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DIFFERENTIAL STRUCTURES IN C^*-ALGEBRAS

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ABSTRACT. The paper elaborates the general approach to the differential structure of C^*-algebras proposed by Blackadar and Cuntz [14]. The smoothness properties of differential Fréchet algebras defined by (not necessarily ℓ^1-summable) differential norms are investigated. They are used, by taking appropriate projective limits and inductive limits, to construct and investigate classes of non-commutative smooth algebras describing differential structures defined by differential norms. A large number of examples are discussed exhibiting unified nature of general theory.

KEYWORDS: *Differential seminorms, derived norms, (strongly) smooth algebras, C^k-completion, C^∞-algebras.*

MSC (2000): 46L80, 46L85, 46L87, 46L89.

1. INTRODCUTION

The present paper contributes to understanding the differential structure in a C^*-algebra $\mathcal{A}$. A fairly general approach based on differential seminorms has been proposed and developed by Blackadar and Cuntz in [14], and the present paper grew out of our efforts to understand [14]. We believe that [14] is an important paper in view of its generality and richness of its results; and its potential has not been fully explored yet. The present paper is a step in this direction. Another general approach related with the one in [14] and based on growth conditions on defining seminorms has been simultaneously proposed in [21]. A differential structure in a C^*-algebra $\mathcal{A}$ is generally specified by specifying an appropriate dense $*$-subalgebra $\mathcal{B}$. Ideally one should have a theorem giving a functional analytic characterization of the subalgebra $C^\infty(M)$ of smooth functions on a (compact) manifold M in the commutative C^*-algebra $C(M)$ of continuous functions on M. Then a non-abelianization of the statement of such a theorem would suggest an appropriate definition of a smooth subalgebra $\mathcal{B}$ of $\mathcal{A}$. This then would lead to non-abelian analogues of various classes of algebras of smooth functions. In the absence of such a theorem, a smooth subalgebra $\mathcal{B}$ is expected to satisfy adhoc smoothness properties like spectral invariance, K-theory isomorphism, closure under appropriate functional calculi and completeness under a suitable locally convex topology. Let us note that recently Alain Connes has obtained a spectral characterization of $C^\infty(M)$ in terms of spectral triple (a non-commutative geometric data) providing a first major step towards such a theorem [18]. Following [14], a differential seminorm on a dense subalgebra $\mathfrak{A}$ of a C^*-algebra $\mathcal{A}$ is a seminorm T from $\mathfrak{A}$ into the space of all ℓ^1-summable non-negative real sequences which is submultiplicative with respect to convolution and whose degree zero part is continuous in the C^*-norm. A derived seminorm on $\mathfrak{A}$ is the quotient of the total seminorm of a differential seminorm. The completion of $\mathfrak{A}$ in the collection of all closable (with respect to C^*-norm) derived norms defines the smooth envelope $\mathcal{S}\mathfrak{A}$; and $\mathfrak{A}$ is smooth if $\mathfrak{A} = \mathcal{S}\mathfrak{A}$. The content of [14] is that a smooth algebra has desired smoothness properties.

In the present paper, we do not require a differential seminorm to be necessarily ℓ^1-summable; and do require it to be involutive. This natural added generality would allow us to develop the Banach $*$-algebras $\mathfrak{A}_{(k)} = C^k(\mathfrak{A}, T)$ and the Fréchet $*$-subalgebra $\mathfrak{A}_\tau = C^\infty(\mathfrak{A}, T)$ of $\mathcal{A}$ which are no-commutative C^k-algebras containing $\mathfrak{A}$ and defined by T. In Section 3, we discuss the smoothness properties of these algebras. In particular, it is shown that they are hermitian Q-algebras spectrally invariant in $\mathcal{A}$ and closed under the holomorphic functional calculus of $\mathcal{A}$ as well as C^∞-functional calculus for selfadjoint elements of $\mathcal{A}$. The spectral invariance of a dense subalgebra $\mathcal{B}$ in $\mathcal{A}$ has long been recognized as an important smoothness property [31] which is closely related with locality [29] and K-theory isomorphism [17]. This has led to the concept of spectral algebras [24], [26] and C^*-spectral algebras [8]. The basic properties of C^*-spectral algebras and of topological algebras [22] [19] that we shall need are summarized in Section 2. Furthermore, we develop the analytic structure and the entire analytic structure defined by a differential norm T. This leads to complete locally convex $*$-algebras $\mathfrak{A}_\tau^\omega = C^\omega(\mathfrak{A}, T)$ and $\mathfrak{A}_\tau^{e\omega} = C_\tau^{e\omega}(\mathfrak{A}, T)$ obtained respectively by the inductive and projective limits of a sequence of analytic Banach $*$-algebras in $C^\infty(\mathfrak{A}, T)$. On the other hand, analytic elements and entire analytic elements of $\mathfrak{A}$ give rise, by completions, to the algebras $\widetilde{\mathfrak{A}^\omega}(T)$ and $\widetilde{\mathfrak{A}^{e\omega}}(T)$ contained respectively in $C^\omega(\mathfrak{A}, T)$ and $C^{e\omega}(\mathfrak{A}, T)$. The basic properties of these algebras are discussed in Section 3. Hereafter we take limits of these locally convex $*$-algebras. There are projective limits and inductive limits taken over suitable families of differential norms and derived norms. We introduce strong analogue of smooth envelope by taking projective limit over all closable differential norms. Section 4 is devoted to smooth algebras as well as strongly smooth algebras. Furthermore, the analytic structure defined by $\mathfrak{A}$ in $\mathcal{A}$ is discussed by taking appropriate limits. Though the resulting algebras exhibit desired smoothness properties, their analyticity is yet to be explored. Permanence properties, smoothness properties and universality of smooth algebras are discussed refining and elaborating the results in [14]. We also show that the algebra $C^\infty(\mathfrak{A}, T)$ is a strongly smooth algebra. In Section 5, we consider the C^k structures for which there are at least two approaches. One initiated in [14] is based on flat differential norms (i.e. those defined by derivations); and this in turn leads us to flat differential norm approach to analyticity (different from the one discussed above). We develop another approach to the C^k structures based on the derived norms of total order $\leqslant k$ which is along the lines of the approach to smooth algebras. This gives rise to C^∞-envelope $C^\infty\mathfrak{A}$. The resulting classes of smooth algebras are also discussed. Finally in Section 6, we discuss examples with a view to unify them in the framework of the present theory. The classical function algebras of smooth functions like $C^k[0,1]$, $C_0^k(\mathbb{R})$ and $S(\mathbb{R})$ (Schwartz space) as well as $CBV[0,1]$ (continuous functions of bounded variations) and Sobolev–Banach algebras $W^{m,p}[0,1]$ are smooth algebras. Given an action α of a Lie group G on a C^*-algebra $\mathcal{A}$, the locally convex algebra $C^\infty(\mathcal{A})$ of C^∞-elements of $\mathcal{A}$ [16] is a smooth topological algebra. In [30], Schweitzer has developed a smooth crossed product $S(G, \mathcal{A}, \alpha)$ which is a Fréchet subalgebra of C^*-crossed product $C^*(G, \mathcal{A}, \alpha)$. We show that when $G = \mathbb{R}$, $S(\mathbb{R}, \mathcal{A}, \alpha)$ is indeed a smooth Fréchet subalgebra of $C^*(\mathbb{R}, \mathcal{A}, \alpha)$. The general case $S(G, \mathcal{A}, \alpha)$ remains to be explored. In particular, the non-commutative torus T_θ^2 is a smooth Fréchet algebra. In the developement of K-theory for Fréchet algebras [27], N.C. Phillips has introduced smooth compact operators K_∞ on the Hilbert space $l^2(\mathbb{Z})$ for what we show to be smooth subalgebra of compact operators K. A non-commutative geometric data specified by either a Fredholm module or a K-cycle on a C^*-normed algebra gives rise to a smooth subalgebra of its C^*-completion. We also exhibit that the smooth structure defined by a pair of bounded self-adjoint operators having a trace class commutator discussed in [20] is indeed a smooth algebra in the sense of the present paper.

2. PRELIMINARIES

A *locally convex* $*$*-algebra* is a $*$-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous in both variables and the involution is continuous. A *Fréchet* $*$*-algebra* is a locally convex $*$-algebra which is a Fréchet space, and the multiplication of a Fréchet $*$-algebra turns out to be jointly continuous. The investigation of locally convex $*$-algebras has begun with those of locally m-convex $*$-algebras [2], [23]. For general theory of topological algebras, we refer to [22], whereas [19] is a recent reference on topological algebras with involution. A locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be *m-convex* if the topology τ is determined by a directed family $\{p_\lambda\}_{\lambda\in\Lambda}$ of algebra $*$-seminorms (that is, $p_\lambda(xy) \leqslant p_\lambda(x)p_\lambda(y)$ and $p_\lambda(x^*) = p_\lambda(x), \forall\lambda \in \Lambda, \forall x, y \in \mathcal{A}$). It is well-known that every complete locally m-convex $*$-algebra is a projective limit of Banach $*$-algebras [23]. Let $\mathcal{A}$ be a locally convex $*$-algebra with identity 1. We define $Sp_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : \nexists(\lambda 1 - x)^{-1} \text{ in } \mathcal{A}\}, r_{\mathcal{A}}(x) = \sup\{|\lambda| : \lambda \in Sp_{\mathcal{A}}(x)\}$. An algebra $*$-seminorm p on $\mathcal{A}$ is said to be *spectral* ([25], [11]) if $\{x \in \mathcal{A} : p(x) < 1\}$ is contained in the set $\mathcal{A}^{\mathrm{qr}}$ of all quasi-regular elements x of $\mathcal{A}$ (i.e. elements x such that $1 - x$ is invertible in $\mathcal{A}$). An algebra $*$-seminorm p on $\mathcal{A}$ is spectral if and only if $r_{\mathcal{A}}(x) \leqslant p(x), \forall x \in \mathcal{A}$ if and only if $r_{\mathcal{A}}(x) = \lim_{n\to\infty} p(x^n)^{1/n}, \forall x \in \mathcal{A}$ if and only if $Sp_{\mathcal{A}}(x) = Sp_{(\mathcal{A}/\ker p)^\sim}(x + \ker p), \forall x \in \mathcal{A}$, where $(\mathcal{A}/\ker p)^\sim$ is the Banach $*$-algebra obtained by the completion of the quotient $*$-algebra $\mathcal{A}/\ker p$ with the norm $\|x + \ker p\|_p = p(x), x \in \mathcal{A}$. In particular, if p is a C^*-seminorm on $\mathcal{A}$, then p is spectral if and only if p equals the Pták function $p_{\mathcal{A}}$ defined by $p_{\mathcal{A}}(x) \equiv r_{\mathcal{A}}(x^*x)^{1/2}, x \in \mathcal{A}$. A locally convex $*$-algebra $\mathcal{A}$ is said to be *spectral (respectively C^*-spectral)* if there exists a non-zero continuous spectral algebra $*$-seminorm (respectively spectral C^*-seminorm) on $\mathcal{A}$. Spectral algebras have been discovered by Palmer [24] as a class of non-normed algebras with satisfactory spectral theory.

To understand the C^*-spectrality of $\mathcal{A}$, we have investigated ([3], 6–10) the notion of spectral invariance and locality of $\mathcal{A}$, spectrality of $*$-representations of $\mathcal{A}$ and stability of $\mathcal{A}$. We denote by CBRep$\mathcal{A}$ (or $R_{\mathrm{c}}(\mathcal{A})$) the family of all continuous bounded operator $*$-representations of $\mathcal{A}$. If CBRep$\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is said to be *representable*. We remark that even a Banach $*$-algebra is not necessarily representable ([15], Example 37.16). Suppose $\mathcal{A}$ is representable. We put $|x| = \sup\{\|\pi(x)\| : \pi \in \text{CBRep}\mathcal{A}\}, x \in \mathcal{A}$. Assume that $|x| < \infty$ for all $x \in \mathcal{A}$. Then $|\cdot|$ is the largest C^*-seminorm on $\mathcal{A}$, and it is called the *Gelfand–Naimark C^*-seminorm* on $\mathcal{A}$. The completion $(\mathcal{A}/\ker|\cdot|)^\sim$ of the quotient $*$-algebra $\mathcal{A}/\ker|\cdot|$ is said to be the *enveloping C^*-algebra* of $\mathcal{A}$ and denoted by $E(\mathcal{A})$. The natural map $j : x \to x + \ker|\cdot|$ of $\mathcal{A}$ into $E(\mathcal{A})$ is a $*$-homomorphism. We remark that in general $|\cdot|$ is not continuous, and so j is not continuous. Let us note that in the absence of the finiteness, there is a notion of *unbounded Gelfand–Naimark C^*-seminorm* leading to the notion of *unbounded C^*-spectral algebras* ([9], [11]). More generally, every representable locally convex $*$-algebra $\mathcal{A}[\tau]$ admits a projective limit of C^*-algebras as a solution of the universal problem for continuous bounded Hilbert space operator representations of $\mathcal{A}$. Let $\Gamma : \{p_\lambda\}_{\lambda\in\Lambda}$ be a family of $*$-seminorms defining τ. Let $P(\mathcal{A})$ be the set of all positive linear functionals on $\mathcal{A}$. For $p \in \Gamma$, let $P_p(\mathcal{A}) = \{f \in P(\mathcal{A}) : |f(x)| \leqslant p(x), \forall x \in \mathcal{A}\}$. Let $P_{\mathrm{c}}(\mathcal{A}) = \bigcup_{p\in\Gamma} P_p(\mathcal{A})$. Let $R_p(\mathcal{A}) = \{\pi \in \text{CBRep}\mathcal{A} : \|\pi(x)\| \leqslant p(x), \forall x \in \mathcal{A}\}$, $r_p(x) = \sup\{\|\pi(x)\| : \pi \in R_p(\mathcal{A})\}, x \in \mathcal{A}$. Then r_p is a continuous C^*-seminorm on $\mathcal{A}$ and $r_p(x) \leqslant p(x), \forall x \in \mathcal{A}$. The *enveloping pro-$C^*$-algebra* $E(\mathcal{A})$ of $\mathcal{A}$ is the Hausdorff completion of $(\mathcal{A}, \{r_p : p \in \Gamma\})$ [3]. In fact, [3] contains construction of an object in a more general class of O^*-algebras that is universal for unbounded operator representations. It is shown in [3] that if $\mathcal{A}$ is a Fréchet m-convex $*$-algebra, then $E(\mathcal{A})$ which by construction is universal for bounded operator representations turns out to be universal also for representations into unbounded operators. We next define the notions of the spectral invariance and the stability of $\mathcal{A}$, and the spectrality of continuous bounded $*$-representations of $\mathcal{A}$. If $\mathcal{A}$ is representable and $Sp_{\mathcal{A}}(x) = Sp_{E(\mathcal{A})}(j(x))$ for each $x \in \mathcal{A}$, then $\mathcal{A}$ is said to be *spectrally invariant*. If for any closed $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$, any continuous bounded $*$-representation π of $\mathcal{B}$ on a Hilbert space $\mathcal{H}_\pi$ admits a dilation to a continuous bounded $*$-representation $\widetilde{\pi}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\widetilde{\pi}}$ in the sense that there exist a Hilbert space $\mathcal{H}_{\widetilde{\pi}}$ containing $\mathcal{H}_\pi$ as a closed subspace and a continuous bounded $*$-representation $\widetilde{\pi}$ of $\mathcal{A}$ on $\mathcal{H}_{\widetilde{\pi}}$ such that $\pi(x) = \widetilde{\pi}(x)\lceil \mathcal{H}_\pi$ for each $x \in \mathcal{B}$, $\mathcal{A}$ is said to be *stable*. A continuous bounded $*$-representation π of $\mathcal{A}$ is said to be *spectral* if $Sp_{\mathcal{A}}(x) = Sp_{C^*(\pi)}(\pi(x))$ for each $x \in \mathcal{A}$, where $C^*(\pi)$ is the C^*-algebra generated by $\pi(\mathcal{A})$. The spectral invariance of $\mathcal{A}$ is characterized in Theorem 2.11 and Theorem 2.15 of [10] by the C^*-spectrality of $\mathcal{A}$, the spectrality of $*$-representations of $\mathcal{A}$ and the stability of $\mathcal{A}$ as follows: The following statements are equivalent:

(i) $\mathcal{A}$ is spectrally invariant and $|\cdot|$ is continuous.
(ii) $|\cdot|$ is a continuous spectral C^*-seminorm.
(iii) $\mathcal{A}$ is C^*-spectral.
(iv) The Pták function $p_{\mathcal{A}}$ is a continuous C^*-seminorm.
(v) There exists a spectral continuous bounded $*$-representation of $\mathcal{A}$.
(vi) $\mathcal{A}$ is spectral and stable.

If this is true, then $\mathcal{A}$ is a Q-algebra (that is, $\mathcal{A}^{\mathrm{qr}}$ is open) and hermitian (that is, $Sp_{\mathcal{A}}(x) \subset \mathbb{R}, \forall x^* = x \in \mathcal{A}$). The spectral invariance of a dense subalgebra $\mathfrak{A}$ of a C^*-algebra $\mathcal{A}$ has been recognized as an important smoothness property [31]; and is closely related with closure under holomorphic functional calculus as well as K-theory isomorphism $K_*(\mathfrak{A}) = K_*(\mathcal{A})$ [17]. At the above stated level of generality, they are discussed in [10], [4]. Following [10], $\mathcal{A}$ is *local* (or *closed under holomorphic functional calculus*) if for each x in $\mathcal{A}$ and for every function f holomorphic in a neighborhood of $Sp_{E(\mathcal{A})}(j(x))$, there exists an element y in $\mathcal{A}$ such that $f(j(x)) = j(y)$; i.e. $\mathcal{A}$ is closed under the holomorphic functional calculus of $E(\mathcal{A})$ via the canonical map j from $\mathcal{A}$ to $E(\mathcal{A})$ defined as $j(x) = x + \mathrm{srad}\mathcal{A}$, where srad$\mathcal{A}$ is the $*$-radical of $\mathcal{A}$. The following is proved in [10]: Let $\mathcal{A}$ be a complete locally convex $*$-algebra. The following are equivalent:

(i) $\mathcal{A}$ is spectrally invariant.
(ii) $\mathcal{A}$ is local and the radical rad$\mathcal{A}$ = $s$rad$\mathcal{A}$.
(iii) $\mathcal{A}$ is spectral and hermitian.

3. DIFFERENTIAL ALGEBRAS AND ANALYTICITY

In [10] we have investigated the properties of C^*-spectrality and spectral invariance of the Fréchet $*$-algebra defined by a differential seminorm, and have left an important open problem. In this section, we shall solve this problem. We shall also discuss C^k-algebras, $1 \leqslant k \leqslant \infty$, structure defined by a differential norm on a C^*-normed algebra $\mathfrak{A}$, as well as the analytic structure theory. Let $\mathfrak{A}[\|\cdot\|]$ be a unital C^*-normed algebra and $\mathcal{A} \equiv \widetilde{\mathfrak{A}}[\|\cdot\|]$ be the C^*-algebra constructed by the completion of $\mathfrak{A}[\|\cdot\|]$.

DEFINITION 3.1. A map T of $\mathfrak{A}$ into the set ω of all complex sequences is said to be a *differential seminorm* on $\mathfrak{A}$ if $T(x) = (T_k(x))_0^\infty \in \omega$ satisfies the following conditions:

(i) $T(x) \geqslant 0$, i.e. $T_k(x) \geqslant 0, \forall x \in \mathfrak{A}$ and $\forall k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.

(ii) $T(x+y) \leqslant T(x) + T(y), \forall x, y \in \mathfrak{A}$.

(iii) $T(\lambda x) = |\lambda| T(x), \forall x \in \mathfrak{A}$ and $\forall \lambda \in \mathbb{C}$.

(iv) $T(xy) \leqslant T(x)T(y)$ (convolution), $\forall x, y \in \mathfrak{A}$; i.e., $\forall k \in \mathbb{N}_0$, we have $T_k(xy) \leqslant \sum_{i+j=k} T_i(x)T_j(y)$.

(v) There exists $c > 0$ such that $T_0(x) \leqslant c\|x\|, \forall x \in \mathfrak{A}$.

We additionally assume that

(vi) $T_k(x^*) = T_k(x)$ for $\forall x \in \mathfrak{A}$ and $\forall k \in \mathbb{N}_0$.

Let us note that the following apparently weaker condition is considered in [21] in place of (iv).

(vi)′ There exist $K_{i,j} > 0, K_{0,0} = 1$ such that $T_k(xy) \leqslant \sum_{i+j=k} K_{i,j} T_i(x) T_j(y)$ for each k.

The powers of a derivation δ naturally define a differential seminorm by $T_k(x) \equiv \left\| \frac{\delta^{(k)}(x)}{k!} \right\|$. It is shown in [14] that a differential seminorm is determined by a linear map with a non-negative norm curvature from $\mathfrak{A}$ into a graded Banach $*$-algebra. Let T be a differential seminorm on $\mathfrak{A}$.

· T is said to be a *differential norm* if $T(x) = 0$ implies $x = 0$.

· T is said to be ℓ^1-*summable* if $T_{\text{tot}}(x) \equiv \sum_{k=0}^{\infty} T_k(x) < \infty$ for each $x \in \mathfrak{A}$.

· T is of *finite order* if for some $n \in \mathbb{N}$ $T_k(x) = 0, \forall k > n, \forall x \in \mathfrak{A}$. Then the smallest such number n is said to be the *order* of T.

Blackadar and Cuntz assumed the ℓ^1-summability in the definition of differential seminorms. We do not require T to be ℓ^1-summable. This added generality allows us to construct the C^k-algebras $C^k(\mathfrak{A}, T)$, $1 \leqslant k \leqslant \infty$ and to consider the analytic structure defined by T. Throughout this paper, we assume that a differential seminorm T satisfies always the following condition: the two topologies defined restrictively by the (semi) norms T_0 and $\|\cdot\|$ are equivalent (denote by $T_0 \sim \|\cdot\|$). Then T is automatically a differential norm. For any $k \in \mathbb{N}_0$, we put $p_k(x) = \sum_{i=0}^{k} T_i(x), \forall x \in \mathfrak{A}$. Then $\{p_k\}$ is a sequence of algebra $*$-norms on $\mathfrak{A}$ satisfying $p_0 \leqslant p_1 \leqslant p_2 \leqslant \cdots$. We denote by $\mathfrak{A}_{(k)}$ (or $C^k(\mathfrak{A}, T)$) the Banach $*$-algebra constructed by the completion of the normed $*$-algebra $\mathfrak{A}[p_k]$ and denote by $\mathfrak{A}_{\tau(T)}$ (or $C^\infty(\mathfrak{A}, T)$) the Fréchet $*$-algebra constructed by the completion of the metrizable locally m-convex $*$-algebra $\mathfrak{A}[\tau(T)]$, where $\tau(T)$ is the locally convex topology on $\mathfrak{A}$ defined by $\{p_k\}$. We denote $\tau(T)$ and $\mathfrak{A}_{\tau(T)}$ by τ and $\mathfrak{A}_\tau$ respectively, when we do not need to distinguish them. We call $\mathfrak{A}_\tau$ the *differential Fréchet $*$-algebra* defined by T. We may consider $\mathfrak{A}_\tau$ as the inverse limit of the Banach $*$-algebras $\{\mathfrak{A}_{(k)}\}$:

$$\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)} = \{x = (x_k) : x_k \in \mathfrak{A}_{(k)} \text{ and } \sigma_{k+1}(x_{k+1}) = x_k, \forall k \in \mathbb{N}_0\},$$

where $\{\sigma_k\}$ is the inverse limit decomposition of $\mathfrak{A}_\tau$ determined by $\{p_k\} : \mathfrak{A}_0 \xleftarrow{\sigma_1} \mathfrak{A}_{(1)} \xleftarrow{\sigma_2} \mathfrak{A}_{(2)} \longleftarrow \cdots \longleftarrow \mathfrak{A}_{(k)} \xleftarrow{\sigma_{k+1}} \mathfrak{A}_{(k+1)} \longleftarrow \cdots$. Since $\|\cdot\| \leqslant p_k$ ($k = 0, 1, 2, \ldots$), the identity map $x \in \mathfrak{A}[p_k] \longrightarrow x \in \mathcal{A}$ extends to continuous $*$-homomorphisms $\varphi_k : \mathfrak{A}_{(k)} \longrightarrow \mathcal{A}$ and $\varphi : \mathfrak{A}_\tau \longrightarrow \mathcal{A}$, and

$$\varphi(x) = \varphi_k(x_k), \forall x = (x_k) \in \mathfrak{A}_\tau. \tag{3.1}$$

Indeed, take an arbitrary $x = (x_k) \in \mathfrak{A}_\tau$. Then there exists a sequence $\{a_n\}$ in $\mathfrak{A}$ which converges to x with respect to τ. Since φ is continuous, $\lim_{n\to\infty} \|\varphi(a_n) - \varphi(x)\| = 0$. For any $k \in \mathbb{N}_0$, $\{a_n\}$ is a Cauchy sequence in $\mathfrak{A}[p_k]$ and $\lim_{n\to\infty} p_k(a_n - x_k) = 0$. Since φ_k is continuous, we have $\lim_{n\to\infty} \|\varphi_k(a_n) - \varphi_k(x_k)\| = 0$. Hence $\varphi(x) = \varphi_k(x_k)$ for all k.

A seminorm α on a C^*-normed algebra $(\mathfrak{A}, \|\cdot\|)$ is *closable* if the map $x \in (\mathfrak{A}, \|\cdot\|) \longrightarrow x \in (\mathfrak{A}, \alpha)$ is closable. We say that a differential seminorm $T = (T_k)$ is *closable* if for each k, $p_k = \sum_{i=0}^{k} T_i$ is closable. If T is closable, then $\mathfrak{A}_{(k)} \subset \mathcal{A}$ and $\mathfrak{A}_\tau \subset \mathcal{A}$ and if T_{tot} is closable, then $\mathfrak{A}_T \subset \mathcal{A}$, where $\mathfrak{A}_T$ is the completion of $\{x \in \mathfrak{A} : T_{\text{tot}}(x) < \infty\}$ in the norm $T_{\text{tot}}(\cdot)$.

To investigate the spectrality of the differential Fréchet $*$-algebra $\mathfrak{A}_\tau$ and to consider the analyticity of differential norms, we introduce analytic algebra $*$-norms defined in [14].

DEFINITION 3.2. An algebra $*$-norm α on $\mathfrak{A}$ is said to be *analytic* if for each finite subset F in $\mathfrak{A}$, with $\|x\| \leqslant 1$ for all $x \in F$, and for each sequence $\{x_s\}$ in F, $\overline{\lim_{s\to\infty}} \left(\frac{1}{s} \log \alpha(x_1 x_2 \cdots x_s)\right) \leqslant 0$. A C^*-normed algebra that is a Banach $*$-algebra with respect to an analytic algebra $*$-norm is said to be an *analytic Banach $*$-algebra*.

The total norm T_{tot} of every differential norm T of finite order is analytic.

The following important theorem has been discussed in [10]. There are some minor errors in the proof therein. We present a simplified proof.

THEOREM 3.3. *The following statements hold:*

(i) $\mathfrak{A}_\tau$ *is a* C^**-spectral algebra with spectral* C^**-seminorm* $\|\cdot\|_\varphi \equiv \|\varphi(\cdot)\|$.

(ii) $\mathfrak{A}_\tau$ *is a hermitian Q-algebra.*

(iii) $\mathfrak{A}_\tau$ *is spectrally invariant in* $\mathcal{A}$ *via the map* φ *and* $E(\mathfrak{A}_\tau) = \mathcal{A}$.

(iv) $\mathfrak{A}_\tau$ *is local, that is, it is closed under holomorphic functional calculus of* $\mathcal{A}$ *in the sense that given* $x \in \mathfrak{A}_\tau$ *and a holomorphic function* f *on* $Sp_{\mathcal{A}}(\varphi(x))$*, there exists an element* y *of* $\mathfrak{A}_\tau$ *such that* $f(\varphi(x)) = \varphi(y)$.

(v) $\mathfrak{A}_\tau$ *and* $\mathcal{A}$ *have the same K-theory, that is,* $K_*(\mathfrak{A}_\tau) = K_*(\mathcal{A})$.

Furthermore, the same results hold for $\mathfrak{A}_{(k)}$.

Proof. Case 1. Assume that T is of finite order, say n, so that $T_k(x) = 0$ for each $x \in \mathfrak{A}$ and $k > n$. Then $T_{\mathrm{tot}} = p_n$ and $\mathfrak{A}_\tau = \mathfrak{A}_{(n)}$ is a Banach $*$-algebra. Since $\mathfrak{T} \equiv \ker\varphi = \{x \in \mathfrak{A}_\tau : \varphi(x) = 0\}$, it follows from [14] that $\mathfrak{T}^n = \{0\}$, which implies $\mathfrak{T} \subset \mathrm{rad}\mathfrak{A}_\tau$. Hence, $x + \mathfrak{T} \longrightarrow x + \mathrm{rad}\mathfrak{A}_\tau$ is a well-defined $*$-homomorphism of the Banach $*$-algebra $\mathfrak{A}_\tau/\mathfrak{T}$ onto the Banach $*$-algebra $\mathfrak{A}_\tau/\mathrm{rad}\mathfrak{A}_\tau$. By standard Banach algebra arguments it is shown that

$$(3.2)\qquad Sp_{\mathfrak{A}_\tau}(x) = Sp_{\mathfrak{A}_\tau/\mathrm{rad}\mathfrak{A}_\tau}(x + \mathrm{rad}\mathfrak{A}_\tau) = Sp_{\mathfrak{A}_\tau/\mathfrak{T}}(x + \mathfrak{T}) = Sp_{\varphi(\mathfrak{A}_\tau)}(\varphi(x))$$

for each $x \in \mathfrak{A}_\tau$ and since the quotient norm of T_{tot} on $\mathfrak{A}_\tau/\mathfrak{T}$ is analytic and $\mathfrak{A}_\tau/\mathfrak{T}$ is isomorphic to the C^*-normed algebra $\varphi(\mathfrak{A}_\tau)$, it follows from Proposition 3.12 of [14] that $Sp_{\mathfrak{A}_\tau/\mathfrak{T}}(x + \mathfrak{T}) = Sp_{\mathcal{A}}(\varphi(x)), \forall x \in \mathfrak{A}_\tau$, which implies by (3.2) that

$$(3.3)\qquad Sp_{\mathfrak{A}_\tau}(x) = Sp_{\mathcal{A}}(\varphi(x)), \quad \forall x \in \mathfrak{A}_\tau.$$

Take an arbitrary C^*-seminorm q on $\mathfrak{A}_\tau$. Let π_q be the bounded $*$-representation of $\mathfrak{A}_\tau$ on a Hilbert space $\mathcal{H}_q$ defined by q. Then we have by (3.3),

$$\begin{aligned} q(x)^2 = q(x^*x) = \|\pi_q(x)^*\pi_q(x)\| &= r_{\mathcal{B}(\mathcal{H}_q)}(\pi_q(x)^*\pi_q(x)) \\ &\leqslant r_{\pi_q(\mathfrak{A}_\tau)}(\pi_q(x)^*\pi_q(x)) \leqslant r_{\mathfrak{A}_\tau}(x^*x) = r_{\mathcal{A}}(\varphi(x)^*\varphi(x)) = \|\varphi(x)\|^2 \end{aligned}$$

for each $x \in \mathfrak{A}_\tau$, which implies that the continuous C^*-seminorm $\|\cdot\|_\varphi \equiv \|\varphi(\cdot)\|$ is the greatest C^*-seminorm on $\mathfrak{A}_\tau$ and hence $\|\cdot\|_\varphi$ equals the Gelfand–Naimark seminorm $|\cdot|$. Hence, $\ker|\cdot| = \mathrm{srad}\mathfrak{A}_\tau = \mathfrak{T} = \ker\varphi$, and $E(\mathfrak{A}_\tau)$ is isomorphic to the C^*-algebra $\mathcal{A}$, which implies by (3.3) that $Sp_{\mathfrak{A}_\tau}(x) = Sp_{\mathcal{A}}(\varphi(x)) = Sp_{E(\mathfrak{A}_\tau)}(j(x)), \forall x \in \mathfrak{A}_\tau$, where j is the $*$-homomorphism of $\mathfrak{A}_\tau$ into $E(\mathfrak{A}_\tau)$ defined by $j(x) = x + \ker|\cdot|, x \in \mathfrak{A}_\tau$. Thus $\mathfrak{A}_\tau$ is spectrally invariant and $E(\mathfrak{A}_\tau) = \mathcal{A}$.

Case 2. Let $T = (T_k)_0^\infty$ be not necessarily of finite order. Consider $\mathfrak{A}_{(k)}$ for each $k \in \mathbb{N}_0$. It is a Banach $*$-algebra with norm p_k which is the total norm $T^{(k)}_{\mathrm{tot}}$ of the differential norm $T^{(k)} = (T_0, T_1, \dots, T_k, 0, 0, \dots)$ of finite order k. Then we have by (3.1) and (3.3), for any $x = (x_k) \in \mathfrak{A}_\tau$,

$$(3.4)\qquad Sp_{\mathfrak{A}_\tau}(x) = \bigcup_k Sp_{\mathfrak{A}_{(k)}}(x_k) = \bigcup_k Sp_{\mathcal{A}}(\varphi_k(x_k)) = \bigcup_k Sp_{\mathcal{A}}(\varphi(x)) = Sp_{\mathcal{A}}(\varphi(x)),$$

which implies similar to Case 1 that $|x| = \|\varphi(x)\| = \|x\|_\varphi$ for each $x \in \mathfrak{A}_\tau$. Hence, $\ker|\cdot| = \mathrm{srad}\mathfrak{A}_\tau = \ker\varphi = \mathfrak{T}$, which implies that $\mathfrak{A}_\tau$ is spectrally invariant and $E(\mathfrak{A}_\tau) = \mathcal{A}$. Furthermore, $\mathfrak{A}_\tau$ is hermitian by (3.4), and since $r_{\mathfrak{A}_\tau}(x) = r_{\mathcal{A}}(\varphi(x)) \leqslant \|\varphi(x)\| = |x|, \forall x \in \mathfrak{A}_\tau$, $|\cdot|$ is a continuous spectral C^*-seminorm on $\mathfrak{A}_\tau$, and further since $\{x \in \mathfrak{A}_\tau : |x| < 1\}$ is an open subset of $\mathfrak{A}_\tau^{\mathrm{qr}}$, $\mathfrak{A}_\tau$ is a Q-algebra. The proofs of remaining assertions are as in Theorem 3.3 of [14]. This completes the proof. ∎

We have conjectured in [10] that if $\mathcal{B}$ denotes either $\mathfrak{A}_{(k)}$ or $\mathfrak{A}_\tau$, then $\mathcal{B}$ is closed under the C^∞-functional calculus of $\mathcal{A}$. We show that this indeed holds.

THEOREM 3.4. *The Fréchet $*$-algebra $\mathfrak{A}_\tau$ is closed under the C^∞-functional calculus of $\mathcal{A}$ via φ in the sense that given $x = x^* \in \mathfrak{A}_\tau$ and a C^∞-function f on $Sp_{\mathcal{A}}(\varphi(x))$, there exists an element y of $\mathfrak{A}_\tau$ such that $f(\varphi(x)) = \varphi(y)$. The Banach $*$-algebra $\mathfrak{A}_{(k)}$ is also closed under the C^∞-functional calculus of $\mathcal{A}$ via φ_k.*

Proof. Step 1. Consider $\mathfrak{A}_{(k)}, \forall k \in \mathbb{N}_0$. Take an arbitrary $x = x^*$ in $\mathfrak{A}_{(k)}$. Since $\varphi_k(x) = \varphi_k(x^*)$ and $Sp_{\mathfrak{A}_{(k)}}(x) = Sp_{\mathcal{A}}(\varphi_k(x))$ by (3.3), it follows that $Sp_{\mathfrak{A}_{(k)}}(x)$ is contained in some bounded closed interval I in $\mathbb{R}$. We show $f(\varphi_k(x)) \in \varphi_k(\mathfrak{A}_{(k)})$ using the Fourier transform. We may consider $f \in C^\infty(I) \subset C^\infty(\mathbb{R})$ and in $L^1(\mathbb{R})$. The functional calculus in the C^*-algebra $\mathcal{A}$ gives

$$f(\varphi_k(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s\varphi_k(x)} \widehat{f}(s)\mathrm{d}s, \tag{3.5}$$

where $\widehat{f}$ is the Fourier transform of f. Since $\mathfrak{A}_{(k)}$ is a Banach $*$-algebra and also closed in holomorphic functional calculus of $\mathcal{A}$ via φ_k (Theorem 3.3), it follows that $\mathrm{e}^{\mathrm{i}sx} \in \mathfrak{A}_{(k)}$ and $\varphi_k(\mathrm{e}^{\mathrm{i}sx}) = \mathrm{e}^{\mathrm{i}s\varphi_k(x)}, \forall s \in \mathbb{R}$ and

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p_k(\mathrm{e}^{\mathrm{i}sx})|\widehat{f}(s)|\mathrm{d}s &\leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K|s|^{N+\varepsilon}|\widehat{f}(s)|\mathrm{d}s \quad \text{([14], p. 274)}\\ &\leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(1+|s|)^{2N}|\widehat{f}(s)|\mathrm{d}s < \infty,\end{aligned}$$

where $K = \sup\{p_k(\mathrm{e}^{\mathrm{i}sx}) : 0 \leqslant s < 1\}$, which implies $y \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sx} \widehat{f}(s)\mathrm{d}s \in \mathfrak{A}_{(k)}$ and by continuity of φ_k

$$\varphi_k(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_k(\mathrm{e}^{\mathrm{i}sx})\widehat{f}(s)\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s\varphi_k(x)}\widehat{f}(s)\mathrm{d}s = f(\varphi_k(x)). \tag{3.6}$$

Step 2. Consider $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)}$. Let $x = x^* \in \mathfrak{A}_\tau$ be given, say $x = (x_k)$ be the associated coherent sequence satisfying $\sigma_{k+1}(x_{k+1}) = x_k$. Let $Sp_{\mathfrak{A}_\tau}(x) \subset I$ and $f \in C^\infty(Sp_{\mathfrak{A}_\tau}(x))$ say $f \in C^\infty(I)$. Since $Sp_{\mathfrak{A}_\tau}(x) = \bigcup_k Sp_{\mathfrak{A}_{(k)}}(x_k)$, we have

$f \in C^\infty(Sp_{\mathfrak{A}_{(k)}}(x_k))$ for $\forall k \in \mathbb{N}_0$. By (3.6) we have

$$y_k = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx_k} \widehat{f}(s) ds \in \mathfrak{A}_{(k)} \quad \text{and} \quad \varphi_k(y_k) = f(\varphi_k(x_k)), \tag{3.7}$$

and

$$\begin{aligned}\sigma_{k+1}(y_{k+1}) &= \sigma_{k+1}\Big(\tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx_{k+1}} \widehat{f}(s) ds\Big) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\sigma_{k+1}(x_{k+1})} \widehat{f(s)} ds \\ &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx_k} \widehat{f}(s) ds = y_k.\end{aligned}$$

Hence, $y = (y_k) \in \mathfrak{A}_\tau$. Furthermore, it follows from (3.1) and (3.7) that $\varphi(y) = \varphi_k(y_k) = f(\varphi_k(x_k)) = f(\varphi(x))$. This completes the proof. ∎

Let T be a differential norm on $\mathfrak{A}$. The completion of $\ell^1(\mathfrak{A}, T) \equiv \{x \in \mathfrak{A} : T_{\text{tot}}(x) < \infty\}$ with respect to the total norm T_{tot} is called the *differential Banach* $*$-*algebra* defined by T and denoted by $\mathfrak{A}_T$. Then $\mathfrak{A}_T$ is contained in the Banach $*$-algebra $b(\mathfrak{A}_\tau) \equiv \ell^1(\mathfrak{A}_\tau, T)$ which is the bounded part of $\mathfrak{A}_\tau$ and $\mathfrak{A}_T = b(\mathfrak{A}_\tau)$ if T is ℓ^1-summable. Suppose that T is closable and ℓ^1-summable but not of finite order. Then $\mathfrak{A}_T = \widetilde{\mathfrak{A}}[T_{\text{tot}}] \subsetneq \mathfrak{A}_\tau$ and $\mathfrak{A}_T$ need not necessarily be an analytic Banach $*$-algebra, and so not necessarily C^*-spectral. With this view we consider the analytic structure defined by a differential norm. Let T be a differential norm on $\mathfrak{A}$. For $t > 0$, we define a differential norm $T(t)$ by $T(t)_k \equiv t^k T_k(x), x \in \mathfrak{A}$. Let $\ell^1(\mathfrak{A}, T(t))$ denote the set of all ℓ^1-summable elements of $\mathfrak{A}$ with respect to $T(t)$, that is,

$$\ell^1(\mathfrak{A}, T(t)) = \Big\{x \in \mathfrak{A} : T(t)_{\text{tot}}(x) \equiv \sum_{k=0}^{\infty} t^k T_k(x) < \infty\Big\}.$$

Then $\ell^1(\mathfrak{A}, T(t))$ is a $*$-subalgebra of $\mathfrak{A}$ and $\ell^1(\mathfrak{A}, T(t))[T(t)_{\text{tot}}]$ is a normed $*$-algebra and $\ell^1(\mathfrak{A}, T(t_1)) \supset \ell^1(\mathfrak{A}, T(t_2))$ and $T(t_1)_{\text{tot}} \leqslant T(t_2)_{\text{tot}}$ for $0 < t_1 < t_2$. Let $\mathfrak{A}_{T(t)}$ be the differential Banach $*$-algebra constructed by the completion of $\ell^1(\mathfrak{A}, T(t))[T(t)_{\text{tot}}]$. We introduce the notions of analytic differential norms and of entire analytic differential norms leading to the notions of analytic structure defined by T.

DEFINITION 3.5. A differential norm T is called *analytic on* $\mathfrak{A}$ if there exists $t_0 > 0$ such that the total norm $T(t_0)_{\text{tot}}$ is analytic on the C^*-normed algebra $\ell^1(\mathfrak{A}, T(t_0))$, and T is called *entire analytic on* $\mathfrak{A}$ if $T(t)_{\text{tot}}$ is analytic on $\ell^1(\mathfrak{A}, T(t))$ for all $t > 0$. The pair $(\mathfrak{A}, T)$ is called *analytic* (respectively *entire analytic*) if $T(t)$ is ℓ^1-summable (that is, $\ell^1(\mathfrak{A}, T(t)) = \mathfrak{A}$) and $T(t)$ is analytic on $\ell^1(\mathfrak{A}, T(t))$ for some $t > 0$ (respectively for all $t > 0$).

The pair $(\mathfrak{A}, T)$ is entire analytic for any differential norm T of finite order. Indeed, this follows since $\ell^1(\mathfrak{A}, T(t)) = \mathfrak{A}$ and the total norm $T(t)_{\text{tot}}$ is analytic on

$\ell^1(\mathfrak{A}, T(t))$ for all $t > 0$ ([14], p. 264). We define analytic (entire analytic) locally convex $*$-algebras defined by a differential norm T. Let T be a differential norm on $\mathfrak{A}$. We put $\mathbb{R}_+^\omega = \{t > 0 : T(t)_{\text{tot}}$ is analytic on $\ell^1(\mathfrak{A}, T(t))\}$. Suppose T is analytic (if and only if $\mathbb{R}_+^\omega \neq \phi$). Then we consider

$$\mathfrak{A}^\omega = \bigcup_{t>l^\omega} \ell^1(\mathfrak{A}, T(t)), \quad \widetilde{\mathfrak{A}^\omega} = \bigcup_{t>l^\omega} \mathfrak{A}_{T(t)},$$

where $l^\omega \equiv \inf \mathbb{R}_+^\omega$. Let $t \in \mathbb{R}_+^\omega$. Then it is easily shown that $T(t)_{\text{tot}}$ extends to an analytic norm on $\mathfrak{A}_{T(t)}$, and so the definition of $\widetilde{\mathfrak{A}^\omega}$ is meaningful. Suppose T is analytic on $\mathfrak{A}$. Then $\mathfrak{A}^\omega$ (respectively $\widetilde{\mathfrak{A}^\omega}$) is the inductive limit algebra of the normed $*$-algebras $\{\ell^1(\mathfrak{A}, T(t)) : t > l^\omega\}$ (respectively the Banach $*$-algebras $\{\mathfrak{A}_{T(t)} : t > l^\omega\}$). By p. 122, and pp. 317–319 of [8], $\widetilde{\mathfrak{A}^\omega}$ is a complete locally m-convex Q-algebra.

The following is easily shown.

LEMMA 3.6. *Suppose $(\mathfrak{A}, T)$ is analytic, that is, $T(t_0)$ is ℓ^1-summable and $T(t_0)_{\text{tot}}$ is analytic on $\ell^1(\mathfrak{A}, T(t_0))$ for some $t_0 > 0$. Then the following hold:*

(i) $\mathfrak{A} = \ell^1(\mathfrak{A}, T(t))$ *and* $T(t)_{\text{tot}}$ *is analytic on* $\ell^1(\mathfrak{A}, T(t))$ *for* $0 < t \leqslant t_0$.

(ii) $l^\omega = 0$, $\mathfrak{A}^\omega = \mathfrak{A}$ *and* $\widetilde{\mathfrak{A}^\omega} = \bigcup_{0<t\leqslant t_0} \widetilde{\mathfrak{A}}[T(t)_{\text{tot}}]$.

Suppose T is entire analytic. We put

$$\widetilde{\mathfrak{A}^{e\omega}} = \bigcap_{t>0} \mathfrak{A}_{T(t)} = \bigcap_{n\in\mathbb{N}} \mathfrak{A}_{T(1/n)}.$$

Then, $\widetilde{\mathfrak{A}^{e\omega}}$ is a projective limit of the sequence of Banach $*$-algebras $\mathfrak{A}_{T(1/n)}, n \in \mathbb{N}$, hence it is a Fréchet $*$-algebras.

THEOREM 3.7. *Let T be a differential norm on $\mathfrak{A}$. Then the following statements hold:*

(i) *Suppose that T is analytic on $\mathfrak{A}$. Then $\widetilde{\mathfrak{A}^\omega}$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_\varphi$, is an hermitian Q-algebra and is local.*

(ii) *Suppose T is entire analytic on $\mathfrak{A}$. Then $\widetilde{\mathfrak{A}^{e\omega}}$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_\varphi$, and is an hermitian Q-algebra. Furthermore, if T is closable, then $\widetilde{\mathfrak{A}^{e\omega}}$ is local.*

Proof. Let $t > 0$ and $\mathcal{A}_t$ the C^*-subalgebra of $\mathcal{A}$ obtained by the Hausdorff completion of $\mathfrak{A}_{T(t)}[\|\cdot\|_\varphi]$. Since $\mathcal{A}_t$ is a closed $*$-subalgebra of the C^*-algebra $\mathcal{A}$, we have

$$Sp_{\mathcal{A}_t}(\varphi(x)) = Sp_{\mathcal{A}}(\varphi(x)), \quad x \in \ell^1(\mathfrak{A}, T(t)). \tag{3.8}$$

(i) Suppose T is analytic on $\mathfrak{A}$. Take an arbitrary $x \in \widetilde{\mathfrak{A}^\omega}$. Then $x \in \mathfrak{A}_{T(t_0)}$ for some $t_0 > l^\omega$. By the definition of l^ω there exists a real number t such that $l^\omega \leqslant t \leqslant t_0$ and $\mathfrak{A}_{T(t)}$ is an analytic Banach $*$-algebra, which by Proposition 3.12 of [14] implies that $Sp_{\mathfrak{A}_{T(t)}}(x) = Sp_{\mathcal{A}_t}(\varphi(x))$, and that $\mathfrak{A}_{T(t)}$ is local, therefore by

(3.8) that

$$Sp_{\widetilde{\mathfrak{A}^{\omega}}}(x) \subset Sp_{\mathfrak{A}_{T(t)}}(x) = Sp_{\mathcal{A}_t}(\varphi(x)) = Sp_{\mathcal{A}}(\varphi(x)).$$

The converse inclusion $Sp_{\widetilde{\mathfrak{A}^{\omega}}}(x) \supset Sp_{\mathcal{A}}(\varphi(x))$ is clear. Hence we have

(3.9) $Sp_{\widetilde{\mathfrak{A}^{\omega}}}(x)=Sp_{\mathfrak{A}_{T(t)}}(x)=Sp_{\mathcal{A}}(\varphi(x)), \quad r_{\widetilde{\mathfrak{A}^{\omega}}}(x)=r_{\mathcal{A}}(\varphi(x))\leqslant\|\varphi(x)\|=\|x\|_{\varphi}.$

Therefore $\|\cdot\|_{\varphi}$ is a spectral C^*-seminorm on $\widetilde{\mathfrak{A}^{\omega}}$. By the locality of $\mathfrak{A}_{T(t)}$, for any holomorphic function f on $Sp_{\widetilde{\mathfrak{A}^{\omega}}}(x) = Sp_{\mathfrak{A}_{T(t)}}(x)$ (by (3.9)) there exists an element y of $\mathfrak{A}_{T(t)} \subset \widetilde{\mathfrak{A}^{\omega}}$ such that $f(\varphi(x)) = \varphi(y)$. Therefore, $\widetilde{\mathfrak{A}^{\omega}}$ is local.

(ii) Suppose T is entire analytic on $\mathfrak{A}$. Let $x \in \widetilde{\mathfrak{A}^{e\omega}}$. We can show similarly to (i) that

$$Sp_{\widetilde{\mathfrak{A}^{e\omega}}}(x) = \bigcup_{t>0} Sp_{\mathfrak{A}_{T(t)}}(x) = \bigcup_{t>0} Sp_{\mathcal{A}_t}(\varphi(x)) = Sp_{\mathcal{A}}(\varphi(x)), \tag{3.10}$$
$$r_{\widetilde{\mathfrak{A}^{e\omega}}}(x) = r_{\mathcal{A}}(\varphi(x)) \leqslant \|\varphi(x)\| = \|x\|_{\varphi}.$$

Hence, $\|\cdot\|_{\varphi}$ is a continuous spectral C^*-seminorm on $\widetilde{\mathfrak{A}^{e\omega}}$. Furthermore, since $\mathfrak{A}_{T(t)}$ is local for all $t > 0$, it follows that for any holomorphic function f on $Sp_{\widetilde{\mathfrak{A}^{e\omega}}}(x)$ there exists an element y_t of $\mathfrak{A}_{T(t)}$ such that $f(\varphi(x)) = \varphi(y_t)$. Suppose T is closable. Then φ is an isomorphism, and so we have $y_t = y_{t'}$ for each $t, t' > 0$. Hence, $y \equiv y_t \in \bigcap_{t>0} \mathfrak{A}_{T(t)} = \widetilde{\mathfrak{A}^{e\omega}}$, and so $\widetilde{\mathfrak{A}^{e\omega}}$ is local. This completes the proof. ∎

A differential norm T on $\mathfrak{A}$ extends uniquely to a differential norm on the Fréchet $*$-algebra $\mathfrak{A}_\tau$. For $t > 0$, let $(\mathfrak{A}_\tau)_{T(t)} \equiv \ell^1(\mathfrak{A}_\tau, T(t))$.

LEMMA 3.8. (i) *$(\mathfrak{A}_\tau)_{T(t)}$ is a Banach $*$-algebra with norm $T_{\mathrm{tot}}(t)(\cdot)$ and $\mathfrak{A}_{T(t)}$ is a $*$-subalgebra of $(\mathfrak{A}_\tau)_{T(t)}$.*

(ii) *For $0 < t_1 < t_2$, $(\mathfrak{A}_\tau)_{T(t_1)} \supset (\mathfrak{A}_\tau)_{T(t_2)}$ and $T(t_1)_{\mathrm{tot}} \leqslant T(t_2)_{\mathrm{tot}}$.*

Proof. (i) It is easily shown that $(\mathfrak{A}_\tau)_{T(t)}[T(t)_{\mathrm{tot}}]$ is a normed $*$-algebra. We claim that $(\mathfrak{A}_\tau)_{T(t)}[T(t)_{\mathrm{tot}}]$ is complete. Indeed, since $c_k p_k(x) \leqslant p_k^t(x) = \sum_{i=0}^{k} T_i(x)t^i \leqslant M_k p_k(x), \forall x \in \mathfrak{A}, \forall k$, where $c_k \equiv \min\{t^i : 0 \leqslant i \leqslant k\}$ and $M_k \equiv \max\{t^i : 0 \leqslant i \leqslant k\}$, the two topologies on $\mathfrak{A}$ defined by p_k^t and p_k are equivalent, and so the completion $\mathfrak{A}^t_{(k)}$ of $\mathfrak{A}[p_k^t] = \mathfrak{A}_{(k)}$ and $\widetilde{\mathfrak{A}}[\{p_k^t\}] = \mathfrak{A}_\tau$. Hence, $(\mathfrak{A}_\tau)_{T(t)} =$ the bounded part of $\mathfrak{A}_\tau$ with respect to $\{p_k^t\}$ and it is a Banach $*$-algebra with norm $T(t)_{\mathrm{tot}}$.

(ii) This is trivial. ∎

DEFINITION 3.9. Let T be a differential norm on $\mathfrak{A}$. Then T is called *analytic on* $\mathfrak{A}_\tau$ if there exists $t_0 > 0$ such that the total norm $T(t_0)_{\mathrm{tot}}$ is analytic on the C^*-normed algebra $(\mathfrak{A}_\tau)_{T(t_0)}$, and T is called *entire analytic on* $\mathfrak{A}_\tau$ if $T(t)_{\mathrm{tot}}$ is analytic on $(\mathfrak{A}_\tau)_{T(t)}$ for all $t > 0$.

Suppose T is analytic on $\mathfrak{A}_\tau$. We put $l_\tau^\omega = \inf\{t > 0 : T(t)_{\mathrm{tot}}$ is analytic on $(\mathfrak{A}_\tau)_{T(t)}\}$. Then T is analytic on $\mathfrak{A}$ and $l^\omega \leqslant l_\tau^\omega$. We define a $*$-subalgebra $\mathfrak{A}_\tau^\omega$ of

$\mathfrak{A}_\tau$ as follows: $\mathfrak{A}_\tau^\omega = \bigcup_{t>l_\tau^\omega} (\mathfrak{A}_\tau)_{T(t)}$ and endow it with the inductive limit topology defined by the inclusions. By Lemma 3.6 $\mathfrak{A}_\tau^\omega$ is the inductive limit algebra of the Banach $*$-algebras $\{(\mathfrak{A}_\tau)_{T(t)} : t > l_\tau^\omega\}$, and so it is a complete locally m-convex Q-algebra ([22], Corollary 10.2, p. 319). Suppose T is entire analytic on $\mathfrak{A}_\tau$. We put $\mathfrak{A}_\tau^{e\omega} = \bigcap_{t>0} (\mathfrak{A}_\tau)_{T(t)} = \bigcap_{n\in\mathbb{N}} (\mathfrak{A}_\tau)_{T(1/n)}$. Then $\mathfrak{A}_\tau^{e\omega}$ is a projective limit of the sequence of Banach $*$-algebras $(\mathfrak{A}_\tau)_{T(1/n)}, n \in \mathbb{N}$, hence it is a Fréchet $*$-algebra. Furthermore, T is entire analytic on $\mathfrak{A}$ and $\widetilde{\mathfrak{A}^{e\omega}}$ is a closed $*$-subalgebra of the Fréchet $*$-algebra $\mathfrak{A}_\tau^{e\omega}$. The following is proved the same way as the proof of Theorem 3.7.

THEOREM 3.10. *Let T be a differential norm on $\mathfrak{A}$. Then the following statements hold:*

(i) *Suppose T is analytic on $\mathfrak{A}_\tau$. Then $\mathfrak{A}_\tau^\omega$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_\varphi$, is an hermitian Q-algebra and is local.*

(ii) *Suppose T is entire analytic on $\mathfrak{A}_\tau$. Then $\mathfrak{A}_\tau^{e\omega}$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_\varphi$ and it is an hermitian Q-algebra. Furthermore, if T is closable, then $\mathfrak{A}_\tau^{e\omega}$ is local.*

4. SMOOTH SUBALGEBRAS OF C^*-ALGEBRAS

In this section we study smooth structure in C^*-algebras. We begin with the definition of derived norms defined in [14]. A differential norm T on $\mathfrak{A}$ is *of total order* $\leqslant k$ if for each T-bounded sequence $\{x_s\}$in $\mathfrak{A}$,

$$\overline{\lim_{s\to\infty}} \frac{\log T_{\text{tot}}(x_1 x_2 \cdots x_s)}{\log s} \leqslant qk.$$

If T is of total order $\leqslant k$ for some k, then T is said to be a differential norm of *total order finite*. By Proposition 3.10 of [14] we have the following.

LEMMA 4.1. *Let T be a differential norm on $\mathfrak{A}$. Consider the following:*

(i) *T is of finite order.*

(ii) *T is of total order finite.*

(iii) *T is analytic, that is, T_{tot} is analytic.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

DEFINITION 4.2. An algebra $*$-norm α on $\mathfrak{A}$ is said to be a *derived norm* if α is the quotient norm of the total norm of a differential norm of total order finite in the sense that there are a C^*-norm decreasing $*$-homomorphism ϕ from a C^*-normed algebra $\mathcal{B}$ with C^*-norm $\|\cdot\|_{\mathcal{B}}$ onto $\mathfrak{A}$ and a differential norm T on $\mathcal{B}$ of total order finite such that α is the quotient norm for T_{tot}, that is,

$$\alpha(x) = \inf\{T_{\text{tot}}(y) : y \in \mathcal{B}, x = \phi(y)\}, \quad x \in \mathfrak{A}.$$

Further, a derived norm α is of *total order* $\leqslant k$ (respectively order $\leqslant k$) if α is the quotient norm of the total norm of a differential norm of total order $\leqslant k$ (respectively order $\leqslant k$).

Here a differential norm T of order $\leqslant k$ means that there exists $m \leqslant k$ such that $T_i = 0$ for all $i > m$.

Since $T_0 \sim \|\cdot\|_{\mathcal{B}}$ and ϕ is C^*-norm decreasing, we have $\|\cdot\| \leqslant \alpha$. A derived norm α is called closable if id : $x \in \mathfrak{A}[\|\cdot\|] \to x \in \mathfrak{A}[\alpha]$ is closable.

LEMMA 4.3. *Let α be a (closable) derived norm on $\mathfrak{A}$. Then:*

(i) *α is analytic.*

(ii) *$\widetilde{\alpha}$ is a (closable) derived norm on $\widetilde{\mathfrak{A}}[\alpha]$, where $\widetilde{\alpha}$ is the natural extension of α to $\widetilde{\mathfrak{A}}[\alpha]$. Hence $\widetilde{\alpha}$ is analytic.*

(iii) *The restriction $\alpha\lceil_{\mathfrak{A}_1}$ of α to each $*$-subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ is a (closable) derived norm on $\mathfrak{A}_1$.*

Furthermore, the following holds:

(iv) *The sum of a finite family of (closable) derived norms is a (closable) derived norm.*

Proof. The statements (i)–(iii) follow immediately from the definition of derived norms (see Section 5 in [14]). The statement (iv) follows from Proposition 5.3 of [14]. ∎

We define the smooth envelope $\mathcal{S}\mathfrak{A}$ of $\mathfrak{A}$ and smooth algebras as in [14]. We define by $\Lambda_{\text{cd}}(\mathfrak{A})$ (Λ_{cd}, simply) the set of all closable derived norms on $\mathfrak{A}$ and it is directed by Lemma 4.3(iv). We denote by $\tau(\Lambda_{\text{cd}}(\mathfrak{A}))$ ($\tau(\Lambda_{\text{cd}})$, simply) the locally convex topology defined by $\Lambda_{\text{cd}}(\mathfrak{A})$.

DEFINITION 4.4. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\text{cd}})]$ of $\mathfrak{A}$ is said to be *smooth envelope* of $\mathfrak{A}$ denoted by $\mathcal{S}\mathfrak{A}$. The algebra $\mathfrak{A}$ is said to be *smooth* if $\mathcal{S}\mathfrak{A} = \mathfrak{A}$, that is, $\mathfrak{A}$ is complete in the topology defined by all closable derived norms on $\mathfrak{A}$.

PROPOSITION 4.5. *The completion of $\mathfrak{A}$ in an arbitrary directed family of closable derived norms is smooth algebra.*

Proof. Let Λ be any directed family of closable derived norms on $\mathfrak{A}$. Take an arbitrary α in Λ. By Lemma 4.3, $\widetilde{\alpha}$ is a closable derived norm on $\widetilde{\mathfrak{A}}[\alpha]$, and by restriction, $\widetilde{\alpha}\lceil\widetilde{\mathfrak{A}}[\Lambda]$ is a closable derived norm on $\widetilde{\mathfrak{A}}[\Lambda]$, where $\widetilde{\mathfrak{A}}[\Lambda]$ is the completion in the topology defined by Λ. This implies that the completion $\mathcal{S}\widetilde{\mathfrak{A}}[\Lambda]$ of all closable derived norms on $\widetilde{\mathfrak{A}}[\Lambda]$ is contained in $\widetilde{\mathfrak{A}}[\Lambda]$. The converse inclusion $\widetilde{\mathfrak{A}}[\Lambda] \subset \mathcal{S}\widetilde{\mathfrak{A}}[\Lambda]$ is trivial. This completes the proof. ∎

We define a C^k-envelope $\mathcal{S}^k\mathfrak{A}$ of $\mathfrak{A}$. We denote by $\Lambda_{\text{cd}}^{\leqslant k}$ the set of all closable derived norms on $\mathfrak{A}$ of total order $\leqslant k$. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\text{cd}}^{\leqslant k})]$ of $\mathfrak{A}$ is said to be the *C^k-envelope* of $\mathfrak{A}$ denoted by $\mathcal{S}^k\mathfrak{A}$. All these $\mathcal{S}^k\mathfrak{A}$ are smooth algebras by

Proposition 4.5 and they have, by Lemma 4.3(iv),

$$\mathfrak{A} \subset \mathcal{S}\mathfrak{A} = \varprojlim_{k\to\infty} \mathcal{S}^k\mathfrak{A} = \varprojlim_{k\to\infty} \varprojlim_{\alpha\in\Lambda_{\mathrm{cd}}^{\leqslant k}} \widetilde{\mathfrak{A}}[\alpha] \subset \cdots \subset \mathcal{S}^{k+1}\mathfrak{A} \subset \mathcal{S}^k\mathfrak{A} \subset \cdots \subset \mathcal{A}.$$

Let Λ be an arbitrary directed family of closable derived norms on $\mathfrak{A}$. By Proposition 4.5 $\widetilde{\mathfrak{A}}[\tau(\Lambda)]$ is a smooth algebra and by Lemma 4.3(ii) the topology $\tau(\Lambda)$ on $\widetilde{\mathfrak{A}}[\tau(\Lambda)]$ is a smooth topology which is weaker than the topology $\tau(\Lambda_{\mathrm{cd}}(\widetilde{\mathfrak{A}}[\tau(\Lambda)]))$ defined by all closable derived norms on $\widetilde{\mathfrak{A}}[\tau(\Lambda)]$, but they do not necessarily coincide. So, we need to define the notion of topologically smooth algebras.

DEFINITION 4.6. A topological $*$-algebra $\mathcal{B}[\tau]$ contained in a C^*-algebra is said to be a *smooth topological algebra* if $\mathcal{B} = \mathcal{S}\mathcal{B}$ and τ coincides with the topology $\tau(\Lambda_{\mathrm{cd}})$ defined by all closable derived norms on $\mathcal{B}$.

A *smooth Banach algebra* (respectively a *smooth Fréchet algebra*) is a smooth algebra $\mathfrak{A} = \mathcal{S}\mathfrak{A}$ whose smooth topology $\tau(\Lambda_{\mathrm{cd}})$ is defined by a norm (respectively is metrizable).

COROLLARY 4.7. (i) $\widetilde{\mathfrak{A}}[\alpha]$ *is a smooth Banach $*$-algebra for a closable derived norm α on $\mathfrak{A}$.*

(ii) *A smooth algebra is an inverse limit of a directed family of smooth Banach $*$-algebras.*

(iii) *The completion of $\mathfrak{A}$ in a countable family of closable derived norms is a smooth Fréchet algebra.*

Proof. By open mapping theorem, $\widetilde{\mathfrak{A}}[\alpha] = \mathcal{S}\mathfrak{A}$, and the topology $\tau(\Lambda_{\mathrm{cd}})$ is defined by α. The assertion (iii) follows by an analogous argument, whereas (ii) follows from (i) and the definition of smooth envelope. ∎

THEOREM 4.8. *Let $\mathfrak{A}$ be a smooth algebra in a C^*-algebra $\mathcal{A}$. Then the following statements hold:*

(i) $\mathfrak{A}$ *is a dense $*$-subalgebra of $\mathcal{A}$ continuously embedded in $\mathcal{A}$.*

(ii) $\mathfrak{A}$ *is a complete locally m-convex $*$-algebra.*

(iii) $\mathfrak{A}$ *is an hermitian Q-algebra.*

(iv) $\mathfrak{A}$ *is spectrally invariant and $E(\mathfrak{A}) = \mathcal{A}$.*

(v) $\mathfrak{A}$ *is closed under holomorphic functional calculus of $\mathcal{A}$, and $\mathcal{A}$ and $\mathfrak{A}$ have the same K-theory.*

(vi) $\mathfrak{A}$ *is closed under the C^∞-functional calculus of self-adjoint elements.*

Proof. The statements (i) and (ii) are trivial. By Lemma 4.3(ii), for any $\alpha \in \Lambda_{\mathrm{cd}}$, $\widetilde{\mathfrak{A}}[\alpha]$ is an analytic Banach $*$-algebra, and so it follows from Proposition 3.12, Proposition 6.4 of [14] that the statements (iii)–(vi) hold for $\widetilde{\mathfrak{A}}[\alpha]$. Since $\mathfrak{A} =$

$\varprojlim_{\alpha\in\Lambda_{\mathrm{cd}}} \widetilde{\mathfrak{A}}[\alpha]$, it is shown similarly to Theorem 3.3 and Theorem 3.4 that the statements (iii)–(vi) hold for $\mathfrak{A}$. ∎

For the further properties of smooth algebras, we have the following. For the sake of completeness, we incorporate relevant results from [14].

PROPOSITION 4.9. *Let $\mathfrak{A}$ be a smooth algebra in the C^*-algebra $\mathcal{A}$. Then the following statements hold:*

(i) *Any closed $*$-subalgebra of $\mathfrak{A}$ is a smooth algebra in the relative topology.*

(ii) *Let J be a closed $*$-ideal of the C^*-algebra $\mathcal{A}$. Then the following hold:*

(a) *$J \cap \mathfrak{A}$ is dense in J.*

(b) *Suppose that $\mathfrak{A}/J\cap\mathfrak{A}$ is complete in the quotient topology τ_{q}. Then $\mathfrak{A}/J\cap\mathfrak{A}$ is a smooth algebra and τ_{q} equals to the topology defined by all closable derived norms on $\mathfrak{A}/J\cap\mathfrak{A}$. In particular, if $\mathfrak{A}$ is a smooth Fréchet algebra in $\mathcal{A}$, then $(\mathfrak{A}/J\cap\mathfrak{A})[\tau_{\mathrm{q}}]$ is a smooth Fréchet algebra.*

(c) *$J\cap\mathfrak{A}$ with the relative topology is a smooth algebra.*

(iii) *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be smooth algebras, and ϕ a $*$-homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. Then the following hold:*

(a) *ϕ is norm decreasing for C^*-norms.*

(b) *ϕ is continuous for the smooth structures.*

(c) *If ϕ is injective, then it is an isometry for C^*-algebras.*

(d) *$\phi(\mathfrak{A}_1)$ is a smooth algebra.*

(iv) *Let $\delta : \mathfrak{A} \to \mathfrak{A}$ be a closable $*$-derivation. Then δ extends to a $*$-derivation $\delta : \mathcal{S}\mathfrak{A} \to \mathcal{S}\mathfrak{A}$. If $\mathfrak{A}$ is smooth, then δ is continuous. Every closable differential norm on a smooth algebra is continuous in the smooth structure.*

Proof. For (b) and (c) in (ii) we remark that Blackadar and Cuntz have shown in Proposition 6.7 of [14] that $\mathfrak{A}/J\cap\mathfrak{A}$ is always a smooth algebra, and $J\cap\mathfrak{A}$ is a smooth algebra if $\mathfrak{A}$ is metrizable. But we think that we need the assumption of the completeness of $(\mathfrak{A}/J\cap\mathfrak{A})[\tau_{\mathrm{q}}]$ in order that the former holds, and the latter holds without the assumption of metrizability. So, we here prove only the statements (ii)(b) and (ii)(c). The other statements are shown in Proposition 6.7, 6.8 of [14] .

(ii)(b) Let α be a derived norm on $\mathfrak{A}$. It is easily shown that α_{q} is a derived norm on $\mathfrak{A}/J\cap\mathfrak{A}$. Suppose α is closable. Then we show that α_{q} is closable. Indeed, take an arbitrary sequence $\{x_n\}$ in $\mathfrak{A}$ such that $\|x_n + J\cap\mathfrak{A}\| \equiv \inf\{\|x_n+y\| : y \in J\cap\mathfrak{A}\} \xrightarrow[n\to\infty]{} 0$ and $\{x_n + J\cap\mathfrak{A}\}$ is α_{q}-Cauchy. We may assume that $\alpha_{\mathrm{q}}((x_{n+1}-x_n)+J\cap\mathfrak{A}) < 1$ for each $n \in \mathbb{N}$. Then there exists a sequence $\{z_n\}$ in $J\cap\mathfrak{A}$ such that:

$$\alpha(x_1+z_1) \leqslant \alpha_{\mathrm{q}}(x_1+J\cap\mathfrak{A}) + \frac{1}{2},$$

$$\alpha((x_n-x_{n-1})+(z_n-z_{n-1})) \leqslant \alpha_{\mathrm{q}}((x_n-x_{n-1})+J\cap\mathfrak{A}) + \frac{1}{2^n}, \quad n = 2,3,\dots.$$

Here we put $y_n = x_n + z_n \in \mathfrak{A}, n \in \mathbb{N}$. Then

$$\|y_n + J \cap \mathfrak{A}\| = \|x_n + J \cap \mathfrak{A}\| \xrightarrow[n\to\infty]{} 0, \quad \alpha(y_m - y_n) \leqslant \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m}, \quad m > n.$$

Since α is closable, we have $\lim_{n\to\infty} \alpha(y_n) = 0$, which implies

$$\lim_{n\to\infty} \alpha_{\mathrm{q}}(x_n + J \cap \mathfrak{A}) \leqslant \lim_{n\to\infty} \alpha(x_n + z_n) = \lim_{n\to\infty} \alpha(y_n) = 0.$$

Thus, α_{q} is closable, therefore τ_{q} is the topology defined by the family $\{\alpha_{\mathrm{q}} : \alpha \in \Lambda_{\mathrm{cd}}\}$ of closable derived norms on $\mathfrak{A}/J \cap \mathfrak{A}$. Since $(\mathfrak{A}/J \cap \mathfrak{A})[\tau_{\mathrm{q}}]$ is complete, it follows from Proposition 4.5 that $(\mathfrak{A}/J \cap \mathfrak{A})[\tau_{\mathrm{q}}]$ is a smooth algebra. To prove that the quotient topology τ_{q} is the smooth topology defined by the family $\Lambda(\mathfrak{A}/J \cap \mathfrak{A})$ of all closable derived norms on $\mathfrak{A}/J \cap \mathfrak{A}$, it is sufficient to prove that each $\rho \in \Lambda_{\mathrm{cd}}(\mathfrak{A}/J \cap \mathfrak{A})$ is τ_{q} continuous. Take any such ρ, and define $\sigma : \mathfrak{A} \to \mathbb{R}, \sigma(x) \equiv \rho(\pi(x))(x \in \mathfrak{A})$, where $\pi : A \to \mathfrak{A}/J \cap \mathfrak{A}, \pi(x) \equiv x + J \cap \mathfrak{A}$ is the quotient map. First we show that σ is a closable derived seminorm on $\mathfrak{A}$. Indeed if (x_n) is a sequence in $\mathfrak{A}$ such that $\sigma(x_n - x_m) \to 0$ and $\|x_n\| \to 0$, then for the quotient C^*-norm $\|\cdot\|_{\mathrm{q}}$ on $\mathfrak{A}/J \cap \mathfrak{A}$, $\|\pi(x_n)\|_{\mathrm{q}} \leqslant \|x_n\| \to 0$ and also $\rho(\pi(x_n) - \pi(x_m)) = \sigma(x_n - x_m) \to 0$. Since ρ is closable, $\pi(x_n) \to 0$ in ρ. Hence $x_n \to 0$ in σ. Thus σ is closable. Notice that σ need not be a norm. By p. 273 of [14], σ is a derived seminorm on $\mathfrak{A}$. Now there exists a C^*-normed algebra $(\mathcal{B}, |\cdot|)$, a surjective norm decreasing $*$-homomorphism $\phi : \mathcal{B} \to \mathfrak{A}$ and a differential seminorm $T = (T_k)$ on $\mathcal{B}$ such that $\sigma(x) = \inf\{T_{\mathrm{tot}}(y) : \phi(y) = x\}$. Define $T' = (T'_k)$ on $\mathcal{B}$ as $T'_0(y) = |y| + |y|'$, where $|y|' = \|x\|$ for $\phi(y) = x$, and $T'_k(y) \equiv T_k(y)$ for all k. Then T' is a differential seminorm on $\mathcal{B}$. For any $x \in \mathfrak{A}$,

$$\theta(x) = \inf\{T'_{\mathrm{tot}}(y) : \phi(y) = x\} = \inf\{|y| + |y|' : \phi(y) = x\} \geqslant \|x\| + \sigma(x) \geqslant \sigma(x).$$

Thus $\theta(x) = 0$ implies that $x = 0$ showing that θ is a derived norm on $\mathfrak{A}$. We show that $\theta \in \Lambda_{\mathrm{cd}}(\mathfrak{A})$, which in turn would imply that $\sigma \in \Lambda_{\mathrm{cd}}(\mathfrak{A})$. For this we need to prove that θ is closable. Let (x_n) be a sequence in $\mathfrak{A}$ such that $\theta(x_n - x_m) \to 0, \|x_n\| \to 0$. Then $\sigma(x_n - x_m) \to 0$ by above, and by the closability of σ, $\inf\{T_{\mathrm{tot}}(y_n) : \phi(y_n) = x_n\} = \sigma(x_n) \to 0$. Hence for each $n \in \mathbb{N}$, there exists $y_n \in \mathcal{B}$ such that $\phi(y_n) = x_n$ and $T_{\mathrm{tot}}(y_n) < \frac{1}{n}$. Thus $T_{\mathrm{tot}}(y_n) \to 0$. Also $|y_n|' = \|x_n\| \to 0$. Thus $\inf\{T'_{\mathrm{tot}}(z) : \phi(z) = x_n\} \leqslant T'_{\mathrm{tot}}(y_n) \to 0$. Hence θ is closable. It follows that σ_{q} is τ_{q}-continuous and

$$\begin{aligned}\sigma_{\mathrm{q}}(\pi(x)) &= \inf\{\sigma(x+z) : z \in J \cap \mathfrak{A}\} = \inf\{\rho(\pi(x) + \pi(z)) : z \in J \cap \mathfrak{A}\} \\ &= \inf\{\rho(\pi(x)) : z \in J \cap \mathfrak{A}\} = \rho(\pi(x)),\end{aligned}$$

for all $x \in \mathfrak{A}$, which implies ρ is τ_{q}-continuous. Suppose that $\mathfrak{A}$ is Fréchet. Then $(\mathfrak{A}/J \cap \mathfrak{A})[\tau_{\mathrm{q}}]$ is complete, and so $(\mathfrak{A}/J \cap \mathfrak{A})[\tau_{\mathrm{q}}]$ is a smooth Fréchet algebra.

(c) Since $J \cap \mathfrak{A}$ is a closed $*$-subalgebra of $\mathfrak{A}$, it follows from (i) that $J \cap \mathfrak{A}$ is a smooth algebra. ∎

We consider the other smooth structures that are useful in constructing examples. We denote by Λ^f_{cd} the set of all closable derived norms on $\mathfrak{A}$ of finite order, and by Λ_{c}^{f} the set of all closable differential norms of finite order. By Lemma 4.1, $\Lambda_{\mathrm{c}}^{f} \subset \Lambda_{\mathrm{cd}}^{f} \subset \Lambda_{\mathrm{cd}}$.

DEFINITION 4.10. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\mathrm{cd}}^{f})]$ of $\mathfrak{A}$ is said to be the *derived strongly smooth envelope* of $\mathfrak{A}$ denoted by $\mathcal{S}_{\mathrm{ds}}\mathfrak{A}$. The algebra $\mathfrak{A}$ is called *derived strongly smooth* if $\mathfrak{A} = \mathcal{S}_{\mathrm{ds}}\mathfrak{A}$. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\mathrm{c}}^{f})]$ of $\mathfrak{A}$ is said to be the *strongly smooth envelope* of $\mathfrak{A}$ denoted by $\mathcal{S}_{\mathrm{s}}\mathfrak{A}$. The algebra $\mathfrak{A}$ is called *strongly smooth* if $\mathfrak{A} = \mathcal{S}_{\mathrm{s}}\mathfrak{A}$.

We can show similarly to the proof of Proposition 4.5 that the completion $\widetilde{\mathfrak{A}}[\tau(\Lambda)]$ of $\mathfrak{A}$ is a derived strongly smooth algebra (respectively a strongly smooth algebra) for each directed family Λ of closable derived norms (respectively closable differential norms) of finite order. The C^k-envelopes $\mathcal{S}_{\mathrm{ds}}^{k}\mathfrak{A}$ and $\mathcal{S}_{\mathrm{s}}^{k}\mathfrak{A}$ of $\mathfrak{A}$ are defined as in case of $\mathcal{S}^k\mathfrak{A}$. We denote by $\Lambda_{\mathrm{cd}}^{\overset{\leqslant k}{d}}$ the set of all closable derived norms of order $\leqslant k$, and by $\Lambda_{\mathrm{c}}^{\overset{\leqslant k}{d}}$ the set of all closable differential norms of order $\leqslant k$. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\mathrm{cd}}^{\overset{\leqslant k}{d}})]$ (respectively $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\mathrm{c}}^{\overset{\leqslant k}{d}})]$) of $\mathfrak{A}$ is said to be the *derived strong C^k-envelope* (respectively the *strong C^k-envelope*) of $\mathfrak{A}$ denoted by $\mathcal{S}_{\mathrm{ds}}^{k}\mathfrak{A}$ (respectively $\mathcal{S}_{\mathrm{s}}^{k}\mathfrak{A}$). All $\mathcal{S}_{\mathrm{ds}}^{k}\mathfrak{A}$ are derived strongly smooth algebras and all $\mathcal{S}_{\mathrm{s}}^{k}\mathfrak{A}$ are strongly smooth algebras, and

$$\left.\begin{array}{lllll}\mathfrak{A} \subset & \mathcal{S}\mathfrak{A} \subset & \cdots & \subset \mathcal{S}^{k+1}\mathfrak{A} \subset \mathcal{S}^{k}\mathfrak{A} \subset \cdots \\ & \cap & & \\ & \mathcal{S}_{\mathrm{ds}}\mathfrak{A} = \varprojlim_{k\to\infty} \mathcal{S}_{\mathrm{ds}}^{k}\mathfrak{A} \subset \cdots & & \subset \mathcal{S}_{\mathrm{ds}}^{k+1}\mathfrak{A} \subset \mathcal{S}_{\mathrm{ds}}^{k}\mathfrak{A} \subset \cdots \\ & \cap & & \quad\cap \\ & \mathcal{S}_{\mathrm{s}}\mathfrak{A} = \varprojlim_{k\to\infty} \mathcal{S}_{\mathrm{s}}^{k}\mathfrak{A} \subset \cdots & & \subset \mathcal{S}_{\mathrm{s}}^{k+1}\mathfrak{A} \subset \mathcal{S}_{\mathrm{s}}^{k}\mathfrak{A} \subset \cdots\end{array}\right\} \subset \mathcal{A}.$$

We do not know the relation between $\mathcal{S}^k\mathfrak{A}$ and $\mathcal{S}_{\mathrm{ds}}^{l}\mathfrak{A}$, $\mathcal{S}_{\mathrm{s}}^{l}\mathfrak{A}$. For strongly smooth algebras we have the following

PROPOSITION 4.11. *The differential Fréchet $*$-algebra $\mathfrak{A}_{\tau(T)}$ defined by a closable differential norm T is a strongly smooth algebra, and $\mathcal{S}_{\mathrm{s}}\mathfrak{A} = \varprojlim_{T\in\Lambda_{\mathrm{c}}} \mathfrak{A}_{\tau(T)}$, where Λ_{c} is the set of all closable differential norms on $\mathfrak{A}$.*

Proof. The topology $\tau(T)$ is defined by the sequence $T^{(n)} = (T_0, T_1, \ldots, T_n, 0, \ldots)$ of closable differential norms of finite order such that $p_n = \sum_{i=0}^{n} T_i = T_{\mathrm{tot}}^{(n)}$. Thus $\mathfrak{A}_{\tau(t)} = \varprojlim_{n} \widetilde{\mathfrak{A}}[T_{\mathrm{tot}}^{(n)}]$ and is a strongly smooth algebra. Furthermore, we have

$$\mathcal{S}_{\mathrm{s}}\mathfrak{A} = \varprojlim_{T\in\Lambda_{\mathrm{c}}^{f}} \widetilde{\mathfrak{A}}[T_{\mathrm{tot}}] = \varprojlim_{T\in\Lambda_{\mathrm{c}}} \mathfrak{A}_{\tau(T)}. \quad \blacksquare$$

We develop the analytic structure defined on a C^*-normed algebra using the totality of differential norms, and this is obtained by taking appropriate limit algebras of the algebras $\widetilde{\mathfrak{A}^{\omega}}$ and $\widetilde{\mathfrak{A}^{\mathrm{e}\omega}}$ considered in the previous Section 3. Though our approach to analyticity is experimental and tentative requiring further refinement, it does lead to several algebras with smoothness properties. Here we consider a family of (entire) analytic differential norms. Hence, for an analytic differential norm T on $\mathfrak{A}$ we denote $\mathfrak{A}^{\omega}$ considered in the previous Section 3 by $\mathfrak{A}^{\omega}(T)$, that is, $\mathfrak{A}^{\omega}(T) = \bigcup_{t>\ell^{\omega}} \ell^1(\mathfrak{A}, T(t)) = \varinjlim_{t>\ell^{\omega}} \ell^1(\mathfrak{A}, T(t))$ and for an entire analytic differential norm denote $\mathfrak{A}^{e\omega}$ by $\mathfrak{A}^{e\omega}(T)$, that is, $\mathfrak{A}^{e\omega}(T) = \bigcap_{t>0} \ell^1(\mathfrak{A}, T(t)) = \varprojlim_{t\to\infty} \ell^1(\mathfrak{A}, T(t))$. Thus $\mathfrak{A}^{\omega}(T)$ is a locally convex $*$-algebra, whereas $\mathfrak{A}^{e\omega}(T)$ is a locally m-convex $*$-algebra.

Here we define the analytic envelope and the entire analytic envelope of $\mathfrak{A}$. We denote by Λ_{ω} (respectively $\Lambda_{e\omega}$) the set of all closable differential norms T on $\mathfrak{A}$ such that $(\mathfrak{A}, T)$ is analytic (respectively entire analytic) in the sense of Definition 3.5. Let $\tau(\Lambda_{e\omega})$ be the locally convex topology on $\mathfrak{A}$ defined by $\{T_{\mathrm{tot}} : T \in \Lambda_{e\omega}\}$ and $\tau(\Lambda_{\omega})$ the locally convex topology on $\mathfrak{A}$ defined by a family $\{\tau^{\omega}(T) : T \in \Lambda_{\omega}\}$ of the inductive limit topologies $\tau^{\omega}(T)$.

DEFINITION 4.12. The completion $\widetilde{\mathfrak{A}}[\tau(\Lambda_{\omega})]$ (respectively $\widetilde{\mathfrak{A}}[\tau(\Lambda_{e\omega})]$) of $\mathfrak{A}$ is said to be the *analytic envelope* (respectively *entire analytic envelope*) of $\mathfrak{A}$ and denoted by $\mathcal{S}_{\omega}\mathfrak{A}$ (respectively $\mathcal{S}_{e\omega}\mathfrak{A}$). The algebra $\mathfrak{A}$ is said to be *analytic* (respectively *entire analytic*) if $\mathfrak{A} = \mathcal{S}_{\omega}\mathfrak{A}$ (respectively $\mathfrak{A} = \mathcal{S}_{e\omega}\mathfrak{A}$).

We remark that Blackadar and Cuntz in [14] have called an *analytic algebra* a C^*-normed algebra that is a Banach $*$-algebra with respect to an analytic algebra $*$-norm. Here we call such a C^*-normed algebra an *analytic Banach $*$-algebra*.

Then $\mathcal{S}_{\omega}\mathfrak{A}$ and $\mathcal{S}_{e\omega}\mathfrak{A}$ are locally convex $*$-algebras and

$$\mathcal{S}_{\omega}\mathfrak{A} = \varprojlim_{T\in\Lambda_{\omega}} \widetilde{\mathfrak{A}^{\omega}}(T) = \varprojlim_{T\in\Lambda_{\omega}} \varinjlim_{t\downarrow 0} \mathfrak{A}_{T(t)}[T(t)_{\mathrm{tot}}],$$

$$\mathcal{S}_{e\omega}\mathfrak{A} = \varprojlim_{T\in\Lambda_{e\omega}} \widetilde{\mathfrak{A}^{e\omega}}(T) = \varprojlim_{T\in\Lambda_{e\omega}} \varprojlim_{t\to\infty} \mathfrak{A}_{T(t)}[T(t)_{\mathrm{tot}}].$$

Concerning $\mathcal{S}\mathfrak{A}, \mathcal{S}_{\mathrm{ds}}\mathfrak{A}, \mathcal{S}_{\mathrm{s}}\mathfrak{A}, \mathcal{S}_{\omega}\mathfrak{A}$ and $\mathcal{S}_{e\omega}\mathfrak{A}$, it is easily shown that the topologies $\tau(\Lambda_{\mathrm{cd}}), \tau(\Lambda^f_{\mathrm{cd}}), \tau(\Lambda_{e\omega})$ and $\tau(\Lambda_{\omega})$ are finer than the topology $\tau(\Lambda^f_{\mathrm{c}})$, and that $\mathcal{S}\mathfrak{A}, \mathcal{S}_{\mathrm{ds}}\mathfrak{A}, \mathcal{S}_{e\omega}\mathfrak{A}$ and $\mathcal{S}_{\omega}\mathfrak{A}$ are contained in $\mathcal{S}_{\mathrm{s}}\mathfrak{A}$. Here if $\mathfrak{A}$ is smooth, then it is derived strongly smooth, smooth, entire analytic and analytic. For completion of $\mathfrak{A}$ with respect to a family of (entire) analytic closable differential norms we have the following.

PROPOSITION 4.13. (i) *Let Λ be a family of closable differential norms T on $\mathfrak{A}$ such that $(\mathfrak{A}, T)$ is entire analytic, and $\tau_{e\omega}(\Lambda)$ the locally convex topology on $\mathfrak{A}$ defined by $\{T(t)_{\mathrm{tot}} : T \in \Lambda, t > 0\}$. Then $\widetilde{\mathfrak{A}}[\tau_{e\omega}(\Lambda)]$ is entire analytic.*

(ii) *Let Λ be a family of closable differential norms T on $\mathfrak{A}$ such that $(\mathfrak{A}, T)$ is analytic, and $\tau_{\omega}(\Lambda)$ the locally convex topology on $\mathfrak{A}$ defined by $\{\tau^{\omega}(T) : T \in \Lambda\}$. Then $\widetilde{\mathfrak{A}}[\tau_{\omega}(\Lambda)]$ is analytic.*

Proof. These are proved in the same way as the result of Proposition 4.5. ∎

The same results as Theorem 4.8 and Proposition 4.9 obtained for the smooth algebra $\mathcal{S}\mathfrak{A}$ hold for $\mathcal{S}_{\mathrm{ds}}\mathfrak{A}$, $\mathcal{S}_{\mathrm{s}}\mathfrak{A}$, $\mathcal{S}_{\mathrm{e}\omega}$ and $\mathcal{S}_{\omega}$.

5. C^k-STRUCTURES

In this section we quickly review the C^k-structures introduced by Blackadar and Cuntz. This leads us to the study of abstract C^∞-algebras. A C^∞-algebra is smooth, but not every smooth algebra is a C^∞-algebra. We investigate this class of smooth subalgebras of C^*-algebras. An *($\mathbb{N}_0$-)graded $*$-algebra* is a $*$-algebra $\mathcal{B}$ equipped with projections $P_k, k \in \mathbb{N}_0$, $P_iP_j = 0, i \neq j$, onto involutive subspaces $\mathcal{B}_k \equiv P_k\mathcal{B}$ of $\mathcal{B}$ such that $\mathcal{B}_i\mathcal{B}_j \subset \mathcal{B}_{i+j}$ and such that $\sum\limits_k P_k = \mathrm{id}$. This implies the important relation $P_k(xy) = \sum\limits_{i+j=k}(P_ix)(P_jy)$, $x, y \in \mathcal{B}$. For a graded algebra $\mathfrak{A}$, let $\partial\mathfrak{A}$ denote the set of elements of $\mathfrak{A}$ of degree $\geqslant 1$. An $\mathbb{N}_0$-graded $*$-algebra $\mathcal{B}$ is said to be a *Banach ($\mathbb{N}_0$)-graded $*$-algebra* if $\mathcal{B}$ is a Banach $*$-algebra, the algebraic direct sum $\bigoplus\limits_k \mathcal{B}_k$ of the $\mathcal{B}_k$ is dense in $\mathcal{B}$, and $\|P_k\| \leqslant 1$, $\forall k \in \mathbb{N}_0$. A *graded Banach $*$-algebra* $\mathcal{B}$ is a Banach $*$-algebra $(\mathcal{B}, \|\cdot\|)$ together with projections $P_k, k \in \mathbb{N}_0$, onto involutive subspaces $\mathcal{B}_k = P_k(\mathcal{B})$ of $\mathcal{B}$ such that $P_iP_j = 0$ for $i \neq j$, $\mathcal{B}_i\mathcal{B}_j \subset \mathcal{B}_{i+j}$, the algebraic direct sum $\sum \mathcal{B}_k$ is dense in $\mathcal{B}$, $\|P_k\| \leqslant 1$, $x = \sum\limits_k P_kx$ in the norm for all $x \in \mathfrak{A}$, and $\sum\limits_k \|P_kx\| < \infty$ for all $x \in \mathcal{B}$. A $*$-homomorphism ϕ of an $\mathbb{N}_0$-graded $*$-algebra $\mathcal{B}_1$ into an $\mathbb{N}_0$-graded $*$-algebra $\mathcal{B}_2$ is said to be *graded* if $\phi(P_k^{\mathcal{B}_1}x) = P_k^{\mathcal{B}_2}\phi(x), \forall x \in \mathcal{B}_1, \forall k \in \mathbb{N}_0$. Let $\mathfrak{A}[\|\cdot\|]$ be a C^*-normed algebra. Let $D\mathfrak{A}$ be the universal graded $*$-algebra over $\mathfrak{A}$ containing $\mathfrak{A}$ and closed under a $*$-derivation. This is the universal $*$-algebra generated by the symbols $d^k(x), x$ in $\mathfrak{A}, k$ in $\mathbb{N}$, that are linear in x and satisfy the relations: $d^k(xy) = \sum\limits_{i+j=k} d^i(x)d^j(y), d^k(x)^* = -d^k(x^*)$. A canonical derivation $\delta : D\mathfrak{A} \to D\mathfrak{A}$ is defined as

$$\delta(d^{i_1}(x_1)d^{i_2}(x_2)\cdots d^{i_n}(x_n)) = \sum_{j=1}^{n}(i_j+1)d^{i_1}(x_1)\cdots d^{i_j+1}(x_j)\cdots d^{i_n}(x_n).$$

Thus, $d^j(x) = \frac{\delta^j(x)}{j!}$. The $*$-algebra $D\mathfrak{A}$ is generated by $\mathfrak{A} \simeq d^0(\mathfrak{A})$ and $\delta^j(\mathfrak{A}), j \in \mathbb{N}$, and $d^0 : x \to d^0(x)$ is a $*$-homomorphism of $\mathfrak{A}$ into $D\mathfrak{A}$. The $*$-algebra $D\mathfrak{A}$ has the following universal properties:

(5.1) If $\phi : \mathfrak{A} \to \mathcal{B}$ is a $*$-homomorphism into a $*$-algebra $\mathcal{B}$ with a $*$-derivation $\delta' : \mathcal{B} \to \mathcal{B}$, then there exists a unique $*$-homomorphism $\psi : D\mathfrak{A} \to \mathcal{B}$ such that $\psi(x) = \phi(x), \forall x \in \mathfrak{A}$ and $\delta' \circ \psi = \psi \circ \delta$.

(5.2) There exists a formal $*$-homomorphism $\mathrm{e}^d : \mathfrak{A} \to D\mathfrak{A}$, $\mathrm{e}^d(x) = \sum\limits_{i=0}^{\infty} d^i(x)$ such that if $\phi : \mathfrak{A} \to \mathcal{B}$ is a $*$-homomorphism into an $\mathbb{N}_0$-graded $*$-algebra

$\mathcal{B}$, then there exists a unique graded $*$-homomorphism $\psi : D\mathfrak{A} \to \mathcal{B}$ such that $\phi = \psi \circ \mathrm{e}^d$.

The homomorphism e^d is well-defined if $D\mathfrak{A}$ is divided by the ideal J_k of elements of degree greater than k, or if $D\mathfrak{A}$ is completed in an appropriate topology. We consider the notion of C^*-graded seminorms [14] on $D\mathfrak{A}$ to define the C^k-structure. An algebra $*$-seminorm α on $D\mathfrak{A}$ is called a *C^*-graded algebra seminorm* if $\exists C > 0 : \alpha(d^0(x)) \leqslant C\|x\|, \forall x \in \mathfrak{A}$, and $\alpha(P_k y) \leqslant \alpha(y), \forall y \in D\mathfrak{A}$ where $P_k \equiv P_k^{D\mathfrak{A}}$ is the projection of $D\mathfrak{A}$ onto the kth degree part of $D\mathfrak{A}$. Let α be a C^*-graded algebra seminorm on $D\mathfrak{A}$. We say that:

- α is of *finite order k* if $\alpha(d^l(x)) = 0, \forall x \in \mathfrak{A}, \forall l > k$, and for some $y \in \mathfrak{A}, \alpha(d^k(y)) \neq 0$.
- α is of *ℓ^1-type on $\mathfrak{A}$* if $\sum_{k=0}^{\infty} \alpha(d^k(x)) < \infty, \forall x \in \mathfrak{A}$.
- α is of *ℓ^1-type on $D\mathfrak{A}$* if $\sum_{k=0}^{\infty} \alpha(P_k z) < \infty, \forall z \in D\mathfrak{A}$.
- α is of *strongly ℓ^1-type on $D\mathfrak{A}$* if $\alpha(z) = \sum_{k=0}^{\infty} \alpha(P_k z), \forall z \in D\mathfrak{A}$.

LEMMA 5.1. *Let α be a C^*-graded algebra seminorm on $D\mathfrak{A}$. Define $T_\alpha = (T_k^\alpha)$ by $T_k^\alpha(x) = \alpha(P_k \mathrm{e}^d(x)) = \alpha(d^k(x)), x \in \mathfrak{A}$. Then T_α is a differential seminorm on $\mathfrak{A}$ such that the following hold:*
(i) *T_α is of finite order k if and only if α is of finite order k.*
(ii) *T_α is ℓ^1-summable if and only if α is of ℓ^1-type on $\mathfrak{A}$.*
T_α is called the differential seminorm on $\mathfrak{A}$ induced by α.

We discuss some technical results describing a relationship between C^*-graded algebra seminorms on $D\mathfrak{A}$ and differential seminorms on $\mathfrak{A}$.

LEMMA 5.2. *α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ if and only if α is induced by a graded $*$-homomorphism of $D\mathfrak{A}$ into a graded $*$-algebra with algebra norm whose restriction to degree zero part is continuous with respect to the C^*-norm on $\mathfrak{A}$.*

Proof. Let α be a C^*-graded algebra seminorm on $D\mathfrak{A}$. We put $N(\alpha) = \{z \in D\mathfrak{A} : \alpha(z) = 0\}$. Then $N(\alpha)$ is a graded $*$-ideal of $D\mathfrak{A}$, that is, $P_k N(\alpha) \subset N(\alpha), \forall k$, and $\mathcal{B}_\alpha \equiv D\mathfrak{A}/N(\alpha)$ is a graded $*$-algebra, the gradation being defined by the maps $P_k^{\mathcal{B}_\alpha} : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$, $P_k^{\mathcal{B}_\alpha}(z + N(\alpha)) \equiv P_k z + N(\alpha), z \in D\mathfrak{A}$. This is well-defined. Now $\mathcal{B}_\alpha$ is a normed $*$-algebra with the norm defined by $\|z + N(\alpha)\|_\alpha = \alpha(z)$. Thus $\mathcal{B}_\alpha$ is a graded $*$-algebra with the algebra norm $\|\cdot\|_\alpha$. Let $\pi_\alpha : D\mathfrak{A} \to \mathcal{B}_\alpha$ be $\pi_\alpha(z) \equiv z + N(\alpha)$. Then clearly π_α is a graded $*$-homomorphism and $\|\pi_\alpha(z)\|_\alpha = \alpha(z)$, $\pi_\alpha(P_k z) = P_k^{\mathcal{B}_\alpha} \pi_\alpha(z), \forall z \in D\mathfrak{A}$. Also, the restriction of π_α to the degree zero part $d^0(\mathfrak{A}) \simeq \mathfrak{A}$ is continuous with respect to the C^*-norm $\|\cdot\|$. The converse is easily shown. ∎

For a characterization of C^*-graded algebra seminorms on $D\mathfrak{A}$ which are of ℓ^1-type on $\mathfrak{A}$, we introduce the notion of flat differential seminorms defined by

Blackadar and Cuntz [14]: A differential seminorm T on $\mathfrak{A}$ is said to be *flat* if T can realized as $T_k(x) = \|P_k^{\mathcal{B}}\phi(x)\|_{\mathcal{B}}$, $k \geqslant 0$, for a $*$-homomorphism $\phi : \mathfrak{A} \to \mathcal{B}$ from $\mathfrak{A}$ into a graded $*$-algebra $\mathcal{B}$ equipped with an algebra seminorm $\|\cdot\|_{\mathcal{B}}$. Suppose that a differential seminorm T is ℓ^1-summable. Then it follows from Theorem 4.4 of [14] that T is flat if and only if there exists a C^*-seminormed algebra $\mathfrak{A}_1[\|\cdot\|_1]$ containing $\mathfrak{A}$ as a $*$-subalgebra and a derivation δ_1 on $\mathfrak{A}_1$ such that $T_0(x) = \|x\|_1, \forall x \in \mathfrak{A}$ and $T_k(x) = \frac{1}{k!}\|\delta_1^k(x)\|_1, \forall x \in \mathfrak{A}$. It follows that there exists a non-zero C^*-graded algebra seminorm on $D\mathfrak{A}$ if and only if there exists a non-zero flat differential seminorm on $\mathfrak{A}$; and there exists a separating family of C^*-graded algebra seminorms on $D\mathfrak{A}$ if and only if $\mathfrak{A}$ is embeded in a pro(jective limit of-) C^*-algebra(s) that admits a separating family of densely defined $*$-derivations.

LEMMA 5.3. *The map $\alpha \to T_\alpha$ is surjective from the set of all C^*-graded algebra seminorms α on $D\mathfrak{A}$ which are of ℓ^1-type on $\mathfrak{A}$ (respectively of order k) onto the set of all flat ℓ^1-summable differential seminorms on $\mathfrak{A}$ (respectively of order k). If T_α is closable, then $\alpha \circ d^j$ is closable for each j.*

Proof. Let α be a C^*-graded algebra seminorm on $D\mathfrak{A}$ of ℓ^1-type on $\mathfrak{A}$. Then e^d is defined on $\mathcal{B}_\alpha$ and $\phi \equiv \pi_\alpha \circ \mathrm{e}^d$ is a $*$-homomorphism of $\mathfrak{A}$ into $\mathcal{B}_\alpha$, where $(\mathcal{B}_\alpha, \pi_\alpha)$ is defined in Lemma 5.2 . Let $T_\alpha = (T_k^\alpha)$ be the differential seminorm on $\mathfrak{A}$ induced by α. Then we have

$$\|P_k^{\mathcal{B}_\alpha}(\phi(x))\|_\alpha = \|P_k(\mathrm{e}^d(x)) + N(\alpha)\|_\alpha = \|d^k(x) + N(\alpha)\|_\alpha = \alpha(d^k(x)) = T_k^\alpha(x),$$

for each $k \in \mathbb{N}$ and $x \in \mathfrak{A}$. Hence T_α is a flat ℓ^1-summable differential seminorm on $\mathfrak{A}$ such that $\alpha(d^k(x)) = T_k^\alpha(x), \forall k \in \mathbb{N}, \forall x \in \mathfrak{A}$. Conversely suppose T is a flat ℓ^1-summable differential seminorm on $\mathfrak{A}$. Then T is realized as $T_k(x) = \|P_k^{\mathcal{B}}\phi(x)\|_{\mathcal{B}}, k \geqslant 0$, for a $*$-homomorphism $\phi : \mathfrak{A} \to \mathcal{B}$ from $\mathfrak{A}$ into a graded $*$-algebra $\mathcal{B}$ equipped with an algebra seminorm $\|\cdot\|_{\mathcal{B}}$. By (5.1) there exists a unique graded $*$-homomorphism $\psi : D\mathfrak{A} \to \mathcal{B}$ such that $\phi = \psi \circ \mathrm{e}^d$. Here we put $\alpha(z) = \|\psi(z)\|_{\mathcal{B}}, z \in D\mathfrak{A}$. Then α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ and $\alpha(d^k(x)) = \|\psi(d^k(x))\|_{\mathcal{B}} = \|\psi(P_k\mathrm{e}^d(x))\|_{\mathcal{B}} = \|P_k^{\mathcal{B}}\psi(\mathrm{e}^d(x))\|_{\mathcal{B}} = \|P_k^{\mathcal{B}}\phi(x)\|_{\mathcal{B}} = T_k(x)$ for each $x \in \mathfrak{A}$. Hence α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is of ℓ^1-type such that $T_\alpha = T$. Thus the map $\alpha \to T_\alpha$ is surjective. This completes the proof. ∎

For C^*-graded algebra seminorms of ℓ^1-type on $D\mathfrak{A}$, we have the following.

LEMMA 5.4. *α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is of ℓ^1-type on $D\mathfrak{A}$ if and only if α is induced by a graded $*$-homomorphism from $D\mathfrak{A}$ into a graded Banach $*$-algebra such that its restriction to the degree zero part is continuous with respect to the C^*-norm on $\mathfrak{A}$. If this is true, then there exists a C^*-graded algebra seminorm α' of strongly ℓ^1-type on $D\mathfrak{A}$ such that α is equivalent to the quotient of α' and the seminorm $\alpha' \circ \mathrm{e}^d$ on $\mathfrak{A}$ is the total seminorm of the differential seminorm defined by α'.*

Proof. Suppose α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is of ℓ^1-type on $D\mathfrak{A}$. Let $\mathcal{B}_\alpha[\|\cdot\|_\alpha]$ and π_α be defined in Lemma 5.2. Since α is of ℓ^1-type on $D\mathfrak{A}$, $\mathcal{B}_\alpha[\|\cdot\|_\alpha]$ is a graded normed $*$-algebra whose completion $\widetilde{\mathcal{B}_\alpha}[\|\cdot\|_\alpha]$ is a graded Banach $*$-algebra, and π_α is a graded $*$-homomorphism of $D\mathfrak{A}$ into $\widetilde{\mathcal{B}_\alpha}[\|\cdot\|_\alpha]$. Conversely suppose α is induced by a graded $*$-homomorphism ψ of $D\mathfrak{A}$ into a graded Banach $*$-algebra $\mathcal{B}[\|\cdot\|_{\mathcal{B}}]$. Then since $\sum\limits_k \alpha(P_k z) = \sum\limits_k \|\psi(P_k z)\|_{\mathcal{B}}$ $= \sum\limits_k \|P_k^{\mathcal{B}}\psi(z)\|_{\mathcal{B}} < \infty$ for each $z \in D\mathfrak{A}$, it follows that α is a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is of ℓ^1-type on $D\mathfrak{A}$.

Let α be a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is of ℓ^1-type on $D\mathfrak{A}$. Since α is of ℓ^1-type on $D\mathfrak{A}$, we may define an algebra $*$-seminorm α' on $D\mathfrak{A}$ by $\alpha'(z) = \sum\limits_{k=0}^{\infty} \alpha(P_k z), z \in D\mathfrak{A}$. Furthermore, we have

$$
\begin{aligned}
&\alpha(z) = \alpha\Big(\sum_k P_k z\Big) \leqslant \sum_k \alpha(P_k z) = \alpha'(z), \quad \forall z \in D\mathfrak{A};\\
&\alpha'(P_k z) = \alpha(P_k z) \leqslant \alpha(z) \leqslant \alpha'(z), \quad \forall z \in D\mathfrak{A};\\
&\alpha'(z) = \sum_k \alpha(P_k z) = \sum_k \alpha'(P_k z), \quad \forall z \in D\mathfrak{A};\\
&\alpha'(x) = \alpha(x) \leqslant C\|x\|, \quad \forall x \in \mathfrak{A}.
\end{aligned}
\tag{5.1}
$$

Hence α' is a C^*-graded algebra seminorm on $D\mathfrak{A}$ which is strongly ℓ^1-type on $D\mathfrak{A}$. By (5.3), $N(\alpha') \subset N(\alpha)$. Conversely take an arbitrary $z \in N(\alpha)$. Then $\alpha(P_k z) = 0, \forall k \in \mathbb{N}$, and so $\alpha'(z) = 0$. Thus, $N(\alpha) \subset N(\alpha')$. Hence, $N(\alpha) = N(\alpha')$. Therefore, on $\mathcal{B}_\alpha = \frac{D\mathfrak{A}}{N(\alpha)}$ both α and α' define norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha'}$ and $\|\cdot\|_\alpha \leqslant \|\cdot\|_{\alpha'}$. Thus we may assume that both α and α' are norms on $D\mathfrak{A}$. Then there exists a continuous $*$-homomorphism $\pi : \widetilde{D\mathfrak{A}}[\alpha'] \to \widetilde{D\mathfrak{A}}[\alpha]$ such that $\pi\lceil D\mathfrak{A} = \mathrm{id}$ and $(\ker \pi) \cap D\mathfrak{A} = \{0\}$. Let $\alpha_0(z) \equiv \alpha(\pi(z)), z \in \widetilde{D\mathfrak{A}}[\alpha']$. Then α_0 is an algebra $*$-seminorm on $\widetilde{D\mathfrak{A}}[\alpha']$ and $\alpha_0(z) = \alpha(z), \forall z \in D\mathfrak{A}$. Now $\ker \pi$ is α'-closed, and so $\frac{\widetilde{D\mathfrak{A}}[\alpha']}{\ker \pi}$ is complete in quotient norm α'_{q} from α'. Furthermore, via π, α'_{q} and α_0 are equivalent. Hence, on $\frac{\widetilde{D\mathfrak{A}}[\alpha']}{\ker \pi}$, α_0 and α'_{q} are equivalent. Thus on $D\mathfrak{A}$, α and α'_{q} are equivalent. Since α is a norm on $D\mathfrak{A}$ and $\alpha \sim \alpha'_{\mathrm{q}}(z) = \inf\{\alpha'(z+u) : z \in \widetilde{D\mathfrak{A}}[\alpha'], \pi(u) = 0\}$, α'_{q} is also a norm on $D\mathfrak{A}$. Now, on $\mathfrak{A}$ define a differential seminorm $T'_j(x) \equiv \alpha'(d^j(x))$. Then $T'_{\mathrm{tot}}(x) = \sum\limits_j \alpha'(d^j(x)) = \alpha'(\mathrm{e}^d(x)), \forall x \in \mathfrak{A}$. This completes the proof. ∎

We first define and investigate the C^k-completion of $\mathfrak{A}$ as defined by Blackdar and Cuntz. Let $\Gamma(D\mathfrak{A})$ be the set of all non-zero C^*-graded algebra norms α on $D\mathfrak{A}$ such that $\alpha\lceil\mathfrak{A} \sim \|\cdot\|$, $\alpha \circ d^j$ is closable for each j, and is $\tau(\Gamma(D\mathfrak{A}))$ the topology on $D\mathfrak{A}$ defined by $\Gamma(D\mathfrak{A})$. We simply denote $\Gamma(D\mathfrak{A})$ by Γ. Suppose

$\Gamma \neq \phi$. Let $k \in \mathbb{N}_0$. Denote by J_k the $*$-ideal of $D\mathfrak{A}$ consisting of all elements of degree $\geqslant k+1$ in $D\mathfrak{A}$.

LEMMA 5.5. *The quotient $D\mathfrak{A}/J_k$ is a graded $*$-algebra with the projections $\{\widehat{P}_j\}$, where $\widehat{P}_j(z+J_k) \equiv P_j z + J_k, z \in D\mathfrak{A}$. For $\alpha \in \Gamma$, let α_{q} denote the quotient norm of α on $D\mathfrak{A}/J_k$. Then $\{\alpha_{\mathrm{q}} : \alpha \in \Gamma\} = \Gamma(D\mathfrak{A}/J_k)$, and so the quotient topology $\tau^k_{\Gamma,\mathrm{q}}$ on $D\mathfrak{A}/J_k$ equals the topology $\tau(\Gamma(D\mathfrak{A}/J_k))$ defined by $\Gamma(D\mathfrak{A}/J_k)$.*

Proof. It is easily shown that $D\mathfrak{A}/J_k$ is a graded $*$-algebra with the projections $\{\widehat{P}_j\}$. Take an arbitrary $\alpha \in \Gamma$. Let $z \in D\mathfrak{A}$. For $0 \leqslant j \leqslant k$ we have $\alpha_{\mathrm{q}}(\widehat{P}_j(z+J_k)) = \alpha_{\mathrm{q}}(P_j z + J_k) \leqslant \alpha(P_j z) = \alpha(P_j(z+y)) \leqslant \alpha(z+y)$ for each $y \in J_k$, which implies $\alpha_{\mathrm{q}}(\widehat{P}_j(z+J_k)) \leqslant \alpha_{\mathrm{q}}(z+J_k)$. For $j \geqslant k+1$ we have $\alpha_{\mathrm{q}}(\widehat{P}_j(z+J_k)) = \alpha_{\mathrm{q}}(J_k) = 0 \leqslant \alpha_{\mathrm{q}}(z+J_k)$. Thus α_{q} is a C^*-graded norm on $D\mathfrak{A}/J_k$. Furthermore, it is easily shown that $\alpha_{\mathrm{q}}\lceil d^0(\mathfrak{A}) + J_k \sim \|\cdot\|$ and $\alpha_{\mathrm{q}}(d^j(\cdot)+J_k)$ is closable by the closability of α. Hence, $\alpha_{\mathrm{q}} \in \Gamma(D\mathfrak{A}/J_k)$. Conversely take an arbitrary $\beta \in \Gamma(D\mathfrak{A}/J_k)$. Then we put $\alpha(z) = \beta(z+J_k), z \in D\mathfrak{A}$. Then it is easily shown that $\alpha \in \Gamma$ and $\alpha_{\mathrm{q}} = \beta$. Thus we have $\{\alpha_{\mathrm{q}} : \alpha \in \Gamma\} = \Gamma(D\mathfrak{A}/J_k)$. ∎

Since the projections P_j of $D\mathfrak{A}$ onto $\mathcal{B}_j \equiv P_j(D\mathfrak{A})$ are $\tau(\Gamma)$-continuous, they extend as the projections $\widetilde{P}_j$ of the completion $\widetilde{D\mathfrak{A}}[\tau(\Gamma)]$ onto the completions $\widetilde{\mathcal{B}}_j$. By Lemma 5.5 we have the following

LEMMA 5.6. (i) *The algebra $\widetilde{D\mathfrak{A}}[\tau(\Gamma)]$ is the inverse limit of Banach graded $*$-algebras $\widetilde{D\mathfrak{A}}[\alpha]$, $\alpha \in \Gamma$, and it is a graded $*$-algebra which is a complete locally m-convex $*$-algebra with continuous projections $\{\widetilde{P}_j\}$ such that $\partial\widetilde{D\mathfrak{A}}[\tau(\Gamma)] = (\partial D\mathfrak{A})^{\sim}[\tau(\Gamma)]$.*

(ii) *The quotient algebra* $\widetilde{D\mathfrak{A}}[\tau(\Gamma)]/\widetilde{J}_k = (D\mathfrak{A}/J_k)^{\sim}[\tau^k_{\Gamma,q}]$ *(completion)*

$$= (D\mathfrak{A}/J_k)^{\sim}[\tau(\Gamma(D\mathfrak{A}/J_k))].$$

This locally m-convex $*$-algebra is simply denoted by $\widetilde{D\mathfrak{A}}^{\leqslant k}$.

(iii) *The algebra $\widetilde{D\mathfrak{A}}^{\leqslant k}$ is the completion of $D\mathfrak{A}^{\leqslant k}$ in the quotient topology.*

We remark that for any $\alpha \in \Gamma$, $\widetilde{D\mathfrak{A}}[\alpha]$ is a Banach graded $*$-algebra, but it need not be a graded Banach $*$-algebra, that is, $\sum_{k=0}^{\infty} \alpha(P_k z) < \infty$, $\forall z \in D\mathfrak{A}$ does not necessarily hold unless α is of ℓ^1-type on $D\mathfrak{A}$.

DEFINITION 5.7. The closure of $\phi_k(\mathfrak{A})$ in $\widetilde{D\mathfrak{A}}^{\leqslant k}$ is called the C^k-*completion* of $\mathfrak{A}$ and is denoted by $C^k\mathfrak{A}$, where $\phi_k : x \in \mathfrak{A} \to d^0(x) + \cdots + d^k(x) + J_k \in D\mathfrak{A}/J_k$.

Following [14], an equivalent approach to the C^k-structure is as follows: Let τ be the topology on $D\mathfrak{A}$ defined by the family of all C^*-graded algebra seminorms on $D\mathfrak{A}$. Let $\widetilde{D\mathfrak{A}}$ be the Hausdorff completion of $(D\mathfrak{A}, \tau)$. Let $\widetilde{D\mathfrak{A}}^{\leqslant k} \equiv \widetilde{D\mathfrak{A}}/\widetilde{J}_k$, $\widetilde{J}_k$ being the ideal of elements of degree $\geqslant k+1$ in $\widetilde{D\mathfrak{A}}$. Notice that $\widetilde{J}_k$ is the Hausdorff completion of the ideal J_k of elements of degree $\geqslant k+1$ in $D\mathfrak{A}$.

Let $\phi : \mathfrak{A} \to \widetilde{D\mathfrak{A}}^{\leqslant k}$ be the map $\phi = j \circ \mathrm{e}^d$ viz. $\phi(x) = d^0(x) + d^1(x) + d^2(x) + \cdots + d^k(x) + \widetilde{J}_k, x \in \mathfrak{A}$. Let $\mathfrak{A}_k$ be the closure of $\phi(\mathfrak{A})$ in $\widetilde{D\mathfrak{A}}^{\leqslant k}$. Then $\mathfrak{A}_k$ is the Hausdorff completion of $\mathfrak{A}$ in the family of all flat, not necessarily closable, differential seminorms of order $\leqslant k$. Now $C^k\mathfrak{A}$ is defined to be $\mathfrak{A}_k/\mathfrak{A}_k \cap \partial\widetilde{D\mathfrak{A}}^{\leqslant k}$ with the quotient topology τ^k, and it is the completion $(\mathfrak{A}, \{T_{\mathrm{tot}} : T \in \Lambda_{\mathrm{cf}}^{\leqslant k}\})^{\sim}$ of $\mathfrak{A}$ in the family $\Lambda_{\mathrm{cf}}^{\leqslant k}$ of all closable flat differential seminorms of order $\leqslant k$. (Notice that here and throughout T is of order $\leqslant k$ means that there exists $m \leqslant k$ such that $T_i = 0$ for all $i > m$.) The following is shown by the closability of each T_α, $\alpha \in \Gamma$ (see Lemma 5.3).

LEMMA 5.8. $C^k\mathfrak{A} \cap \partial\widetilde{D\mathfrak{A}}^{\leqslant k} = \{0\}$.

THEOREM 5.9. *The algebra $C^k\mathfrak{A}$ is the completion of $\mathfrak{A}$ in the family of all flat closable differential norms of order $\leqslant k$.*

Proof. We show that the $*$-isomorphism $\phi_k : x \in \mathfrak{A}[\tau(\Delta_{\mathrm{cf}}^{\leqslant k})] \to \phi_k(x) \equiv d^0(x) + \cdots + d^k(x) + J_k \in \widetilde{D\mathfrak{A}}^{\leqslant k}$ is bicontinuous. Indeed, take an arbitrary $\beta \in \Gamma(D\mathfrak{A}/J_k)$, we put $\alpha(z) = \beta(z + J_k), z \in D\mathfrak{A}$. By Lemma 5.5 α is a C^*-graded algebra norm on $D\mathfrak{A}$ of order $\leqslant k$. Hence it follows from Lemma 5.3 that $T_\alpha \in \Lambda_{\mathrm{cf}}^{\leqslant k}$ and $\beta(\phi_k(x)) = \alpha(d^0(x) + \cdots + d^k(x)) \leqslant \sum_{j=0}^{k} T_j^\alpha(x) = (T_\alpha)_{\mathrm{tot}}(x)$ for each $x \in \mathfrak{A}$. Conversely, take an arbitrary $T \in \Lambda_{\mathrm{cf}}^{\leqslant k}$. By Lemma 5.3 $T = T_\alpha$ for $\alpha \in \Gamma$ of order $\leqslant k$. Then, for $0 \leqslant j \leqslant k, \forall x \in \mathfrak{A}$ and $\forall y \in J_k$ we have

$$T_j(x) = \alpha(d^j(x)) = \alpha(P_j(d^0(x) + \cdots + d^k(x) + y)) \leqslant \alpha(d^0(x) + \cdots + d^k(x) + y),$$

which implies that $\frac{1}{k+1} T_{\mathrm{tot}}(x) \leqslant \alpha_{\mathrm{q}}(\phi_k(x))$. Thus ϕ_k is bicontinuous by Lemma 5.5, and $\widetilde{\mathfrak{A}}[\Lambda_{\mathrm{cf}}^{\leqslant k}] \underset{\phi_k}{\simeq} C^k\mathfrak{A}$. This completes the proof. ∎

DEFINITION 5.10. The algebra $C^\infty\mathfrak{A} \equiv \bigcap_{k \in \mathbb{N}_0} C^k\mathfrak{A}$ with the projective topology τ^∞ from $\{C^k\mathfrak{A}[\tau_{\Lambda_{\mathrm{cf}}^{\leqslant k}}] : k \in \mathbb{N}_0\}$ is called the *C^∞-completion* of $\mathfrak{A}$, and it is the completion of $\mathfrak{A}$ in the family $\Lambda_{\mathrm{cf}}^{f}$ of all flat closable differential seminorms of finite order. If $\mathfrak{A} = C^\infty\mathfrak{A}$, that is, $\mathfrak{A}$ is complete in the family $\Lambda_{\mathrm{cf}}^{f}$ of all flat closable differential norms of finite order, then $\mathfrak{A}$ is called a *C^∞-algebra*.

By Theorem 4.4 of [14], the projective topology τ^∞ on $C^\infty\mathfrak{A}$ is defined by the norms $\left\{ \sum_{j=0}^{k} \frac{\|\delta^j(x)\|_{D_\delta}}{j!} : k \in \mathbb{N}_0, \delta \text{ is a closable } *\text{-derivation of } C^\infty\mathfrak{A} \right\}$, where D_δ is a normed $*$-algebra containing $\mathfrak{A}$ with norm $\|\cdot\|_{D_\delta}$ whose restriction to $\mathfrak{A}$ is equivalent to the C^*-norm $\|\cdot\|$. The next two results describe some properties of

C^k-completions. We leave open the important problem: When is the embeding id : $C^{k+1}\mathfrak{A} \to C^k\mathfrak{A}$ compact?

PROPOSITION 5.11. (i) $\mathfrak{A} \subset \mathcal{S}\mathfrak{A} \subset \mathcal{S}_{\mathrm{s}}\mathfrak{A} \subset C^\infty\mathfrak{A} = \varprojlim_{k\to\infty} C^k\mathfrak{A}[\tau_{\Lambda_{\mathrm{cf}}^{\leqslant k}}] = \varprojlim_{T\in\Lambda_{\mathrm{cf}}^{<\infty}} \widetilde{\mathfrak{A}}[T_{\mathrm{tot}}] \subset C^{k+1}\mathfrak{A} \subset C^k\mathfrak{A} \subset \mathcal{A}$.
If $\mathfrak{A}$ is a C^∞-algebra, then it is a strongly smooth algebra.

(ii) *The algebras $C^k\mathfrak{A}$ and $C^\infty\mathfrak{A}$ are strongly smooth algebras.*

(iii) *The completion of a C^*-normed algebra $\mathfrak{A}$ with respect to a countable family of closable flat differential seminorms of finite order is a Fréchet C^∞-algebra.*

(iv) *If $\mathfrak{A}$ has finitely many generators, then $C^\infty\mathfrak{A}$ is a Fréchet C^∞-algebra.*

Proof. (i) The statement (i) follows from Theorem 5.9 and the definitions of $\mathcal{S}\mathfrak{A}, \mathcal{S}_{\mathrm{s}}\mathfrak{A}$ and $C^\infty\mathfrak{A}$, which implies (ii). The statements (iii) and (iv) are shown similarly to Corollary 4.7. ∎

The following is essentially in [14].

THEOREM 5.12. *Let $k \in \mathbb{N}$. The following statements hold:*

(i) *$C^\infty\mathfrak{A}$ and $C^k\mathfrak{A}$ are complete locally m-convex $*$-algebras continuously embedded as dense $*$-subalgebras of $\mathcal{A}$.*

(ii) *$C^\infty\mathfrak{A}$ and $C^k\mathfrak{A}$ are hermitian Q-algebras.*

(iii) *$C^\infty\mathfrak{A}$ and $C^k\mathfrak{A}$ are spectrally invariant in $\mathcal{A}$ and $E(C^\infty\mathfrak{A}) = E(C^k\mathfrak{A}) = \mathcal{A}$.*

(iv) *$C^\infty\mathfrak{A}$ and $C^k\mathfrak{A}$ are C^*-spectral algebras.*

(v) *$C^\infty\mathfrak{A}$ and $C^k\mathfrak{A}$ are closed under holomorphic functional calculus and under C^∞-functional calculus for self-adjoint elements.*

(vi) $K_*(C^\infty\mathfrak{A}) = K_*(C^k\mathfrak{A}) = K_*(\mathcal{A})$.

(vii) (a) *Any $*$-homomorphism φ of $\mathfrak{A}$ onto a C^*-normed algebra $\mathcal{B}$ extends to a continuous $*$-homomorphism of $C^\infty\mathfrak{A}$ onto $C^\infty\mathcal{B}$.*

(b) *Any closable $*$-derivation δ of $\mathfrak{A}$ extends to a closable $*$-derivation of $C^\infty\mathfrak{A}$.*

6. EXAMPLES

EXAMPLE 6.1. (i) Let $\mathcal{A} \equiv C[0,1]$ be the C^*-algebra of all continuous functions on $[0,1]$ with the supremum norm $\|\cdot\|_\infty$. Let $\mathfrak{A} \equiv C^\infty[0,1]$ be the Fréchet $*$-algebra of all infinitely differentiable functions on $[0,1]$. On $\mathfrak{A}$, we define $T = (T_k)_0^\infty$ as $T_k(f) = \frac{\|f^{(k)}\|_\infty}{k!}, f \in \mathfrak{A}$. Then T is a closable differential norm on $\mathfrak{A}$ such that $T_0(f) = \|f\|_\infty$. The algebra $\mathfrak{A}_{(k)}$ equals the Banach $*$-algebra $C^k[0,1]$ of functions on $[0,1]$ having k continuous derivatives, and $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)} = \varprojlim C^k[0,1] = C^\infty[0,1]$. Hence $C^\infty[0,1]$ is a strongly smooth algebra. In fact $C^\infty[0,1]$ is a C^∞-algebra.

(ii) Let $\mathcal{A} \equiv C_0(\mathbb{R})$ be the C^*-algebra of all continuous functions on $\mathbb{R}$ that vanish at infinity with the supremum norm $\|\cdot\|_\infty$. We put $\mathfrak{A} \equiv C_0^\infty(\mathbb{R}) \equiv \{f \in$

$C^\infty(\mathbb{R}) : f^{(i)} \in C_0(\mathbb{R}), 0 \leqslant i < \infty\}$. The algebra $\mathfrak{A}$ is a Fréchet $*$-algebra. On $\mathfrak{A}$, we define $T = (T_k)_0^\infty$ as $T_k(f) = \frac{\|f^{(k)}\|_\infty}{k!}, \forall f \in \mathfrak{A}$. Then T is a closable differential norm on $\mathfrak{A}$. The algebra $\mathfrak{A}_{(k)}$ equals the Banach $*$-algebra $C_0^k(\mathbb{R}) \equiv \{f \in C^{(k)}(\mathbb{R}) : f^{(i)} \in C_0(\mathbb{R}) : 0 \leqslant i \leqslant k\}$, and the differential Fréchet $*$-algebra $\mathfrak{A}_\tau = \varprojlim C_0^k(\mathbb{R}) = C_0^\infty(\mathbb{R})$. Hence $C_0^\infty(\mathbb{R})$ is a strongly smooth algebra. In fact, $C_0^\infty(\mathbb{R})$ is a C^∞-algebra.

(iii) Let $\mathcal{A} \equiv C_0(\mathbb{R})$ and $\mathfrak{A} = \mathcal{S}(\mathbb{R})$ be the Schwartz space of all infinitely differentiable rapidly decreasing functions on $\mathbb{R}$. The algebra $\mathfrak{A}$ has the Schwartz topology which is defined by the family $\{\|f\|_{m,i}\}$ of seminorms: $\|f\|_{m,i} = \sup_{x\in\mathbb{R}}(1+|x|)^m|f^{(i)}(x)|$. Then $\mathfrak{A}$ is a Fréchet $*$-algebra with pointwise multiplication (as well as with convolution which we do not consider here). On $\mathfrak{A}$, we define $T = (T_m)_0^\infty$ as $T_m(f) = \sum_{i=0}^{m} \frac{\|f\|_{m,i}}{i!}$, for $\forall f \in \mathfrak{A}$. Then T is a closable differential norm on $\mathfrak{A}$ such that $T_0(f) = \|f\|_\infty$. The algebra $\mathfrak{A}_\tau$ equals the Schwartz space $\mathcal{S}(\mathbb{R})$. Hence $\mathcal{S}(\mathbb{R})$ is a strongly smooth algebra, and the Schwartz topology is the topology defined by all derived norms on $\mathcal{S}(\mathbb{R})$. Notice that $\mathcal{S}(\mathbb{R})$ is not a C^∞-algebra. Indeed, the differential norm $T'=(T'_k)_{k=0}^\infty$, $T'_k(f) = \frac{\|f^{(k)}\|_\infty}{k!}$ on $\mathcal{S}(\mathbb{R})$ is flat and closable, and the completion of $\mathcal{S}(\mathbb{R})$ in T' is $C_0^\infty(\mathbb{R})$. Thus the C^∞-envelope of $\mathcal{S}(\mathbb{R})$ is $C^\infty(\mathcal{S}(\mathbb{R})) = C_0^\infty(\mathbb{R})$.

(iv) Let $\mathcal{A}= C[0,1]$ and $\mathfrak{A}$ the Banach $*$-algebra $CBV[0,1]$ of real valued continuous functions of bounded variations on [0,1] equipped with the norm $\|f\| = \|f\|_\infty + V(f)$, $V(f)$ being the total variation of f on [0,1]. For all f, g in $\mathfrak{A}$, $V(fg) \leqslant \|f\|_\infty V(g) + \|g\|_\infty V(f)$. Thus $T_0(f) = \|f\|_\infty, T_1(f) = V(f)$ defines on $\mathfrak{A}$ a differential norm $T = \{T_0, T_1\}$ of order 1 for which $T_{\text{tot}} = \|\cdot\|$. Thus $\mathfrak{A}$ is a smooth Banach algebra. It follows that if $f \in CBV[0,1]$, and $g : [0,1] \to [0,1]$ is smooth, then $g \circ f$ is of bounded variation. Analogously the Banach $*$-algebra $AC_p[0,1], 1 \leqslant p < \infty$ consisting of functions f in [0,1] such that the derivative f' exists almost everywhere on [0,1] with $f' \in L^p[0,1]$ and with the norm $\|f\| = \|f\|_\infty + \|f'\|_p$ is a differential Banach algebra defined by the differential norm $T(f) = \{\|f\|_\infty, \|f'\|_p\}$ of order 1. Consider the Sobolev–Banach algebra $W^{m,p}[0,1] = \{f \in C^{(m-1)}[0,1] : f^{(m-1)}$ is absolutely continuous on $[0,1], f^{(m)} \in L^p[0,1]\}$ with the norm $\|f\| \equiv \sum_{j=0}^{m-1} \frac{\|f^{(j)}\|_\infty}{j!} + \|f'\|_p$. The differential norm $T = (T_k)$ of order m defined as $T_0(f) = \|f\|_\infty, T_j(f) = \frac{\|f^{(j)}\|_\infty}{j!}$ $(1 \leqslant j \leqslant m-1), T_m(f) = \|f^{(m)}\|_p$ makes $W^{m,p}[0,1]$ to be a differential Banach $*$-algebra. It also follows that the algebra $\mathrm{Lip}[0,1]$ of Lipschitz functions on $[0,1]$ is also a smooth subalgebra of $C[0,1]$.

EXAMPLE 6.2. (i) Let $\mathcal{A}$ be a C^*-algebra with C^*-norm $\|\cdot\|$. Let $\{\delta_1, \delta_2, \ldots, \delta_d\}$ be closed $*$-derivations on $\mathcal{A}$. Then the C^n-elements $(1 \leqslant n < \infty)$ and the

C^∞-elements of $\mathcal{A}$ are defined [16] as follows:

$$C^n(\mathcal{A}) \equiv \{x \in \mathcal{A} \,:\, x \in \text{ Dom } (\delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}) \text{ for all } n\text{-tuples}\{\delta_{i_1},\delta_{i_2},\ldots,\delta_{i_n}\} \text{ in } \{\delta_1,\delta_2,\ldots,\delta_d\}\},$$
$$C^\infty(\mathcal{A}) \equiv \bigcap\{C^n(\mathcal{A}) : n \in \mathbb{N}\}.$$

$C^n(\mathcal{A})$ is a Banach $*$-algebra with the norm

$$\|x\|_n = \|x\| + \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,i_k=1}^{d} \frac{1}{k!}\|\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)\|.$$

Thus $C^\infty(\mathcal{A}) = \varprojlim C^n(\mathcal{A})$ is a Fréchet $*$-algebra. Assume that $C^\infty(\mathcal{A})$ is dense in $\mathcal{A}$. Let $\mathfrak{A} \equiv C^\infty(\mathcal{A})$. On $\mathfrak{A}$, we define $T = (T_k)_0^\infty$ as

$$T_0(x) = \|x\|, \quad T_k(x) = \sum_{i_1,i_2,\ldots,i_k=1}^{d} \frac{1}{k!}\|\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)\|, \quad \text{for } \forall x \in \mathfrak{A}, \forall k \in \mathbb{N}.$$

Then T is a closable differential norm on $\mathfrak{A}$. Since $p_n \equiv \sum_{k=0}^{n} T_k$ and $\|\cdot\|_n$ are equivalent, we have $\mathfrak{A}_{(n)} = \widetilde{\mathfrak{A}}[p_n] = C^n(\mathcal{A})$, and $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(n)} = \mathfrak{A}$. Hence $C^\infty(\mathcal{A})$ is a differential Fréchet algebra, hence is a strongly smooth algebra.

(ii) We give an example satisfying the assumption that $C^\infty(\mathcal{A})$ is dense in $\mathcal{A}$. Let G be a Lie group acting on $\mathcal{A}$ by α. Let Δ denote the set of all infinitesimal generators of actions of one-parameter subgroups of G on $\mathcal{A}$, viz.

$$\Delta = \Big\{ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\alpha_{u(t)}\big|_{t=0} : t \mapsto u(t) \text{ is a continuous homomorphism of } \mathbb{R} \text{ into } G \Big\}.$$

Then Δ consists of derivations, and by a theorem of Malliavin ([16], p. 40), it is a finite dimensional vector space having basis, say $\{\delta_1,\delta_2,\ldots,\delta_d\}$, where $d = \dim\Delta$. By Proposition 2.2.1 of [16], $C^n(\mathcal{A})$ and $C^\infty(\mathcal{A})$ are dense in $\mathcal{A}$. Now the C^∞-elements of $\mathcal{A}$ defined by the Lie group action α is

$$C^\infty(\mathcal{A},\alpha) = \{x \in \mathcal{A} : g \in G \to \alpha_g(x) \in \mathcal{A} \text{ is smooth}\}$$

which is the set $C^\infty(\mathcal{A})$ defined above. Thus $C^\infty(\mathcal{A},\alpha)$ is a smooth subalgebra of $\mathcal{A}$, in fact it is a differential Fréchet algebra.

(iii) It follows from (ii) above that the non-commutative torus T_θ^n are differential Fréchet algebras, hence are strongly smooth. We elaborate the case $n = 2$. Let θ be any real number. We identify the functions $F \in C(T)$, T unit circle, with continuous periodic functions $f : \mathbb{R} \to \mathbb{C}$ of period 1 by $f(t) = F(\mathrm{e}^{2\pi \mathrm{i}t})$. Let α be the automorphism $\alpha(f)(t) = f(t+\theta)$ defined by the rotation of T by the angle $2\pi\theta$. The *torus algebra* $\mathcal{A}_\theta^2$ is the crossed product of $C(T)$ by $\mathbb{Z}$, and is realized as the C^*-algebra of operators on $L^2(T)$ defined by the unitaries u and v, $u\xi(t) = \mathrm{e}^{2\pi \mathrm{i}t}\xi(t), v\xi(t) = \xi(t+\theta)$. The universal C^*-algebra generated by the two unitaries u,v satisfying $vu = \mathrm{e}^{2\pi\mathrm{i}\theta}uv$ is isomorphic to $\mathcal{A}_\theta^2$. Thus any $a \in \mathcal{A}_\theta^2$ is expressed as $a = \sum_{r,s\in\mathbb{Z}} a_{rs}u^r v^s$ with an appropriate scalar sequence $\{a_{rs}\}$. The

C^*-algebra $\mathcal{A}_\theta^2$ is abelian if and only if θ is an integer, and then $\mathcal{A}_\theta^2$ is isomorphic to $C(T^2)$ (which in turn is identified with the 2-torus T^2). The *non-commutative torus* T_θ^2 is the dense subalgebra of $\mathcal{A}_\theta^2$ defined as $T_\theta^2 = \Big\{a = \sum\limits_{r,s\in\mathbb{Z}} a_{rs}u^r v^s : \{a_{rs}\} \in \mathcal{S}(\mathbb{Z}^2)\Big\}$. The Lie group T^2 acts on $\mathcal{A}_\theta^2$ by $\alpha_\theta(z,w)(u^r v^s) \equiv z^r w^s u^r v^s$, and $T_\theta^2 = C^\infty(\mathcal{A}_\theta^2, \alpha_\theta)$, the C^∞-elements of $\mathcal{A}_\theta^2$ defined by α_θ. It follows that T_θ^2 is a differential Fréchet algebra, hence is strongly smooth, and the well known smoothness properties of T_θ^2 follow. Notice that $C^\infty(T_\theta^2) = T_\theta^2$ is a C^∞-algebra.

(iv) Here is an example of a different kind of differential structure. This is a multivariate analogue of an example in [14]. Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathfrak{A}$ be a $*$-algebra of operators on $\mathcal{H}$. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ be an n-tuple of strongly commuting self-adjoint operators on $\mathcal{H}$, possibly unbounded. Let E be the joint spectral measure of $\mathcal{D}$ defined on Borel subsets of the joint spectrum $\sigma(\mathcal{D})$. Then for each $j = 1, 2, \ldots, n$, $D_j = \int\limits_{\sigma(\mathcal{D})} \lambda_j \mathrm{d}E_\lambda = \int\limits_{\mathbb{R}^n} \lambda_j \mathrm{d}E_\lambda$. For $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, let

$$\mathfrak{A}_k = \{x \in \mathfrak{A} : xE_\lambda\mathcal{H} \subset E_{\lambda+k}\mathcal{H}, x^*E_\lambda\mathcal{H} \subset E_{\lambda+k}\mathcal{H} \text{ for all } \lambda \in \mathbb{R}^n\}.$$

Assume that $\mathfrak{A} = \bigcup\limits_k \mathfrak{A}_k$. Let $\delta_j(x) \equiv i[D_j, x], 1 \leqslant j \leqslant n$. Then $\{\delta_1, \delta_2, \ldots, \delta_n\}$ is a commuting family of closable derivations on $\mathfrak{A}$. Define, for each $x \in \mathfrak{A}$,

$$r_k(x) \equiv \sup_{\lambda\in\mathbb{R}^n} \{\|(1 - E_{\lambda+k})xE_\lambda\|, \|(1 - E_{\lambda+k})x^*E_\lambda\|\},$$
$$\rho_m(x) \equiv \sup\{k_1^{m_1} k_2^{m_2} \cdots k_n^{m_n} r_k(x) : k_1, k_2, \ldots, k_n \text{ in } \mathbb{R}_+\}.$$

Then $T = \{T_0, T_1\}, T_0(x) \equiv c\|x\|, T_1(x) \equiv \rho_m(x), x \in \mathfrak{A}$ is a differential seminorm of order 1. Another differential seminorm $T' = (T'_k)$ is defined as follows. $T'_0(x) = \|x\|, T'_k(x) = \frac{1}{k!} \sum\limits_{|m|=k, m\in\{\mathbb{N}\cup(0)\}^n} \|\delta_1^{m_1}\delta_2^{m_2}\cdots\delta_n^{m_n}(x)\|$.

EXAMPLE 6.3. We consider the differential structure defined by a smooth operator algebra crossed product by the action of a Lie group as developed in [31]. Let $\mathcal{A}$ be a C^*-algebra. Let G be a Lie group acting by an action α on $\mathcal{A}$. Let $C^*(G, \mathcal{A}, \alpha)$ be the crossed product C^*-algebra. The smooth crossed product $\mathcal{S}(G, \mathcal{A}, \alpha)$ is envisaged as a smooth version of the crossed product construction and is primarily constructed to produce examples of subalgebras of $C^*(G, \mathcal{A}, \alpha)$ having desired smoothness properties like spectral invariance and K-theory isomorphism. Our contention is that there must exists a smooth structure (in the sense of the present paper) lurking behind this construction responsible for the smoothness properties exhibited by $\mathcal{S}(G, \mathcal{A}, \alpha)$. We exhibit this with the particular case of $\mathbb{R}$. The general case requires a separate treatment. Let α be a strongly continuous action of $\mathbb{R}$ by $*$-automorphisms of $\mathcal{A}$. Let $C_c(\mathbb{R}, \mathcal{A})$ be the set of all continuous $\mathcal{A}$-valued functions on $\mathbb{R}$ having compact supports. It is a

$*$-algebra with twisted convolution: $(f * g)(r) = \int_{\mathbb{R}} f(s)\alpha_s(g(r-s))\mathrm{d}s, f^*(r) = \alpha_r(f(-r)^*)$. It is a normed $*$-algebra with the norm: $\|f\|_1 = \int_{\mathbb{R}} \|f(s)\|\mathrm{d}s$. The Banach $*$-algebra $L^1(\mathbb{R}, \mathcal{A}, \alpha)$ is the completion of $C_c(\mathbb{R}, \mathcal{A})$ in the norm $\|\cdot\|_1$, and the crossed product C^*-algebra $C^*(\mathbb{R}, \mathcal{A}, \alpha)$ is the enveloping C^*-algebra of $L^1(\mathbb{R}, \mathcal{A}, \alpha)$. Let $\mathfrak{A} \equiv \mathcal{S}(\mathbb{R}, \mathcal{A}, \alpha)$ be the set of all $\mathcal{A}$-valued Schwartz functions on $\mathbb{R}$. $\mathfrak{A}$ is called the smooth Schwartz crossed product of $\mathcal{A}$ by $\mathbb{R}$. $\mathfrak{A}$ is a dense $*$-subalgebra of $C^*(\mathbb{R}, \mathcal{A}, \alpha)$. It is shown by [28] that the topology $\tau_{\mathcal{S}}$ of $\mathfrak{A}$ is also determined by the following sequence of seminorms: $\|f\|_n = \sum_{i+j=n} \int_{\mathbb{R}} (1 + |r|)^i \|f^{(j)}(r)\|\mathrm{d}r, \forall f \in \mathfrak{A}$, and $\mathfrak{A}$ is a Fréchet $*$-algebra. The seminorms $\{\|\cdot\|_n\}$ satisfy the following: $\|f * g\|_n \leqslant \|f\|_n \|g\|_n$, $\|f\|_0 = \|f\|$, $\|f * g\|_n \leqslant 2^{n+1} \sum_{i+j=n} \|f\|_i \|g\|_j$. Thus $S(\mathbb{R}, \mathcal{A}, \alpha)$ is a differential Fréchet algebra, hence a smooth subalgebra of $C^*(\mathbb{R}, \mathcal{A}, \alpha)$. We believe that $\mathcal{S}(\mathbb{R}, \mathcal{A}, \alpha)$ is not a C^∞-algebra. Now let $C^\infty(\mathcal{A}, \alpha)$ be the Fréchet $*$-subalgebra of $\mathcal{A}$ consisting of all C^∞-elements of $\mathcal{A}$ defined by the action α. Let $\mathcal{S}(\mathbb{R}, C^\infty(\mathcal{A}), \alpha)$ be the smooth Fréchet algebra crossed product (twisted convolution) which is isomorphic to the completed projective tensor product $\mathcal{S}(\mathbb{R}) \otimes C^\infty(\mathcal{A})$. It is a Fréchet $*$-algebra in $C^*(\mathbb{R}, \mathcal{A}, \alpha)$, and its topology is defined by the seminorms

$$\|f\|_{n,l} = \sum_{i+j=n} \int_{\mathbb{R}} (1 + |r|)^i \|f^{(j)}(r)\|_l \mathrm{d}r,$$

where $\|f^{(j)}(r)\|_l = \sum_{k=0}^{l} \frac{1}{k!} \|\delta^k(\alpha_s((\frac{\mathrm{d}}{\mathrm{d}x})^j f(r))|_{s=0}\|$. The algebra $\mathcal{S}(\mathbb{R}, C^\infty(\mathcal{A}), \alpha)$ is a smooth subalgebra of $C^*(\mathbb{R}, \mathcal{A}, \alpha)$ in the sense of the present paper, and is an inverse limit of differential Fréchet algebras.

EXAMPLE 6.4. We consider the smooth version of compact operators introduced in [27]. Let ℓ^2 be the Hilbert space of all functions $a : \mathbb{Z} \to \mathbb{C}$ such that $\sum_{i \in \mathbb{Z}} |a(i)|^2 < \infty$, and let $\mathcal{K}(\ell^2)$ be the C^*-algebra of all compact operators on ℓ^2. Let $\mathfrak{A} \equiv \mathcal{S}(\mathbb{Z}^2)$ denote the Schwartz space of double sequences $s = \{s(m,n)\}$ on $\mathbb{Z}^2$ satisfying, for each $\nu \in \mathbb{N}$, $p_\nu(s) \equiv \sum_{m,n \in \mathbb{Z}} (1 + |m| + |n|)^\nu |s(m,n)| < \infty$. Then $\mathfrak{A}$ is a locally m-convex Fréchet $*$-algebra with the product defined by $(st)(m,n) = \sum_{j \in \mathbb{Z}} s(m,j)t(j,n)$, and with the topology $\tau_{\mathcal{S}}$ defined by the family $\{p_\nu : \nu \in \mathbb{N}\}$ of seminorms satisfying for each ν, $p_\nu(st) \leqslant p_\nu(s)p_\nu(t)$ for all s, t in $\mathcal{S}(\mathbb{Z}^2)$ [27]. Each $s \in \mathfrak{A}$ defines a bounded operator T_s on ℓ^2 by $(T_s a)(m) = \sum_{j \in \mathbb{Z}} s(m,j)a(j) (a \in \mathcal{S}(\mathbb{Z}))$ and extended continuously to ℓ^2. This gives an isomorphism of $\mathfrak{A}$ onto $\{T_s : s \in \mathfrak{A}\}$. By this embedding, $\mathfrak{A}$ is a dense $*$-subalgebra of $\mathcal{K}(\ell^2)$. On $\mathfrak{A}$, we define $T = (T_k)_0^\infty$ as $T_0(s) = \|T_s\|$, (operator norm) , $T_k(s) = \frac{1}{k!} \sum_{m,n\in\mathbb{Z}} (1+|m|+|n|)^k |s(m,n)|$. Then T is a closable differential norm on $\mathfrak{A}$ and the topology defined by $\{p_\nu\}$ coincides with the differential Fréchet topology defined by T. It follows that $\mathcal{S}(\mathbb{Z}^2)$ is a strongly smooth algebra.

EXAMPLE 6.5. (i) The Fredholm module of non-commutative geometry leads to a certain differential norm naturally. Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$. A *Fredholm module* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is as above and D is a bounded selfadjoint operator on $\mathcal{H}$ such that $[D, x]$ is a compact operator for each $x \in \mathcal{A}$. It is said to be *p-summable* if for $x \in \mathcal{A}$, $[D, x] \in C^p(\mathcal{H})$, where $C^p(\mathcal{H})$ is the set of Schatten class operators on $\mathcal{H}$. Given a Fredholm module, we define a derivation $\delta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $\delta(x) = i[D, x]$. We put $\mathcal{A}_p = \{x \in \mathcal{A} : \delta(x) \in C^p(\mathcal{H})\}$. $\mathcal{A}_p$ is called the *p-summable part of* $\mathcal{A}$. We assume that $\mathcal{A}_p$ is dense in $\mathcal{A}$. The algebra $\mathcal{A}_p$ is a Banach $*$-algebra with the norm $|x|_p = \|x\| + \|\delta(x)\|_p$, where $\|\cdot\|_p$ is the Schatten p-norm. Now on $\mathcal{A}_p$ we define $T_0(x) = \|x\|$, $T_1(x) = \|\delta(x)\|_p$. Then we have $T_1(xy) \leqslant T_0(x)T_1(y) + T_0(y)T_1(x)$. Thus $T = \{T_0, T_1\}$ is a closable differential norm of order 1 on $\mathcal{A}_p$, and $\mathcal{A}_p$ is a differential Banach $*$-algebra. Hence $\mathcal{A}_p$ is a strongly smooth algebra. Let $\mathcal{A}_{1+} = \bigcap_{1<p\leqslant\infty} \mathcal{A}_p$, a Fréchet locally convex $*$-algebra with the topology τ_{1+} defined by $\{|\cdot|_p : 1 < p \leqslant \infty\}$. One can consider the Fréchet algebra $\mathcal{A}_{p+} \equiv \bigcap_{p<r\leqslant\infty} \mathcal{A}_r$. One gets a chain of smooth algebras

$$\mathcal{A}_1 \subset \mathcal{A}_{1+} \subset \mathcal{A}_2 \subset \mathcal{A}_{2+} \subset \cdots \subset \mathcal{A}_p \subset \mathcal{A}_{p+} \subset \mathcal{A}_{p+1} \subset \cdots \subset \mathcal{A}_0 \subset \mathcal{A}$$

where $\mathcal{A}_0 \equiv \bigcup_{1\leqslant p<\infty} \mathcal{A}_p$ which is a complete locally convex $*$-algebra with the inductive limit topology. In fact, each $\mathcal{A}_p$ is an ideal in $\mathcal{A}_0$, and by [22], $\mathcal{A}_0$ is a locally m-convex $*$-algebra. It follows that these algebras have desired smoothness properties.

(ii) Let $(\mathcal{A}, \mathcal{H}, D)$ be an *unbounded Fredholm module* wherein $\mathcal{A}$ is a C^*-algebra of operators on $\mathcal{H}$ and D is an unbounded self-adjoint operator such that $\mathcal{A}_0 \equiv \{x \in \mathcal{A} : x(\mathrm{dom}D) \subset \mathrm{dom}(D), [x, D] \text{ is bounded }\}$ is dense in $\mathcal{A}$ and for all λ not in $Sp(D)$, $(\lambda 1 - D)^{-1}$ is compact. Let δ be the derivation $\delta(x) = i[x, D]$ assumed closable. The algebra $\mathcal{A}_0$ represents Lipschitz regularity, and is a Banach $*$-algebra with norm $|x| = \|x\| + \|[x, D]\|$. A differential seminorm $T_0(x) \equiv \|x\|, T_1(x) \equiv \|[x, D]\|$ is defined making $\mathcal{A}_0$ a smooth algebra.

EXAMPLE 6.6. Given almost commuting bounded operators on a Hilbert space, Helton and Howe [20] constucted a certain operator algebra exhibiting certain smoothness properties. We show that this algebra is indeed a smooth algebra in the sense of the present paper. Let X and Y be bounded selfadjoint operators on a Hilbert space $\mathcal{H}$. Assume that X and Y are almost commutative in the sense that the commutator $[X, Y] \in C^1(\mathcal{H})$, where $C^1(\mathcal{H})$ is the set of trace class operators on $\mathcal{H}$. Let $\mathfrak{A}$ be the C^*-algebra generated by X and Y, and let $P\mathfrak{A}$ be the $*$-algebra of all complex polynomials in X and Y. Let $\mathfrak{A}_1$ denote the maximal subset of $\mathfrak{A}$ with respect to the property that for any pair R, S in $\mathfrak{A}_1$, $[R, S] \in C^1(\mathcal{H})$. Then by [20] $\mathfrak{A}_1$ is a $*$-subalgebra of $\mathfrak{A}$, $P\mathfrak{A} \subset \mathfrak{A}_1 \subset \mathfrak{A}$, hence $\mathfrak{A}_1$ is dense in $\mathfrak{A}$, and $\mathfrak{A}_1$ is closed under Wick order power series functional calculus and Wick order C^∞-functional calculus for self-adjoint elements. First we consider the topology suggested in [20]. Let τ_1 be the locally m-convex $*$-algebra topology on $\mathfrak{A}_1$ defined by the family of norms $\{\|\cdot\|_{1,A} : A = A^* \in \mathfrak{A}_1\}$, where $\|B\|_{1,A} = \|B\| + \|[A, B]\|_1$. Notice that for each $A = A^*$ in $\mathfrak{A}_1$, $T_0(B) \equiv \|B\|, T_1(B) \equiv \|[A, B]\|_1$ defines a differential norm of order 1 on $\mathfrak{A}_1$. Further T_{tot} and T_1 are closable. The topology τ_1 can be regarded as describing the differential structure of order 1 defined by X and Y. We show that $(\mathfrak{A}_1, \tau_1)$ is complete. Let $\widetilde{\mathfrak{A}}_{1,A}$ be the completion of $\mathfrak{A}_1$ in the norm $\|\cdot\|_{1,A}$. Then $\varprojlim_A \widetilde{\mathfrak{A}}_{1,A} = \widetilde{\mathfrak{A}}_1$ which is the completion $(\mathfrak{A}_1, \tau_1)^\sim$. First we show that for any $A \in \mathfrak{A}_1$, $\widetilde{\mathfrak{A}}_{1,A} \subset \{T \in \mathfrak{A} : [T, A] \in \mathcal{C}^1(\mathcal{H})\}$. Let $(B_n) \subset \mathfrak{A}_1$ be a $\|\cdot\|_{1,A}$ - Cauchy sequence. Then there exists $B \in \mathfrak{A}$ such that $\|B_n - B\| \to 0, \|[B_n, A] - [B, A]\|_1 \to 0$, and $[B, A] \in \mathcal{C}^1(\mathcal{H})$. It follows that $\widetilde{\mathfrak{A}}_1$ is contained in $\{T \in \mathfrak{A} : [A, T] \in \mathcal{C}^1(\mathcal{H})\}$. Now let T_1, T_2 be in $\widetilde{\mathfrak{A}}_1$. Then for all $A \in \mathfrak{A}_1$, $[A, T_1]$ and $[A, T_2]$ are in $\mathcal{C}^1(\mathcal{H})$. By the maximality of $\mathfrak{A}_1$, $\mathfrak{A}_1 = \mathfrak{A}_1 \cup \{T_1\} = \mathfrak{A}_1 \cup \{T_2\}$ showing that both T_1 and T_2 are in $\mathfrak{A}_1$, and by above, $[T_1, T_2] \in \mathcal{C}^1(\mathcal{H})$. Thus $\widetilde{\mathfrak{A}}_1 \subset \mathfrak{A}$ has the property that for all T_1, T_2 in $\widetilde{\mathfrak{A}}_1$, $[T_1, T_2] \in \mathcal{C}^1(\mathcal{H})$. The maximality of $\mathfrak{A}_1$ implies that $\mathfrak{A}_1 = \widetilde{\mathfrak{A}}_1$ showing that $\mathfrak{A}_1$ is complete in τ_1. To show that $\mathfrak{A}_1$ is a smooth subalgebra of $\mathfrak{A}$, it is sufficient to show that $\mathfrak{A}_1$ is complete in the smooth topology τ defined by the family of all closable derived norms. Since each $\|\cdot\|_{1,A}, A \in \mathfrak{A}_1$ is closable and a derived norm (being the total norm of a differential norm of order 1), we have that $\tau_1 \leqslant \tau$. Let (T_α) be a Cauchy net in $\mathfrak{A}_1$ in the topology τ. Then there exists $T \in \mathfrak{A}_1$ such that $T_\alpha \to T$ in τ_1. Then $S_\alpha \equiv T_\alpha - T \to 0$ in $\|\cdot\|$, and for each closable derived norm β on $\mathfrak{A}_1$, S_α is β-Cauchy, hence $S_\alpha \to S$ in β by the closability of β, for some $S = S(\beta)$ in $\mathfrak{A}_1$. Since (S_α) is τ-Cauchy (as so is (T_α)), there exists S in the completion $(\mathfrak{A}_1, \tau)^\sim \subset \mathfrak{A}$ such that $S_\alpha \to S$ in τ, and so in each β. Thus $S(\beta) = S$ for each closable derived norm β on $\mathfrak{A}_1$. Now as $S_\alpha \to S$ in τ, $S_\alpha \to S$ in τ_1, and so $S_\alpha \to S$ in $\|\cdot\|$. Thus $S = 0$ and $T_\alpha \to T$ in τ. Since $T \in \mathfrak{A}_1$, it follows that $(\mathfrak{A}_1, \tau)$ is complete. Thus $\mathfrak{A}_1$ is a smooth subalgebra of $\mathfrak{A}$. The desired smoothness properties of $\mathfrak{A}_1$ follow.

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