J. OPERATOR THEORY
64:2(2010), 387–399

ESSENTIAL NORM OF OPERATORS ON WEIGHTED BERGMAN SPACES OF INFINITE ORDER

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Communicated by Kenneth R. Davidson

ABSTRACT. We obtain a formula for the essential norm of any operator between weighted Bergman spaces of infinite order. Then we apply it to obtain or estimate essential norms of operators acting on Bloch type spaces and to differences of composition operators or Toeplitz operators on some weighted Bergman spaces.

KEYWORDS: *Weighted Bergman spaces of infinite order, essential norm, composition operator, Toeplitz operator.*

MSC (2000): 47B25.

1. INTRODUCTION

We denote by $H(D)$ the space of holomorphic functions on the open unit disk D and consider the weighted Bergman spaces of infinite order

$$H_v^\infty = \Big\{ f \in H(D) : \|f\|_v := \sup_{z\in D} v(z)|f(z)| < \infty \Big\},$$
$$H_v^0 = \Big\{ f \in H_v^\infty : \lim_{|z|\to 1} v(z)|f(z)| = 0 \Big\},$$

endowed with the norm $\|\cdot\|_v$, where v is a positive continuous weight on the open unit disk D in the complex plane. These spaces appear in the study of growth conditions of analytic functions and have been studied in various articles, see e.g. [21], [22], [3], [12], [13], [2], [9].

To characterize compactness of (weighted) composition operators, Toeplitz operators and differences of (weighted) composition operators acting on various Banach spaces of analytic functions has recently been the object of investigations of several authors. For instance, K. Madigan and A. Matheson [14] characterized compactness of composition operators on the Bloch and the little Bloch space. Later A. Montes-Rodríguez calculated the essential norm for composition operators on these two spaces [16] and also on H_v^∞ and H_v^0 [17]. J. Moorhouse [15] completely answered the question of characterizing compact differences of composition operators acting on standard weighted Bergman spaces A^2_α. The corresponding problem for the Hardy space H^2 seems still to be open. Further, T. Yu [24] has characterized very recently the compactness of operators on the weighted Bergman space A^1_ψ and applied it to Toeplitz operators. Other papers dealing with this topic are e.g. [4], [5], [7] and [15]. See also [6], [11] for results on differences of (weighted) composition operators on H^∞_v and H^0_v.

The purpose of this paper is to calculate the essential norms of all operators acting on weighted Bergman spaces of infinite order, on Bloch type spaces and on the weighted Bergman space A^1_ψ. A Banach–Steinhaus type theorem and a sequence of operators approximating the identity constructed by Montes-Rodríguez in [17] are the crucial tools in the proof of the main result. As applications of the main formula (Theorem 3.1) we obtain several known results by specializing to certain types of operators. In particular, we also improve the main theorem due to Yu in [24] by using an approach based on the Dixmier–Ng theorem [18].

See the books of C. Cowen and B. MacCluer [8] and J. Shapiro [20] for discussions of composition operators on classical spaces of analytic functions.

2. PRELIMINARIES

A weight v is radial if it satisfies $v(z) = v(|z|)$ for every $z \in D$. Throughout this paper we assume that all weights v are radial, non-increasing with respect to $|z|$ and satisfy that $\lim_{|z|\to 1^-} v(z) = 0$. Many results on weighted spaces of analytic functions and on operators between them have to be given in terms of the so-called *associated weights* (see [2]) and in terms of the weights. For a weight v the associated weight $\widetilde{v}$ is defined as follows $\widetilde{v}(z) := (\sup\{|f(z)| : f \in H^\infty_v, \|f\|_v \leqslant 1\})^{-1}$. The associated weights $\widetilde{v}$ are also positive, continuous, radial, non-increasing with respect to $|z|$ and satisfy that $\lim_{|z|\to 1^-} \widetilde{v}(z) = 0$. Furthermore, for each $z \in D$ there is $f_z \in H^\infty_v$, $\|f_z\|_v \leqslant 1$, such that $f_z(z) = \frac{1}{\widetilde{v}(z)}$. We set $X \simeq Y$ to mean that the Banach space X is isomorphic to the Banach space Y. It is well known that $H^\infty_{\widetilde{v}} \simeq H^\infty_v$ and $H^0_{\widetilde{v}} \simeq H^0_v$ isometrically. See [2] for details concerning these facts. The norm topology of H^∞_v is finer than the topology τ_0 of uniform convergence on the compact subsets of D. This implies that the closed unit ball $B_{H^\infty_v}$ of H^∞_v is τ_0-compact, and therefore a result of Dixmier–Ng [18] gives that

$$G^\infty_v := \{l \in (H^\infty_v)^* : l|B_{H^\infty_v} \text{ is } \tau_0\text{-continuous}\}$$

endowed with the norm induced by $(H^\infty_v)^*$ is a Banach space which is a predual of H^∞_v. In fact, the map $f \mapsto [l \mapsto l(f)]$ is an onto isometric isomorphism between H^∞_v and $(G^\infty_v)^*$. Since all weights v are assumed to be radial and $\lim_{|z|\to 1^-} v(z) = 0$, the closed unit ball of H^0_v is τ_0-dense in the closed unit ball $B_{H^\infty_v}$. Hence the restriction map $l \mapsto l|_{H_v^0}$ gives rise to an isometric isomorphism $G_v^\infty \simeq (H_v^0)^*$. See [3] for details. For every $z \in D$ the evaluation map $\delta_z : H_v^\infty \to \mathbb{C}$, $\delta_z(f) := f(z)$, is non-zero and $\|\delta_z\| = \frac{1}{\tilde{v}(z)}$. Since the H_v^0 is τ_0-dense in H_v^∞ it follows that $\delta_z : H_v^\infty \to \mathbb{C}$ is the bidual map of its restriction $\delta_z : H_v^0 \to \mathbb{C}$ and that $\tilde{v}$ can be written $\tilde{v}(z) = (\sup\{|f(z)| : f \in H_v^0, \|f\|_v \leqslant 1\})^{-1}$. Clearly $\delta_z \in G_v^\infty$. Moreover, the set $C := \{\delta_z : z \in D\}$ is a total set, that is, its linear span is norm dense in G_v^∞. This follows easily from the Hahn-Banach theorem. Recall also that the map $z \in D \mapsto \delta_z \in G_v^\infty$ is analytic.

Following A. Shields and D. Williams [21] the radial weight v will be called *normal* if there exist $k > \varepsilon > 0$ and $r_0 < 1$, such that

$$\frac{v(r)}{(1-r)^\varepsilon} \searrow 0 \quad \text{and} \quad \frac{v(r)}{(1-r)^k} \nearrow \infty \quad (r_0 \leqslant r, r \to 1^-). \tag{2.1}$$

Of course k and ε are not uniquely determined by v. Further the pair of radial functions $\{v, \psi\}$ will be called *normal* if v is normal and if, for some k satisfying (2.1), there exists $\alpha > k-1$ such that $v(z)\psi(z) = (1-|z|^2)^\alpha$. Let L_ψ^1 denote the Banach space of all measurable functions f on D with norm $\|f\|_\psi = \int_D |f(z)|\psi(|z|)\mathrm{d}A(z)$, where $\mathrm{d}A(z)$ denotes the normalized Lebesgue measure on D. By the condition $\alpha > k-1$ the measure $\psi(|z|)\mathrm{d}A(z)$ is finite on D, that is, $\int_0^1 \psi(r)r\mathrm{d}r < \infty$. The weighted Bergman space A_ψ^1 is the Banach space consisting of all analytic functions on D that also are in L_ψ^1. For the standard weight v_p with $p = \alpha$, we have that $\psi = 1$ and A_ψ^1 is the classical Bergman space A^1. For a given normal pair $\{v, \psi\}$ we will use the following pairing between H_v^∞ and A_ψ^1 :

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha \mathrm{d}A(z).$$

Under the above pairing A. Shields and D. Williams ([21], Theorem 2) have obtained the following isomorphisms

$$(H_v^0)^* \simeq A_\psi^1 \quad \text{and} \quad (A_\psi^1)^* \simeq H_v^\infty.$$

For $z, x \in D$, let

$$K_z^\alpha(x) = \frac{1+\alpha}{(1-\bar{z}x)^{2+\alpha}}.$$

Then the reproducing kernel K_z^α is in both H_v^0 and A_ψ^1 and $g(z) = \langle K_z^\alpha, g\rangle$ for all $g \in A_\psi^1$. Moreover, $f(z) = \langle f, K_z^\alpha\rangle$ for all $f \in H_v^\infty$. For these facts see Lemma 10 in [21]. W. Lusky [12], [13] has characterized the spaces H_v^0 which are isomorphic to c_0 in terms of the weight v. This is the case for all normal weights. For example the standard weights $v_p(z) = (1-|z|^2)^p$, $0 < p < \infty$, are normal. For normal weights v, we have that v and $\tilde{v}$ are equivalent.

Let $\{v, \psi\}$ be a normal pair. In [21] A. Shields and D. Williams also showed that there is a bounded projection Q^α from L^1_ψ onto A^1_ψ given by

$$Q^\alpha(f)(z) = \langle f, K^\alpha_z \rangle = \int_D f(x)\overline{K^\alpha_z(x)}(1-|x|^2)^\alpha \mathrm{d}A(x).$$

For $u \in L^\infty(D)$, the Toeplitz operator with symbol u is defined by $T^\alpha_u(f) = Q^\alpha(uf)$ for $f \in A^1_\psi$. Clearly T^α_u is bounded on A^1_ψ.

Let $\varphi : D \to D$ be an analytic mapping and $\phi \in H(\mathbb{D})$. Each such pair φ, ϕ induces via composition and multiplication a linear weighted composition operator $C_{\varphi,\phi}(f) = (f \circ \varphi)\phi$. If $\phi = 1$, then $C_{\varphi,1} = C_\varphi$ is the usual composition operator. Whenever $\varphi(0) = 0$, the Littlewood Subordination Principle yields that C_φ is bounded on A^1_ψ. For boundedness and compactness of composition operators acting on weighted Bergman spaces see [10] and [23].

The following result due to Montes-Rodríguez ([17], Proposition 2.1) is crucial in the proof of the main result:

(A) *There exists a sequence $(L_n)_n$ of compact operators on H^0_v such that:*

(i) for any $0 < t < 1$, $\lim_n \sup_{\|f\|_v \leqslant 1} \sup_{|z| \leqslant t} |(\mathrm{Id} - L_n)(f)(z)| = 0$, and

(ii) $\limsup_n \|\mathrm{Id} - L_n\| \leqslant 1$.

Let X and Y be Banach spaces. The essential norm $\|T\|_{\mathrm{e}}$ of a bounded operator $T \in \mathcal{L}(X, Y)$ is the distance in the operator norm from T to the compact operators, that is, $\|T\|_{\mathrm{e}} = \inf\{\|T - K\| : K \text{ is compact}\}$. Thus the essential norm of T equals zero if and only if T is compact.

3. THE ESSENTIAL NORM FORMULAE

Our main result is the following formula of the essential norm of a bounded operator T from H^0_v into H^0_w. However, let us first notice that for all operators $T : H^\infty_v \to H^\infty_w$ we have

$$\|T\| = \sup_{z \in D} \frac{\|T^*(\delta_z)\|}{\|\delta_z\|}.$$

Indeed, for $f \in H^\infty_v$ and $z \in D$,

$$\widetilde{w}(z)|T(f)z| = \widetilde{w}(z)|T^*(\delta_z)f| \leqslant \|\delta_z\|^{-1}\|T^*(\delta_z)\| \, \|f\|_v,$$

so $\|T\| \leqslant \sup_{z \in D} \|T^*(\delta_z)\| \|\delta_z\|^{-1}$. The other inequality is obvious.

THEOREM 3.1. *Let v, w be weights and let $T : H^0_v \to H^0_w$ be a bounded operator. Then*

$$\|T\|_{\mathrm{e}} = \limsup_{|z| \to 1^-} \frac{\|T^*(\delta_z)\|}{\|\delta_z\|}.$$

Proof. Since $H^0_{\widetilde{w}} \simeq H^0_w$ isometrically, we calculate the essential norm of the bounded operator $T : H^0_v \to H^0_{\widetilde{w}}$. Put $\ell := \limsup_{|z|\to 1^-} \frac{\|T^*(\delta_z)\|}{\|\delta_z\|}$. Select a sequence $(z_n) \subset D$, $|z_n| \to 1$ such that $\ell = \lim_n \frac{\|T^*(\delta_{z_n})\|}{\|\delta_{z_n}\|}$. Notice that $\left(\frac{\delta_{z_n}}{\|\delta_{z_n}\|}\right)_n$ is a weak*-null sequence in G^∞_w since for any $f \in H^0_w \simeq H^0_{\widetilde{v}}$, $\lim_n \left\langle \frac{\delta_{z_n}}{\|\delta_{z_n}\|}, f \right\rangle = \lim_n \frac{f(z_n)}{\|\delta_{z_n}\|} = \lim_n f(z_n)\widetilde{w}(z_n) = 0$. For any compact operator $K \in L(H^0_v, H^0_{\widetilde{w}})$,

$$\|T - K\| = \|T^* - K^*\| \geqslant \left\| T^*\left(\frac{\delta_{z_n}}{\|\delta_{z_n}\|}\right) - K^*\left(\frac{\delta_{z_n}}{\|\delta_{z_n}\|}\right) \right\|.$$

Further, $\lim_n K^*\left(\frac{\delta_{z_n}}{\|\delta_{z_n}\|}\right) = 0$, since K^* is both a compact and a weak*-weak* continuous operator. Therefore, $\|T - K\| \geqslant \lim_n \left\| T^*\left(\frac{\delta_{z_n}}{\|\delta_{z_n}\|}\right)\right\| = \ell$. Hence, $\|T\|_{\mathrm{e}} \geqslant \ell$.

For the reverse inequality, recall the sequence $(L_n)_n$ from (A). Then $\|T\|_{\mathrm{e}} \leqslant \|T - T \circ L_n\|$ for all n. Next we estimate $\limsup_n \|T - T \circ L_n\|$. Fix $0 < t < 1$ and f in the unit ball of H^0_v. Then

$$|[T \circ (\mathrm{Id} - L_n)](f)(z)| = |\langle \delta_z, [T \circ (\mathrm{Id} - L_n)](f)\rangle| = |\langle (\mathrm{Id} - L_n^*) \circ T^*(\delta_z), f\rangle|.$$

Recall that $(H^0_v)^* \simeq G^\infty_v$, thus the sequence $(\mathrm{Id} - L_n^*)_n \subset \mathcal{L}(G^\infty_v)$ is bounded according to (A) (ii) and converges to zero on every point in the total set $C = \{\delta_x : x \in D\} \subset G^\infty_v$ because of (A) (i). So we appeal to the Banach–Steinhaus type theorem ([19], III 4.5) to conclude that $(\mathrm{Id} - L_n^*)_n$ converges to zero uniformly on compact subsets of G^∞_v. In particular on the image $\{T^*(\delta_z) : |z| \leqslant t\} \subset G^\infty_v$ of the compact set $\{\delta_z : |z| \leqslant t\} \subset G^\infty_w$, that is,

$$0 = \lim_n \sup_{\{|z|\leqslant t\}} \|(\mathrm{Id} - L_n^*)(T^*(\delta_z))\| = \lim_n \sup_{\{\|f\|_v\leqslant 1\}} \sup_{\{|z|\leqslant t\}} |\langle (\mathrm{Id} - L_n^*) \circ T^*(\delta_z), f\rangle|,$$

where the last equality holds because of the isometry between $(H^0_v)^*$ and G^∞_v. Therefore since the weights are bounded,

$$\lim_n \sup_{\{\|f\|_v\leqslant 1\}} \sup_{\{|z|\leqslant t\}} |[T \circ (\mathrm{Id} - L_n)](f)(z)|\widetilde{w}(z) = 0.$$

On the other hand, for $|z| > t$,

$$\begin{aligned}\sup_{\{\|f\|_v\leqslant 1\}} |[T \circ (\mathrm{Id} - L_n)](f)(z)|\widetilde{w}(z) &= \sup_{\{\|f\|_v\leqslant 1\}} \frac{1}{\|\delta_z\|} |\langle T^*(\delta_z), (\mathrm{Id} - L_n)(f)\rangle| \\ &\leqslant \frac{1}{\|\delta_z\|} \|T^*(\delta_z)\| \, \|\mathrm{Id} - L_n\| \\ &\leqslant \|\mathrm{Id} - L_n\| \cdot \sup_{|z|>t} \frac{\|T^*(\delta_z)\|}{\|\delta_z\|}.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_n \|T - T\circ L_n\| &= \limsup_n \sup_{\{\|f\|_v\leqslant 1\}} \sup_{\{|z|<1\}} |[T\circ(\mathrm{Id}-L_n)](f)(z)|\widetilde{w}(z)\\ &= \limsup_n \sup_{\{\|f\|_v\leqslant 1\}} \sup_{\{|z|>t\}} |[T\circ(\mathrm{Id}-L_n)](f)(z)|\widetilde{w}(z)\\ &\leqslant \limsup_n \Big(\|\mathrm{Id}-L_n\|\cdot \sup_{|z|>t}\frac{\|T^*(\delta_z)\|}{\|\delta_z\|}\Big) \leqslant \sup_{|z|>t}\frac{\|T^*(\delta_z)\|}{\|\delta_z\|}.\end{aligned}$$

Finally to obtain the claimed inequality, let t tend to 1. ∎

REMARK 3.2. The result of Theorem 3.1 is valid for any linear operator, bounded or not, since for unbounded operators the essential norm is ∞.

By Theorem 2.2 of [17] the weighted composition operator $C_{\varphi,\phi} : H_v^0 \to H_w^0$ is bounded if and only if $\phi \in H_w^0$ and $\limsup_{|z|\to 1^-} \frac{w(z)|\phi(z)|}{\widetilde{v}(\varphi(z))} < \infty$. See also [7] for a similar result.

COROLLARY 3.3 ([17]). *Let v, w be weights and let $C_{\varphi,\phi} : H_v^0 \to H_w^0$ be a bounded weighted composition operator. Then*

$$\|C_{\varphi,\phi}\|_{\mathrm{e}} = \limsup_{|z|\to 1^-} \frac{\widetilde{w}(z)|\phi(z)|}{\widetilde{v}(\varphi(z))}.$$

Proof. It suffices to check that $C^*_{\varphi,\phi}(\delta_z) = \phi(z)\delta_{\varphi(z)}$. ∎

In order to consider differences of composition operators we need the pseudohyperbolic metric $\rho(z,w)$ for $z, w \in D$ which is defined by $\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$. Moreover, a weight v is said to satisfy the Lusky condition (L1) if

$$\inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0.$$

Several conditions equivalent to (L1) can be seen in [9] and [11]. The standard weights v_p, $p > 0$, are weights which have (L1). The following result has previously been directly proved by J. Bonet et al. in [6].

In the sequel we will write $A \preceq B$ if there is a positive constant c, not depending on properties of A and B, such that $A \leqslant cB$. We write $A \approx B$ whenever $A \preceq B$ and $B \preceq A$.

COROLLARY 3.4. *Let $C_{\varphi_1} - C_{\varphi_2} : H_v^0 \to H_w^0$ be a bounded difference of two composition operators and v, w weights such that v satisfies* (L1). *Then,*

$$\|C_{\varphi_1} - C_{\varphi_2}\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \max\left\{\frac{1}{\widetilde{v}(\varphi_1(z))}, \frac{1}{\widetilde{v}(\varphi_2(z))}\right\}\widetilde{w}(z)\rho(\varphi_1(z),\varphi_2(z)).$$

Proof. Since, $C^*_{\varphi_1} - C^*_{\varphi_2}(\delta_z) = \delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}$, we apply Theorem 3.1 to get

$$\|C_{\varphi_1} - C_{\varphi_2}\|_{\mathrm{e}} = \limsup_{|z|\to 1^-} \frac{\|\delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}\|}{\|\delta_z\|}.$$

Then, by Lemma 1 in [6] (see also Lemma 14 in [9]), there is a constant C such that

$$\|\delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}\| \leqslant C \max\left\{\frac{1}{\widetilde{v}(\varphi_1(z))}, \frac{1}{\widetilde{v}(\varphi_2(z))}\right\}\rho(\varphi_1(z), \varphi_2(z)),$$

so the validity of the upper bound follows now. For the lower bound, it suffices to recall for each $z \in D$ the function $f_{\varphi_1(z)} \in H^\infty_v$, and to consider the function

$$f_1(x) = f_{\varphi_1(z)}(x)\frac{x - \varphi_2(z)}{1 - \overline{\varphi_2(z)}x}, \quad x \in D.$$

Then $f_1 \in H^\infty_v$ and $\|f_1\|_v \leqslant 1$. Also $|f_1(\varphi_1(z))| = \frac{\rho(\varphi_1(z),\varphi_2(z))}{\widetilde{v}(\varphi_1(z))}$ and $f_1(\varphi_2(z)) = 0$. Hence $\|\delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}\| \geqslant \frac{\rho(\varphi_1(z),\varphi_2(z))}{\widetilde{v}(\varphi_1(z))}$. By exchanging the subindexes we obtain $\|\delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}\| \geqslant \frac{\rho(\varphi_1(z),\varphi_2(z))}{\widetilde{v}(\varphi_2(z))}$. Thus

$$\|\delta_{\varphi_1(z)} - \delta_{\varphi_2(z)}\| \geqslant \rho(\varphi_1(z), \varphi_2(z)) \max\left\{\frac{1}{\widetilde{v}(\varphi_1(z))}, \frac{1}{\widetilde{v}(\varphi_1(z))}\right\}. \quad \blacksquare$$

COROLLARY 3.5. *Assume that v, w are weights such that H^0_w is isomorphic to c_0. Let $T : H^\infty_v \to H^\infty_w$ be a bounded operator such that $T(H^0_v) \subset H^0_w$ and whose restrictions to bounded subsets of H^∞_v are $\tau_0 - \tau_0$ continuous. Then*

$$\|T\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \frac{\|T^*(\delta_z)\|}{\|\delta_z\|}.$$

Proof. To see that $T = (T_{|_{H^0_v}})^{**}$, it suffices to remark that T is weak*-weak* continuous, which in turn follows from $T = (T^*_{|_{G^\infty_w}})^*$, an identity guaranteed by the $\tau_0 - \tau_0$ continuity. Since H^0_w is isomorphic to c_0, we may use S. Axler et al. [1] to obtain that $\|T\|_{\mathrm{e}} = \|(T_{|_{H^0_v}})^*\|_{\mathrm{e}} \approx \|T_{|_{H^0_v}}\|_{\mathrm{e}}$. Then apply Theorem 3.1. $\blacksquare$

We shall now apply Theorem 3.1 to bounded operators on Bloch type spaces. Let $\mathcal{B}_p$, $0 < p < \infty$, denote the Bloch type spaces of functions $f \in H(D)$ satisfying $f(0) = 0$ and $\|f\|_{\mathcal{B}_p} := \sup_{z\in D}(1 - |z|^2)^p|f'(z)| < \infty$. The spaces $\mathcal{B}_p$ and the subspaces $\mathcal{B}^0_p = \left\{f \in \mathcal{B}_p : \lim_{|z|\to 1^-}(1 - |z|^2)^p|f'(z)| = 0\right\}$ become Banach spaces under the norm $\|\cdot\|_{\mathcal{B}_p}$. Let us put $\delta'_z(f) := f'(z)(1 - |z|^2)^p$ for $z \in D$ and $f \in \mathcal{B}_p$.

COROLLARY 3.6. (i) *Let $T : \mathcal{B}^0_p \to \mathcal{B}^0_p$ be a bounded operator. Then*

$$\|T\|_{\mathrm{e}} = \limsup_{|z|\to 1^-} \|T^*(\delta'_z)\|.$$

(ii) *Let* $T : \mathcal{B}_p \to \mathcal{B}_p$ *be a bounded operator such that* $\mathcal{B}_p^0$ *is an invariant subspace and whose restrictions to the bounded subsets are* $\tau_0 - \tau_0$ *continuous. Then*

$$\|T\|_e \approx \limsup_{|z|\to 1^-} \|T^*(\delta'_z)\|.$$

Proof. Put $S_p : \mathcal{B}_p \to H^\infty_{v_p}$ defined by $S_p(f)(z) = f'(z)$. Note that $f \in \mathcal{B}_p^0$ if and only if $f' \in H^0_{v_p}$, and also that S_p is an onto isometry. Either in (i) or (ii) if we put $L = S_p \circ T \circ S_p^{-1}$, we have $\|L\|_e = \|T\|_e$. And also $T^*(\delta'_z) = (S_p^* \circ L^*)\left(\frac{\delta_z}{\|\delta_z\|}\right)$, indeed, since $L \circ S_p = S_p \circ T$ and $\|\delta_z\| = (1-|z|^2)^{-p}$,

$$(S_p^* \circ L^*)\left(\frac{\delta_z}{\|\delta_z\|}\right) = T^* \circ S_p^*\left(\frac{\delta_z}{\|\delta_z\|}\right) = T^*((1-|z|^2)^p S_p^*(\delta_z)) = T^*(\delta'_z).$$

Since S_p^* is also an isometry, $\|T^*(\delta'_z)\| = \left\|L^*\left(\frac{\delta_z}{\|\delta_z\|}\right)\right\|$. Now for (i), just apply Theorem 3.1. For (ii), observe that $H^0_{v_p}$ is an invariant subspace for L and, moreover, that L is $\tau_0 - \tau_0$ continuous on bounded sets, because both S_p and S_p^{-1} preserve the $\tau_0 - \tau_0$ continuity. Then use Corollary 3.5. ∎

It is easily checked that $C^*_\varphi(\delta'_z) = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p \varphi'(z)\delta'_{\varphi(z)}$ for a composition operator C_φ. Then Theorem 1 in [14] and Theorem 2.1 in [16] follow straightforward from the above corollary. The essential norm of composition operators acting on Bloch type spaces has also been calculated in [17]. Note also that since $\mathcal{B}_p^0 \simeq c_0$, any weakly compact operator $T : \mathcal{B}_p^0 \to \mathcal{B}_p^0$ is compact.

COROLLARY 3.7. *Assume that* v, w *are weights such that* H^0_w *is isomorphic to* c_0. *Let* $S : G^\infty_w \to G^\infty_v$ *be a bounded operator. If* $S^*(H^0_v) \subset H^0_w$, *then*

$$\|S\|_e \approx \limsup_{|z|\to 1^-} \frac{\|S(\delta_z)\|}{\|\delta_z\|}.$$

In particular, S is compact and $S^*(H^0_v) \subset H^0_w$, *if and only if* $\lim_{|z|\to 1^-} \frac{\|S(\delta_z)\|}{\|\delta_z\|} = 0$.

Proof. Put $T := S^*_{|_{H^0_v}}$. It is easy to see that $T^* = S$. Then, apply Theorem 3.1 to T and recall that $\|T\|_e \approx \|S\|_e$ by Theorem 3 in [1] and the assumption $H^0_w \simeq c_0$. This gives the essential norm (up to equivalence) of S. Moreover, if $\lim_{|z|\to 1^-} \frac{\|S(\delta_z)\|}{\|\delta_z\|} = 0$, then for any $f \in H^0_v$, $\lim_{|z|\to 1^-} |S^*(f)(z)|\widetilde{w}(z) = \lim_{|z|\to 1^-} |\langle f, S(\delta_z)\rangle| \frac{1}{\|\delta_z\|} = 0$. Thus, $S^*(f) \in H^0_{\widetilde{w}} \simeq H^0_w$ and S is obviously compact. ∎

The last statement of the above corollary includes the main result in [24], Theorem 1.

Let us now reformulate the above result for bounded operators on A^1_ψ. To do this, we assume that $\{v, \psi\}$ is a normal pair, so $H^0_v \simeq c_0$. Let us denote by J the isomorphism between G^∞_v and A^1_ψ. Observe that $J(\delta_z) = K^\alpha_z$. If $T : A^1_\psi \to A^1_\psi$

is a bounded operator, we consider the bounded operator $S := J^{-1} \circ T \circ J$. Then $\|S\|_{\mathrm{e}} \approx \|T\|_{\mathrm{e}}$. Further, $J \circ S = T \circ J$, hence

$$J \circ S\Big(\frac{\delta_z}{\|\delta_z\|}\Big) = T \circ J\Big(\frac{\delta_z}{\|\delta_z\|}\Big) = \frac{T(K_z^\alpha)}{\|K_z\|_\psi}\frac{\|K_z^\alpha\|_\psi}{\|\delta_z\|}, \quad \text{therefore}$$

$$\frac{1}{\|J^{-1}\|}\Big\|S\Big(\frac{\delta_z}{\|\delta_z\|}\Big)\Big\| \leqslant \Big\|J \circ S\Big(\frac{\delta_z}{\|\delta_z\|}\Big)\Big\| = \frac{\|T(K_z)\|_\psi}{\|K_z^\alpha\|_\psi}\frac{\|K_z^\alpha\|_\psi}{\|\delta_z\|} \leqslant \|J\|\frac{\|T(K_z^\alpha)\|_\psi}{\|K_z^\alpha\|_\psi}.$$

Thus

$$\limsup_{|z|\to 1^-} \frac{\|T(K_z^\alpha)\|_\psi}{\|K_z^\alpha\|_\psi} \approx \limsup_{|z|\to 1^-} \frac{\|S(\delta_z)\|}{\|\delta_z\|}.$$

Since $S^*(H_v^0) \subset H_v^0$ if and only if $T^*(H_v^0) \subset H_v^0$, Corollary 3.7 gives the following estimate of the essential norm of T.

COROLLARY 3.8. *Let $\{v, \psi\}$ be a normal pair and let $T : A_\psi^1 \to A_\psi^1$ be a bounded operator. If $T^*(H_v^0) \subset H_v^0$, then*

$$\|T\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \frac{\|T(K_z^\alpha)\|_\psi}{\|K_z^\alpha\|_\psi}.$$

Next, we apply this corollary to Toeplitz operators thereby improving Yu's result ([24], Theorem 2), and also to composition operators on the weighted Bergman space A_ψ^1.

COROLLARY 3.9. *Assume that $\{v, \psi\}$ is a normal pair.*

(i) *Let $u \in L^\infty$. For the Toeplitz operator $T_u^\alpha \in \mathcal{L}(A_\psi^1, A_\psi^1)$ we have that*

$$\|T_u^\alpha\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \frac{\|T_u^\alpha(K_z^\alpha)\|_\psi}{\|K_z^\alpha\|_\psi}.$$

(ii) *For any pair of composition operators $C_{\varphi_1}, C_{\varphi_2} \in \mathcal{L}(A_\psi^1, A_\psi^1)$ we have that*

$$\|C_{\varphi_1} - C_{\varphi_2}\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \frac{\|(C_{\varphi_1} - C_{\varphi_2})(K_z^\alpha)\|_\psi}{\|K_z^\alpha\|_\psi}.$$

Proof. The result follows from the above corollary once one realizes that H_v^0 is mapped into H_v^0 by the dual map. This invariance was proved by Yu in the case of the Toeplitz operators. Following his approach, we next show the invariance for the dual of composition operators: Let $f \in H_v^0$. Note that $C_\varphi(K_z^\alpha)(x) = \frac{1+\alpha}{(1-\bar{z}\varphi(x))^{2+\alpha}}$. Then

$$\begin{aligned} v(z)C_\varphi^*(f)(z) &= v(z)\langle C_\varphi^*(f), K_z^\alpha\rangle = v(z)\langle f, C_\varphi(K_z^\alpha)\rangle \\ &= v(z)\int_D f(x)\overline{C_\varphi(K_z^\alpha)(x)}(1-|x|^2)^\alpha \mathrm{d}A(x) \end{aligned}$$

$$= v(z) \int\limits_{D\setminus rD} f(x)\overline{C_\varphi(K_z^\alpha)(x)}v(x)\psi(x)\mathrm{d}A(x)$$
$$+ v(z)\int\limits_{rD} f(x)\frac{1+\alpha}{(1-z\overline{\varphi}(x))^{2+\alpha}}v(x)\psi(x)\mathrm{d}A(x).$$

The first integral is estimated by

$$\sup_{|x|\geqslant r}|f(x)v(x)|\int\limits_D v(z)\|C_\varphi\|\ \|K_z^\alpha\|_\psi\psi(x)\mathrm{d}A(x)\leqslant \sup_{|x|\geqslant r}|f(x)v(x)|\|C_\varphi\|C\int\limits_D\psi(x)\mathrm{d}A(x),$$

where C is the constant with $\|K_z^\alpha\|_\psi v(z) \leqslant C$ for all $z \in D$. Given $\varepsilon > 0$, we may pick r close enough to 1 so that the first summand is less than $\frac{\varepsilon}{2}$. The second integral may be estimated by $\|f\|_v \int\limits_{rD} \frac{1+\alpha}{(1-c_r)^{2+\alpha}}\psi(w)\mathrm{d}A(w)$ where $c_r < 1$ is a bound got from the compactness of $\varphi(r\overline{D})$. Thus by letting $|z| \to 1$, the second summand will be less than $\frac{\varepsilon}{2}$. So, $C_\varphi^* f \in H_v^0$. ∎

Corollary 3.9 (ii) should be compared to Moorhouse's result [15] that $C_{\varphi_1} - C_{\varphi_2}$ is compact on the standard weighted Bergman spaces A_α^2, $\alpha > -1$, if and only if $\left\|(C_{\varphi_1} - C_{\varphi_2})^*\left(\frac{K_z^\alpha}{\|K_z^\alpha\|_{A_\alpha^2}}\right)\right\|_{A_\alpha^2} \to 0$ as $|z| \to 1^-$.

This corollary leads us to estimate the essential norm of composition operators acting on the Bergman space A^1. We can do it since $A_\psi^1 = A^1$ for $\psi = 1$, $v_p(z) = (1-|z|^2)^\alpha$ and $\alpha = p > 0$. In this case, $\|K_z^\alpha\|_{A^1} \approx \|\delta_z\| = (1-|z|^2)^{-\alpha}$.

Recall that by Proposition 2.4 in [23], we have

$$\|C_\varphi(K_z^\alpha)\|_{A^1} \approx |K_z^\alpha(\varphi(0))| + \int\limits_D |K_z(x)|^{-1}|(K_z^\alpha)'(x)|^2 N_{\varphi,2}(x)\mathrm{d}A(x),$$

where $N_{\varphi,2}$ is the generalized counting function $N_{\varphi,2}(x) = \sum\limits_{z\in\varphi^{-1}(x)}(\log(1/|z|))^2$ for $x \in D\setminus\{\varphi(0)\}$.

COROLLARY 3.10. *The following formula holds for any* $C_\varphi \in \mathcal{L}(A^1, A^1)$*:*

$$\|C_\varphi\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-}\frac{N_{\varphi,2}(z)}{(1-|z|^2)^2}.$$

Proof. Following the argument in the proof of Theorem 4.3 in [23], we check that for $|z|$ close enough to 1, one has

$$\|C_\varphi(K_z^\alpha)\|_{A^1} \succeq \frac{N_{\varphi,2}(z)}{(1-|z|^2)^{2+\alpha}};\quad \text{hence}\quad \|C_\varphi\|_{\mathrm{e}} \succeq \limsup_{|z|\to 1^-}\frac{N_{\varphi,2}(z)}{(1-|z|^2)^2}.$$

Put $L := \limsup_{|z|\to 1^-} \frac{N_{\varphi,2}(z)}{(1-|z|^2)^2} < \infty$. For any $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that $\frac{N_{\varphi,2}(x)}{(1-|x|^2)^2} \leqslant L + \varepsilon$ for all $|x| \geqslant r_\varepsilon$. Therefore,

$$\begin{aligned}\int\limits_{|x|\geqslant r_\varepsilon} & |K_z^\alpha(x)|^{-1}|(K_z^\alpha)'(x)|^2 N_{\varphi,2}(x)\mathrm{d}A(x) \\ &\leqslant \int\limits_{|x|\geqslant r_\varepsilon} |K_z^\alpha(x)|^{-1}|(K_z^\alpha)'(x)|^2(L+\varepsilon)(1-|x|^2)^2\mathrm{d}A(x) \\ &\preceq (L+\varepsilon)\int\limits_D |K_z^\alpha(x)|^{-1}|(K_z^\alpha)'(x)|^2(\log(1/|x|))^2\mathrm{d}A(x) \preceq (L+\varepsilon)\|K_z^\alpha\|_{A^1},\end{aligned}$$

where we have used Lemma 2.3 in [23] and the fact that $\log(1/|x|)$ is comparable to $(1-|x|^2)$ for $r_\varepsilon \leqslant |x| < 1$. Then,

$$\begin{aligned}\frac{\|C_\varphi(K_z^\alpha)\|_{A^1}}{\|K_z^\alpha\|_{A^1}} \approx \frac{|K_z^\alpha(\varphi(0))|}{\|K_z^\alpha\|_{A^1}} &+ \frac{1}{\|K_z^\alpha\|_{A^1}}\int\limits_{r_\varepsilon D} |K_z^\alpha(x)|^{-1}|(K_z^\alpha)'(x)|^2 N_{\varphi,2}(x)\mathrm{d}A(x) \\ &+ \frac{1}{\|K_z^\alpha\|_{A^1}}\int\limits_{|x|\geqslant r_\varepsilon} |K_z^\alpha(x)|^{-1}|(K_z^\alpha)'(x)|^2 N_{\varphi,2}(x)\mathrm{d}A(x) \\ \preceq \frac{|K_z^\alpha(\varphi(0))|}{\|K_z\|_{A^1}} &+ \frac{1}{\|K_z^\alpha\|_{A^1}}\int\limits_{r_\varepsilon D}|K_z^\alpha(x)|^{-1}|(K_z^\alpha)(x)|^2 N_{\varphi,2}(x)\mathrm{d}A(x) + L + \varepsilon.\end{aligned}$$

The first term in the above sum goes to 0 whenever $|z| \to 1$. For the second one,

$$\begin{aligned}\limsup_{|z|\to 1^-}(1-|z|^2)^\alpha|z|^2 &\int\limits_{r_\varepsilon D} \frac{N_{\varphi,2}(x)}{|1-\bar{z}x|^{4+\alpha}}\mathrm{d}A(x) \\ &\leqslant \limsup_{|z|\to 1^-}(1-|z|^2)^\alpha|z|^2 \int\limits_{r_\varepsilon D} \frac{N_{\varphi,2}(x)}{(1-r_\varepsilon)^{4+\alpha}}\mathrm{d}A(x) = 0;\end{aligned}$$

$$\text{hence} \quad \|C_\varphi\|_{\mathrm{e}} \approx \limsup_{|z|\to 1^-} \frac{\|C_\varphi(K_z^\alpha)\|_{A^1}}{\|K_z^\alpha\|_{A^1}} \preceq L + \varepsilon; \ \text{thus} \quad \|C_\varphi\|_{\mathrm{e}} \preceq L. \quad \blacksquare$$

As a further application of Corollary 3.9 (i) we obtain an upper bound of the essential norm of the Toeplitz operators T_u^α on A^1 for $\alpha > 0$ and $u \in L^\infty(D)$. This can be done in the following way:

$$\|T_u^\alpha(K_z^\alpha)\|_{A^1} = \int\limits_D \Big| \int\limits_D u(y)K_z^\alpha(y)\overline{K_x^\alpha(y)}\Big|(1-|y|^2)^\alpha \mathrm{d}A(y)|\mathrm{d}A(x).$$

Then we obtain exchanging the order of integration and using $(1-|y|^2)^{-\alpha} \approx \int\limits_D \frac{1}{|1-\overline{y}x|^{2+\alpha}} \mathrm{d}A(x)$ (see [25]), that

$$\|T_u^\alpha(K_z^\alpha)\|_{A^1} \preceq \int\limits_D |u(y)|\ |K_z^\alpha(y)| \mathrm{d}A(y).$$

For fixed $r > 0$

$$\limsup_{|z|\to 1} \frac{\|(T_u^\alpha(K_z^\alpha)\|_{A^1}}{\|K_z^\alpha\|_{A^1}} \preceq \limsup_{|z|\to 1} \int\limits_{|y|>r} \frac{|u(y)|(1-|z|^2)^\alpha}{|1-\overline{y}z|^{2+\alpha}} \mathrm{d}A(y) \quad \text{and}$$

$$1 \approx \limsup_{|z|\to 1} \int\limits_{|y|>r} \frac{(1-|z|^2)^\alpha}{|1-\overline{y}z|^{2+\alpha}} \mathrm{d}A(y).$$

Therefore we get the upper bound,

$$\|T_u^\alpha\|_{\mathrm{e}} \preceq \limsup_{|y|\to 1} |u(y)|.$$

Acknowledgements. Pablo Galindo's research supported partially by Project MEC-FEDER MTM 2007 064521. Mikael Lindström's research supported partially by Magnus Ehrnrooths stiftelse and Project MEC-FEDER MTM 2007 064521 .

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Received January 16, 2008.