] C. RADIN, M. WOLFF, Space tilings and local isomorphism, *Geom. Dedicata* **42**(1992), 355–360.

[22] J.N. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin 1980.

[23] M. RØRDAM, F. LARSEN, N.J. LAUSTSEN, *An Introduction to K-Theory for C^*-algebras*, London Math. Soc. Stud. Texts, vol. 49, Cambridge Univ. Press, Cambridge 2000.

[24] M. RØRDAM, Classification of nuclear C^*-algebras, *Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras*, Encyclopaedia Math. Sci., vol. 126, Springer-Verlag, Berlin 2002, pp. 1–145.

[25] L. SADUN, Tiling spaces are inverse limits, *J. Math. Phys.* **44**(2003), 5410–5414.

[26] B. SOLOMYAK, Nonperiodicity implies unique composition for self-similar translationally finite tilings, *Discrete Comput. Geom.* **20**(1998), 265–279.

[27] M.F. WHITTAKER, Groupoid C^*-algebras of the pinwheel tiling, Ph.D. Dissertation, Univ. of Victoria, Victoria 2005.

MICHAEL F. WHITTAKER, MATHEMATICS AND STATISTICS, UNIVERSITY OF VICTORIA, PO BOX 3060, STN CSC VICTORIA B.C., CANADA V8W 3R4; *Current address:* SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, UNIVERSITY OF WOLLONGONG, WOLLONGONG NSW 2522, AUSTRALIA

E-mail address: mfwhittaker@gmail.com