

THE GAP BETWEEN UNBOUNDED REGULAR OPERATORS

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ABSTRACT. We study and compare the gap and the Riesz topologies of the space of all unbounded regular operators on Hilbert C^* -modules. We show that the space of all bounded adjointable operators on Hilbert C^* -modules is an open dense subset of the space of all unbounded regular operators with respect to the gap topology. The restriction of the gap topology on the space of all bounded adjointable operators is equivalent to the topology which is generated by the usual operator norm. The space of regular selfadjoint Fredholm operators on Hilbert C^* -modules over the C^* -algebra of compact operators is path-connected with respect to the gap topology, however, the result may not be true for some Hilbert C^* -modules.

KEYWORDS: *Hilbert C^* -modules, unbounded operators, gap topology, C^* -algebras of compact operators, Fredholm operators.*

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1. INTRODUCTION

Hilbert C^* -modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C^* -algebra. The theory of these modules, together with bounded and unbounded operators, is not only rich and attractive in its own right but forms an infrastructure for some of the most important research topics in operator algebras. They play an important role in the modern theory of C^* -algebra, in KK-theory, in non-commutative geometry and in quantum groups.

A (left) *pre-Hilbert C^* -module* over a C^* -algebra \mathcal{A} is a left \mathcal{A} -module E equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathcal{A}$, $(x, y) \mapsto \langle x, y \rangle$, which is \mathcal{A} -linear in the first variable x and has the properties:

$$\begin{aligned}\langle x, y \rangle &= \langle y, x \rangle^*, & \langle ax, y \rangle &= a \langle x, y \rangle \quad \text{for all } a \text{ in } \mathcal{A}, \\ \langle x, x \rangle &\geq 0 \quad \text{with equality only when } x = 0.\end{aligned}$$

A pre-Hilbert \mathcal{A} -module E is called a *Hilbert \mathcal{A} -module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. If E, F are two Hilbert \mathcal{A} -modules

then the set of all ordered pairs of elements $E \oplus F$ from E and F is a Hilbert \mathcal{A} -module with respect to the \mathcal{A} -valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_F$. It is called the direct *orthogonal sum of E and F* . A Hilbert \mathcal{A} -submodule E of a Hilbert \mathcal{A} -module F is an orthogonal summand if $E \oplus E^\perp = F$, where E^\perp denotes the orthogonal complement of E in F . The papers [9], [8], [24], some chapters in [14], [25], and the book by E.C. Lance [19] are used as standard sources of reference.

As a convention, throughout the present paper we assume \mathcal{A} to be an arbitrary C^* -algebra (i.e. not necessarily unital), we also assume $\mathcal{K}(H)$ to be the C^* -algebra of all compact operators on an arbitrary Hilbert space H . Since we deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $\text{Dom}(\cdot)$, $\text{Ker}(\cdot)$ and $\text{Ran}(\cdot)$ for domain, kernel and range of operators, respectively.

Suppose E, F are Hilbert \mathcal{A} -modules. We denote the set of all \mathcal{A} -linear maps $T : E \rightarrow F$ for which there is a map $T^* : F \rightarrow E$ such that the equality $\langle Tx, y \rangle = \langle x, T^*y \rangle$ holds for any $x \in E, y \in F$ by $B(E, F)$. The operator T^* is called the *adjoint operator* of T . $B(E, E)$ is denoted by $B(E)$.

Unbounded regular operators were first introduced by Baaj and Julg in [4] and later they were studied more by Woronowicz and Napiórkowski in [27], [26]. Lance gave a brief indication in his book [19] about unbounded regular operators on Hilbert C^* -modules. An operator t from a Hilbert \mathcal{A} -module E to another Hilbert \mathcal{A} -module F is said to be *regular* if

- (i) t is closed and densely defined,
- (ii) its adjoint t^* is also densely defined, and
- (iii) the range of $1 + t^*t$ is dense in E .

Note that if we set $\mathcal{A} = \mathbb{C}$ i.e. if we take E, F to be Hilbert spaces, then this is exactly the definition of a densely defined closed operator, except that in that case, both the second and the third condition follow from the first one. In the frame work of Hilbert C^* -modules, one needs to add these extra conditions in order to get a reasonably good theory. The reader is encouraged to study the publications [11], [12], [18], [23] for more detailed information about unbounded operators on Hilbert C^* -modules.

The *gap topology* is induced by the metric $d(t, s) = \|P_{G(t)} - P_{G(s)}\|$ where $P_{G(t)}$ and $P_{G(s)}$ are projections onto the graphs of densely defined closed operators t, s , respectively. The gap topology on the space of all densely defined closed operators has been studied systematically in the book [16] and in the seminal paper by Cordes and Labrousse in [7]. Recently the gap topology on the space of all unbounded selfadjoint Fredholm operators has been reconsidered in [6], [15], [21].

We study the gap topology of the space of all unbounded regular operators on arbitrary Hilbert C^* -modules. We also introduce a strictly stronger topology

than the gap topology of the space of all unbounded regular operators, which is called Riesz topology. We show that the space of all bounded adjointable operators on Hilbert C^* -modules is an open dense subset of the space of all unbounded regular operators with respect to the gap topology. Moreover the restriction of the gap topology on the space of all bounded adjointable operators is equivalent to the topology which is generated by the usual operator norm. The gap metric will help us to find an isometric operation preserving map of the space of all densely defined closed operators on Hilbert C^* -modules over the C^* -algebra of compact operators onto the space of all densely defined closed operators on a suitable Hilbert space. This fact together with a result of Cordes and Labrousse [7] give us the opportunity to characterize the path-connected components of the space of unbounded regular Fredholm operators on Hilbert $\mathcal{K}(H)$ -modules with respect to the gap topology. Indeed, every two unbounded regular Fredholm operators are homotopic if and only if they have the same index.

2. PRELIMINARIES

In this section we would like to recall some definitions and simple facts about regular operators on Hilbert \mathcal{A} -modules. For details see Chapters 9 and 10 of [19], and the papers [11], [18], [27], [26]. Then we will introduce and compare the Riesz and gap topologies of the space of all unbounded regular operators.

Let E, F be Hilbert \mathcal{A} -modules, we will use the notation $t : \text{Dom}(t) \subseteq E \rightarrow F$ to indicate that t is an \mathcal{A} -linear operator whose domain $\text{Dom}(t)$ is a dense submodule of E (not necessarily identical with E) and whose range is in F . A densely defined operator $t : \text{Dom}(t) \subseteq E \rightarrow F$ is called *closed* if its graph $G(t) = \{(x, tx) : x \in \text{Dom}(t)\}$ is a closed submodule of the Hilbert \mathcal{A} -module $E \oplus F$. If t is closable, the operator $s : \text{Dom}(s) \subseteq E \rightarrow F$ with the property $G(s) = \overline{G(t)}$ is called the *closure* of t denoted by $s = \bar{t}$. A densely defined operator $t : \text{Dom}(t) \subseteq E \rightarrow F$ is called *adjointable* if it possesses a densely defined map $t^* : \text{Dom}(t^*) \subseteq F \rightarrow E$ with the domain

$$\text{Dom}(t^*) = \{y \in F : \text{there is } z \in E \text{ such that } \langle tx, y \rangle_F = \langle x, z \rangle_E \text{ for all } x \in \text{Dom}(t)\}$$

which satisfies the property $\langle tx, y \rangle_F = \langle x, t^*y \rangle_E$ for any $x \in \text{Dom}(t)$, $y \in \text{Dom}(t^*)$. This property implies that t^* is a closed \mathcal{A} -linear map. A densely defined closed \mathcal{A} -linear map $t : \text{Dom}(t) \subseteq E \rightarrow F$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. We denote the set of all regular operators from E to F by $R(E, F)$. Then $R(E, E)$ is denoted by $R(E)$. A densely defined operator t which has a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented $E \oplus F$ (cf. [11]). If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $Q_t = (1 + t^*t)^{-1/2}$, $R_t = (1 + t^*t)^{-1} = Q_t^2$ and $F_t = tQ_t$, then $\text{Ran}(Q_t) = \text{Dom}(t)$, $0 \leq Q_t \leq 1$ in $B(E, E)$ and $F_t \in B(E, F)$ (cf. Chapter 9 of [19]). The bounded operator F_t is called

the bounded transform (or Woronowicz transform) of the regular operator t . The map $t \rightarrow F_t$ defines a bijection

$$R(E, F) \rightarrow \{T \in B(E, F) : \|T\| \leq 1 \text{ and } \text{Ran}(1 - T^*T) \text{ is dense in } E\},$$

(cf. Theorem 10.4 of [19]). This map is adjoint-preserving, i.e. $F_t^* = F_{t^*}$, and for the bounded transform $F_t = tQ_t = t(1 + t^*t)^{-1/2}$ we have $\|F_t\| \leq 1$ and $t = F_tQ_t^{-1} = F_t(1 - F_t^*F_t)^{-1/2}$. A regular operator $t \in R(E)$ is called selfadjoint if $t^* = t$. Obviously a regular operator t is selfadjoint if and only if its bounded transform F_t is selfadjoint.

COROLLARY 2.1. *Let $T \in R(E, F)$ be a regular operator and F_T be its bounded transform. Then $T \in B(E, F)$ if and only if $\|F_T\| < 1$.*

Proof. For each $T \in R(E, F)$ the bounded adjointable operator $Q_T : E \rightarrow \text{Ran}(Q_T) = \text{Dom}(T) \subseteq E$ is invertible and satisfies $Q_T^2 = 1 - F_T^*F_T$. Therefore $T \in B(E, F)$ if and only if $\text{Dom}(T) = E$, if and only if $\|F_T\| < 1$. ■

There is a natural metric on the set of regular operators, the so-called gap metric. Let $t \in R(E, F)$ then $E \oplus F = G(t) \oplus V(G(t^*))$, where $V \in B(E \oplus F, F \oplus E)$ is defined by $V(x, y) = (y, -x)$. The orthogonal projection $P_{G(t)} : E \oplus F \rightarrow E \oplus F$ can be described through the following matrix

$$(2.1) \quad P_{G(t)} = \begin{pmatrix} R_t & t^*R_{t^*} \\ tR_t & 1 - R_{t^*} \end{pmatrix} \in B(E \oplus F).$$

It follows from (9.7) of [19] and the equalities $F_tF_t^* = 1 - R_t^*$ and $(F_tQ_t)^* = (tR_t)^* = t^*R_{t^*}$.

DEFINITION 2.2. Let $t, s \in R(E, F)$, then the gap metric on the space of all unbounded regular operators is defined by $d(t, s) = \|P_{G(t)} - P_{G(s)}\|$ where $P_{G(t)}$ and $P_{G(s)}$ are orthogonal projections onto $G(t)$ and $G(s)$, respectively. The topology induced by this metric is called *gap topology*.

Let E, F be two Hilbert \mathcal{A} -modules and operators t, s be in $R(E, F)$. An equivalent picture of the gap metric is now definable by using (2.1) as well as the fact that $(tR_t)^* = t^*R_{t^*}$. Indeed, the following metric, which is again denoted by d , is uniformly equivalent to the gap metric

$$(2.2) \quad d(t, s) = \sup\{\|R_t - R_s\|, \|R_{t^*} - R_{s^*}\|, \|tR_t - sR_s\|\}.$$

REMARK 2.3. Let $t \in R(E, F)$ be a regular operator and F_t be its bounded transform. For every bounded adjointable operator S in $B(E, F)$ of norm $\|S\| \leq 1$ the operator $\mathcal{F}(S) := 1 - S^*S$ is a positive operator. we also have

$$R_t = Q_t^2 = 1 - F_t^*F_t = \mathcal{F}(F_t), \quad R_{t^*} = Q_{t^*}^2 = 1 - F_tF_t^* = \mathcal{F}(F_t^*), \quad tR_t = tQ_t^2 = F_tQ_t = F_t\mathcal{F}(F_t)^{1/2}.$$

Therefore we can reformulate the gap metric (2.2) via the bounded transforms of regular operators t and s as follows:

$$(2.3) \quad d(t, s) = \sup\{\|\mathcal{F}(F_t) - \mathcal{F}(F_s)\|, \|\mathcal{F}(F_t^*) - \mathcal{F}(F_s^*)\|, \|F_t\mathcal{F}(F_t)^{1/2} - F_s\mathcal{F}(F_s)^{1/2}\|\}.$$

The Riesz topology of unbounded selfadjoint operators on Hilbert spaces has been investigated in [6], [17], [21]. Their works motivate us for the following definition.

DEFINITION 2.4. Let $t, s \in R(E, F)$ then the Riesz metric on the space of all unbounded regular operators is defined by $\sigma(t, s) = \|F_t - F_s\|$. The topology induced by this metric is called *Riesz topology*.

LEMMA 2.5. *The Riesz topology is stronger than the gap topology on $R(E, F)$.*

Proof. Let $\{t_n\}$ be a sequence in $R(E, F)$ that is convergent to a regular operator t with respect to the Riesz topology, i.e. $\sigma(t_n, t) = \|F_{t_n} - F_t\| \rightarrow 0$. By elementary methods and continuity of the function $g(x) = \sqrt{x}$ on $[0, +\infty)$ we can deduce

$$\begin{aligned} \|\mathcal{F}(F_{t_n}) - \mathcal{F}(F_t)\| &= \|F_{t_n}^* F_{t_n} - F_t^* F_t\| \rightarrow 0, & \|\mathcal{F}(F_{t_n}^*) - \mathcal{F}(F_t^*)\| &= \|F_{t_n} F_{t_n}^* - F_t F_t^*\| \rightarrow 0, \\ \|\mathcal{F}(F_{t_n})^{1/2} - \mathcal{F}(F_t)^{1/2}\| &\rightarrow 0, & \|F_{t_n} \mathcal{F}(F_{t_n})^{1/2} - F_t \mathcal{F}(F_t)^{1/2}\| &\rightarrow 0. \end{aligned}$$

Therefore (2.3) implies that

$$d(t_n, t) = \sup\{\|\mathcal{F}(F_{t_n}) - \mathcal{F}(F_t)\|, \|\mathcal{F}(F_{t_n}^*) - \mathcal{F}(F_t^*)\|, \|F_{t_n} \mathcal{F}(F_{t_n})^{1/2} - F_t \mathcal{F}(F_t)^{1/2}\|\} \rightarrow 0,$$

i.e. the sequence $\{t_n\}$ is gap convergent to the regular operator t . ■

COROLLARY 2.6. *Let $\mathcal{B} = \{T \in B(E, F) : \|T\| \leq 1 \text{ and } \text{Ran}(1 - T^*T) \text{ is dense in } E\}$, then the map*

$$(\mathcal{B}, \|\cdot\|) \rightarrow (R(E, F), d), \quad F_t \mapsto t = F_t(1 - F_t^*F_t)^{-1/2}$$

is bijective and continuous.

Bijectivity and continuity of the map are obtained from Theorem 10.4 of [19] and Lemma 2.5. Suppose E is a Hilbert \mathcal{A} -module, $UB(E)$ and $SR(E)$ denote unitary elements of $B(E)$ and selfadjoint elements of $R(E)$.

REMARK 2.7. Let E be a Hilbert \mathcal{A} -module and let $t \in SR(E)$. According to Lemmata 9.7, 9.8 of [19], the operators $t \pm i$ are bijection (see also Proposition 6 of [18]). Then

$$\begin{aligned} c_t : SR(E) &\longrightarrow \mathcal{C} = \{U \in UB(E) : 1 - U \text{ has dense range}\}, \\ t &\mapsto c_t = (t - i)(t + i)^{-1}, \end{aligned}$$

is a bijection which is called the *Cayley transform* of t , cf. Theorem 10.5 of [19]. The Cayley transform c_t can be written as $c_t = 1 - 2i(t + i)^{-1}$. Thus $(t + i)^{-1} - (s + i)^{-1} = \frac{1}{2}(c_t - c_s)$, for each $t, s \in SR(E)$.

COROLLARY 2.8. *On the space $SR(E)$ the gap metric is uniformly equivalent to the metric \tilde{d} given by*

$$(2.4) \quad \tilde{d}(t, s) := \|(t + i)^{-1} - (s + i)^{-1}\| = \frac{1}{2}\|c_t - c_s\|, \quad \text{for all } t, s \in SR(E).$$

Proof. For each $t, s \in SR(E)$, the expression (2.2) can be written as follows:

$$(2.5) \quad d(t, s) = \sup\{\|R_t - R_s\|, \|tR_t - sR_s\|\}.$$

On the other hand the operators $t \pm i$ are bijective, hence the identities $(t - i)^{-1} = (t + i)(t^2 + 1)^{-1} = tR_t + iR_t$, $(t + i)^{-1} = (t - i)(t^2 + 1)^{-1} = tR_t - iR_t$ hold, which yield

$$(2.6) \quad R_t = \frac{1}{2i}((t - i)^{-1} - (t + i)^{-1}), \quad tR_t = \frac{1}{2}((t - i)^{-1} + (t + i)^{-1}).$$

Now from (2.4), (2.5) and (2.6) we infer that $\frac{1}{2}\tilde{d}(t, s) \leq d(t, s) \leq \tilde{d}(t, s)$, for all $t, s \in SR(E)$. ■

The following example attributed to Fuglede is used to show that the Riesz and gap metrics are different, cf. [6], [21].

EXAMPLE 2.9. Let \mathcal{A} be unital C^* -algebra and $H_{\mathcal{A}}$ be the standard Hilbert \mathcal{A} -module which is countably generated, for $j \in \mathbb{N}$, by orthonormal basis $\xi_j = (0, \dots, 0, 1, 0, \dots, 0)$. For every integer $n \geq 0$ we define $t_n : \text{Dom}(t_n) = \{\sum \lambda_j \xi_j : \sum j^2 |\lambda_j|^2 < +\infty\} \subseteq H_{\mathcal{A}} \rightarrow H_{\mathcal{A}}$ by

$$t_n(\xi_j) = \begin{cases} j\xi_j & \text{if } j \neq n, \\ -j\xi_j & \text{if } j = n. \end{cases}$$

The sequence t_n of selfadjoint regular operator converges to the selfadjoint regular operator t_0 in gap topology. To see this, we apply (2.4) and get

$$\begin{aligned} \tilde{d}(t_n, t_0) &= \|(t_n + i)^{-1} - (t_0 + i)^{-1}\| = \|(t_n + i)^{-1}\xi_n - (t_0 + i)^{-1}\xi_n\| \\ &= \left| \frac{1}{i - n} - \frac{1}{i + n} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

But

$$\sigma(t_n, t_0) \geq \|F_{t_n}\xi_n - F_{t_0}\xi_n\| = \frac{2n}{\sqrt{1 + n^2}} \rightarrow 2.$$

In view of Corollary 2.8 this shows that the Riesz topology is strictly stronger than the gap topology.

3. ON THE GAP TOPOLOGY

Recall that every bounded adjointable operator is regular, that is, for Hilbert C^* -modules E, F , the space $B(E, F)$ can be regarded as a subset of $R(E, F)$. We show that the space of all bounded adjointable operators on Hilbert C^* -modules is an open dense subset of the space of all unbounded regular operators with respect to the gap topology. Then we can conclude that the space of odd bounded adjointable operators is a dense subset of odd unbounded regular operators. The author believe that these results are new even in the case of Hilbert spaces.

LEMMA 3.1. *Let E, F be Hilbert C^* -modules. Then the space $B(E, F)$ is open in $R(E, F)$ with respect to the gap metric d .*

Proof. Let $S \in B(E, F)$ then $\|F_S\|^2 < 1$ by Corollary 2.1, and so, there is a real number δ such that $0 < \delta < 1 - \|F_S\|^2 < 1$. We claim $\{T \in R(E, F) : d(T, S) < \delta\} \subseteq B(E, F)$. Let T be a (possibly unbounded) operator in $R(E, F)$ and $d(T, S) < \delta$, then

$$\|F_T^*F_T\| - \|F_S^*F_S\| \leq \|F_T^*F_T - F_S^*F_S\| = \|\mathcal{F}(F_T) - \mathcal{F}(F_S)\| \leq d(T, S) < \delta.$$

That is, $\|F_T\|^2 = \|F_T^*F_T\| < \delta + \|F_S\|^2 < 1$, and T is therefore bounded by Corollary 2.1. ■

PROPOSITION 3.2. *The metric which is given by the usual norm of bounded operator and the gap metric d are equivalent on the space of all bounded adjointable operators.*

A similar result has been proved in the case of Hilbert spaces in Addendum of [7]. Our argument seems to be shorter.

Proof. Let $T, S \in B(E, F)$, we use the expression (2.2) to show that there exist real numbers M_1, M_2 such that $M_2\|T - S\| \leq d(T, S) \leq M_1\|T - S\|$. Since $\|R_T\|, \|R_S\| \leq 1$, we have

$$\begin{aligned} \|R_T - R_S\| &\leq \|R_T\| \|S^*S - T^*T\| \|R_S\| \leq \|S^*(S - T) + (S^* - T^*)T\| \\ &\leq \|S^*\| \|S - T\| + \|S^* - T^*\| \|T\| = (\|T\| + \|S\|)\|T - S\|, \\ \|TR_T - SR_S\| &= \|(T - S)R_T + S(R_T - R_S)\| \leq \|T - S\| + \|S\|\|R_T - R_S\| \\ &\leq \|T - S\|(1 + \|S\|(\|S\| + \|T\|)). \end{aligned}$$

Similarly $\|R_{T^*} - R_{S^*}\| \leq (\|T\| + \|S\|)\|T - S\|$. Therefore the above inequalities imply that $d(T, S) \leq M_1\|T - S\|$, where $M_1 = \max\{\|T\| + \|S\|, 1 + \|S\|(\|S\| + \|T\|)\}$. We know that $T - S = (T - S)R_T R_T^{-1}$ and $(T - S)R_T = TR_T - SR_S + S(R_S - R_T)$, so we obtain

$$\begin{aligned} \|T - S\| &= \|(T - S)R_T\| \|R_T^{-1}\| \leq (1 + \|T\|^2)\|(T - S)R_T\| \\ &\leq (1 + \|T\|^2)(\|TR_T - SR_S\| + \|S\|\|R_S - R_T\|) \\ &\leq (1 + \|T\|^2)(d(T, S) + \|S\|d(T, S)) \leq (1 + \|T\|^2)(1 + \|S\|)d(T, S). \end{aligned}$$

Therefore $M_2\|T - S\| \leq d(T, S)$, for $M_2 = [(1 + \|T\|^2)(1 + \|S\|)]^{-1}$. ■

REMARK 3.3. Let $t \in R(E, F)$ and F_t be its bounded transform. Then $1 - F_t^*F_t$ has dense range if and only if $c1 - F_t^*F_t$ has dense range for each real number $c \geq 1$ (cf. Lemma 10.1 and Corollary 10.2 of [19]).

THEOREM 3.4. *Let E, F be Hilbert C^* -modules, then $B(E, F)$ is an open dense subset of the space $R(E, F)$ with respect to the gap topology.*

Proof. Let $t \in R(E, F)$ and F_t be its bounded transform. We set $P_n = \frac{n}{n+1}F_t$, for all $n \in \mathbb{N}$. Then for every $n \geq 1$ the operator P_n is bounded and satisfies

$\|P_n\| < 1$. The operator $1 - F_t^*F_t$ has dense range and so is $1 - P_n^*P_n$ (cf. Remark 3.3). By Theorem 10.4 of [19], for any natural number n there exists a regular operator T_n such that $F_{T_n} = P_n = \frac{n}{n+1}F_t$. Therefore $\|F_{T_n}\| = \|P_n\| < 1$, so T_n will be in $B(E, F)$. We also have $\sigma(T_n, t) = \|F_{T_n} - F_t\| = \|\frac{n}{n+1}F_t - F_t\| \rightarrow 0$. Recall that the Riesz topology was stronger than the gap topology, that is $d(T_n, t) \rightarrow 0$. $B(E, F)$ is therefore dense in $R(E, F)$ with respect to the gap topology. Openness of $B(E, F)$ was given in Lemma 3.1. ■

COROLLARY 3.5. *The uniform structures induced by the gap metric and by the operator norm on the space of bounded adjointable operators are different. This follows from the fact that the metric which is given by the usual norm of bounded operator is complete while the gap metric on the set of bounded adjointable operators is not complete.*

LEMMA 3.6. *Let E be a Hilbert \mathcal{A} -module and $t \in R(E)$. Then $\widehat{t} : \text{Dom}(\widehat{t}) = \text{Dom}(t) \times \text{Dom}(t^*) \subseteq E \oplus E \rightarrow E \oplus E$, $\widehat{t} = \begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix}$ is a selfadjoint regular operator on the Hilbert \mathcal{A} -module $E \oplus E$.*

Proof. t and t^* are densely defined closed operators and so is \widehat{t} . For each $(x, y), (u, v) \in \text{Dom}(\widehat{t})$ we have

$$\begin{aligned} \langle \widehat{t}(x, y), (u, v) \rangle &= \langle (t^*y, tx), (u, v) \rangle = \langle t^*y, u \rangle + \langle tx, v \rangle = \langle y, t^{**}u \rangle + \langle x, t^*v \rangle \\ &= \langle x, t^*v \rangle + \langle y, tu \rangle = \langle (x, y), (t^*v, tu) \rangle. \end{aligned}$$

Consequently, $\text{Dom}(\widehat{t}) = \text{Dom}(\widehat{t}^*)$ and $\widehat{t}^* = \widehat{t}$. The operator t is regular and so is t^* , therefore the range of the operator

$$1 + \widehat{t}^*\widehat{t} = 1 + \widehat{t}^2 = \begin{pmatrix} 1 + t^*t & 0 \\ 0 & 1 + tt^* \end{pmatrix}$$

is dense in $E \oplus E$. That is, \widehat{t} is a selfadjoint regular operator on $E \oplus E$. ■

Let E be a Hilbert \mathcal{A} -module, then the operators of the sets

$$SR(E \oplus E)_{\text{odd}} := \left\{ \begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix} : t \in R(E) \right\}, \quad SB(E \oplus E)_{\text{odd}} := \left\{ \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} : T \in B(E) \right\},$$

are called *odd unbounded regular* and *odd bounded adjointable* operators on the $\mathbb{Z}/2$ -graded Hilbert \mathcal{A} -module $E \oplus E$, respectively. Odd operators appear in Kasparov’s KK-theory.

PROPOSITION 3.7. *The map which associates to a regular operator $t \in R(E)$ the selfadjoint operator $\widehat{t} = \begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix}$ is an isometric map from $R(E)$ onto $SR(E \oplus E)_{\text{odd}}$ with respect to the gap metric, i.e. $d(t, s) = d(\widehat{t}, \widehat{s})$, for all $t, s \in R(E)$.*

Proof. Clearly the map $t \rightarrow \widehat{t}$ is a bijection from $R(E)$ onto $SR(E \oplus E)_{\text{odd}}$. So it is enough to check that the map preserves the gap distance. For this end we

have

$$\begin{aligned}
 1 + \widehat{t}^* \widehat{t} &= 1 + \widehat{t}^2 = \begin{pmatrix} 1 + t^*t & 0 \\ 0 & 1 + tt^* \end{pmatrix}, \\
 (1 + \widehat{t}^2)^{-1} &= \begin{pmatrix} (1 + t^*t)^{-1} & 0 \\ 0 & (1 + tt^*)^{-1} \end{pmatrix}, \\
 \widehat{t}(1 + \widehat{t}^2)^{-1} &= \begin{pmatrix} 0 & t^*(1 + tt^*)^{-1} \\ t(1 + t^*t)^{-1} & 0 \end{pmatrix}.
 \end{aligned}$$

Therefore we have $\|R_{\widehat{t}} - R_{\widehat{s}}\| = \sup\{\|R_t - R_s\|, \|R_{t^*} - R_{s^*}\|\}$ and $\|tR_t - sR_s\| = \|\widehat{t}R_{\widehat{t}} - \widehat{s}R_{\widehat{s}}\|$, for $t, s \in R(E)$. We get

$$\begin{aligned}
 d(t, s) &= \sup\{\|R_t - R_s\|, \|R_{t^*} - R_{s^*}\|, \|tR_t - sR_s\|\} \\
 &= \sup\{\|R_{\widehat{t}} - R_{\widehat{s}}\|, \|\widehat{t}R_{\widehat{t}} - \widehat{s}R_{\widehat{s}}\|\} = d(\widehat{t}, \widehat{s}). \quad \blacksquare
 \end{aligned}$$

COROLLARY 3.8. *Let E be a Hilbert C^* -module, then $SB(E \oplus E)_{\text{odd}}$ is dense in $SR(E \oplus E)_{\text{odd}}$ with respect to the gap topology.*

By an arbitrary C^* -algebra of compact operators \mathcal{A} we mean that $\mathcal{A} = c_0 - \bigoplus_{i \in I} \mathcal{K}(H_i)$, i.e. \mathcal{A} is a c_0 -direct sum of elementary C^* -algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i , $i \in I$ cf. Theorem 1.4.5 of [2]. Hilbert C^* -modules over C^* -algebras of compact operators are generally neither self-dual nor countably generated. However, they share many of their properties with Hilbert spaces. Generic properties of these Hilbert C^* -modules have been studied by several authors, e.g. [1], [5], [11], [12], [13], [22], [23], [24]. Let \mathcal{A} be a C^* -algebra, then \mathcal{A} is an arbitrary C^* -algebra of compact operators if and only if for every pair of Hilbert \mathcal{A} -modules E, F , every densely defined closed operator $t : \text{Dom}(t) \subseteq E \rightarrow F$ is regular (see [11]).

The following results are borrowed from [5], [13]. Let $\mathcal{A} = c_0 - \bigoplus_{i \in I} \mathcal{K}(H_i)$ be a C^* -algebra of compact operators and E be a Hilbert \mathcal{A} -module. For each $i \in I$ consider the associated submodule $E_i = \overline{\text{span}\{\mathcal{K}(H_i)E\}}$. Obviously, $\{E_i\}$ is a family of pairwise orthogonal closed submodule of E and it is well known (cf. [24]) that E admits a decomposition into the direct orthogonal sum $E = \bigoplus_{i \in I} E_i$ as well as $E \oplus E = \bigoplus_{i \in I} (E_i \oplus E_i)$. Suppose t is a densely defined closed operator on E and $t_i := t|_{\text{Dom}(t) \cap E_i}$, then $G(t) = \bigoplus_{i \in I} G(t_i)$. This enable us to reduce our attention to the case of a Hilbert C^* -module over an elementary C^* -algebra $\mathcal{K}(H)$. Let $e \in \mathcal{K}(H)$ be an arbitrary minimal projection and E be a $\mathcal{K}(H)$ -module. Suppose $E_e := eE = \{ex : x \in E\}$, then E_e is a Hilbert space with respect to the inner product $(\cdot, \cdot) = \text{trace}(\langle \cdot, \cdot \rangle)$, which is introduced in [5]. Let $B(E)$ and $B(E_e)$ be C^* -algebras of all bounded adjointable operators on Hilbert $\mathcal{K}(H)$ -module E and Hilbert space E_e , respectively. Bakić and Guljaš have shown that the map $\Psi : B(E) \rightarrow B(E_e)$, $\Psi(T) = T|_{E_e}$ is a $*$ -isomorphism of C^* -algebras ([5], Theorem 5).

Suppose $R(E)$ and $R(E_e)$ are the spaces of densely defined closed operators on Hilbert $\mathcal{K}(H)$ -module E and Hilbert space E_e , respectively. Then $\psi : R(E) \rightarrow R(E_e)$, $\psi(t) = t|_{E_e}$ is a bijection operation preserving map of $R(E)$ onto $R(E_e)$, cf. Theorem 2.4 of [13]. By the restriction $t|_{E_e}$ of an operator $t \in R(E)$ we mean the restriction of t onto the subspace $e\text{Dom}(t)$, where $e\text{Dom}(t) \subseteq E_e$ and $\overline{e\text{Dom}(t)} = E_e$.

THEOREM 3.9. *Let E be a Hilbert $\mathcal{K}(H)$ -module and $e \in \mathcal{K}(H)$ be any minimal projection. We equip $R(E)$ and $R(E_e)$ with the gap metric, then $\psi : R(E) \rightarrow R(E_e)$, $\psi(t) = t_e$ is an isometric operation preserving map of $R(E)$ onto $R(E_e)$*

Proof. ψ is a bijection operation preserving map of $R(E)$ onto $R(E_e)$ by Theorem 2.4 of [13]. Let t be a regular operator on E and $t_e := t|_{E_e}$, then $(R_t)|_{E_e}$ and $(tR_t)|_{E_e}$ are bounded operators on the Hilbert space E_e and $(R_t)|_{E_e} = R_{t_e}$ and $(tR_t)|_{E_e} = t_e R_{t_e}$. For $t, s \in R(E)$, the equalities $\|R_t - R_s\| = \|R_{t_e} - R_{s_e}\|$, $\|R_{t^*} - R_{s^*}\| = \|R_{t_e^*} - R_{s_e^*}\|$, $\|tR_t - sR_s\| = \|t_e R_{t_e} - s_e R_{s_e}\|$ hold by utilizing Theorem 5 of [5]. Therefore $d(t, s) = d(\psi(t), \psi(s))$ as we required. ■

The above theorem lifts the properties of the gap metric from the space of densely defined closed operators on Hilbert spaces to the space of densely defined closed operators on Hilbert C^* -modules over C^* -algebras of compact operators.

4. CONNECTIVITY

Unbounded Fredholm operators has been studied systematically in the papers [6], [7], [15] and the book [16]. In this section we use Theorem 3.9 to classify the path-components of the set of regular Fredholm operators in Hilbert $\mathcal{K}(H)$ -modules.

Suppose E is a Hilbert \mathcal{A} -module. Recall that a bounded operator $T \in B(E)$ is said to be *Fredholm* (or \mathcal{A} -Fredholm) if there exists $G \in B(E)$ such that $GT = TG = 1 \bmod K(E)$. Consider a regular operator t on E . An adjointable bounded operator $G \in B(E)$ is called a *pseudo left inverse* of t if Gt is closable and its closure \overline{Gt} satisfies $\overline{Gt} \in B(E)$ and $\overline{Gt} = 1 \bmod K(E)$. Analogously G is called a *pseudo right inverse* if tG is closable and its closure \overline{tG} satisfies $\overline{tG} \in B(E)$ and $\overline{tG} = 1 \bmod K(E)$. The regular operator t is called *Fredholm* (or \mathcal{A} -Fredholm), if it has a pseudo left as well as a pseudo right inverse. The regular operator t is Fredholm if and only if F_t is, cf. [15], [22]. For the general theory of bounded and unbounded Fredholm operators on Hilbert C^* -modules we refer to [14], [15], [25]. Let $\text{Fred}R(E)$ denote the space of regular Fredholm operators on E , equipped with the gap topology, and let $\text{Fred}SR(E)$ denote the subspace consisting of the set selfadjoint regular Fredholm operators.

Every two operators in $\text{Fred}R(E)$ are called *homotopic* if they are in the same path-connected component of $\text{Fred}R(E)$. It is natural to ask for a characterization

of the path-components of $\text{Fred}R(E)$. This question was completely answered by Cordes and Labrousse in [7] in the case of Hilbert spaces.

THEOREM 4.1. *Let H be a Hilbert space of infinite dimension. Every two operators in $\text{Fred}R(H)$ have the same index if and only if they are homotopic, i.e. they can be connected by a continuous path in $\text{Fred}R(H)$.*

Suppose E is a Hilbert $\mathcal{K}(H)$ -module and t is a regular operator on E . Then t is Fredholm if and only if the range of t is a closed submodule of E and both $\dim_{\mathcal{K}}\text{Ker}(t)$, $\dim_{\mathcal{K}}\text{Ker}(t^*)$ are finite. In this case we can define an index of t formally, i.e. we can define:

$$\text{ind}t = \dim_{\mathcal{K}}\text{Ker}(t) - \dim_{\mathcal{K}}\text{Ker}(t^*).$$

We refer to the publications [5], [20], [22] for the proof of the preceding results. More information about orthonormal Hilbert bases can be found in [1], [10]. Apply Theorems 3.9, 4.1 to get the following fact.

COROLLARY 4.2. *Let E is a Hilbert $\mathcal{K}(H)$ -module. Every two operators in $\text{Fred}R(E)$ have the same index if and only if they are homotopic, i.e. they can be connected by a continuous path in $\text{Fred}R(E)$.*

COROLLARY 4.3. *Suppose E is a Hilbert $\mathcal{K}(H)$ -module. The space $\text{Fred}SR(E)$ is path-connected and the space $\text{Fred}R(E)$ is not path-connected.*

For the proof, just recall that any element of $\text{Fred}SR(E)$ has zero index and then apply Corollary 4.2.

We close the paper with the notification that the previous corollary may fail for some other C^* -algebra of coefficients. To find an example one can use a result due to Joachim ([15], Theorem 3.5). Indeed, Joachim's theorem is a remarkable generalization of a result due to Atiyah and Singer [3] which describes the space of regular (respectively, selfadjoint regular) Fredholm operators on standard Hilbert C^* -modules over unital C^* -algebra of coefficients. Mingo also gave a description of path-components in the set of bounded Fredholm operators on standard Hilbert modules [20].

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