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s-NUMBERS OF ELEMENTARY OPERATORS ON C^*-ALGEBRAS

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ABSTRACT. We study s-functions of elementary operators acting on C^*-algebras. The main results are the following: If τ is any tensor norm and $A, B \in \mathbf{B}(\mathcal{H})$ are such that the sequences $s(A), s(B)$ of their singular numbers belong to a tensor stable Calkin space $\mathfrak{i}$ then the sequence of approximation numbers of $A \otimes_\tau B$ belongs to $\mathfrak{i}$. If $\mathcal{A}$ is a C^*-algebra, $\mathfrak{i}$ is a tensor stable Calkin space, s is an s-number function, and $a_i, b_i \in \mathcal{A}$, $i = 1, 2, \dots, m$ are such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}$, $i = 1, 2, \dots, m$ for some faithful representation π of $\mathcal{A}$ then $\mathrm{s}\Big(\sum\limits_{i=1}^{m} M_{a_i,b_i}\Big) \in \mathfrak{i}$. The converse implication holds if and only if the ideal of compact elements of $\mathcal{A}$ has finite spectrum. We also prove a quantitative version of a result of Ylinen.

KEYWORDS: *Elementary operator, s-numbers, C^*-algebra.*

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INTRODUCTION

Let $\mathcal{A}$ be a C^*-algebra. If $a, b \in \mathcal{A}$ we denote by $M_{a,b}$ the operator on $\mathcal{A}$ given by $M_{a,b}(x) = axb$. An operator $\Phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if $\Phi = \sum\limits_{i=1}^{m} M_{a_i,b_i}$ for some $a_i, b_i \in \mathcal{A}$, $i = 1, \dots, m$.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathbf{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. A theorem of Fong and Sourour [11] asserts that an elementary operator Φ on $\mathbf{B}(\mathcal{H})$ is compact if and only if there exists a representation $\sum\limits_{i=1}^{m} M_{A_i,B_i}$ of Φ such that the *symbols* A_i, B_i, $i = 1, \dots, m$, of Φ are compact operators. If, instead of $\mathbf{B}(\mathcal{H})$, one has a C^*-algebra $\mathcal{A}$, the role of the compact operators is played by the compact elements: recall that an element a of $\mathcal{A}$ is called *compact* if the operator $M_{a,a} : \mathcal{A} \to \mathcal{A}$ is compact. Ylinen [27] showed that an element $a \in \mathcal{A}$ is compact if and only if there exists a faithful $*$-representation π of $\mathcal{A}$ such that the operator $\pi(a)$ is compact.

The result of Fong and Sourour was extended by Mathieu [19] who showed that if $\mathcal{A}$ is a prime C^*-algebra, then an elementary operator Φ on $\mathcal{A}$ is compact if and only if there exist compact elements $a_i, b_i \in \mathcal{A}$, $i = 1, \ldots, m$, such that $\Phi = \sum_{i=1}^{m} M_{a_i,b_i}$. Recently Timoney [25] extended this result to general C^*-algebras.

In this paper we investigate quantitative aspects of the above results. It is well-known that a bounded operator on a Banach space is compact if and only if its Kolmogorov numbers form a null sequence. In our approach we use the more general notion of an s-function introduced by Pietsch and the theory of ideals of $\mathbf{B}(\mathcal{H})$ developed by von Neumann, Schatten, Calkin and others. A detailed study of these notions is presented in the monographs [21], [5], [12] and [23]. Our analysis rests on the classical description of the ideals of $\mathbf{B}(\mathcal{H})$ in terms of subspaces of c_0 satisfying a certain closure property [4], a result that has inspired many of the developments in the area thereafter. Recently, advances in the study of ideals of $\mathbf{B}(\mathcal{H})$ were made in [9], [13], [14], [15], [16].

In Section 1 of the paper we recall the definitions of Calkin spaces and the basic properties of s-functions.

Weiss considered in [26] a property for ideals of $\mathbf{B}(\mathcal{H})$, called "tensor product closure property", or "tensor stability". In Section 2 we study the analogous property for Calkin spaces. We give a necessary and sufficient condition for the tensor stability of a singly generated Calkin space. We also provide a sufficient condition for the tensor stability of a Lorentz sequence space.

If $a, b \in \mathcal{A}$ and $\mathcal{C}$ is a C^*-subalgebra of $\mathcal{A}$ such that $M_{a,b}(\mathcal{C}) \subseteq \mathcal{C}$, we let $M_{a,b}^{\mathcal{C}}$ be the operator on $\mathcal{C}$ given by $M_{a,b}^{\mathcal{C}}(x) = axb$. In Section 3 we prove inequalities relating the s-number functions of the operators $M_{a,b}$ and $M_{a,b}^{\mathcal{C}}$. These results are used subsequently in Section 5.

In Section 4 we study elementary operators acting on $\mathbf{B}(\mathcal{H})$. Some of our results can be presented in a more general setting. Namely, we show that if τ is any tensor norm and $A, B \in \mathbf{B}(\mathcal{H})$ are such that $s(A), s(B)$ belong to a tensor stable Calkin space $\mathfrak{i}$ then the sequence of approximation numbers of the operator $A \otimes_\tau B$ acting on the Banach space tensor product $\mathcal{H} \otimes_\tau \mathcal{H}$ belongs to $\mathfrak{i}$. A result of this type for $\mathfrak{i} = \ell_{p,q}$ was proved by König in [17] who used it to establish results concerning tensor stability of s-number ideals in Banach spaces. We also show that if Φ is an elementary operator on $\mathbf{B}(\mathcal{H})$, $\mathfrak{i}$ is a tensor stable Calkin space and s is an s-function then $\mathrm{s}(\Phi) \in \mathfrak{i}$ if and only if there exist $A_i, B_i \in \mathbf{B}(\mathcal{H})$, $i = 1, \ldots, m$, such that $\Phi = \sum_{i=1}^{m} M_{A_i,B_i}$ and $s(A_i), s(B_i) \in \mathfrak{i}$. It is well known that all s-functions coincide for operators acting on Hilbert spaces. It follows from our result that if Φ is an elementary operator on $\mathbf{B}(\mathcal{H})$, $\mathfrak{i}$ is a tensor stable Calkin space and s, s' are s-number functions, then the sequence $\mathrm{s}(\Phi)$ belongs to $\mathfrak{i}$ if and only if the sequence $\mathrm{s}'(\Phi)$ belongs to $\mathfrak{i}$.

In Section 5 we study elementary operators acting on C^*-algebras. We show that if $\mathcal{A}$ is a C^*-algebra, $\mathfrak{i}$ is a tensor stable Calkin space, s is an s-number function, and $a_i, b_i \in \mathcal{A}$, $i = 1, \ldots, m$, are such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}$, $i = 1, \ldots, m$ for some faithful representation π of $\mathcal{A}$, then $\mathrm{s}\Big(\sum\limits_{i=1}^{m} M_{a_i,b_i}\Big) \in \mathfrak{i}$. The converse implication holds if and only if the ideal of compact elements of $\mathcal{A}$ has finite spectrum. Finally, we prove that if $a \in \mathcal{A}$ and the sequence $\mathrm{d}(M_{a,a})$ of Kolmogorov numbers of $M_{a,a}$, belongs to $\mathfrak{i}$ for some Calkin space $\mathfrak{i}$ then $s(\rho(a))^2 \in \mathfrak{i}$, where ρ is the reduced atomic representation of $\mathcal{A}$. This result may be viewed as a quantitative version of the aforementioned result of Ylinen.

1. CALKIN SPACES AND s-FUNCTIONS

In this section we recall some notions and results concerning the ideal structure of the algebra of all bounded linear operators acting on a separable Hilbert space. We also recall the definition of an s-function.

We will denote by $\mathbf{B}$ the class of all bounded linear operators between Banach spaces. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, we will denote by $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$ we set $\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X}, \mathcal{X})$. Ideals of $\mathbf{B}(\mathcal{X})$ or, more generally, of a normed algebra $\mathcal{A}$, will be proper, two-sided and not necessarily norm closed. By $\mathbf{K}(\mathcal{X})$ (respectively $\mathbf{F}(\mathcal{X})$) we denote the ideal of all compact (respectively finite rank) operators on $\mathcal{X}$. By $\|T\|$ we denote the operator norm of a bounded linear operator T. We denote by ℓ_∞ the space of all bounded complex sequences, by c_0 the space of all sequences in ℓ_∞ converging to 0 and by c_{00} the space of all sequences in c_0 that are eventually zero. The space of all p-summable complex sequences is denoted by ℓ_p. For a subspace $\jmath$ of ℓ_∞, we let $\jmath^+$ be the subset of $\jmath$ consisting of all sequences with non-negative terms. We denote by $c_0^\star$ the subset of c_0^+ consisting of all decreasing sequences.

If $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty$ and $\boldsymbol{\beta} = (\beta_n)_{n=1}^\infty$ are sequences of real numbers, we write $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ if $\alpha_n \leqslant \beta_n$ for each $n \in \mathbb{N}$. For every $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty \in c_0$, we let $\boldsymbol{\alpha}^\star = (\alpha_n^\star)_{n=1}^\infty \in c_0^\star$ be the decreasing rearrangement of the sequence $(|\alpha_n|)_{n=1}^\infty$ including multiplicities, that is, the sequence given by

$$\alpha_1^\star = \max\{|\alpha_n| : n \in \mathbb{N}\},$$
$$\alpha_1^\star + \cdots + \alpha_n^\star = \max\Big\{\sum_{i\in I} |\alpha_i| : I \subseteq \mathbb{N}, |I| = n\Big\}.$$

A *Calkin space* [23] is a subspace $\mathfrak{i}$ of c_0 possessing the following property:

$$\text{If } \boldsymbol{\alpha} \in \mathfrak{i},\ \boldsymbol{\beta} \in c_0 \text{ and } \boldsymbol{\beta}^\star \leqslant \boldsymbol{\alpha}^* \text{ then } \boldsymbol{\beta} \in \mathfrak{i}.$$

Let $\mathcal{H}$ be a separable Hilbert space. If $e, f \in \mathcal{H}$ we denote by $f^* \otimes e$ the rank one operator on $\mathcal{H}$ given by $f^* \otimes e(x) = (x, f)e$, $x \in \mathcal{H}$. If $T \in \mathbf{K}(\mathcal{H})$, there exist orthonormal sequences $(f_n)_{n=1}^\infty \subseteq \mathcal{H}$ and $(e_n)_{n=1}^\infty \subseteq \mathcal{H}$ and a unique sequence

$(s_n(T))_{n=1}^{\infty} \in c_0^{\star}$ such that

$$T = \sum_{n=1}^{\infty} s_n(T) f_n^* \otimes e_n,$$

where the series converges in norm. Such a decomposition of T is called a *Schmidt expansion*. The elements of the sequence $s(T) = (s_n(T))_{n=1}^{\infty}$ are called the *singular numbers* of T.

For every ideal $\mathcal{I} \subseteq \mathbf{B}(\mathcal{H})$, we set

$$\mathfrak{i}(\mathcal{I}) = \{\boldsymbol{\alpha} \in c_0 : \text{there exists } T \in \mathcal{I} \text{ such that } \boldsymbol{\alpha}^{\star} = s(T)\};$$

conversely, for every Calkin space $\mathfrak{i}$ we set

$$\mathcal{I}(\mathfrak{i}) = \{T \in \mathbf{B}(\mathcal{H}) : s(T) \in \mathfrak{i}\}.$$

The following classical result of Calkin [4] describes the ideal structure of $\mathbf{B}(\mathcal{H})$ in terms of Calkin spaces (for a proof of the formulation given here see Theorem 2.5 of [23]).

THEOREM 1.1 ([4]). *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. The mapping $\mathcal{I} \mapsto \mathfrak{i}(\mathcal{I})$ is an isomorphism from the lattice of all ideals of $\mathbf{B}(\mathcal{H})$ onto the lattice of all Calkin spaces with inverse $\mathfrak{i} \mapsto \mathcal{I}(\mathfrak{i})$.*

There are several equivalent ways of working with ideals of $\mathbf{B}(\mathcal{H})$ (ideals, characteristic sets and Calkin spaces are some of them). We have chosen to work with Calkin spaces since in Section 5 we consider operators acting on a general C^*-algebra and Calkin's classification of ideals is not valid in this context.

We now recall Pietsch's definition of s-functions. A map s which assigns to every operator $T \in \mathbf{B}$ a sequence of non-negative real numbers $\mathrm{s}(T) = (\mathrm{s}_1(T), \mathrm{s}_2(T), \dots)$ is called an *s-function* if the following are satisfied:

(i) $\|T\| = \mathrm{s}_1(T) \geqslant \mathrm{s}_2(T) \geqslant \cdots$, for $T \in \mathbf{B}$.
(ii) $\mathrm{s}_n(S+T) \leqslant \mathrm{s}_n(S) + \|T\|$, for $S, T \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$.
(iii) $\mathrm{s}_n(RST) \leqslant \|R\| \, \|T\| \mathrm{s}_n(S)$, for $T \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$, $S \in \mathbf{B}(\mathcal{Y}, \mathcal{Z})$, $R \in \mathbf{B}(\mathcal{Z}, \mathcal{W})$.
(iv) If $\operatorname{rank}(T) < n$ then $\mathrm{s}_n(T) = 0$.
(v) $\mathrm{s}_n(I_n)$=1, where I_n is the identity operator on ℓ_2^n. (Here ℓ_2^n is the n-dimensional complex Hilbert space.)

An s-function s is said to be *additive* if $\mathrm{s}_{m+n-1}(S+T) \leqslant \mathrm{s}_m(S) + \mathrm{s}_n(T)$ for all $m, n \in \mathbb{N}$ and all $S, T \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$.

We give below the definition of some s-functions which will be used in the sequel. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$.

(a) The sequence $\mathrm{a}(T) = (\mathrm{a}_n(T))_{n=1}^{\infty}$ of *approximation numbers* of T is given by

$$\mathrm{a}_n(T) = \inf\{\|T - A\| : A \in \mathbf{B}(\mathcal{X}, \mathcal{Y}),\ \operatorname{rank}(A) < n\}.$$

(b) The sequence $\mathrm{d}(T) = (\mathrm{d}_n(T))_{n=1}^{\infty}$ of *Kolmogorov numbers* of T is given by

$$\mathrm{d}_n(T) = \inf_{V} \sup_{\|x\| \leqslant 1} \inf_{y \in V} \|Tx - y\|,$$

where the infimum is taken over all subspaces V of $\mathcal{Y}$ with $\dim V < n$.

(c) The sequence $\mathrm{h}(T)$ of *Hilbert numbers* of T is given by

$$\mathrm{h}_n(T) = \sup s_n(ATB)$$

where the supremum is taken over all contractions $A \in \mathbf{B}(\mathcal{Y}, \mathcal{H})$, $B \in \mathbf{B}(\mathcal{K}, \mathcal{X})$ and Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

The approximation, the Kolmogorov and the Hilbert s-functions are additive [21]. Moreover, for every s-function s, every operator $T \in \mathbf{B}$ and every n we have $\mathrm{h}_n(T) \leqslant \mathrm{s}_n(T) \leqslant \mathrm{a}_n(T)$ [21].

A well-known result of Pietsch ([21], Theorem 11.3.4) implies that if s is an s-function, $\mathcal{H}$ is a separable Hilbert space and $T \in \mathbf{K}(\mathcal{H})$ then $\mathrm{s}_n(T)$ is equal to the n^{th}-singular number $s_n(T)$ of T.

2. TENSOR STABLE CALKIN SPACES

In this section we present some results concerning tensor stable Calkin spaces. We characterize the tensor stable principal Calkin spaces and show that certain Lorentz sequence spaces are tensor stable.

If $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^{\infty}, \boldsymbol{\beta} = (\beta_n)_{n=1}^{\infty} \in c_0$, we define the sequence $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = (\gamma_n)_{n=1}^{\infty} \in c_0^{\star}$ by

$$\begin{aligned} &\gamma_1 = \max\{|\alpha_i \beta_j| : (i,j) \in \mathbb{N} \times \mathbb{N}\}, \\ &\gamma_1 + \cdots + \gamma_n = \max\Big\{ \sum_{(i,j) \in I} |\alpha_i \beta_j| : I \subseteq \mathbb{N} \times \mathbb{N},\ |I| = n \Big\}. \end{aligned}$$

The sequence $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is the rearrangement of the double sequence $(|\alpha_n \beta_m|)_{n,m=1}^{\infty}$ in decreasing order including multiplicities.

DEFINITION 2.1. Let $\mathfrak{i}$ and $\mathfrak{j}$ be Calkin spaces. We let $\mathfrak{i} \otimes \mathfrak{j}$ be the smallest Calkin space containing the sequences $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$, where $\boldsymbol{\alpha} \in \mathfrak{i}$ and $\boldsymbol{\beta} \in \mathfrak{j}$. A Calkin space $\mathfrak{i}$ is said to be *tensor stable* if $\mathfrak{i} \otimes \mathfrak{i} = \mathfrak{i}$.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Weiss [26] defined the *tensor product closure property* for ideals of $\mathbf{B}(\mathcal{H})$. An ideal $\mathcal{I}$ of $\mathbf{B}(\mathcal{H})$ has this property if $S \otimes T \in \mathcal{I}$ whenever $S, T \in \mathcal{I}$. Here, the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ is identified with $\mathcal{H}$ in a natural way. It is easy to see that an ideal $\mathcal{I} \subseteq \mathbf{B}(\mathcal{H})$ has the tensor product closure property if and only if $\mathfrak{i}(\mathcal{I})$ is a tensor stable Calkin space.

We would like to note that tensor stability may be considered in the more general context of the study of ideals of $\mathbf{B}(\mathcal{H})$ as it appears in [9]. More specifically, a related property called arithmetic mean stability is studied there and several applications are obtained in the papers of Kaftal and Weiss [13], [14], [15], [16].

We will need the following lemma.

LEMMA 2.2. *If $\boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}' \in c_0^+$ are such that $\boldsymbol{\alpha} \leqslant \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \leqslant \boldsymbol{\beta}'$ then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}' \otimes \boldsymbol{\beta}'$.*

Proof. Clearly $\alpha_m\beta_n \leqslant \alpha'_m\beta'_n$ for every m, n. So to prove the lemma it suffices to prove that if $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ then $\boldsymbol{\alpha}^\star \leqslant \boldsymbol{\beta}^\star$. Consider an injection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\alpha_n^\star = \alpha_{\pi(n)}$. Clearly $\alpha_1^\star \leqslant \beta_1^\star$. Suppose that $\alpha_i^\star \leqslant \beta_i^\star$ for $i = 1, \dots, n-1$. If $\alpha_n^\star > \beta_n^\star$ then $\alpha_i^\star > \beta_n^\star$ for every $i = 1, \dots, n$ and hence $\beta_{\pi(1)}, \dots, \beta_{\pi(n)}$ are strictly greater than $\beta_n^\star$ so the number of all $i \in \mathbb{N}$ such that $\beta_i > \beta_n^\star$ is greater than $n-1$, a contradiction. ∎

NOTATION 2.3. (i) If $\boldsymbol{\alpha}_n = (\alpha_i^n)_{i=1}^{k_i}$, $n \in \mathbb{N}$, are finite sequences, we set

$$(\boldsymbol{\alpha}_n)_{n=1}^\infty = (\alpha_1^1, \dots, \alpha_{k_1}^1, \alpha_1^2, \dots, \alpha_{k_2}^2, \alpha_1^3, \dots, \alpha_{k_3}^3, \dots).$$

(ii) If $\vartheta \in \mathbb{C}$ and $r \in \mathbb{N}$ we set $(\vartheta)_r = (\underbrace{\vartheta, \dots, \vartheta}_{r \text{ times}})$.

(iii) If $r \in \mathbb{N}$ we let $\mathbf{r} = (\underbrace{1, 1, \dots, 1}_{r \text{ times}}, 0, 0, \dots)$.

(iv) If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are sequences of real numbers we write $\boldsymbol{\alpha} \lesssim \boldsymbol{\beta}$ if there exists a constant $C > 0$ such that $\boldsymbol{\alpha} \leqslant C\boldsymbol{\beta}$.

Observe that if $(M_n)_{n=0}^\infty$, $(N_n)_{n=0}^\infty$ are sequences of non negative integers then

$$((\vartheta^n)_{M_n})_{n=0}^\infty \otimes ((\vartheta^n)_{N_n})_{n=0}^\infty = ((\vartheta^n)_{\sum_{k+l=n} M_k N_l})_{n=0}^\infty. \tag{2.1}$$

LEMMA 2.4. *Let $\mathfrak{i}$ be a Calkin space. If $\boldsymbol{\alpha} \in \mathfrak{i}$ and $\boldsymbol{\beta} \in c_{00}$ then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \in \mathfrak{i}$.*

Proof. Let $\boldsymbol{\alpha}^\star = (\alpha_n^\star)_{n=1}^\infty$. Clearly, if $r \in \mathbb{N}$ then

$$\mathbf{r} \otimes \boldsymbol{\alpha} = ((\alpha_1^\star)_r, (\alpha_2^\star)_r, \dots, (\alpha_n^\star)_r, \dots).$$

It follows easily from the definition of a Calkin space that $\mathbf{r} \otimes \boldsymbol{\alpha} \in \mathfrak{i}$. Since $\boldsymbol{\beta}^\star \in c_{00}^+$, there exists $r \in \mathbb{N}$ such that $\boldsymbol{\beta}^\star \lesssim \mathbf{r}$. By Lemma 2.2, $\boldsymbol{\beta} \otimes \boldsymbol{\alpha} = \boldsymbol{\beta}^\star \otimes \boldsymbol{\alpha} \lesssim \mathbf{r} \otimes \boldsymbol{\alpha}$. Hence, $\boldsymbol{\beta} \otimes \boldsymbol{\alpha} \in \mathfrak{i}$. ∎

The following notation will be used in the sequel.

NOTATION 2.5. Let $\boldsymbol{\alpha}=(\alpha_m)_{m=1}^\infty \in c_0^\star$ and $\vartheta\in(0,1)$. For every $n=0,1,\dots$, set

$$\mathcal{K}_n^{(\vartheta)}(\boldsymbol{\alpha}) = \{m : \vartheta^{n+1} < \alpha_m \leqslant \vartheta^n\}, \quad K_n^{(\vartheta)}(\boldsymbol{\alpha}) = |\mathcal{K}_n^{(\vartheta)}(\boldsymbol{\alpha})|,$$

$$\widetilde{K}_n^{(\vartheta)}(\boldsymbol{\alpha}) = \sum_{i=0}^n K_i^{(\vartheta)}(\boldsymbol{\alpha}), \quad M_n^{(\vartheta)}(\boldsymbol{\alpha}) = \sum_{i+j=n} K_i^{(\vartheta)}(\boldsymbol{\alpha}) K_j^{(\vartheta)}(\boldsymbol{\alpha}),$$

$$\widetilde{M_n}^{(\vartheta)}(\boldsymbol{\alpha}) = \sum_{i=0}^n M_i^{(\vartheta)}(\boldsymbol{\alpha}), \quad K_{-1}^{(\vartheta)}(\boldsymbol{\alpha}) = \widetilde{K}_{-1}^{(\vartheta)}(\boldsymbol{\alpha}) = M_{-1}^{(\vartheta)}(\boldsymbol{\alpha}) = \widetilde{M}_{-1}^{(\vartheta)}(\boldsymbol{\alpha}) = 0.$$

LEMMA 2.6. *Let $\vartheta \in (0,1)$, $\boldsymbol{\alpha} = (\alpha_m)_{m=1}^\infty \in c_0^\star$, and $\boldsymbol{\beta} = (\beta_m)_{m=1}^\infty \in c_0^\star$. Assume that $\alpha_1, \beta_1 \leqslant 1$. Then $\boldsymbol{\alpha} \lesssim \boldsymbol{\beta}$ if and only if there exists a positive integer r such* *that for every* $n \in \mathbb{N} \cup \{0\}$,

$$\widetilde{K}_n^{(\vartheta)}(\boldsymbol{\alpha}) \leqslant \widetilde{K}_{n+r}^{(\vartheta)}(\boldsymbol{\beta}).$$

Proof. Set $\widetilde{K}_n = \widetilde{K}_n^{(\vartheta)}(\boldsymbol{\alpha})$ and $\widetilde{L}_n = \widetilde{K}_n^{(\vartheta)}(\boldsymbol{\beta})$. Suppose that $\boldsymbol{\alpha} \lesssim \boldsymbol{\beta}$ and let $C > 0$ be such that $\alpha_m \leqslant C\beta_m$, for every $m \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $\vartheta^r C \leqslant 1$. Then $\beta_{\widetilde{K}_n} \geqslant C^{-1}\alpha_{\widetilde{K}_n} > C^{-1}\vartheta^{n+1} \geqslant \vartheta^{n+1+r}$. Thus, $\widetilde{K}_n \leqslant \widetilde{L}_{n+r}$.

Conversely, suppose that there exists $r \in \mathbb{N}$ such that $\widetilde{K}_n \leqslant \widetilde{L}_{n+r}$, for every $n \in \mathbb{N} \cup \{0\}$. Fix $m \in \mathbb{N}$ and let n and k be such that $m = \widetilde{K}_{n-1} + k$ and $1 \leqslant k \leqslant K_n$. Since $\widetilde{K}_{n-1} < m \leqslant \widetilde{K}_n \leqslant \widetilde{L}_{n+r}$ we have

$$\alpha_m \leqslant \vartheta^n = \vartheta^{-r-1}\vartheta^{n+r+1} \leqslant \vartheta^{-r-1}\beta_{\widetilde{L}_{n+r}} \leqslant \vartheta^{-r-1}\beta_m.$$

Thus, $\boldsymbol{\alpha} \lesssim \boldsymbol{\beta}$. ∎

If $\boldsymbol{\alpha} \in c_0$ we let $\langle \boldsymbol{\alpha} \rangle$ denote the smallest Calkin space containing $\boldsymbol{\alpha}$. A Calkin space of the form $\langle \boldsymbol{\alpha} \rangle$ will be called *principal*. Note that a Calkin space is principal precisely when it is the Calkin space of a principal ideal of $\mathbf{B}(\mathcal{H})$.

The proof of the following lemma is straightforward.

LEMMA 2.7. *If* $\boldsymbol{\alpha} \in c_0$ *then*

$$\langle \boldsymbol{\alpha} \rangle = \{\boldsymbol{\beta} \in c_0 : \text{there exists } r \in \mathbb{N} \text{ such that } \boldsymbol{\beta}^\star \lesssim \mathbf{r} \otimes \boldsymbol{\alpha}\}.$$

THEOREM 2.8. *Let* $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty \in c_0^\star$ *with* $\alpha_1 \leqslant 1$ *and* $\vartheta \in (0,1)$. *The following are equivalent:*

(i) *The principal Calkin space* $\langle \boldsymbol{\alpha} \rangle$ *is tensor stable.*

(ii) $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \in \langle \boldsymbol{\alpha} \rangle$.

(iii) *There exists* $r \in \mathbb{N}$ *and* $C > 0$ *such that* $\widetilde{M}_n^{(\vartheta)}(\boldsymbol{\alpha}) \leqslant C\widetilde{K}_{n+r}^{(\vartheta)}(\boldsymbol{\alpha})$, *for every* $n \in \mathbb{N} \cup \{0\}$.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Let $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \langle \boldsymbol{\alpha} \rangle$. By Lemma 2.7, there exist positive integers m, n such that $\boldsymbol{\beta} \lesssim \mathbf{m} \otimes \boldsymbol{\alpha}$ and $\boldsymbol{\gamma} \lesssim \mathbf{n} \otimes \boldsymbol{\alpha}$. By Lemma 2.2,

$$\boldsymbol{\beta} \otimes \boldsymbol{\gamma} \lesssim (\mathbf{m} \otimes \boldsymbol{\alpha}) \otimes (\mathbf{n} \otimes \boldsymbol{\alpha}) = (\mathbf{mn}) \otimes (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}).$$

Since $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \in \langle \boldsymbol{\alpha} \rangle$, using Lemma 2.7 again we conclude that $\boldsymbol{\beta} \otimes \boldsymbol{\gamma} \in \langle \boldsymbol{\alpha} \rangle$ and so $\langle \boldsymbol{\alpha} \rangle$ is tensor stable.

(i) $\Leftrightarrow$ (iii) Set $K_n = K_n^{(\vartheta)}(\boldsymbol{\alpha})$ and $\widetilde{\boldsymbol{\alpha}} = ((\vartheta^n)_{K_n})_{n=0}^\infty$; clearly, $\langle \boldsymbol{\alpha} \rangle = \langle \widetilde{\boldsymbol{\alpha}} \rangle$. By the previous paragraph, $\langle \boldsymbol{\alpha} \rangle$ is tensor stable if and only if $\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\alpha}} \in \langle \widetilde{\boldsymbol{\alpha}} \rangle$. By Lemma 2.7, $\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\alpha}} \in \langle \widetilde{\boldsymbol{\alpha}} \rangle$ if and only if there exists a positive integer m such that $\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\alpha}} \lesssim \mathbf{m} \otimes \widetilde{\boldsymbol{\alpha}}$. Since $\widetilde{K}_n^{(\vartheta)}(\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\alpha}}) = \widetilde{M}_n^{(\vartheta)}(\widetilde{\boldsymbol{\alpha}})$ (see equation (2.1)) and $\widetilde{K}_n^{(\vartheta)}(\mathbf{m} \otimes \widetilde{\boldsymbol{\alpha}}) = m\widetilde{K}_n^{(\vartheta)}(\widetilde{\boldsymbol{\alpha}})$, the conclusion follows from Lemma 2.6. ∎

COROLLARY 2.9. *Let* $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty \in c_0^\star$ *with* $\alpha_1 \leqslant 1$ *and* $\vartheta \in (0,1)$. *Suppose that* $C > 0$ *is a constant such that*

$$K_{n+j}^{(\vartheta)}(\boldsymbol{\alpha}) \geqslant C(\widetilde{K}_n^{(\vartheta)}(\boldsymbol{\alpha}))^2 \tag{2.2}$$

for all $j \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then $\langle \boldsymbol{\alpha} \rangle$ is a tensor stable Calkin space.

Proof. Set $K_n = K_n^{(\vartheta)}(\boldsymbol{\alpha})$, $\widetilde{K}_n = \widetilde{K}_n^{(\vartheta)}(\boldsymbol{\alpha})$ and $\widetilde{M}_n = \widetilde{M}_n^{(\vartheta)}(\boldsymbol{\alpha})$. Let r be a positive integer such that $rC \geqslant 1$. Since $(\widetilde{K}_n)^2 \geqslant \widetilde{M}_n$, it follows that

$$\widetilde{K}_{n+r} \geqslant K_{n+1} + \cdots + K_{n+r} \geqslant rC\widetilde{M_n} \geqslant \widetilde{M_n}, \quad n \in \mathbb{N},$$

and hence condition (iii) of Theorem 2.8 holds. ∎

We next give some examples.

EXAMPLE 2.10. (i) It follows from Theorem 2.8(iii) that for every $\vartheta \in (0,1)$, the principal Calkin space $\langle (\vartheta^n)_{n=0}^{\infty} \rangle$ is not tensor stable. This example was first given in [26].

(ii) Let $\lambda > 0$ and $\boldsymbol{\alpha} = (n^{-\lambda})_{n=1}^{\infty}$. Then the principal Calkin space $\langle \boldsymbol{\alpha} \rangle$ is not tensor stable. To show this, let $\mu = \lambda^{-1}$ and $\vartheta = e^{-1}$. Let $K_n, M_n, \widetilde{K}_n, \widetilde{M}_n$ be the positive integers associated with the sequence $(n^{-\lambda})_{n=1}^{\infty}$ and ϑ (Notation 2.5). Since

$$\frac{1}{2}[e^{(j+1)\mu} - e^{j\mu}] \leqslant K_j \leqslant [e^{(j+1)\mu} - e^{j\mu}],$$

there exist constants $C_1, C_2 > 0$ such that, for every j, we have

$$C_2 e^{j\mu} \leqslant K_j \leqslant C_1 e^{j\mu}.$$

It follows that $M_n = \sum\limits_{i+j=n} K_i K_j \geqslant (n+1)C_2^2 e^{n\mu}$. Let r be a positive integer. Then

$$\widetilde{K}_{n+r} \leqslant C_1 \sum_{i=0}^{n+r} (e^{\mu})^i = \frac{C_1(e^{(n+r+1)\mu} - 1)}{e^{\mu} - 1} \quad \text{and} \quad \widetilde{M}_n \geqslant C_2^2 \int_0^n (t+1)e^{\mu t}\mathrm{d}t \geqslant C_3 n e^{n\mu}$$

for some $C_3 > 0$. Thus,

$$\lim_{n \to +\infty} \frac{\widetilde{M}_n}{\widetilde{K}_{n+r}} = +\infty$$

for each $r \in \mathbb{N}$. By Theorem 2.8, $\langle \boldsymbol{\alpha} \rangle$ is not tensor stable.

It follows from the characterization of the symmetrically normable principal ideals due to Allen and Shen [1] that the principal ideal $\langle T \rangle$ of $\mathbf{B}(\mathcal{H})$ generated by an operator T with $s(T) = (n^{-\lambda})_{n=1}^{\infty}$, $\lambda \in (0,1)$, is symmetrically normed. However, as we have shown, the principal Calkin space $\langle (n^{-\lambda})_{n=1}^{\infty} \rangle$, for $\lambda \in (0,1)$, is not tensor stable.

(iii) Let $\boldsymbol{\alpha} = \left(\frac{1}{\log_2 m}\right)_{m=2}^{\infty}$. Then the Calkin space $\langle \boldsymbol{\alpha} \rangle$ is tensor stable. To see this, consider the integers K_n for $n = 0, 1, \ldots$, associated with the sequence $\left(\frac{1}{\log_2 m}\right)_{m=2}^{\infty}$ and $\vartheta = \frac{1}{2}$ (Notation 2.5). We have that $K_n = 2^{2^{n+1}} - 2^{2^n}$. Since

$$(K_0 + \cdots + K_n)^2 = (2^{2^{n+1}} - 2)^2$$

it follows from Corollary 2.9 that $\langle \boldsymbol{\alpha} \rangle$ is tensor stable.

In the sequel we examine the tensor stability of a class of Calkin spaces, namely, the Lorentz sequence spaces. We recall their definition [18]. Let $1 \leqslant p < \infty$ and let $\mathbf{w} = (w_n)_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $w_1 = 1$, $\lim\limits_{n\to\infty} w_n = 0$ and $\sum\limits_{n=1}^{\infty} w_n = \infty$. We shall call such a $\mathbf{w}$ a *weight sequence*. The linear space $\ell_{\mathbf{w},p}$ of all sequences $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^{\infty}$ of complex numbers such that

$$\|\boldsymbol{\alpha}\|_{\mathbf{w},p} \overset{\text{def}}{=} \sup_{\pi} \left\{ \Big(\sum_{n=1}^{\infty} w_n \, |\alpha_{\pi(n)}|^p \Big)^{1/p} \right\} < \infty,$$

where π ranges over all the permutations of $\mathbb{N}$, is a Banach space under the norm $\|\cdot\|_{\mathbf{w},p}$, called a *Lorentz sequence space*.

If $\boldsymbol{\alpha} \in \ell_{\mathbf{w},p}$ then one easily sees that

$$\|\boldsymbol{\alpha}\|_{\mathbf{w},p} = \Big(\sum_{n=1}^{\infty} w_n (\alpha_n^{\star})^p \Big)^{1/p}.$$

If $w_n = n^{p/q-1}$ with $0 < p < q$ we obtain the classical $\ell_{q,p}$ spaces of Lorentz.

THEOREM 2.11. *Let $\mathbf{w} = (w_n)_{n=1}^{\infty}$ be a weight sequence such that there exists a constant $C > 0$ with $w_{mn} \leqslant C w_m w_n$ for every $m, n \in \mathbb{N}$. Then for every $p \geqslant 1$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \ell_{\mathbf{w},p}$ we have that*

$$\|\boldsymbol{\alpha} \otimes \boldsymbol{\beta}\|_{\mathbf{w},p} \leqslant C^{1/p} \, \|\boldsymbol{\alpha}\|_{\mathbf{w},p} \, \|\boldsymbol{\beta}\|_{\mathbf{w},p}.$$

In particular, $\ell_{\mathbf{w},p}$ is a tensor stable Calkin space.

Proof. We may assume that $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^{\infty}$ and $\boldsymbol{\beta} = (\beta_n)_{n=1}^{\infty}$ are positive decreasing sequences with $\alpha_1, \beta_1 \leqslant 1$. Fix ϑ with $0 < \vartheta < 1$. For every $n \in \mathbb{N} \cup \{0\}$, let $K_n = K_n^{(\vartheta)}(\boldsymbol{\alpha})$, $L_n = K_n^{(\vartheta)}(\boldsymbol{\beta})$, $\widetilde{M}_n = \sum\limits_{0 \leqslant i+j \leqslant n} K_i L_j$ and $K_{-1} = L_{-1} = \widetilde{M}_{-1} = 0$.

Let

$$\widetilde{\boldsymbol{\alpha}} = ((\vartheta^n)_{K_n})_{n=0}^{\infty}, \quad \widetilde{\boldsymbol{\beta}} = ((\vartheta^n)_{L_n})_{n=0}^{\infty}.$$

Then

$$\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\beta}} = ((\vartheta^n)_{\widetilde{M}_n - \widetilde{M}_{n-1}})_{n=0}^{\infty}.$$

For every $n, i, k, l \in \mathbb{N} \cup \{0\}$ such that $0 \leqslant i \leqslant n$, $1 \leqslant k \leqslant K_i$ and $1 \leqslant l \leqslant L_{n-i}$ we set

$$\phi_n(i,k,l) = \widetilde{M}_{n-1} + \sum_{j=0}^{i-1} K_j L_{n-j} + kl. \tag{2.3}$$

Also, for every i, $1 \leqslant k \leqslant K_i$ and $1 \leqslant l \leqslant L_i$ we set

$$\psi(i,k) = \sum_{j=0}^{i-1} K_j + k, \quad \psi'(i,l) = \sum_{j=0}^{i-1} L_j + l.$$

We observe that for every positive integer r, $(\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\beta}})_r = \vartheta^n$ if and only if $r = \widetilde{M}_{n-1} + s$ with $1 \leqslant s \leqslant \sum\limits_{i+j=n} K_i L_j$ and therefore $(\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\beta}})_r = \vartheta^n$ if and only if there exist $n, i, k, l \in \mathbb{N} \cup \{0\}$ such that $0 \leqslant i \leqslant n$, $1 \leqslant k \leqslant K_i$, $1 \leqslant l \leqslant L_{n-i}$ and $r = \phi_n(i,k,l)$. So,

$$\|\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p}^p = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} \sum_{k=1}^{K_i} \sum_{l=1}^{L_{n-i}} w_{\phi_n(i,k,l)} \Big) \vartheta^{np}. \tag{2.4}$$

Also, $\widetilde{\alpha}_r = \vartheta^i$ if and only if $r = \sum_{j=-1}^{i-1} K_j + k$ for some k with $1 \leqslant k \leqslant K_i$ and $\widetilde{\beta}_r = \vartheta^{i'}$ if and only if $r = \sum_{j=-1}^{i'-1} L_j + l$ for some l with $1 \leqslant l \leqslant L_{i'}$. So,

$$\|\widetilde{\boldsymbol{\alpha}}\|_{\mathbf{w},p}^p = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{K_n} w_{\psi(n,k)} \Big) \vartheta^{np}, \quad \|\widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p}^p = \sum_{n=0}^{\infty} \Big(\sum_{l=1}^{L_n} w_{\psi'(n,l)} \Big) \vartheta^{np}$$

and

$$\|\widetilde{\boldsymbol{\alpha}}\|_{\mathbf{w},p}^p \, \|\widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p}^p = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} \sum_{k=1}^{K_i} \sum_{l=1}^{L_{n-i}} w_{\psi(i,k)} w_{\psi'(n-i,l)} \Big) \vartheta^{np}. \tag{2.5}$$

But

$$\begin{aligned}
\psi(i,k)\, \psi'(n-i,l) &= \Big(\sum_{j=0}^{i-1} K_j + k \Big) \Big(\sum_{j=0}^{n-i-1} L_j + l \Big) \\
&= \sum_{j=0}^{i-1} \sum_{j'=0}^{n-i-1} K_j L_{j'} + k \sum_{j=0}^{n-i-1} L_j + l \sum_{j=0}^{i-1} K_j + kl \\
&\leqslant \sum_{j=0}^{i-1} \sum_{j'=0}^{n-i-1} K_j L_{j'} + K_i \sum_{j=0}^{n-i-1} L_j + L_{n-i} \sum_{j=0}^{i-1} K_j + kl \\
&\leqslant \widetilde{M}_{n-1} + kl \leqslant \phi_n(i,k,l).
\end{aligned}$$

By the monotonicity of the weight sequence $\mathbf{w}$ we have

$$w_{\phi_n(i,k,l)} \leqslant w_{\psi(i,k)\, \psi'(n-i,l)} \leqslant C w_{\psi(i,k)}\, w_{\psi'(n-i,l)}. \tag{2.6}$$

Finally, by (2.4), (2.5) and (2.6),

$$\begin{aligned}
\|\boldsymbol{\alpha} \otimes \boldsymbol{\beta}\|_{\mathbf{w},p} &\leqslant \|\widetilde{\boldsymbol{\alpha}} \otimes \widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p} \leqslant C^{1/p} \|\widetilde{\boldsymbol{\alpha}}\|_{\mathbf{w},p} \, \|\widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p} \\
&= C^{1/p} \frac{1}{\vartheta^2} \, \|\vartheta \widetilde{\boldsymbol{\alpha}}\|_{\mathbf{w},p} \, \|\vartheta \widetilde{\boldsymbol{\beta}}\|_{\mathbf{w},p} \leqslant C^{1/p} \frac{1}{\vartheta^2} \, \|\boldsymbol{\alpha}\|_{\mathbf{w},p} \, \|\boldsymbol{\beta}\|_{\mathbf{w},p}.
\end{aligned}$$

Letting $\vartheta \to 1$ we obtain

$$\|\boldsymbol{\alpha} \otimes \boldsymbol{\beta}\|_{p,\mathbf{w}} \leqslant C^{1/p} \, \|\boldsymbol{\alpha}\|_{\mathbf{w},p} \, \|\boldsymbol{\beta}\|_{\mathbf{w},p}. \qquad \blacksquare$$

3. s-NUMBERS OF RESTRICTIONS

Let $\mathcal{A}$ be a C^*-algebra. If $a, b \in \mathcal{A}$ we denote by $M_{a,b}$ the operator on $\mathcal{A}$ given by $M_{a,b}(x) = axb$. An operator $\Phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if $\Phi = \sum\limits_{i=1}^{m} M_{a_i,b_i}$ for some $a_i, b_i \in \mathcal{A}$, $i = 1, \ldots, m$.

If $\mathcal{C}$ is a C^*-subalgebra of $\mathcal{A}$ such that $M_{a,b}(\mathcal{C}) \subseteq \mathcal{C}$ we will denote by $M^{\mathcal{C}}_{a,b}$ the operator $\mathcal{C} \to \mathcal{C}$ defined by $M^{\mathcal{C}}_{a,b}(x) = axb$. In this section we prove inequalities concerning the s-number functions of the operators $M_{a,b}$ and $M^{\mathcal{C}}_{a,b}$.

It is well-known that every closed two-sided ideal $\mathcal{J}$ of $\mathcal{A}$ is an M-ideal, that is, that there exists a projection $\eta : \mathcal{A}^* \to \mathcal{J}^\perp$, where $\mathcal{J}^\perp$ is the annihilator of $\mathcal{J}$ in $\mathcal{A}^*$, such that for every $\varphi \in \mathcal{A}^*$,

$$\|\varphi\| = \|\eta(\varphi)\| + \|\varphi - \eta(\varphi)\|$$

(see e.g. Theorem 11.4 of [7]). A functional $\varphi \in \mathcal{A}^*$ is called a *Hahn–Banach extension* of $\phi \in \mathcal{J}^*$ if it is an extension of ϕ and $\|\varphi\| = \|\phi\|$. If $\mathcal{J}$ is an M-ideal of $\mathcal{A}$ then every $\phi \in \mathcal{J}^*$ has a unique Hahn–Banach extension in $\mathcal{A}^*$ denoted by $\widetilde{\phi}$. Thus, if we identify $\mathcal{J}^*$ with the subspace $\{\widetilde{\phi} : \phi \in \mathcal{J}^*\}$ of $\mathcal{A}^*$ then

$$\mathcal{A}^* = \mathcal{J}^* \oplus_{\ell_1} \mathcal{J}^\perp;$$

hence $\|\widetilde{\phi} + \psi\| = \|\phi\| + \|\psi\|$ for all $\phi \in \mathcal{J}^*$, $\psi \in \mathcal{J}^\perp$. Given $T \in \mathbf{B}(\mathcal{J})$ let $\widehat{T} : \mathcal{A}^* \to \mathcal{A}^*$ be given by

$$\widehat{T}(\widetilde{\phi} + \psi) = \widetilde{T^*(\phi)},$$

where $\phi \in \mathcal{J}^*$ and $\psi \in \mathcal{J}^\perp$. We identify $\mathcal{A}$ with a subspace of $\mathcal{A}^{**}$ via the canonical embedding and denote by $\widetilde{T} : \mathcal{A} \to \mathcal{A}^{**}$ the restriction of $\widehat{T}^*$ to $\mathcal{A}$.

LEMMA 3.1. (i) *If $T \in \mathbf{B}(\mathcal{J})$ then $\widetilde{T}$ extends T and $\|\widetilde{T}\| = \|T\|$.*
(ii) *The map $T \to \widetilde{T}$ is linear.*

Proof. The second assertion is easily verified. We show (i). Let $x \in \mathcal{J}$ and $f \in \mathcal{A}^*$. Then $f = \widetilde{\phi} + \psi$ with $\phi \in \mathcal{J}^*$ and $\psi \in \mathcal{J}^\perp$. We have

$$\begin{aligned}\widetilde{T}(x)(f) &= \widehat{T}^*(x)(f) = \widehat{T}(f)(x) = \widetilde{T^*(\phi)}(x) = T^*(\phi)(x)\\ &= \phi(Tx) = \widetilde{\phi}(Tx) = f(Tx) = T(x)(f).\end{aligned}$$

Hence, $\widetilde{T}$ is an extension of T and so $\|T\| \leqslant \|\widetilde{T}\|$.

We show that $\|\widetilde{T}\| \leqslant \|T\|$. Let $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Then $f = \widetilde{\phi} + \psi$ with $\phi \in \mathcal{J}^*$ and $\psi \in \mathcal{J}^\perp$. We have

$$\begin{aligned}|\widetilde{T}(x)(f)| &= |\widehat{T}^*(x)(f)| = |\widehat{T}(f)(x)| = |\widetilde{T^*(\phi)}(x)| \leqslant \|\widetilde{T^*(\phi)}\|\|x\|\\ &= \|T^*(\phi)\|\|x\| \leqslant \|T^*\|\|\phi\|\|x\| = \|T^*\|\|\widetilde{\phi}\|\|x\| \leqslant \|T^*\|\|f\|\|x\|.\end{aligned}$$

Hence, $\|\widetilde{T}\| \leqslant \|T^*\| = \|T\|$ and the proof is complete. ∎

Let $\mathcal{X}$ be a reflexive Banach space and $\iota : \mathcal{J}^* \to \mathcal{A}^*$ be the map defined by $\iota(\phi) = \widetilde{\phi}$, $\phi \in \mathcal{J}^*$. Clearly, $\|\iota\| = 1$.

Let $T : \mathcal{J} \to \mathcal{X}$. Write $T^\sharp : \mathcal{A} \to \mathcal{X}$ for the restriction of $(\iota \circ T^*)^*$ to $\mathcal{A}$.

LEMMA 3.2. *The operator $T^\sharp$ extends T and $\|T^\sharp\| = \|T\|$.*

Proof. Let $a \in \mathcal{J}$ and $g \in \mathcal{X}^*$. We have that

$$\begin{aligned} T^\sharp(a)(g) &= (\iota \circ T^*)^*(a)(g) = a(\iota(T^*(g))) = \iota(T^*(g))(a) = \widetilde{T^*(g)}(a) \\ &= T^*(g)(a) = g(Ta) = T(a)(g). \end{aligned}$$

Hence $T^\sharp$ extends T and so $\|T\| \leqslant \|T^\sharp\|$. On the other hand,

$$\|T^\sharp\| \leqslant \|(\iota \circ T^*)^*\| = \|\iota \circ T^*\| \leqslant \|\iota\|\|T^*\| = \|T\|. \quad \blacksquare$$

LEMMA 3.3. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{J} \subseteq \mathcal{A}$ be a closed two sided ideal and $\Phi : \mathcal{A} \to \mathcal{A}$ be a bounded operator which leaves $\mathcal{J}$ invariant. Let $\Phi_0 : \mathcal{J} \to \mathcal{J}$ be the operator given by $\Phi_0(x) = \Phi(x)$. Then $h_n(\Phi_0) \leqslant h_n(\Phi)$, for each $n \in \mathbb{N}$.*

Proof. Write $\iota_0 : \mathcal{J} \to \mathcal{A}$ for the inclusion map. In the supremum below, $\mathcal{H}$ and $\mathcal{K}$ are arbitrary Hilbert spaces. Using Lemma 3.2 we have that

$$\begin{aligned} h_n(\Phi_0) &= \sup\{s_n(A\Phi_0 B) : B \in \mathbf{B}(\mathcal{H}, \mathcal{J}), A \in \mathbf{B}(\mathcal{J}, \mathcal{K}) \text{ contractions}\} \\ &= \sup\{s_n(A^\sharp \Phi(\iota_0 B)) : B \in \mathbf{B}(\mathcal{H}, \mathcal{J}), A \in \mathbf{B}(\mathcal{J}, \mathcal{K}) \text{ contractions}\} \\ &\leqslant \sup\{s_n(A_1 \Phi B_1) : B_1 \in \mathbf{B}(\mathcal{H}, \mathcal{A}), A_1 \in \mathbf{B}(\mathcal{A}, \mathcal{K}) \text{ contractions}\} \\ &= h_n(\Phi). \quad \blacksquare \end{aligned}$$

If $\mathcal{X}$ is a Banach space, $c \in \mathcal{X}$ and $\phi \in \mathcal{X}^*$ we denote by $\phi \otimes c$ the operator on $\mathcal{X}$ given by $\phi \otimes c(x) = \phi(x)c$. We denote by $\mathbf{F}_n(\mathcal{X})$ the set of all operators F on $\mathcal{X}$ of rank less than or equal to n. It is well-known that $\mathbf{F}_n(\mathcal{X}) = \Big\{ \sum\limits_{i=1}^n \phi_i \otimes c_i :$ $\phi_i \in \mathcal{X}^*, c_i \in \mathcal{X}, i = 1, 2, \ldots, n\Big\}$.

LEMMA 3.4. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{J}$ be a closed two-sided ideal of $\mathcal{A}$.*

(i) *Assume that $a, b \in \mathcal{J}$. Then $\widetilde{M^{\mathcal{J}}_{a,b}}(x) = M_{a,b}(x)$ for every $x \in \mathcal{A}$.*

(ii) *Let $\phi_i \in \mathcal{J}^*, c_i \in \mathcal{J}$, $i = 1, \ldots, n$, and F be the operator on $\mathcal{J}$ given by $F = \sum\limits_{i=1}^n \phi_i \otimes c_i$. Then $\widetilde{F}(x) = \Big(\sum\limits_{i=1}^n \widetilde{\phi}_i \otimes c_i\Big)(x)$ for every $x \in \mathcal{A}$.*

Proof. (i) Let $S = M_{a,b}$, $T = M^{\mathcal{J}}_{a,b}$, and $\phi \in \mathcal{J}^*$. First note that $S^*(\widetilde{\phi})$ is an extension of $T^*(\phi)$. Indeed, for every $x \in \mathcal{J}$ we have that

$$S^*(\widetilde{\phi})(x) = \widetilde{\phi}(Sx) = \phi(Tx) = T^*(\phi)(x).$$

We show that $S^*(\widetilde{\phi})$ is the Hahn–Banach extension of $T^*(\phi)$. To this end, let $x \in \mathcal{A}$ and $\{u_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{J}$ be a contractive approximate unit for $\mathcal{J}$. Then for each $x \in \mathcal{A}$,

$au_\lambda xb \to_\lambda axb$ in norm and hence $\phi(au_\lambda xb) \to_\lambda \phi(axb)$. We thus have that

$$\begin{aligned}|S^*(\widetilde{\phi})(x)| &= |\widetilde{\phi}(Sx)| = |\widetilde{\phi}(axb)| = |\phi(axb)| = \lim_{\lambda\in\Lambda} |\phi(au_\lambda xb)| \\ &= \lim_{\lambda\in\Lambda} |T^*(\phi)(u_\lambda x)| \leqslant \|T^*(\phi)\|\|x\|.\end{aligned}$$

It follows that $\|S^*(\widetilde{\phi})\| \leqslant \|T^*(\phi)\|$. Since $S^*(\widetilde{\phi})$ extends $T^*(\phi)$, we have that $S^*(\widetilde{\phi})$ is the Hahn–Banach extension of $T^*(\phi)$, that is, $S^*(\widetilde{\phi}) = \widetilde{T^*(\phi)}$.

Let $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Then $f = \widetilde{\phi} + \psi$ with $\phi \in \mathcal{J}^*$ and $\psi \in \mathcal{J}^\perp$, and

$$\begin{aligned}\widetilde{T}(x)(f) &= \widehat{T}^*(x)(f) = \widetilde{T^*(\phi)}(x) = S^*(\widetilde{\phi})(x) = \widetilde{\phi}(Sx) \\ &= (\widetilde{\phi} + \psi)(Sx) = S^*(f)(x) = S(x)(f).\end{aligned}$$

(ii) By Lemma 3.1(ii), it suffices to show the statement in the case $F = \phi_1 \otimes c_1$, where $\phi_1 \in \mathcal{J}^*$ and $c_1 \in \mathcal{J}$. Let $\phi \in \mathcal{J}^*$. We have that $\widetilde{F^*(\phi)}(x) = \widetilde{\phi}_1(x)\phi(c_1)$ for every $x \in \mathcal{A}$. Indeed, the functional $x \to \widetilde{\phi}_1(x)\phi(c_1)$ extends $F^*(\phi)$ and has norm equal to the norm of $F^*(\phi)$ since $\|\phi_1\| = \|\widetilde{\phi}_1\|$. Let $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$. We have $f = \widetilde{\phi} + \psi$, where $\phi \in \mathcal{J}^*$ and $\psi \in \mathcal{J}^\perp$. Then

$$\begin{aligned}\widetilde{F}(x)(f) &= \widehat{F}^*(x)(f) = \widehat{F}(f)(x) = \widehat{F}(\widetilde{\phi} + \psi)(x) = \widetilde{F^*(\phi)}(x) \\ &= \widetilde{\phi}_1(x)\phi(c_1) = \widetilde{\phi}_1(x)\widetilde{\phi}(c_1) = \widetilde{\phi}_1(x)f(c_1) = (\widetilde{\phi}_1 \otimes c_1)(x)(f). \quad \blacksquare\end{aligned}$$

The following theorem is the main result of this section.

THEOREM 3.5. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{J}$ be a closed two-sided ideal of $\mathcal{A}$ and $a, b \in \mathcal{J}$. Then for every $n \in \mathbb{N}$ we have that*

$$\mathrm{h}_n(M^{\mathcal{J}}_{a,b}) \leqslant \mathrm{h}_n(M_{a,b}) \leqslant \mathrm{a}_n(M_{a,b}) \leqslant \mathrm{a}_n(M^{\mathcal{J}}_{a,b}).$$

Proof. The first inequality follows from Lemma 3.3 while the second one is trivial. In what follows the operators $\widetilde{F}$ for $F \in \mathbf{F}_{n-1}(\mathcal{J})$ and $\widetilde{M^{\mathcal{J}}_{a,b}}$ are considered as operators from $\mathcal{A}$ to $\mathcal{A}$; this is possible by Lemma 3.4. It follows from Lemmas 3.1 and 3.4 that for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathrm{a}_n(M_{a,b}) &= \inf\{\|M_{a,b} - G\| : G \in \mathbf{F}_{n-1}(\mathcal{A})\} \\ &\leqslant \inf\{\|M_{a,b} - \widetilde{F}\| : F \in \mathbf{F}_{n-1}(\mathcal{J})\} \\ &= \inf\{\|\widetilde{M^{\mathcal{J}}_{a,b}} - \widetilde{F}\| : F \in \mathbf{F}_{n-1}(\mathcal{J})\} \\ &= \inf\{\|M^{\mathcal{J}}_{a,b} - F\| : \ F \in \mathbf{F}_{n-1}(\mathcal{J})\} = \mathrm{a}_n(M^{\mathcal{J}}_{a,b}). \quad \blacksquare\end{aligned}$$

We close the section with a lemma which will be used in the proof of Theorem 5.6.

LEMMA 3.6. *Let $\mathcal{B} \subseteq \mathbf{B}(\mathcal{H})$ be a C^*-algebra, $\mathcal{A} = \overline{\mathcal{B}}^{\mathrm{wot}}$ and $A \in \mathcal{A}$. Assume that $A \in \mathcal{B}$. Then $\mathrm{d}(M_{A,A}) \leqslant \mathrm{d}(M^{\mathcal{B}}_{A,A})$.*

Proof. Set $\mathrm{d}(M_{A,A}^{\mathcal{B}}) = (d_n)_{n=1}^{\infty}$. Let $\varepsilon > 0$ and $\mathcal{F} \subseteq \mathcal{B}$ be a linear space such that $\dim \mathcal{F} < n$ and

$$\inf_{F \in \mathcal{F}} \|AXA - F\| < d_n + \varepsilon$$

for each contraction $X \in \mathcal{B}$. It suffices to show that $\inf_{F \in \mathcal{F}} \|AYA - F\| \leqslant d_n + \varepsilon$ for each contraction $Y \in \mathcal{A}$. Suppose this is not the case and let $Y \in \mathcal{A}$ be a contraction such that $\|AYA - F\| > d_n + \varepsilon$, for each $F \in \mathcal{F}$. By the Kaplansky Density theorem, there exists a net $(X_\nu)_\nu \subseteq \mathcal{B}$ of contractions such that $X_\nu \to_\nu Y$ in the weak operator topology. Let $F_\nu \in \mathcal{F}$ be such that $\|AX_\nu A - F_\nu\| < d_n + \varepsilon$. We have that $\|F_\nu\| \leqslant d_n + \varepsilon + 1$ for each ν, and hence we may assume without loss of generality that $F_\nu \to F_0$ in norm. We thus have $AX_\nu A - F_\nu \to AYA - F_0$ weakly. It follows that

$$\|AYA - F_0\| \leqslant \liminf \|AX_\nu A - F_\nu\| \leqslant d_n + \varepsilon,$$

a contradiction. ∎

4. ELEMENTARY OPERATORS ON $\mathbf{B}(\mathcal{H})$

In this section we obtain estimates for the s-numbers of an elementary operator acting on $\mathbf{B}(\mathcal{H})$ in terms of the singular numbers of its symbols. We formulate some of our results using tensor products. Recall [21] that a *cross norm* τ is a norm defined simultaneously on all algebraic tensor products $\mathcal{X} \otimes \mathcal{Y}$ of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\tau(x \otimes y) = \|x\|\|y\|$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. By $\mathcal{X} \otimes_\tau \mathcal{Y}$ we denote the completion of the algebraic tensor product with respect to τ. A *tensor norm* is a cross norm τ such that for every $A \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathbf{B}(\mathcal{X}', \mathcal{Y}')$ the linear operator $A \otimes B : \mathcal{X} \otimes \mathcal{X}' \to \mathcal{Y} \otimes \mathcal{Y}'$ given by $A \otimes B(x \otimes x') = Ax \otimes Bx'$ is bounded with respect to τ and the norm of its extension $A \otimes_\tau B \in \mathbf{B}(\mathcal{X} \otimes_\tau \mathcal{X}', \mathcal{Y} \otimes_\tau \mathcal{Y}')$ satisfies the inequality $\|A \otimes_\tau B\| \leqslant \|A\| \, \|B\|$.

In Theorem 4.2 below we give an upper bound for the approximation numbers of the operator $A \otimes_\tau B$ in terms of the sequence $s(A) \otimes s(B)$. We will need the following lemma due to König ([17], Lemma 2).

LEMMA 4.1. *Let τ be a tensor norm, $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $A \in \mathbf{B}(\ell_2, \mathcal{X})$, $B \in \mathbf{B}(\ell_2, \mathcal{Y})$ and $(P_k)_{k=0}^n$, $(Q_k)_{k=0}^n$ be families of mutually orthogonal projections acting on ℓ_2. Then*

$$\Big\| \sum_{k=0}^{n} AP_k \otimes_\tau BQ_k \Big\|_{\ell_2 \otimes_\tau \ell_2 \to \mathcal{X} \otimes_\tau \mathcal{Y}} \leqslant \max_{0 \leqslant k \leqslant n} \{\|AP_k\| \, \|BQ_k\|\}.$$

THEOREM 4.2. *Let $\mathcal{H}$ be a Hilbert space, $A, B \in \mathbf{K}(\mathcal{H})$ and τ be a tensor norm. Then*

$$\mathrm{a}(A \otimes_\tau B) \leqslant 6.75 \, s(A) \otimes s(B). \tag{4.1}$$

Consequently, if $\mathfrak{i}$ *and* $\mathfrak{j}$ *are Calkin spaces,* $s(A) \in \mathfrak{i}$ *and* $s(B) \in \mathfrak{j}$ *and* s *is any s-function then* $\mathrm{s}(A \otimes_\tau B) \in \mathfrak{i} \otimes \mathfrak{j}$.

Proof. If $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty$ is a bounded sequence we write $D_{\boldsymbol{\alpha}} \in \mathbf{B}(\ell_2)$ for the diagonal operator given by $D_{\boldsymbol{\alpha}}((x_n)_{n=1}^\infty) = (\alpha_n x_n)_{n=1}^\infty$ for $(x_n)_{n=1}^\infty \in \ell_2$. It suffices to prove the theorem in the case where A and B are diagonal operators in $\mathbf{B}(\ell_2)$. Indeed, suppose that (4.1) holds in this case. By polar decomposition, there exist partial isometries $U_A, U_B : \mathcal{H} \to \ell_2$, $V_A, V_B : \ell_2 \to \mathcal{H}$ and diagonal operators $D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\beta}} : \ell_2 \to \ell_2$, where $\boldsymbol{\alpha} = s(A), \boldsymbol{\beta} = s(B)$, such that $A = V_A D_{\boldsymbol{\alpha}} U_A$ and $B = V_B D_{\boldsymbol{\beta}} U_B$. Then

$$A \otimes_\tau B = (V_A \otimes_\tau V_B)(D_{\boldsymbol{\alpha}} \otimes_\tau D_{\boldsymbol{\beta}})(U_A \otimes_\tau U_B),$$

and hence

$$\begin{aligned}\mathrm{a}(A \otimes_\tau B) &\leqslant \|V_A \otimes_\tau V_B\| \, \mathrm{a}(D_{\boldsymbol{\alpha}} \otimes_\tau D_{\boldsymbol{\beta}}) \, \|U_A \otimes_\tau U_B\| \leqslant \mathrm{a}(D_{\boldsymbol{\alpha}} \otimes_\tau D_{\boldsymbol{\beta}}) \\ &\leqslant 6.75 \, \boldsymbol{\alpha} \otimes \boldsymbol{\beta} = 6.75 \, s(A) \otimes s(B).\end{aligned}$$

So let $A = D_{\boldsymbol{\alpha}} : \ell_2 \to \ell_2$, $B = D_{\boldsymbol{\beta}} : \ell_2 \to \ell_2$, where $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^\infty, \boldsymbol{\beta} = (\beta_n)_{n=1}^\infty$ are non-negative decreasing sequences. We may further assume that $\alpha_1, \beta_1 \leqslant 1$. Set $\mathrm{a}_n = \mathrm{a}_n(A \otimes_\tau B)$ and fix ϑ with $0 < \vartheta < 1$.

In what follows we use Notation 2.5. For every $n \in \mathbb{N} \cup \{0\}$ let

$$K_n = K_n^{(\vartheta)}(\boldsymbol{\alpha}), \quad L_n = K_n^{(\vartheta)}(\boldsymbol{\beta}), \quad \widetilde{M}_n = \sum_{0 \leqslant i+j \leqslant n} K_i L_j, \quad \widetilde{M}_{-1} = 0,$$

$$P_n = \sum_{i \in \mathcal{K}_n^{(\vartheta)}(\boldsymbol{\alpha})} e_i^* \otimes e_i, \quad Q_n = \sum_{i \in \mathcal{K}_n^{(\vartheta)}(\boldsymbol{\beta})} e_i^* \otimes e_i,$$

where $(e_n)_{n=0}^\infty$ is the standard basis of ℓ_2.

Let $A_n = AP_n$, $B_n = BQ_n$ and $E_n = \sum_{0 \leqslant k+l \leqslant n} A_k \otimes_\tau B_l$. Clearly, $\|A_n\| \leqslant \vartheta^n$, $\|B_n\| \leqslant \vartheta^n$ and $\operatorname{rank} E_n \leqslant \widetilde{M}_n$. Moreover,

$$A = \sum_{n=0}^\infty A_n, \quad B = \sum_{n=0}^\infty B_n, \quad A \otimes_\tau B = \sum_{n,m=0}^\infty A_m \otimes_\tau B_n,$$

where the series are absolutely convergent in the norm topology. Hence,

$$\mathrm{a}_{\widetilde{M}_n+1}(A \otimes_\tau B) \leqslant \|A \otimes_\tau B - E_n\| \leqslant \sum_{N=n+1}^\infty \Big\| \sum_{k+l=N} A_k \otimes_\tau B_l \Big\|.$$

By Lemma 4.1,

$$\Big\| \sum_{k+l=N} A_k \otimes_\tau B_l \Big\| = \Big\| \sum_{k=0}^N A P_k \otimes_\tau B Q_{N-k} \Big\| \leqslant \max_{0 \leqslant k \leqslant N} \|A_k\| \|B_{N-k}\| \leqslant \max_{0 \leqslant k \leqslant N} \vartheta^k \vartheta^{N-k} = \vartheta^N$$

and so

$$\mathrm{a}_{\widetilde{M}_n+1} \leqslant \sum_{N=n+1}^\infty \vartheta^N = \frac{1}{1-\vartheta} \, \vartheta^{n+1}. \tag{4.2}$$

By the monotonicity of the approximation numbers, Lemma 2.2, (2.1) and (4.2) we obtain

$$\begin{aligned}
(\mathrm{a}_n)_{n=1}^{\infty} &= ((\mathrm{a}_j)_{j=\widetilde{M}_n+1}^{\widetilde{M}_{n+1}})_{n=-1}^{\infty} \leqslant ((\mathrm{a}_{\widetilde{M}_n+1})_{\widetilde{M}_{n+1}-\widetilde{M}_n})_{n=-1}^{\infty} \\
&\leqslant \frac{1}{1-\vartheta}((\vartheta^{n+1})_{\widetilde{M}_{n+1}-\widetilde{M}_n})_{n=-1}^{\infty} = \frac{1}{1-\vartheta}((\vartheta^{n})_{\sum_{i+j=n} K_i L_j})_{n=0}^{\infty} \\
&= \frac{1}{1-\vartheta}((\vartheta^{n})_{K_n})_{n=0}^{\infty} \otimes ((\vartheta^{n})_{L_n})_{n=0}^{\infty} \\
&= \frac{1}{\vartheta^2(1-\vartheta)}((\vartheta^{n+1})_{K_n})_{n=0}^{\infty} \otimes ((\vartheta^{n+1})_{L_n})_{n=0}^{\infty} \\
&\leqslant \frac{1}{\vartheta^2(1-\vartheta)} s(A) \otimes s(B).
\end{aligned}$$

The minimal value of $\frac{1}{\vartheta^2(1-\vartheta)}$ for $\vartheta \in (0,1)$ is 6.75, and so

$$\mathrm{a}(A \otimes_\tau B) \leqslant 6.75\, s(A) \otimes s(B). \quad ∎$$

Theorems 4.2 and 2.11 yield the following corollary.

COROLLARY 4.3. *Let* $\mathbf{w} = (w_n)_{n=1}^{\infty}$ *be a weight sequence with* $w_{mn} \leqslant C w_m w_n$ *for all* m,n *and let* $A, B \in \mathbf{K}(\mathcal{H})$ *be operators with* $s(A), s(B) \in \ell_{\mathbf{w},p}$. *Then*

$$\|\mathrm{a}(A \otimes_\tau B)\|_{\mathbf{w},p} \leqslant 6.75\, C^{1/p}\, \|s(A)\|_{\mathbf{w},p}\, \|s(B)\|_{\mathbf{w},p}.$$

Consider the weight sequence $\mathbf{w} = (w_n)_{n=1}^{\infty}$, where $w_n = \frac{(1+\ln n)^\gamma}{n^\alpha}$. If $0 < \alpha \leqslant 1$ and $\gamma \geqslant 0$, then $w_{mn} \leqslant w_m w_n$ for all m, n. Hence Corollary 4.3 extends results of H. König ([17], Proposition 3) and F. Cobos and L.M. Fernàndez-Cabrera [6].

For the rest of the paper, we will be concerned with elementary operators. Let A, B be compact operators in $\mathbf{B}(\mathcal{H})$. We recall that $M_{A,B}$ is the operator $\mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$ defined by $M_{A,B}(X) = AXB$ and $M_{A,B}^{\mathbf{K}(\mathcal{H})}$ is the operator $\mathbf{K}(\mathcal{H}) \to \mathbf{K}(\mathcal{H})$ defined by $M_{A,B}^{\mathbf{K}(\mathcal{H})}(X) = AXB$. Theorems 3.5 and 4.2 imply the following corollary.

COROLLARY 4.4. *Let* A, B *be compact operators in* $\mathbf{B}(\mathcal{H})$. *Then*

$$\mathrm{a}(M_{A,B}) \leqslant \mathrm{a}(M_{A,B}^{\mathbf{K}(\mathcal{H})}) \leqslant 6.75\, s(A) \otimes s(B).$$

Proof. For every $x \in \mathcal{H}$ we denote by f_x the functional on $\mathcal{H}$ defined by $f_x(y) = \langle y, x\rangle$. The *conjugate space* $\overline{\mathcal{H}}$ of $\mathcal{H}$ is defined to be the set $\{f_x : x \in \mathcal{H}\}$ with vector space operations $f_x + f_y = f_{x+y}$, $\lambda f_x = f_{\overline{\lambda} x}$ and inner product given by $\langle f_x, f_y\rangle = \overline{\langle x, y\rangle}$. For every $A \in \mathbf{B}(\mathcal{H})$ we denote by $\overline{A} \in \mathbf{B}(\overline{\mathcal{H}})$ the operator defined by $\overline{A}(f_x) = f_{Ax}$.

Note that the map $A \mapsto \overline{A}$ is a surjective conjugate linear isometry and that $s(A) = s(\overline{A})$, for every compact operator A.

Let ε be the injective tensor norm. The mapping $F : \overline{\mathcal{H}} \otimes \mathcal{H} \to \mathbf{B}(\mathcal{H})$ given by $F\Big(\sum\limits_{i=1}^{n} f_{x_i} \otimes y_i\Big) = \sum\limits_{i=1}^{n} x_i^* \otimes y_i$ is a linear isometry ([24], Chapter IV, Theorem 2.5) of $\overline{\mathcal{H}} \otimes_\varepsilon \mathcal{H}$ onto $\mathbf{K}(\mathcal{H})$.

We define $\widetilde{F} : \mathbf{B}(\overline{\mathcal{H}} \otimes \mathcal{H}) \to \mathbf{B}(\mathbf{K}(\mathcal{H}))$ by $\widetilde{F}(T) = F \circ T \circ F^{-1}$. Clearly $\widetilde{F}$ is a surjective linear isometry and $\widetilde{F}(T)$ is given by $\widetilde{F}(T)(x^* \otimes y) = F(T(f_x \otimes y))$ for $x, y \in \mathcal{H}$.

For every $\overline{A} \in \mathbf{B}(\overline{\mathcal{H}})$, where $A \in \mathbf{B}(\mathcal{H})$, and every $B \in \mathbf{B}(\mathcal{H})$ we have that

$$\widetilde{F}(\overline{A} \otimes_\varepsilon B) = M_{B^*,A}^{\mathbf{K}(\mathcal{H})}. \tag{4.3}$$

Indeed, for every $x, y \in \mathcal{H}$,

$$\begin{aligned}\widetilde{F}(\overline{A} \otimes_\varepsilon B)(x^* \otimes y) &= F(\overline{A} \otimes_\varepsilon B)(f_x \otimes y) = F(\overline{A} f_x \otimes By) = F(f_{Ax} \otimes By)\\ &= (Ax)^* \otimes By = B(x^* \otimes y)A^* = M_{B,A^*}^{\mathbf{K}(\mathcal{H})}(x^* \otimes y).\end{aligned}$$

So if $A, B \in \mathbf{B}(\mathcal{H})$ by (4.3) and Theorems 3.5 and 4.2 we have that

$$\begin{aligned}\mathrm{a}(M_{A,B}) &\leqslant \mathrm{a}(M_{A,B}^{\mathbf{K}(\mathcal{H})}) = \mathrm{a}(\widetilde{F}(\overline{B} \otimes_\varepsilon A^*)) = \mathrm{a}(\overline{B} \otimes_\varepsilon A^*)\\ &\leqslant 6.75\, s(\overline{B}) \otimes s(A^*) = 6.75\, s(A) \otimes s(B). \quad \blacksquare\end{aligned}$$

PROPOSITION 4.5. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathbf{B}(\mathcal{H})$ such that $\mathbf{K}(\mathcal{H}) \subseteq \mathcal{A}$. Let $A_i, B_i \in \mathcal{A}$, $i = 1, \dots, m$, and $\Phi = \sum\limits_{i=1}^{m} M_{A_i,B_i}$. If the operators A_i (respectively, B_i), $i = 1, \dots, m$, are linearly independent then there exists $r \in \mathbb{N}$ and a constant $C > 0$ such that for every n and for every $i = 1, \dots, m$,*

$$s_{rn-r+1}(A_i) \leqslant C\,\mathrm{h}_n(\Phi) \quad (\textit{respectively}, \ s_{rn-r+1}(B_i) \leqslant C\,\mathrm{h}_n(\Phi)).$$

In particular, if $\mathfrak{i}$ is a Calkin space and $\mathrm{h}(\Phi) \in \mathfrak{i}$ then $s(A_i) \in \mathfrak{i}$ (respectively, $s(B_i) \in \mathfrak{i}$) for every $i = 1, \dots, m$.

Proof. We will only consider the case where the operators B_i, $i = 1, \dots, m$, are linearly independent. The other case can be treated similarly.

By Lemma 1 of [11], there exist $r \in \mathbb{N}$ and $\xi_i, \eta_i \in \mathcal{H}$, $i = 1, \dots, r$, such that

$$\sum_{j=1}^{r} \langle B_i \eta_j, \xi_j \rangle = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2, \dots, m. \end{cases}$$

Let $\phi_j : \mathcal{H} \to \mathcal{A}$, $j = 1, \dots, r$ be the operators given by $\phi_j(\xi) = \xi_j^* \otimes \xi$, $\psi_j : \mathcal{A} \to \mathcal{H}$, $j = 1, \dots, r$ be the operators given by $\psi_j(B) = B\eta_j$ and

$$S = \sum_{j=1}^{r} \psi_j \circ \Phi \circ \phi_j = \sum_{i=1}^{m} \sum_{j=1}^{r} \psi_j \circ M_{A_i,B_i} \circ \phi_j.$$

For $\xi \in \mathcal{H}$ we have

$$(\psi_j \circ M_{A_i,B_i} \circ \phi_j)(\xi) = \psi_j(A_i\phi_j(\xi)B_i) = \psi_j(A_i(\xi_j^* \otimes \xi)B_i)$$
$$= \psi_j((B_i^*\xi_j)^* \otimes A_i\xi) = \langle \eta_j, B_i^*\xi_j\rangle A_i\xi = \langle B_i\eta_j, \xi_j\rangle A_i\xi$$

and hence

$$S = \sum_{i=1}^{m} \Big(\sum_{j=1}^{r} \langle B_i\eta_j, \xi_j\rangle \Big) A_i = A_1.$$

By the additivity of the singular numbers, we have that

$$s_{rn-r+1}(A_1) \leqslant \sum_{j=1}^{r} s_n(\psi_j \circ \Phi \circ \phi_j), \quad n \in \mathbb{N}.$$

Let $C = r \max_{j=1,\dots,r} \|\psi_j\|\|\phi_j\|$. Then $s_n(\psi_j \circ \Phi \circ \phi_j) \leqslant \|\psi_j\|\|\phi_j\| h_n(\Phi)$ and so $s_{nr-r+1}(A_1) \leqslant C\mathrm{h}_n(\Phi)$, $n \in \mathbb{N}$.

Finally, by the monotonicity of s-numbers, we have that

$$s(A_1) = (s_n(A_1))_{n=1}^{\infty} = ((s_{nr-r+1+k}(A_1))_{k=0}^{r-1})_{n=1}^{\infty}$$
$$\leqslant ((s_{nr-r+1}(A_1))_r)_{n=1}^{\infty} \leqslant C((\mathrm{h}_n(\Phi))_r)_{n=1}^{\infty}.$$

If $\mathfrak{i}$ is a Calkin space and $\mathrm{h}(\Phi) \in \mathfrak{i}$, Lemma 2.4 implies that $((h_n(\Phi))_r)_{n=1}^{\infty} \in \mathfrak{i}$. It follows that $s(A_1) \in \mathfrak{i}$. Similarly, $s(A_i) \in \mathfrak{i}$, $i = 2, \dots, m$. ∎

The following theorem is the main result of this section.

THEOREM 4.6. *Let Φ be an elementary operator on $\mathbf{B}(\mathcal{H})$ (respectively on $\mathbf{K}(\mathcal{H})$), $\mathfrak{i}$ be a tensor stable Calkin space and* s *be an s-function. Then* $\mathrm{s}(\Phi) \in \mathfrak{i}$ *if and only if there exist $m \in \mathbb{N}$ and $A_i, B_i \in \mathbf{B}(\mathcal{H})$, $i = 1, \dots, m$, such that $\Phi = \sum_{i=1}^{m} M_{A_i,B_i}$ and $s(A_i), s(B_i) \in \mathfrak{i}$ for $i = 1, \dots, m$.*

Proof. We prove the theorem in the case where Φ is an elementary operator on $\mathbf{B}(\mathcal{H})$. The proof in the case where Φ is an elementary operator on $\mathbf{K}(\mathcal{H})$ is similar.

Suppose that $\mathrm{s}(\Phi) \in \mathfrak{i}$. Let $\Phi = \sum_{i=1}^{m} M_{A_i,B_i}$ be a representation of Φ where m is minimal. Then A_i (respectively B_i), $i = 1, \dots, m$, are linearly independent. Since $\mathrm{h}(\Phi) \leqslant \mathrm{s}(\Phi)$ we have that $\mathrm{h}(\Phi) \in \mathfrak{i}$. By Proposition 4.5, $s(A_i), s(B_i) \in \mathfrak{i}$ for every $i = 1, \dots, m$.

Conversely, suppose that $\Phi = \sum_{i=1}^{m} M_{A_i,B_i}$ where $s(A_i), s(B_i) \in \mathfrak{i}$ for every $i = 1, \dots, m$. Since $\mathfrak{i}$ is tensor stable, Corollary 4.4 implies that $\mathrm{a}(M_{A_i,B_i}) \in \mathfrak{i}$. By the additivity of the approximation numbers, $\mathrm{a}(\Phi) \in \mathfrak{i}$ and so $\mathrm{s}(\Phi) \in \mathfrak{i}$. ∎

Theorem 4.4 provides an upper bound for the the approximation numbers of $M_{A,B}$ in terms of the sequence $s(A) \otimes s(B)$. In the following proposition we obtain a lower bound for the Hilbert numbers of $M_{A,B}$ in terms of the sequence

$s(A)\otimes s(B)$. For $1\leqslant p<\infty$ we denote by $(\mathcal{S}_p,\|\cdot\|_p)$ the Schatten p-class, that is, the space of all operators $A\in\mathbf{K}(\mathcal{H})$ such that $s(A)\in\ell^p$, where the norm is given by $\|A\|_p=\Big(\sum\limits_{n=1}^{\infty}|s(A)|^p\Big)^{1/p}$. If $\boldsymbol{\alpha}=(\alpha_n)_{n=1}^{\infty}$ and $\boldsymbol{\beta}=(\beta_n)_{n=1}^{\infty}$ are sequences of complex numbers we denote by $\boldsymbol{\alpha\beta}$ the sequence $(\alpha_n\beta_n)_{n=1}^{\infty}$.

PROPOSITION 4.7. *Let $A,B\in\mathbf{K}(\mathcal{H})$. The following hold:*

(i) *If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are sequences of unit norm in ℓ_4^+ then* $\mathrm{h}(M_{A,B})\geqslant(\boldsymbol{\lambda}s(A))\otimes(\boldsymbol{\mu}s(B))$.

(ii) *If $\boldsymbol{\lambda}$ is a sequence of unit norm in ℓ_2^+ then* $\mathrm{h}(M_{A,B})\geqslant(\boldsymbol{\lambda}s(A))\otimes s(B)$ *and* $\mathrm{h}(M_{A,B})\geqslant s(A)\otimes(\boldsymbol{\lambda}s(B))$.

In particular,

$$\mathrm{h}_n(M_{A,B})\geqslant\frac{(s(A)\otimes\ s(B))(n)}{\sqrt{n}},\quad n\in\mathbb{N}. \tag{4.4}$$

Proof. (i) Let $A,B\in\mathbf{K}(\mathcal{H})$ have norm one and $A^*=U|A^*|$ and $B=V|B|$ be the polar decompositions of A^* and B, respectively. Let $s(A)=(\alpha_n)_{n=1}^{\infty}$, $s(B)=(\beta_n)_{n=1}^{\infty}$ and

$$|A^*|=\sum_{i=1}^{\infty}\alpha_i e_i^*\otimes e_i\quad\text{and}\quad|B|=\sum_{j=1}^{\infty}\beta_j f_j^*\otimes f_j$$

be Schmidt expansions of $|A^*|$ and $|B|$, respectively. Let $\mathcal{K}$ be the closed subspace of $\mathcal{S}_2$ spanned by the family $\{f_i^*\otimes e_j,i,j\}$ and $F:\mathcal{K}\to\mathbf{B}(\mathcal{H})$ be the map given by $F(X)=UXV^*$. Clearly, $\|F\|\leqslant 1$.

Consider sequences $\boldsymbol{\lambda}=(\lambda_i),\boldsymbol{\mu}=(\mu_j)\in\ell_4^+$ of unit norm and let $D_{\boldsymbol{\lambda}},D_{\boldsymbol{\mu}}\in\mathbf{B}(\mathcal{H})$ be the operators given by

$$D_{\boldsymbol{\lambda}}=\sum_{i=1}^{\infty}\lambda_i e_i^*\otimes e_i,\quad D_{\boldsymbol{\mu}}=\sum_{j=1}^{\infty}\mu_j f_j^*\otimes f_j.$$

Let $G:\mathbf{B}(\mathcal{H})\to\mathcal{K}$ be the operator given by $G(Y)=D_{\boldsymbol{\lambda}}YD_{\boldsymbol{\mu}}$. Since

$$\|D_{\boldsymbol{\lambda}}YD_{\boldsymbol{\mu}}\|_2\leqslant\|D_{\boldsymbol{\lambda}}\|_4\|D_{\boldsymbol{\mu}}\|_4\|Y\|\leqslant\|Y\|$$

the operator G is well defined and $\|G\|\leqslant 1$. The family $\{f_i^*\otimes e_j,i,j\}$ is an orthonormal basis of $\mathcal{K}$ and

$$(G\circ M_{A,B}\circ F)(f_i^*\otimes e_j)=\lambda_j\alpha_j\mu_i\beta_i f_i^*\otimes e_j.$$

It follows that

$$\mathrm{h}_n(M_{A,B})\geqslant s_n(G\circ M_{A,B}\circ F)=(\boldsymbol{\lambda}s(A)\otimes\boldsymbol{\mu}s(B))(n)$$

and (i) is proved. The proof of (ii) is similar.

We show inequality (4.4). Let $s(A)\otimes s(B)=(\nu_n)_{n\in\mathbb{N}}$ and $\pi:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$, $\pi(n)=(i_n,j_n)$ be a bijection such that $\nu_n=\alpha_{i_n}\beta_{j_n}$. We set $\boldsymbol{\lambda}=(\lambda_i)_{i=1}^{\infty}$, $\boldsymbol{\mu}=$

$(\mu_j)_{j=1}^{\infty}$ where

$$\lambda_i = \begin{cases} \frac{1}{\sqrt[4]{n}} & \text{if } i \in \{i_1,\dots,i_n\}, \\ 0 & \text{if } i \notin \{i_1,\dots,i_n\}; \end{cases} \qquad \mu_j = \begin{cases} \frac{1}{\sqrt[4]{n}} & \text{if } i \in \{j_1,\dots,j_n\}, \\ 0 & \text{if } i \notin \{j_1,\dots,j_n\}. \end{cases}$$

We have that $(\boldsymbol{\lambda} s(A) \otimes \boldsymbol{\mu} s(B))(k) = \dfrac{1}{\sqrt{n}}\nu_k$ for every $k = 1,\dots,n$, and so $\mathrm{h}_n(M_{A,B}) \geqslant \frac{1}{\sqrt{n}}\nu_n$. ∎

It follows from Theorem 4.6 that if the s-numbers of the symbols of an elementary operator Φ belong to a tensor stable Calkin space $\mathfrak{i}$ then the s-numbers of Φ also belong to $\mathfrak{i}$. In what follows we show that this is not true without the assumption that $\mathfrak{i}$ is tensor stable.

PROPOSITION 4.8. *Let $\vartheta \in (0,1)$ and $\mathfrak{i}$ be the principal Calkin space generated by the sequence $\boldsymbol{\vartheta} = (\vartheta^{n-1})_{n=1}^{\infty}$. Then there exists $A \in \mathbf{B}(\mathcal{H})$ such that $s(A) \in \mathfrak{i}$ and $\mathrm{h}(M_{A,A}) \notin \mathfrak{i}$.*

Proof. Let $A \in \mathbf{B}(\mathcal{H})$ be such that $s(A) = \boldsymbol{\vartheta}$. We will show that $\mathrm{h}(M_{A,A}) \notin \mathfrak{i}$. By Proposition 4.7 it suffices to show that the sequence $\boldsymbol{\alpha} = \big(\frac{1}{n}(\boldsymbol{\vartheta} \otimes \boldsymbol{\vartheta})(n)\big)_{n=1}^{\infty}$ does not belong to $\mathfrak{i}$, or (by Lemma 2.7) that for every $r \in \mathbb{N}$, $\boldsymbol{\alpha} \not\lesssim \mathbf{r} \otimes \boldsymbol{\vartheta}$.

Suppose that there exist $r_0 \in \mathbb{N}$ and $C > 0$ such that $\boldsymbol{\alpha} \leqslant C\mathbf{r_0} \otimes \boldsymbol{\vartheta}$. Let $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^{\infty}$ and $\mathbf{r}_0 \otimes \boldsymbol{\vartheta} = (\beta_n)_{n=1}^{\infty}$. Then for every m we have that

$$\beta_{r_0 m} = \vartheta^{m-1} \quad \text{and} \quad \alpha_{m(m+1)/2} = \frac{2}{m(m+1)}\vartheta^{m-1}.$$

So, if r is an even positive integer and $n(r) = \frac{rr_0(rr_0+1)}{2}$ we have that

$$\frac{2}{rr_0(rr_0+1)}\vartheta^{rr_0-1} = \alpha_{n(r)} \leqslant C\beta_{n(r)} = C\vartheta^{r(r_0 r+1)/2-1},$$

which leads to a contradiction. ∎

5. ELEMENTARY OPERATORS ON C^*-ALGEBRAS

Let $\mathcal{A}$ be a C^*-algebra. Recall that an element $a \in \mathcal{A}$ is called *compact* if the operator $M_{a,a} : \mathcal{A} \to \mathcal{A}$ is compact. We denote by $\mathcal{K}(\mathcal{A})$ the closed two-sided ideal of all compact elements of $\mathcal{A}$. The *spectrum* of $\mathcal{A}$ is the set of unitary equivalence classes of non-zero irreducible representations of $\mathcal{A}$. We will need two lemmas which follow from Section 5.5 of [20].

LEMMA 5.1. *Let $(\rho, \mathcal{H}) = (\bigoplus_{i\in I}\rho_i, \bigoplus_{i\in I}\mathcal{H}_i)$ be the reduced atomic representation of $\mathcal{A}$ where $\{(\rho_i, \mathcal{H}_i), i \in I\}$ is a maximal family of unitarily inequivalent irreducible representations of $\mathcal{A}$. Let $J = \{i \in I : \rho_i(\mathcal{K}(\mathcal{A})) \neq \{0\}\}$. Let σ_i be the restriction of ρ_i to $\mathcal{K}(\mathcal{A})$. Then the representation $\sigma = (\bigoplus_{i\in J}\sigma_i, \bigoplus_{i\in J}\mathcal{H}_i)$ is the reduced atomic representation of $\mathcal{K}(\mathcal{A})$.*

LEMMA 5.2. *Let $\mathcal{A}$ be a C^*-algebra such that $\mathcal{A} = \mathcal{K}(\mathcal{A})$ and $\sigma = (\bigoplus_{i\in J}\sigma_i, \bigoplus_{i\in J}\mathcal{H}_i)$ be the reduced atomic representation of $\mathcal{A}$. Then $\mathcal{A}$ has finite spectrum if and only if J is finite. In this case, $\sigma(\mathcal{A}) = \sum\limits_{i\in J}\mathbf{K}(\mathcal{H}_i)$.*

THEOREM 5.3. *Let $\mathcal{A}$ be a C^*-algebra, $\mathfrak{i}$ be a tensor stable Calkin space and s be an s-function. Let Φ be a compact elementary operator on $\mathcal{A}$.*

(i) *Suppose that*

$$\Phi = \sum_{i=1}^{m} M_{a_i,b_i} \quad a_i, b_i \in \mathcal{A},\ i = 1,\dots,m, \tag{5.1}$$

and that π is a faithful representation of $\mathcal{A}$ such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}, i = 1,\dots,m$. Then $\mathrm{s}(\Phi) \in \mathfrak{i}$.

(ii) *Suppose that $\mathcal{K}(\mathcal{A})$ has finite spectrum and that $\mathrm{s}(\Phi) \in \mathfrak{i}$. Then there exist a representation $\sum\limits_{i=1}^{m} M_{a_i,b_i}, a_i, b_i \in \mathcal{A},\ i = 1,\dots,m$, of Φ and a faithful representation π of $\mathcal{A}$ such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}$, $i = 1,\dots,m$.*

Proof. (i) Since $\mathrm{s}_n(\Phi) \leqslant \mathrm{a}_n(\Phi)$ for each n, it suffices to show that $\mathrm{a}(\Phi) \in \mathfrak{i}$. By the additivity of the approximation numbers we have that $\mathrm{a}_{nm-m+1}(\Phi) \leqslant \sum\limits_{i=1}^{m} \mathrm{a}_n(M_{a_i,b_i})$. If $\mathrm{a}(M_{a_i,b_i}) \in \mathfrak{i}$ for each $i = 1,\dots,m$, Lemma 2.4 implies that $\mathrm{a}(\Phi) \in \mathfrak{i}$. Thus, we may assume that $\Phi = M_{a,b}$, where $a, b \in \mathcal{A}$.

Let $\pi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ be a faithful representation such that $s(\pi(a)), s(\pi(b)) \in \mathfrak{i}$. Set $A = \pi(a)$ and $B = \pi(b)$. We denote by $M_{A,B}$ the corresponding elementary operator acting on $\pi(\mathcal{A})$. Clearly, A and B are compact operators and $\mathrm{a}(\Phi) = \mathrm{a}(M_{A,B})$. Let $\mathcal{J} = \pi(\mathcal{A}) \cap \mathbf{K}(\mathcal{H})$. By Theorem 3.5,

$$\mathrm{a}_n(M_{A,B}) \leqslant \mathrm{a}_n(M^{\mathcal{J}}_{A,B}), \quad \text{for every } n \in \mathbb{N}.$$

Let $\mathcal{H}_0$ the closure of $\mathcal{J}\mathcal{H}$. Then there exist positive integers $\{m_i\}_{i\in I}$ and Hilbert spaces $\{H_i\}_{i\in I}$ such that:

(i) $\mathcal{H}_0 = \sum\limits_{i\in I}(\underbrace{\mathcal{H}_i \oplus \cdots \oplus \mathcal{H}_i}_{m_i \text{ times}})$.

(ii) the C^*-algebra $\mathcal{J}$ is equal to a c_0-direct sum $\bigoplus\limits_{i\in I}(\mathbb{C}I_{m_i} \otimes \mathbf{K}(\mathcal{H}_i))$ where I_{m_i} is the the identity operator on a Hilbert space of dimension m_i ([3], Section 1.4).

Let $\Theta : \mathcal{J} \to \mathbf{K}(\mathcal{H}_0)$ be the canonical injection. Let P_i be the orthogonal projection from $\mathcal{H}_0$ onto $(\underbrace{\mathcal{H}_i \oplus \cdots \oplus \mathcal{H}_i}_{m_i \text{ times}})$ and $\Delta_1 : \mathbf{K}(\mathcal{H}_0) \to \sum\limits_{i\in I} P_i\mathbf{K}(\mathcal{H}_0)P_i$ the operator given by $\Delta_1(X) = \sum\limits_{i\in I} P_iXP_i$.

An element $Y \in \sum\limits_{i\in I} P_i\mathbf{K}(\mathcal{H}_0)P_i$ may be written as

$$Y = \sum_{i\in I} Y^i$$

where $Y^i \in P_i\mathbf{K}(\mathcal{H}_0)P_i$ is an $m_i \times m_i$ matrix $Y^i_{p,q}$ with coefficients in $\mathbf{K}(\mathcal{H}_i)$. Let $\Delta_2 : \sum_{i\in I} P_i\mathbf{K}(\mathcal{H}_0)P_i \to \mathcal{J}$ be the operator defined as follows: If $Y^i \in P_i\mathbf{K}(\mathcal{H}_0)P_i$, then $\Delta_2(Y^i)$ is the diagonal $m_i \times m_i$ matrix with all its diagonal entries equal to $\frac{1}{m_i}\sum_{p=1}^{m_i} Y^i_{p,p}$. Set $\Delta = \Delta_2\Delta_1$. We have that $M^{\mathcal{J}}_{A,B} = \Delta \circ M^{\mathbf{K}(\mathcal{H}_0)}_{A,B} \circ \Theta$ where $M^{\mathbf{K}(\mathcal{H}_0)}_{A,B}$ is the corresponding elementary operator acting on $\mathbf{K}(\mathcal{H}_0)$. Clearly, Δ is a contraction. Thus,

$$\mathrm{a}_n(M^{\mathcal{J}}_{A,B}) \leqslant \|\Delta\|\ \mathrm{a}_n(M^{\mathbf{K}(\mathcal{H})}_{A,B})\ \|\Theta\| \leqslant \mathrm{a}_n(M^{\mathbf{K}(\mathcal{H})}_{A,B}).$$

By Corollary 4.4, $\mathrm{a}(\Phi) \in \mathfrak{i}$.

(ii) We identify $\mathcal{A}$ with $\rho(\mathcal{A})$ where $(\rho, \mathcal{H}) = (\bigoplus_{i\in I}\rho_i, \bigoplus_{i\in I}\mathcal{H}_i)$ is the reduced atomic representation of $\mathcal{A}$. By Theorem 3.1 of [25], there exist $A_{0j}, B_{0j} \in \mathcal{K}(\mathcal{A})$, $j = 1, \dots, m$, such that $\Phi = \sum_{j=1}^{m} M_{A_{0j},B_{0j}}$. Since $\mathrm{h}_n(\Phi) \leqslant \mathrm{s}_n(\Phi)$ for each n, we may assume that $\mathrm{s} = \mathrm{h}$. Let $\Phi_0 : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ be the operator defined by $\Phi_0(X) = \Phi(X)$. By Lemma 3.3, $\mathrm{h}(\Phi_0) \in \mathfrak{i}$. Consequently, the C^*-algebra $\mathcal{K}(\mathcal{A})$ and the operator Φ_0 satisfy our assumptions. Thus we may assume that $\mathcal{A} = \mathcal{K}(\mathcal{A})$.

By Lemmas 5.1 and 5.2, $\mathcal{K}(\mathcal{A}) = \bigoplus_{i\in I_0} \mathbf{K}(\mathcal{H}_i)$ where I_0 is a finite subset of I. Let $i \in I_0$. Clearly, $\mathbf{K}(\mathcal{H}_i)$ is invariant by Φ. Let $\Phi_i : \mathbf{K}(\mathcal{H}_i) \to \mathbf{K}(\mathcal{H}_i)$ be the operator defined by $\Phi_i(X) = \Phi(X)$. The operator Φ_i is an elementary operator on $\mathbf{K}(\mathcal{H}_i)$. By Theorem 3.5, $\mathrm{h}(\Phi_i) \in \mathfrak{i}$. By Theorem 4.6, there exists a representation $\sum_{j=1}^{m_i} M_{A_{ij},B_{ij}}$ of Φ_i where $A_{ij}, B_{ij} \in \mathbf{K}(\mathcal{H}_i)$ and $s(A_{ij}), s(B_{ij}) \in \mathfrak{i}$. Considering A_{ij} and B_{ij} as operators on $\mathcal{H}$ we obtain that $\Phi = \sum_{i=1}^{k}\sum_{j=1}^{m_i} M_{A_{ij},B_{ij}}$ is a representation with the required properties. ∎

Part (ii) of Theorem 5.3 does not hold if we do not assume that $\mathcal{K}(\mathcal{A})$ has finite spectrum. In fact, we have the following:

THEOREM 5.4. *Let $\mathcal{A}$ be a C^*-algebra. The following are equivalent:*

(i) *$\mathcal{K}(\mathcal{A})$ has finite spectrum.*

(ii) *Let s be an s-function, $\mathfrak{i}$ be a tensor stable Calkin space and Φ be a compact elementary operator on $\mathcal{A}$. Assume that $\mathrm{s}(\Phi) \in \mathfrak{i}$. Then there exist a representation $\sum_{i=1}^{m} M_{a_i,b_i}$ of Φ and a faithful representation π of $\mathcal{A}$ such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}$ for every $i = 1, \dots, m$.*

Proof. The implication (i) ⇒ (ii) follows from Theorem 5.3. We prove that (ii) implies (i). Suppose that $\mathcal{K}(\mathcal{A}) \neq \{0\}$ and that $\mathcal{K}(\mathcal{A})$ does not have finite spectrum. We will show that for every $p > 2$ there exists an elementary operator Φ on $\mathcal{A}$ such that:

(a) $\mathrm{a}(\Phi) \in \ell_p$, and

(b) whenever π is a faithful representation of $\mathcal{A}$ and $\Phi = \sum_{i=1}^{m} M_{c_i,d_i}$, $c_i, d_i \in \mathcal{K}(\mathcal{A})$, there exists i, $1 \leqslant i \leqslant m$, such that $s(\pi(c_i)) \notin \ell_p$ or $s(\pi(d_i)) \notin \ell_p$.

Let σ be the reduced atomic representation of $\mathcal{K}(\mathcal{A})$. Then

$$\sigma(\mathcal{K}(\mathcal{A})) = \bigoplus_{j\in J} \mathbf{K}(\mathcal{H}_j).$$

It follows from Lemma 5.2 that J is infinite. Choose an infinite countable subfamily $\{\mathcal{H}_j\}_{j=1}^{\infty}$ of the family J. For each $j \in \mathbb{N}$, consider a unit vector $e_j \in \mathcal{H}_j$.

Let r_j be the projection of $\mathcal{K}(\mathcal{A})$ such that $\sigma(r_j){=}e_j^*\otimes e_j$ and $(\lambda_j)_{j=1}^{\infty}$ be a decreasing sequence of positive real numbers belonging to ℓ_{2p} but not to ℓ_p. We set

$$c = \sum_{j=1}^{\infty} \lambda_j r_j, p_k = \sum_{j=1}^{k} r_j \quad \text{and} \quad \Phi = M_{c,c} \in \mathbf{B}(\mathcal{A}).$$

We will show that $\mathrm{a}(\Phi) \in \ell_p$.

Let ρ be the reduced atomic representation of $\mathcal{A}$. Let $c_n = \sum_{i=1}^{n} \lambda_i r_i$. It follows from Lemma 5.1 that

$$M_{\rho(c_n),\rho(c_n)}(\rho(a)) = \sum_{i=1}^{n} \sigma_i(r_i)\rho_i(a)\sigma_i(r_i)$$

and hence the operator $M_{\rho(c_n),\rho(c_n)}$ is an operator of rank n. It also follows from Lemma 5.1 that $M_{\rho(c),\rho(c)} - M_{\rho(c_n),\rho(c_n)} = M_{\rho(c-c_n),\rho(c-c_n)}$. Hence,

$$\mathrm{a}_n(M_{c,c}) = \mathrm{a}_n(M_{\rho(c),\rho(c)}) \leqslant \|\rho(c - c_{n-1})\|^2 \leqslant \lambda_n^2,$$

and so $\mathrm{a}(\Phi) \in \ell_p$.

Assume that there exist a faithful representation $\pi\colon \mathcal{A} \to \mathbf{B}(\mathcal{H})$ and elements $a_i, b_i \in \mathcal{K}(\mathcal{A})$ for $i = 1, \dots, m$, such that $s(\pi(a_i))$, $s(\pi(b_i)) \in \ell_p, i = 1, \dots, m$, and $\Phi = \sum_{i=1}^{m} M_{a_i,b_i}$. We have $\Phi(p_k) = cp_kc = \sum_{i=1}^{m} a_i p_k b_i$. Hence $\pi(c)\pi(p_k)\pi(c) = \sum_{i=1}^{m} \pi(a_i)\pi(p_k)\pi(b_i)$ and by continuity

$$\pi(c)P\pi(c) = \sum_{i=1}^{m} \pi(a_i)P\pi(b_i), \tag{5.2}$$

where $P = \sum_{j=1}^{\infty} \pi(r_j)$ is the sot-limit of the sequence $(\pi(p_k))_{k=1}^{\infty}$.

It follows from (5.2) that $\pi(c)P\pi(c) \in \mathcal{S}_{p/2}$. On the other hand,

$$\pi(c)P\pi(c) = \sum_{j=1}^{\infty} \lambda_j^2 \pi(r_j).$$

It follows that $(\lambda_j^2) \in \ell_{p/2}$ and so $(\lambda_j) \in \ell_p$, a contradiction. ∎

We note the following corollary of Theorem 5.3.

COROLLARY 5.5. *Let $\mathcal{A}$ be a C^*-algebra such that $\mathcal{K}(\mathcal{A})$ has finite spectrum, $\mathfrak{i}$ be a tensor stable Calkin space and s be an s-function. Let Φ be an elementary operator on $\mathcal{A}$ such that $\mathrm{s}(\Phi) \in \mathfrak{i}$. Then Φ is a linear combination of positive elementary operators Φ_j, $j = 1,2,3,4$ such that, $\mathrm{s}(\Phi_j) \in \mathfrak{i}$ for every $j = 1,2,3,4$.*

Proof. By assertion (ii) of Theorem 5.3, there exist a representation $\sum_{i=1}^{m} M_{a_i,b_i}$, $a_i, b_i \in \mathcal{A}$, $i = 1,\dots,m$, of Φ and a faithful representation π of $\mathcal{A}$ such that $s(\pi(a_i)), s(\pi(b_i)) \in \mathfrak{i}$, $i = 1,\dots,m$. Let $\Phi^{\pm}(x) = \frac{1}{4}\sum_{i=1}^{m}(a_i \pm b_i^*)x(a_i^* \pm b_i)$ and $\Psi^{\pm}(x) = \frac{1}{4}\sum_{i=1}^{m}(a_i \pm ib_i^*)x(a_i^* \mp ib_i)$. Clearly, all operators $\Phi^{\pm}$, $\Psi^{\pm}$ are positive. By assertion (i) of Theorem 5.3, $\mathrm{s}(\Phi^{\pm}), \mathrm{s}(\Psi^{\pm}) \in \mathfrak{i}$. A straightforward verification shows that $\Phi = \Phi^+ - \Phi^- + i(\Psi^+ - \Psi^-)$. The proof is complete. ∎

We close this section by proving a result which may be viewed as a quantitative version of a result of Ylinen [27].

THEOREM 5.6. *Let $\mathcal{A}$ be a C^*-algebra, $a \in \mathcal{A}$ and $\mathfrak{i}$ be a Calkin space. Assume that $\mathrm{d}(M_{a,a}) \in \mathfrak{i}$. Then $s(\rho(a))^2 \in \mathfrak{i}$ where $(\rho, \mathcal{H})$ is the reduced atomic representation of $\mathcal{A}$.*

Proof. Since $\mathrm{d}(M_{a,a}) \in \mathfrak{i}$, the operator $M_{a,a}$ is compact and it follows from [27] that $\rho(a)$ is compact.

Let $(\rho, \mathcal{H}) = (\bigoplus_{i\in I}\rho_i, \bigoplus_{i\in I}\mathcal{H}_i)$. Set $\mathcal{C} = \bigoplus_{i\in I}\mathbf{B}(\mathcal{H}_i)$.

Let $\Phi : \mathcal{C} \to \mathcal{C}$ be the operator defined by $\Phi(X) = \rho(a)X\rho(a)$. Since $\overline{\rho(\mathcal{A})}^{\mathrm{wot}} = \mathcal{C}$, Lemma 3.6 implies that $\mathrm{d}(\Phi) \leqslant \mathrm{d}(M_{\rho(a),\rho(a)})$ and so $\mathrm{d}(\Phi) \in \mathfrak{i}$.

Let $\rho(a) = UA$ be the polar decomposition of $\rho(a)$ and $A = \sum_{k=1}^{\infty}\lambda_k e_k^* \otimes e_k$ be a Schmidt expansion of A. Define

$$\alpha : \ell_\infty \to \mathcal{C} \quad \text{by } \alpha((x_l)_{l=1}^{\infty}) = \sum_{l=1}^{\infty} x_l e_l^* \otimes e_l, \text{ and}$$
$$\beta : \mathcal{C} \to \ell_\infty \quad \text{by } \beta(X) = (\langle Xe_l, e_l\rangle)_{l=1}^{\infty}.$$

Consider the map $\Psi : \ell_\infty \to \ell_\infty$ defined by

$$\Psi((x_l)_{l=1}^{\infty}) = \beta(U^*\Phi(\alpha((x_l)_{l=1}^{\infty})U^*)).$$

Since α and β are contractions we have $\mathrm{d}(\Psi) \leqslant \mathrm{d}(\Phi)$ and so $\mathrm{d}(\Psi) \in \mathfrak{i}$. A direct calculation shows that $\Psi((x_l)_{l=1}^{\infty}) = (\lambda_l^2 x_l)_{l=1}^{\infty}$. It follows ([21], Theorem 11.11.3) that $\mathrm{d}(\Psi) = (\lambda_l^2)_{l=1}^{\infty}$. Hence, $s(A)^2 \in \mathfrak{i}$. ∎

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