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QUASI-DIAGONAL FLOWS

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ABSTRACT. We introduce two notions for flows (or one-parameter automorphism groups) on quasi-diagonal C^*-algebras, quasi-diagonal and pseudo-diagonal flows; the former being apparently stronger than the latter. We derive basic facts about these flows and give various examples. In addition we extend results of Voiculescu from quasi-diagonal C^*-algebras to these flows.

KEYWORDS: *C^*-algebra, flow, approximately inner, crossed product, KMS state, quasi-diagonal, pseudo-diagonal, Weyl–von Neumann theorem.*

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1. INTRODUCTION

Flows on C^*-algebras have been studied for some time; basic facts on flows and their generators, from the perspectives of functional analysis, spectral analysis, and Hilbert space representation theory, etc. are described in [5], [6]. But there remain many problems pertaining to C^*-algebras. For example we still lack clear and useful criteria which distinguish various kinds of flows, e.g. approximately inner flows and, in the case of AF algebras, (approximate) AF flows. (See [21] for some results for flows on AF algebras.) We hope to contribute towards clarification of the situation by introducing other properties of flows which appear to have close bearing on these features at least in the case of simple C^*-algebras.

A bounded operator T on a separable Hilbert space $\mathcal{H}$ is called *quasi-diagonal* if there is an increasing sequence (E_n) of finite-rank projections on $\mathcal{H}$ such that $\lim\limits_n E_n = 1$ strongly and $\|[E_n, T]\| \to 0$. This notion is extended to a norm-closed $*$-algebra A of bounded operators: In case A is separable A is called quasi-diagonal if there is such a sequence (E_n) and $\|[E_n, T]\| \to 0$ for all $T \in A$. If A is a separable C^*-algebra, then A is called quasi-diagonal if there is a faithful representation π of A such that $\pi(A)$ is quasi-diagonal. (See [25], [10] for more details.) Easy examples of quasi-diagonal C^*-algebras include AF algebras and commutative C^*-algebras. We mimic this notion in application to flows on C^*-algebras in two ways.

DEFINITION 1.1. Given a Hilbert space $\mathcal{H}$, let A be a norm-closed $*$-algebra of bounded operators on $\mathcal{H}$ and U a unitary flow on $\mathcal{H}$ such that $U_t x U_t^* \in A$ for $t \in \mathbb{R}$ and $t \mapsto U_t x U_t^*$ is norm-continuous for any $x \in A$.

We call (A, U) *quasi-diagonal* if for any finite set $\mathcal{F}$ of A, any finite set ω of $\mathcal{H}$ and $\varepsilon > 0$ there is a finite-rank projection E on $\mathcal{H}$ such that $\|[E, x]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$, $\|(1-E)\xi\| \leqslant \varepsilon\|\xi\|$ for $\xi \in \omega$ and $\|[E, U_t]\| < \varepsilon$ for $t \in [-1, 1]$. We call (A, U) *pseudo-diagonal* if for any finite set $\mathcal{F}$ of A, any finite set ω of $\mathcal{H}$, and $\varepsilon > 0$ there is a finite-rank projection E on $\mathcal{H}$ and a unitary flow V on $E\mathcal{H}$ such that $\|[E, x]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$, $\|(1-E)\xi\| \leqslant \varepsilon\|\xi\|$ for $\xi \in \omega$ and $\|EU_t x U_t^* E - V_t ExE V_t^*\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1, 1]$.

Let A be a C^*-algebra and let α be a flow on A. We call α *quasi-diagonal* (respectively *pseudo-diagonal*) if (A, α) has a covariant representation (π, U) on a Hilbert space $\mathcal{H}_\pi$, with π faithful and non-degenerate, such that $(\pi(A), U)$ is quasi-diagonal (respectively pseudo-diagonal).

In the above definition π is required to be non-degenerate. But this is not essential. A direct proof will be given in the beginning of Section 2 but this also follows from Theorems 1.5 and 1.6 below. Thus we immediately obtain the following result. (We do not know if a similar statement is true or false for approximately inner flows.)

COROLLARY 1.2. *Let α be a quasi-diagonal (respectively pseudo-diagonal) flow on a C^*-algebra A and B an α-invariant C^*-subalgebra of A. Then $\alpha|_B$ is quasi-diagonal (respectively pseudo-diagonal).*

Let H denote the self-adjoint generator of U in the above definition. In general H is unbounded. If Q is a bounded operator on $\mathcal{H}$ then $[Q, H]$ is defined to be bounded if $Q\mathcal{D}(H) \subset \mathcal{D}(H)$ and $QH - HQ$ is bounded on $\mathcal{D}(H)$ (and extends to a bounded operator on $\mathcal{H}$). We may replace the condition $\|[E, U_t]\| < \varepsilon$ for $t \in [-1, 1]$ in the definition of quasi-diagonality by the seemingly stronger condition $\|[E, H]\| < \varepsilon$. The opposite implication can be seen from the proposition given below. Using this we conclude that quasi-diagonality implies pseudo-diagonality since if $\|[E, H]\| < \varepsilon$ and we set $V_t = \mathrm{e}^{\mathrm{i}tEHE}$ then $\|EU_t\pi(x)U_t^* - V_t E\pi(x)EV_t^*\| \leqslant 2\varepsilon\|\pi(x)\|$ for any $x \in A$.

PROPOSITION 1.3. *Let U be a flow on $\mathcal{H}$ and H the self-adjoint generator of U. For any $\varepsilon > 0$ there is a $\delta > 0$ satisfying the following condition.*

If E is a projection such that $\|[E, U_t]\| < \delta$ for $t \in [-1, 1]$, then there is a projection F on $\mathcal{H}$ such that $\|E - F\| < \varepsilon$ and $\|[F, H]\| < \varepsilon$.

Proof. Note that it follows from the above estimate on $\|[E, U_t]\|$ that

$$\|[E, U_t]\| < \delta(1 + |t|)$$

for all $t \in \mathbb{R}$. In addition to this estimate we use the fact that $t \mapsto U_t E U_t^*$ is continuous in the strong operator topology (or in norm).

Let f be a non-negative C^∞ function on $\mathbb{R}$ such that $\mathrm{supp}(f) \subset [1/3, 4/3]$ and $f(t) = 1$ for $t \in [2/3, 1]$. Define $\widehat{f}$ by $\widehat{f}(p) = (2\pi)^{-1} \int \mathrm{e}^{-\mathrm{i}pt} f(t)\mathrm{d}t$ and set $C = \int |t\widehat{f}(t)|\mathrm{d}t < \infty$. Let g be a non-negative C^∞ function on $\mathbb{R}$ such that the support of g is compact, $\int g(t)\mathrm{d}t = 1$ and $\int |g'(t)|\mathrm{d}t < \varepsilon/C$. Set $D = \int g(t)(1 + |t|)\mathrm{d}t$. Assuming $\delta D < \varepsilon/2 < 1/3$ we define

$$Q = \int g(t) U_t E U_t^* \mathrm{d}t \ .$$

Then $0 \leqslant Q \leqslant 1$, $\|Q - E\| < \varepsilon/2$ and $\|[H, Q]\| < \varepsilon/C$, where $\mathrm{i}[H, Q]$ is identified with $-\int g'(t) U_t E U_t^* \mathrm{d}t$. Since $\mathrm{Sp}(Q) \subset [0, \varepsilon/2) \cup (1 - \varepsilon/2, 1]$ it follows that $F = f(Q) = \int \widehat{f}(t)\mathrm{e}^{\mathrm{i}tQ}\mathrm{d}t$ is a projection satisfying $\|F - Q\| < \varepsilon/2$. It also follows that $\|[H, F]\| \leqslant \|[H, Q]\| \int |t\widehat{f}(t)|\mathrm{d}t < \varepsilon$. Since $\|E - F\| < \varepsilon$, this concludes the proof. (See [5] for the norm estimates used here.) ∎

We note that a covariant representation (ρ, V) of (A, α) naturally induces a representation $\rho \times V$ of the crossed product $A \times_\alpha \mathbb{R}$ on the representation space $\mathcal{H}_\rho$ of ρ. We denote by $\mathcal{K}(\mathcal{H}_\rho)$ the compact operators on $\mathcal{H}_\rho$.

By extending Voiculescu's theorem [22] to accommodate the flow we establish the following:

THEOREM 1.4. *Let α be a quasi-diagonal (respectively pseudo-diagonal) flow on A. If (ρ, V) is a covariant representation of A such that $\rho \times V$ is a faithful representation of $A \times_\alpha \mathbb{R}$ and $\mathrm{Ran}(\rho \times V) \cap \mathcal{K}(\mathcal{H}_\rho) = \{0\}$ then $(\rho(A), V)$ is quasi-diagonal (respectively pseudo-diagonal).*

Mimicking the corresponding result due to Voiculescu [24] we shall give characterizations of quasi-diagonal and pseudo-diagonal flows.

If A and B are C^*-algebras then a linear map ϕ of A into B is called *positive* if $\phi(A_+) \subset B_+$ and *completely positive* (or CP) if $\phi_n = \mathrm{id} \otimes \phi \colon M_n \otimes A \to M_n \otimes B$ is positive for all n.

THEOREM 1.5. *Let α be a flow on a C^*-algebra A. Then the following conditions are equivalent:*

(i) *α is quasi-diagonal.*

(ii) *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there is a finite-dimensional C^*-algebra B, a flow β on B and a CP map ϕ of A into B such that $\|\phi\| \leqslant 1$, $\|\phi(x)\| \geqslant (1 - \varepsilon)\|x\|$ and $\|\phi(x)\phi(y) - \phi(xy)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$, and $\|\beta_t\phi - \phi\alpha_t\| < \varepsilon$ for $t \in [-1, 1]$.*

(iii) *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there is a covariant representation (π, U) and a finite-rank projection E on $\mathcal{H}_\pi$ such that $\|E\pi(x)E\| \geqslant \|x\| - \varepsilon$ and $\|[E, \pi(x)]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $\|[E, U_t]\| < \varepsilon$ for $t \in [-1, 1]$.*

When A is separable it follows from the proof of (iii) $\Rightarrow$ (i) that there is a covariant representation (π, U) of (A, α) on a separable Hilbert space such that $(\pi(A), U)$ is quasi-diagonal. This fact will be used in the proof of Theorem 1.4 above.

THEOREM 1.6. *Let α be a flow on a C^*-algebra A. Then the following conditions are equivalent:*

(i) *α is pseudo-diagonal.*

(ii) *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there is a finite-dimensional C^*-algebra B, a flow β on B and a* CP *map ϕ of A into B such that $\|\phi\| \leqslant 1$, $\|\phi(x)\| \geqslant (1-\varepsilon)\|x\|$ and $\|\phi(x)\phi(y) - \phi(xy)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$, and $\|\beta_t\phi(x) - \phi\alpha_t(x)\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1, 1]$.*

(iii) *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there is a covariant representation (π, U), a finite-rank projection E on $\mathcal{H}_\pi$ and a unitary flow V on $E\mathcal{H}_\pi$ such that $\|E\pi(x)E\| \geqslant (1-\varepsilon)\|x\|$ and $\|[E, \pi(x)]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $\|EU_t\pi(x)U_t^*E - V_tE\pi(x)EV_t^*\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1, 1]$.*

In the above theorems the finite-dimensional C^*-algebra B can be assumed to be a matrix algebra M_k for some $k \in \mathbb{N}$.

If A is separable and α is a pseudo-diagonal flow let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of A whose union is dense in A and choose, for each $(\mathcal{F}_n, n^{-1})$ in place of $(\mathcal{F}, \varepsilon)$, a CP map ϕ_n into M_{k_n} and a flow $\beta^{(n)}$ on M_{k_n} as specified in condition (ii) of the above theorem. Thus we can define a non-continuous flow β on the direct product $B = \prod_n M_{k_n}$ by $\beta_t(x) = \prod_n \beta_t^{(n)}(x_n)$ for $x = (x_n) \in B$ and a CP map ϕ of A into B by $\phi(x) = (\phi_n(x))_n$. Let $I = \bigoplus_n M_{k_n}$, which is the ideal of B consisting of sequences converging to zero, and let Q denote the quotient map of B onto B/I. Then it follows that $\psi = Q\phi$ is an isomorphism of A into B/I satisfying $\psi\alpha_t = \beta_t\psi$. A separable C^*-algebra is an MF algebra if it can be embedded into $\prod_n M_{k_n} / \bigoplus_n M_{k_n}$ for some (k_n) (see [2] for MF algebras). We may call the flow α an *MF flow* since it satisfies the intertwining property with β. It might be interesting to explore this class of flows.

We will show that if α is an approximately inner flow on a quasi-diagonal C^*-algebra then α is pseudo-diagonal (Proposition 2.17). We will also show that if α is a pseudo-diagonal flow on a unital C^*-algebra then α has KMS states for all inverse temperatures (Proposition 2.8).

If A is an AF algebra and α is an (approximate) AF flow then it follows that α is quasi-diagonal (Proposition 2.18). If A is an AF algebra which has a faithful family of type I quotients then any flow on A is quasi-diagonal (Proposition 2.25).

Let α (respectively β) be a flow on a C^*-algebra A (respectively B). We say that (B, β) homotopically dominates (A, α) if there are homomorphisms $\phi \colon A \to B$ and $\psi \colon B \to A$ and a homotopy $\{\chi_s : s \in [0, 1]\}$ of homomorphisms of A into A such that $\phi\alpha_t = \beta_t\phi$, $\psi\beta_t = \alpha_t\psi$, $\chi_s\alpha_t = \alpha_t\chi_s$, $\chi_0 = \psi\phi$ and $\chi_1 = \mathrm{id}_A$. The main result of Voiculescu's paper [24] has the following analogue:

THEOREM 1.7. *Suppose that (B, β) homotopically dominates (A, α). If β is quasi-diagonal then α is quasi-diagonal.*

This implies that if α is a flow on a C^*-algebra A then the flow $\alpha \otimes \mathrm{id}$ on $A \otimes C_0[0,1)$ is quasi-diagonal. (The family of endomorphisms $\phi_s, s \in [0,1]$ of $A \otimes C_0[0,1)$ defined by $\phi_s(x)(t) = x(st)$ commutes with the flow $\alpha \otimes \mathrm{id}$ and satisfies $\phi_1 = \mathrm{id}$ and $\phi_0 = 0$. This also follows directly from Proposition 2.14.) Thus approximate innerness does not follow from quasi-diagonality without further conditions on the C^*-algebra. Another result of this type is that if α is a flow on a quasi-diagonal C^*-algebra A then the flow β on $A \otimes C[0,1]$ defined by $\beta_t(x)(s) = \alpha_{st}(x(s))$ is quasi-diagonal (Proposition 2.15).

We note that we have not been able to give the pseudo-diagonal version of the above theorem. We also note that we do not know if quasi-diagonality is strictly stronger than pseudo-diagonality or not.

Let u be an α-cocycle, i.e. let u denote a continuous function from $\mathbb{R}$ into the unitary group of $M(A)$ such that $t \mapsto u_t$ is continuous in the strict topology and $u_s\alpha_s(u_t) = u_{s+t}$ for $s, t \in \mathbb{R}$. If A is unital then the multiplier algebra $M(A)$ is just A and the strict topology is the norm topology. We say the flow $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ is a cocycle perturbation of α. We note that quasi-diagonality (respectively pseudo-diagonality) is stable under cocycle perturbations (Propositions 2.2 and 2.5). We also note that if B is an α-invariant hereditary C^*-subalgebra of A which generates A as an ideal then $\alpha|_B$ is quasi-diagonal (respectively pseudo-diagonal) if and only if α is quasi-diagonal (respectively pseudo-diagonal) (Corollary 2.7).

In Section 2 we will give the above basic facts on quasi-diagonal and pseudo-diagonal flows and some examples including the proof of Theorem 1.7. For example the rotation flow on the continuous functions on the unit circle is not quasi-diagonal (and not even pseudo-diagonal) but the rotation flow on the continuous functions on the unit disk is quasi-diagonal. In Section 3 we generalize Voiculescu's Weyl–von Neumann theorem [22] to cover the present situation and thereby prove Theorem 1.4. In Section 4 we deal with the adaptation of Voiculescu's results in [24] to prove Theorems 1.5 and 1.6.

2. QUASI-DIAGONAL AND PSEUDO-DIAGONAL FLOWS

Let α be a flow on a C^*-algebra A. The definition of quasi-diagonality, or pseudo-diagonality, of α required the representation π in the covariant representation (π, U) of (A, α) to be faithful and non-degenerate. But the non-degeneracy of π is not essential by the following argument.

First this is evident if A is unital. Therefore we assume that A is not unital.

Secondly, let π be a faithful degenerate representation of A on a Hilbert space $\mathcal{H}$ and U a unitary flow on $\mathcal{H}$ such that $\mathrm{Ad}\, U_t\pi(x) = \pi\alpha_t(x)$ for all $x \in A$. Let P be the projection onto the closure of $\pi(A)\mathcal{H}$. Note that $U_tP = PU_t$ and let us denote by UP the unitary flow $t \mapsto U_tP$ on $P\mathcal{H}$.

Suppose that $(\pi(A), U)$ is pseudo-diagonal. We shall show that the restriction of the pair to $P\mathcal{H}$ is pseudo-diagonal. For a finite subset $\mathcal{F}$ of A, a finite subset

ω of $P\mathcal{H}$ and $\varepsilon > 0$ we choose a finite-rank projection E on $\mathcal{H}$ and a unitary flow V on $E\mathcal{H}$ which satisfy the conditions of the definition. Let $\mathcal{K}_1$ be the subspace $(1-P)E\mathcal{H}$. We find a subspace $\mathcal{K}_2$ of $P\mathcal{H}$ with the same dimension as $\mathcal{K}_1$ such that $\mathcal{K}_2$ is orthogonal to $PE\mathcal{H}$ and $\|\pi\alpha_t(x)|_{\mathcal{K}_2}\| \leqslant (\varepsilon/2)\|x\|$ for $x \in \mathcal{F} \cup \mathcal{F}^*$ and $t \in [-1,1]$. Let W_1 be a unitary from $\mathcal{K}_1$ onto $\mathcal{K}_2$ and denote by P_i the projection onto $\mathcal{K}_i$ for $i = 1,2$. Regarding W_1 as $W_1 = W_1P_1$, let $W = W_1 + W_1^* + (1 - P_1 - P_2)$, which is a unitary on $\mathcal{H}$, and let $F = WEW^*$. Since $WE\mathcal{H} \subset W(1-P)E\mathcal{H} + WPE\mathcal{H} \subset P_2W(1-P)E\mathcal{H} + PE\mathcal{H}$, it follows that $F \leqslant P$. Since $\pi(x)WE = \pi(x)W_1E + \pi(x)(1-P_2)E = \pi(x)P_2(W_1-1)E + \pi(x)E$ we obtain $\|\pi(x)WE - \pi(x)E\| < \varepsilon$, which implies that $\|F\pi\alpha_t(x)F - WV_tW^*F\pi(x)FWV_t^*W^*\| \leqslant 5\varepsilon\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1,1]$. The other properties follow easily. Thus the pair F and $t \mapsto WV_tW^*$ satisfies the required conditions for $(\pi(A)P, UP)$.

Now suppose that $(\pi(A), U)$ is quasi-diagonal. Let $(\overline{\pi}, \overline{U})$ be the direct sum of (π, χ_pU) over all rational numbers p, on the representation space $\overline{\mathcal{H}} = \bigoplus_p \mathcal{H}$, where χ_pU is the unitary flow $t \mapsto \mathrm{e}^{\mathrm{i}pt}U_t$. Let P be the projection onto the closure of $\pi(A)\mathcal{H}$ as before and let $\overline{P}$ be the projection onto the closure of $\overline{\pi}(A)\overline{\mathcal{H}}$, i.e. $\overline{P} = \bigoplus_p P$. We shall show that $(\overline{\pi}(A)\overline{P}, \overline{UP})$ is quasi-diagonal.

From now on we use π, U, P to denote $\overline{\pi}, \overline{U}, \overline{P}$. We have now assumed that $\pi \times U$ is faithful besides $(\pi(A), U)$ being quasi-diagonal. Let H be the self-adjoint generator of U. For a finite subset $\mathcal{F}$ of A, a finite subset ω of $P\mathcal{H}$ and $\varepsilon > 0$ we choose a finite-rank projection E on $\mathcal{H}$ such that $\|[E,H]\| < \varepsilon$ holds in addition to the other conditions in the definition. There is a finite-dimensional subspace $\mathcal{K}_1$ of $(1-P)\mathcal{H}$ such that $\mathcal{K}_1 \supset (1-P)E\mathcal{H}$ and $\|[P_1,H]\| < \varepsilon/2$, where P_1 is the projection onto $\mathcal{K}_1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of P_1HP_1 in increasing order. We choose a finite-rank projection P_2 such that $P_2 \leqslant P$, $P_2PE = 0$, $\|P_2\pi(x)\|, \|\pi(x)P_2\| \leqslant (\varepsilon/2)\|x\|$ for $x \in \mathcal{F}$, $\|[P_2,H]\| < \varepsilon/2$ and the increasing list of eigenvalues of P_2HP_2 are arbitrarily close to $\lambda_1, \lambda_2, \ldots, \lambda_n$. (In particular P_1 and P_2 have the same rank.) This is possible by the lemma below which uses faithfulness of $\pi \times U$. Then we choose a unitary W_1 of $\mathcal{K}_1$ onto $P_2\mathcal{H}$ such that $W_1P_1HP_1 \approx P_2HP_2W_1$. We set $W = W_1 + W_1^* + (1 - P_1 - P_2)$. Then $F = WEW^* \leqslant P$ and $\|U_tW - WU_t\| \leqslant \varepsilon|t|$ (by making $W_1P_1HP_1 \approx P_2HP_2W_1$ precise). This implies that F satisfies the required conditions.

LEMMA 2.1. *Suppose that A is non-unital and let π, U, P be as above. For any finite subset $\mathcal{F}$ of A, $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ there exists a unit vector $\xi \in P\mathcal{H}$ such that $\|\pi(x)\xi\| < \varepsilon$ for $x \in \mathcal{F}$ and $\|U_t\xi - \mathrm{e}^{\mathrm{i}\lambda t}\xi\| \leqslant \varepsilon|t|$.*

Proof. Let $z = \sum_{x \in \mathcal{F}} x^*x$ and let P_H denote the spectral measure for H. Suppose that there is an $\varepsilon > 0$ such that $\langle \xi, \pi(z)\xi \rangle \geqslant \varepsilon$ for any unit vector ξ in $P_H(\lambda - \varepsilon, \lambda + \varepsilon)P\mathcal{H}$. Let $\widehat{f}$ be a non-negative C^∞-function on $\mathbb{R}$ such that $f \neq 0$ and $\mathrm{supp}(\widehat{f}) \subset (\lambda - \varepsilon, \lambda + \varepsilon)$. Since $\pi \times U$ is faithful $\lambda(f)(z - \varepsilon)\lambda(f)^* \geqslant 0$, where

$\lambda(f) = \int f(t)\lambda_t dt$ is a multiplier of $A \times_\alpha \mathbb{R}$ such that $\pi(\lambda(f)) = \widehat{f}(H)$. Applying $\widehat{\alpha}_p$ and taking the integral over p implies that $\int |f(t)|^2 \alpha_t(z) dt$ is invertible, which contradicts that A is non-unital. (See 7.8 of [20] for more details.) ∎

In order for α to be quasi-diagonal or pseudo-diagonal the C^*-algebra A must be quasi-diagonal. Moreover, it follows that if α is quasi-diagonal then the crossed product $A \times_\alpha \mathbb{R}$ is quasi-diagonal. (The pair (π, U) gives a representation $\pi \times U$ of $A \times_\alpha \mathbb{R}$, which may not be faithful, such that $\pi \times U(A \times_\alpha \mathbb{R})$ is quasi-diagonal. As a faithful representation of $A \times_\alpha \mathbb{R}$ is required in the definition of quasi-diagonality we may take the direct sum of $\pi \times \chi_p U$ over all rationals p as in the previous paragraph.)

If α is a flow on an AF algebra A then the crossed product $A \times_\alpha \mathbb{R}$ is AF-embeddable; in particular it is quasi-diagonal. (We learned this fact from M. Izumi; the argument uses the fact that the crossed product of A by $\alpha|_{\mathbb{Z}}$ is AF-embeddable, due to [23] and [9].)

As we shall see α need not be quasi-diagonal, nor pseudo-diagonal, even if $A \times_\alpha \mathbb{R}$ is quasi-diagonal.

PROPOSITION 2.2. *Let α be a flow on A and let u be an α-cocycle. Then α is quasi-diagonal if and only if $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ is quasi-diagonal.*

Proof. If A is unital this follows straightforwardly. Suppose that A does not have a unit and that α is quasi-diagonal. Thus we assume that A acts on a Hilbert space $\mathcal{H}$ non-degenerately and there is a unitary flow U such that $\alpha_t(x) = U_t x U_t^*$ for $x \in A$ and (A, U) is quasi-diagonal. Let $\mathcal{F}$ be a finite subset of A and ω a finite subset of $\mathcal{H}$. Then we choose $p, e \in A$ such that $0 \leqslant p \leqslant e \leqslant 1$, $ep = p$, $\|x - pxp\| \approx 0$ for $x \in \mathcal{F}$, $\|p\xi - \xi\| \approx 0$ for $\xi \in \omega$ and $\|\alpha_t(e) - e\| \approx 0$ for $t \in [-1, 1]$. We choose an α-cocycle v in $A + \mathbb{C}1$ such that $\|(u_t - v_t)e\| \approx 0$, $t \in [-1, 1]$, where $t \mapsto v_t$ is continuous in norm [17]. We choose a finite-rank projection E such that $\|[E, x]\| \approx 0$ for $x \in \mathcal{F} \cup \{p, e\} \cup \{v_t : t \in [-1, 1]\}$, $\|(1 - E)\xi\| \approx 0$ for $\xi \in \Omega$ and $\|[E, H]\| \approx 0$. By the lemma below there is a subprojection F of E such that $Fp \approx Ep$, $Fe \approx F$, and $\|[F, H]\| \approx 0$. Since $\|x - pxp\| \approx 0$ for $x \in \mathcal{F}$ we have $\|[F, x]\| \approx \|[E, x]\| \approx 0$ for $x \in \mathcal{F}$. Since $u_t e \approx v_t e$ and $eu_t \approx ev_t$ we have $\|[F, u_t]\| \approx \|[F, v_t]\| \approx 0$ for $t \in [-1, 1]$, which implies that $\|[F, u_t U_t]\| \approx 0$ for $t \in [-1, 1]$. Further we have $\|(1 - F)\xi\| \approx \|(1 - E)\xi\| \approx 0$ for $\xi \in \omega$. ∎

LEMMA 2.3. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds.*

If $e, p \in A$ and a finite-rank projection E satisfy $0 \leqslant p \leqslant e \leqslant 1$, $ep = p$, $\|\alpha_t(e) - e\| < \delta$ for $t \in [-1, 1]$, $\|[E, H]\| < \delta$, $\|[E, e]\| < \delta$ and $\|[E, p]\| < \delta$, then there is a finite rank projection F such that $F \leqslant E$, $\|Ep - Fp\| < \varepsilon$, $\|Fe - F\| < \varepsilon$ and $\|[F, H]\| < \varepsilon$.

Proof. Let $e' = EeE$, $p' = EpE$, and $H' = EHE$. Since $\|[e^{itH}, e']\| \approx 0$ for $t \in [-1, 1]$ and $\|(1 - E)HE\| \approx 0$ we conclude that $\|[e^{itH'}, e']\| \approx 0$ for $t \in [-1, 1]$. Then Lin's theorem [19], [4] for almost commuting self-adjoint e' and H'

in $\mathcal{B}(E\mathcal{H})$ tells us that there is a self-adjoint h in $\mathcal{B}(E\mathcal{H})$ such that $h \approx e'$ and $\|[q, H']\| \approx 0$ uniformly for any spectral projection q of h. Since $p'e' \approx p'$we deduce that $p'h \approx p'$. Let F be the spectral projection of h corresponding to $[1-\varepsilon/2, 1]$. From the lemma below and $p' = EpE \approx Ep$ we may suppose that $\|Fp - Ep\| < \varepsilon$. Since $\|Fh - F\| \leqslant \varepsilon/2$ and $h \approx EeE$ we may also suppose that $\|Fe - F\| < \varepsilon$. Since $[F, H] = [F, H'] + FH(1-E) - (1-E)HF$ it follows that $\|[F, H]\| \leqslant \|[F, H']\| + \|(1-E)HE\|$, which we may suppose is smaller than ε. ∎

LEMMA 2.4. *For any $\varepsilon, \varepsilon' > 0$ there is a $C > 0$ such that the following holds.*

For any $h, p \in A_{\mathrm{sa}}$ such that $0 \leqslant h \leqslant 1$, $0 \leqslant p \leqslant 1$ and $\|hp - p\| < \delta$ the spectral projection F of h corresponding to $[1-\varepsilon, 1]$ satisfies $\|Fp - p\| < \varepsilon' + C\delta$.

Proof. Fix a continuous function f on $[0,1]$ such that $0 \leqslant f \leqslant 1$, $f(1) = 1$ and $\mathrm{supp}(f) \subset [1-\varepsilon, 1]$ and choose a polynomial $q(t) = \sum_{k=1}^{n} c_k t^k$ with $q(1) = 1$ such that $|f(t) - q(t)| < \varepsilon'$ for $t \in [0,1]$. Since $\|q(h)p - p\| \leqslant \sum_{k=1}^{n} |c_k| k\delta \equiv C\delta$ and $\|f(h) - q(h)\| < \varepsilon'$ it follows that $\|(1-F)p\| = \|(1-F)(1-f(h))p\| < \varepsilon' + C\delta$, where F denotes the spectral projection of h corresponding to $[1-\varepsilon, 1]$. ∎

PROPOSITION 2.5. *Let α be a flow on A and u an α-cocycle. Then α is pseudo-diagonal if and only if $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ is pseudo-diagonal.*

Proof. If A is unital this follows straightforwardly. Suppose that A does not have a unit and that α is pseudo-diagonal. Thus we assume that A acts on a Hilbert space $\mathcal{H}$ non-degenerately and there is a unitary flow U such that $\alpha_t(x) = U_t x U_t^*$, $x \in A$ and (A, U) is pseudo-diagonal. Let u be an α-cocycle in $M(A)$. Further let $\mathcal{F}$ be a finite subset of A and ω a finite subset of $\mathcal{H}$. Then, by the lemma below, we choose $p, e \in A$ such that $0 \leqslant p \leqslant e \leqslant 1$, $ep = p$, $\|x - pxp\| \approx 0$ for $x \in \mathcal{F}$, $\|p\xi - \xi\| \approx 0$ for $\xi \in \omega$, $\|\alpha_t(e) - e\| \approx 0$ for $t \in [-1,1]$ and $\|[e, u_t]\| \approx 0$ for $t \in [-1,1]$. For u and e we choose an α-cocycle v in $A + \mathbb{C}1$ such that $\|(u_t - v_t)e\| \approx 0$ for $t \in [-1,1]$ [17]. We then express v_t as $wU_t^{(h,\alpha)}\alpha_t(w^*)$, where $w \in \mathcal{U}(A + \mathbb{C}1)$, $h = h^* \in A + \mathbb{C}1$ and $U_t^{(h,\alpha)}$ denotes the α-cocycle defined by $\frac{\mathrm{d}}{\mathrm{d}t}U_t^{(h,\alpha)} = U_t^{(h,\alpha)}\alpha_t(\mathrm{i}h)$:

$$U_t^{(h,\alpha)} = 1 + \sum_{n=1}^{\infty} \int_{\Omega_n(t)} \alpha_{t_1}(\mathrm{i}h)\alpha_{t_2}(\mathrm{i}h)\cdots\alpha_{t_n}(\mathrm{i}h)\mathrm{d}t_1\cdots\mathrm{d}t_n$$

where $\Omega_n(t) = \{(t_1, t_2, \ldots, t_n) : 0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t\}$ for $t \geqslant 0$ and similarly for $t < 0$ (see Lemma 1.1 of [14]).

Let $\mathcal{G} = \mathcal{F} \cup \{p, e\} \cup \{\alpha_t(h), \alpha_t(w) : t \in [-1,1]\}$. Then $\mathcal{G}$ is a compact subset of $A + \mathbb{C}1$. Since (A, U) is pseudo-diagonal, we choose, for $\mathcal{G}$ and ω, a finite-rank projection E and a unitary flow V on $E\mathcal{H}$ such that $\|[E, x]\| \approx 0$ for $x \in \mathcal{G}$, $\|(1-E)\xi\| \approx 0$ for $\xi \in \omega$ and $\|E\alpha_t(x)E - V_t ExE V_t^*\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{G}$

and $t \in [-1,1]$. Set $\beta_t = \operatorname{Ad} V_t$ on $\mathcal{B}(E\mathcal{H})$. From the above expression for $U_t^{(h,\alpha)}$ we note that

$$Ev_t = EwU^{(h,\alpha)}\alpha_t(w^*) \approx EwEU_t^{(EhE,\beta)}\beta_t(EwE)^*$$

for $t \in [-1,1]$. Thus replacing EwE by a close unitary in $\mathcal{B}(E\mathcal{H})$ we obtain a β-cocycle b in $\mathcal{B}(E\mathcal{H})$ such that $Ev_t \approx Eb_t$ for $t \in [-1,1]$. It follows that $E\operatorname{Ad} v_t\alpha_t(x)E \approx \operatorname{Ad} b_t\beta_t(ExE)$ for $x \in \mathcal{F}$ and $t \in [-1,1]$. Since $\operatorname{Ad} b_t\beta_t(EeE) \approx E\operatorname{Ad} b_t\alpha_t(e)E \approx E\operatorname{Ad} v_t(e)E \approx E\operatorname{Ad} u_t(e)E \approx EeE$ for $t \in [-1,1]$, by Lemma 2.3, there is a subprojection F of E such that $Fp \approx Ep$, $Fe \approx F$ and $\|[F,H']\| \approx 0$, where H' is the self-adjoint generator of $t \mapsto b_tV_t$.

Since $\|x - pxp\| \approx 0$ for $x \in \mathcal{F}$, we have $\|[F,x]\| \approx \|[E,x]\| \approx 0$ for $x \in \mathcal{F}$ and since $p\xi \approx \xi$ for $\xi \in \omega$ we have $\|(1-F)\xi\| \approx \|(1-E)\xi\| \approx 0$ for $\xi \in \omega$. We conclude that $F\operatorname{Ad} u_t\alpha_t(x)F \approx F\operatorname{Ad} v_t\alpha_t(x)F \approx F\operatorname{Ad} \mathrm{e}^{\mathrm{i}tH'}(ExE)F \approx \operatorname{Ad} \mathrm{e}^{\mathrm{i}tFH'F}(FxF)$ for $x \in \mathcal{F}$. ∎

LEMMA 2.6. *Let A be a non-unital C^*-algebra and α a flow on A. Let u be an α-cocycle in $M(A)$. Then there exists an approximate identity $(e_\mu)_{\mu\in I}$ in A such that* $\max\{\|\alpha_t(e_\mu) - e_\mu\| : t \in [-1,1]\}$ *and* $\max\{\|[e_\mu, u_t]\| : t \in [-1,1]\}$ *converge to zero. Moreover one may assume that there is another approximate identity $(p_\mu)_{\mu\in I}$ with the same index set I satisfying the same conditions as (e_μ) and $e_\mu p_\mu = p_\mu$ for $\mu \in I$.*

Proof. Define a flow γ on $M_2 \otimes A$ by $\gamma_t(e_{12} \otimes x) = e_{12} \otimes \alpha_t(x)u_t^*$ for $x \in A$. (Thus $\gamma_t(e_{11} \otimes x) = e_{11} \otimes \alpha_t(x)$ and $\gamma_t(e_{22} \otimes x) = e_{22} \otimes \operatorname{Ad} u_t\alpha_t(x)$.) We choose an approximate identity (f_μ) in $M_2 \otimes A$ such that $\max\{\|\gamma_t(f_\mu) - f_\mu\| : t \in [-1,1]\} \to 0$. By taking a net in the convex combinations of $\{f_\mu\}$ we may further suppose that $\|[e_{ij} \otimes 1, f_\mu]\| \to 0$. Then we define $e_\mu \in A$ by

$$1 \otimes e_\mu = \frac{1}{2}\sum_i (e_{i1} \otimes 1)f_\mu(e_{1i} \otimes 1),$$

which is almost equal to f_μ. Thus it follows that $\|\gamma_t(1 \otimes e_\mu) - 1 \otimes e_\mu\| \leqslant \|\gamma_t(1 \otimes e_\mu) - \gamma_t(f_\mu)\| + \|\gamma_t(f_\mu) - f_\mu\| + \|f_\mu - 1 \otimes e_\mu\|$, which converges to zero uniformly in t on $[-1,1]$. Since $\gamma_t(1 \otimes e_\mu) = e_{11} \otimes \alpha_t(e_\mu) + e_{22} \otimes u_t\alpha_t(e_\mu)u_t^*$, this completes the proof for the first part. To prove the additional assertion, we choose two continuous functions f, g from $[0,1]$ onto $[0,1]$ such that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$ and $fg = g$. Then the pair $f(e_\mu)$ and $g(e_\mu)$ satisfy $f(e_\mu)g(e_\mu) = g(e_\mu)$. One can prove that $f(e_\mu)$ (respectively $g(e_\mu)$) is an approximate identity satisfying the required properties. ∎

COROLLARY 2.7. *Let α be a flow on A and let B be an α-invariant hereditary C^*-subalgebra of A such that B generates A as an ideal. Then α is quasi-diagonal (respectively pseudo-diagonal) if and only if $\alpha|_B$ is quasi-diagonal (respectively pseudo-diagonal).*

Proof. The "only if" part follows from the definition even if B is an arbitrary α-invariant C^*-subalgebra of A. (See also Theorems 1.5 and 1.6.)

Suppose that $\alpha|_B$ is quasi-diagonal (respectively pseudo-diagonal). If A is separable (or has a strictly positive element), then $B \otimes \mathcal{K}$ and $A \otimes \mathcal{K}$ are isomorphic with each other (see [7]), where $\mathcal{K}$ is the separable C^*-algebra of compact operators. Under this identification, $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{K}$ is a cocycle perturbation of $\alpha|_B \otimes \mathrm{id}$ (see [17]). Thus the "if" part follows from Propositions 2.2 and 2.5 in the separable case.

Suppose that A is not separable. Let $\mathcal{F}$ be a finite subset of A. Since the linear span of ABA is dense in A, there is a countable subset $\mathcal{G}$ of AB such that the closed linear span of $\{xy^* : x, y \in \mathcal{G}\}$ contains $\mathcal{F}$. Let A_1 be the α-invariant C^*-subalgebra of A generated by $\mathcal{G}$. Then $A_1 \supset \mathcal{F}$. Since $\alpha_s(x)^*\alpha_t(y) \in A_1 \cap B$ for $x, y \in \mathcal{G}$, the hereditary C^*-subalgebra $B_1 = A_1 \cap B$ of A_1 is essential, i.e. it generates A_1 as an ideal of A_1. Since $\alpha|_{B_1}$ is quasi-diagonal (respectively pseudo-diagonal), it follows that $\alpha|_{A_1}$ is quasi-diagonal (respectively pseudo-diagonal). Since $\mathcal{F}$ is arbitrary this completes the proof. ∎

Recall that pseudo-diagonality follows from quasi-diagonality.

PROPOSITION 2.8. *Suppose that α is a pseudo-diagonal flow on a unital C^*-algebra A. Then α has a* KMS *state for all inverse temperatures including $\pm\infty$.*

Proof. Let $\mathcal{F}$ be a finite subset $\mathcal{F}$ of A and $\varepsilon > 0$. For each $(\mathcal{F}, \varepsilon)$ we have a flow β on a finite-dimensional C^*-algebra B and a CP map ϕ of A into B such that $\phi(1) = 1$, $\|\phi(x)\| \geqslant (1-\varepsilon)\|x\|$ and $\|\phi(x)\phi(y) - \phi(xy)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$ and $\|\beta_t\phi(x) - \phi\alpha_t(x)\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1, 1]$. Here we have replaced the condition $\|\phi\| \leqslant 1$ by $\phi(1) = 1$ since A is unital. To justify this we note that we may assume that $1 \in \mathcal{F}$, which entails that $\|\phi(1)^2 - \phi(1)\| \leqslant \varepsilon$ and $\|\beta_t(\phi(1)) - \phi(1)\| \leqslant \varepsilon$ for $t \in [-1, 1]$. By functional calculus for small ε we obtain a projection p from $\phi(1)$. Since $\|\beta_t(p) - p\|$ is of order ε for $t \in [-1, 1]$ we can perturb β by a β-cocycle which differs from 1 on $[-1, 1]$ by up to order ε and suppose that $\beta_t(p) = p$. Replacing B by pBp and ϕ by $q\phi(\,\cdot\,)q$ with $q = (p\phi(1)p)^{-1/2}$ and restricting β we can assume that ϕ is unital. Since $\|q - p\|$ is of order ε we could start with a smaller ε to obtain the right estimates.

There is a self-adjoint $h \in B$ such that $\beta_t = \mathrm{Ad}\, e^{ith}$. Fix $\gamma \in \mathbb{R}$ and define a state φ on B by

$$\varphi(Q) = \frac{\mathrm{Tr}(e^{-\gamma h}Q)}{\mathrm{Tr}(e^{-\gamma h})},$$

where Tr is a trace on B. Then φ is a KMS state on B with respect to β at inverse temperature γ.

Let $f_{(\mathcal{F},\varepsilon)} = \varphi\phi$ be a state on A where φ and ϕ depend on $(\mathcal{F}, \varepsilon)$. Let f be a weak*-limit point of $f_{(\mathcal{F},\varepsilon)}$, where the set X of $(\mathcal{F}, \varepsilon)$ is a directed set in an obvious way. We fix a Banach limit ψ on $L^\infty(X)$ such that $f(x)$ is the ψ limit of $(\mathcal{F}, \varepsilon) \mapsto f_{(\mathcal{F},\varepsilon)}(x)$ for $x \in A$. Note that $f(x\alpha_t(y))$ is the ψ limit of $(\mathcal{F}, \varepsilon) \mapsto \varphi(\phi(x\alpha_t(y)))$, which is close to $\varphi(\phi(x)\beta_t\phi(y))$ around ∞. Thus one can conclude that f is a KMS state at γ.

A similar proof works for a KMS state for $\gamma = \pm\infty$ (or a ground state and ceiling state). See [5], [6] for more details on KMS states. ∎

We may call such a state $f_{\mathcal{F},\varepsilon}$ on A as above a *local KMS state* (depending also on the choice of B, ϕ, β, h and Tr on B) and a KMS state f on A obtained as a limit of local KMS states *locally approximable*. It follows that the locally approximable KMS states at an inverse temperature form a closed convex cone. It may be natural to ask whether all the KMS states are locally approximable for a pseudo-diagonal flow on some C^*-algebra. An easy example of such will be given later.

We remind the reader that if α is approximately inner then we obtain the same conclusion as in the above proposition [5]. The proof is similar. Since there is a flow on a unital AF algebra which has no KMS states for $\gamma > 0$, we know that there is a flow, on a unital AF algebra, which is not pseudo-diagonal. Obvious examples of non-pseudo-diagonal flows are as follows:

EXAMPLE 2.9. Let Ω be a compact Hausdorff space and α a flow of homeomorphisms of Ω such that no point of Ω is fixed under α. We denote by the same symbol α the flow of the C^*-algebra on $A = C(\Omega)$ which naturally arises as $\alpha_t(f)(\omega) = f(\alpha_{-t}(\omega))$ for $f \in C(\Omega)$ and $\omega \in \Omega$. Then the flow α is not pseudo-diagonal since if α has a KMS state for non-zero inverse temperature then α acts trivially on $\pi(A)''$, where π is the associated GNS representation of A, (since $\pi(A)''$ is commutative) and this implies the existence of fixed points under α in Ω.

EXAMPLE 2.10. Define a flow α on the C^*-algebra $C_0(\mathbb{R})$ by $\alpha_t(f)(s) = f(s-t)$. Then α is not pseudo-diagonal. If one defines self-adjoint operators P and Q on $L^2(\mathbb{R})$ by $P\xi(s) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\xi(s)$ and $Q\xi(s) = s\xi(s)$ then there is a finite sequence $f_1, f_2, \ldots, f_n$ in $C_0(\mathbb{R})$ and $\varepsilon > 0$ such that if a finite-rank projection E on $L^2(\mathbb{R})$ satisfies $\|Ef_i(Q)E\| \geqslant (1-\varepsilon)\|f_i\|$ and $\|[E, f_i(Q)]\| \leqslant \varepsilon\|f_i\|$ for $i = 1, \ldots, n$ then $\|[E,P]\| > \varepsilon$. (This statement appears considerably stronger than the statement that if E is a finite-rank projection on $L^2(\mathbb{R})$ such that $\|E\Omega_0\| > 1/2$ then $\|[E,Q]\| + \|[E,P]\| > 1/3$, where $\Omega_0 = \pi^{-1/4}\mathrm{e}^{-s^2/2}$ is the *vacuum* vector.)

First consider the paranthetic assertion and note that $(P - \mathrm{i}Q)(P + \mathrm{i}Q) \geqslant 1$ and $(P - \mathrm{i}Q)\Omega_0 = 0$. Assuming there is such a projection E with $\|E\Omega_0\| > 1/2$ let $T = E(P + \mathrm{i}Q)E$ and $\gamma = \|[E,P]\| + \|[E,Q]\| < 1$. Then $T\xi = [E, P + \mathrm{i}Q]\xi + (P + \mathrm{i}Q)\xi$ for $\xi \in EL^2(\mathbb{R})$ and this implies that $\|T\xi\| \geqslant (1-\gamma)\|\xi\|$. Since $T^*E\Omega_0 = E(P - \mathrm{i}Q)E\Omega_0 = E[P - \mathrm{i}Q, E]\Omega_0$ we deduce that $\|T^*E\Omega_0\| \leqslant \gamma$. Since $\|T^{-1}\| = \|(T^*)^{-1}\|$ (as operators on the finite-dimensional subspace $EL^2(\mathbb{R})$) it follows that $\|E\Omega_0\|/\gamma \leqslant (1-\gamma)^{-1}$ or $\gamma \geqslant \|E\Omega_0\|/(\|E\Omega_0\| + 1) > 1/3$. (This assertion is related to the Heisenberg uncertainty principle.)

To establish the principal assertion of Example 2.10 we may add an identity to $C_0(\mathbb{R})$, i.e. we may consider α as acting on the continuous functions on $\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$. We define a unitary $u \in C(\mathbb{R}^+)$ by $u(t) = 1$ for $|t| \geqslant 1$ and

$u(t) = \mathrm{e}^{\mathrm{i}\pi(t+1)}$ for $t \in (-1,1)$. Note that $\alpha_t(u) = \mathrm{e}^{-\mathrm{i}b_t}u$ for $t \geqslant 0$, where b_t is a continuous function on $\mathbb{R}$ with $\mathrm{supp}(b_t) = [-1, 1+t]$ such that $b_t(s) = 1+s$ for $s \in [-1,-1+t]$, $b_t(s) = t$ for $s \in (-1+t, 1)$ and $b_t(s) = 1+t-s$ for $s \in [1, 1+t]$.

We fix $t_0 \in (0,1/2)$ and $\varepsilon \in (0,1/6)$ and introduce $f, g, h \in C_0(\mathbb{R})$ as in Lemma 2.12 below. In particular it follows that $f\alpha_t(g) = \alpha_t(g)$ and $fb_t = b_t$ for $t \in [0,t_0]$, $(u-1)g = u-1$, and $b_{t_0}h = t_0 h$. By applying Theorem 1.6 to $u, f, g, h, b_t \in [0,t_0]$ etc. we obtain a unital CP map ϕ of $C(\mathbb{R}^+)$ into M_n for some n and a flow β on M_n such that $\phi(u^*)\phi(u) \approx 1$, $\phi(g)(\phi(u)-1) \approx \phi(u)-1 \approx (\phi(u)-1)\phi(g)$, $\phi(\alpha_t(g)) \approx \beta_t(\phi(g))$ and

$$\phi(\alpha_t(u)) \approx \phi(\mathrm{e}^{-\mathrm{i}b_t})\phi(u) \approx \mathrm{e}^{-\mathrm{i}\phi(b_t)}\phi(u) \approx \beta_t(\phi(u))$$

for $t \in [0,t_0]$. In addition ϕ and β satisfy the assertion made in Lemma 2.12. We construct the spectral projections $F, G, H \in M_n$ out of $\phi(f), \phi(g), \phi(h)$ corresponding to $[1-\delta, 1]$ with a small $\delta > 0$ as in Lemma 2.12. In particular this ensures that $G(\phi(u)-1) \approx \phi(u)-1$ and $F\beta_t(G) \approx \beta_t(G)$ and $F\phi(b_t) \approx \phi(b_t)$ for $t \in [0,t_0]$. By slightly modifying F we can suppose that $GF = G$. By the polar decomposition of $G\phi(u)G + 1 - G \approx \phi(u)$ we obtain a unitary $W \in M_n$ such that $W = GWG + 1 - G$ and $\mathrm{e}^{-\mathrm{i}\phi(b_t)}W \approx \beta_t(W)$. Let V be a unitary flow in M_n such that $\beta_t = \mathrm{Ad}\, V_t$. Since $FV_tGF \approx V_tG$ there is a unitary $Y_t \in FM_nF$ such that $Y_tG \approx V_tG$. We may suppose that $t \in [0,t_0] \mapsto Y_t$ is continuous with $Y_0 = F$. Since $W(F-G) = F-G$ and $F\beta_t(G) \approx \beta_t(G)$ we deduce that $F\beta_t(W)F \approx Y_tWY_t^*$ where W is now regarded as a unitary in FM_nF. By using $F\mathrm{e}^{-\mathrm{i}\phi(b_t)}F \approx \mathrm{e}^{-\mathrm{i}F\phi(b_t)F}$ we thus deduce that $Y_tWY_t^*W^* \approx \mathrm{e}^{-\mathrm{i}F\phi(b_t)F}$ in M_n for $t \in [0,t_0]$. Hence there is a self-adjoint $d_t \in FM_nF$ such that $d_t \approx 0$ and $Y_tWY_t^*W^* = \mathrm{e}^{-\mathrm{i}F\phi(b_t)F}\mathrm{e}^{\mathrm{i}d_t}$, where $t \mapsto d_t$ is continuous. Since $\det(Y_tWY_t^*W^*) = 1$ we obtain $-\mathrm{Tr}(F\phi(b_t)F) + \mathrm{Tr}(d_t) \in 2\pi\mathbb{Z}$. Since $Y_tWY_t^*W^* = 1$, $b_t = 0$ and $d_t = 0$ at $t = 0$ it follows that $\mathrm{Tr}(F\phi(b_t)F) = \mathrm{Tr}(d_t)$. Note that $\mathrm{Tr}(F\phi(b_t)F) \geqslant t\mathrm{Tr}(F\phi(h)F) \geqslant t(1-\delta)\mathrm{Tr}(FHF)$ and $\mathrm{Tr}(FHF)$ is almost greater than $\dim(F)/3$ (by Lemma 2.12). Since $|\mathrm{Tr}(d_t)| \leqslant \|d_t\|\dim(F) \approx 0$ this gives a contradiction for some t away from 0. (See [11], [3] for similar arguments.) ∎

LEMMA 2.11. *For any $\varepsilon_1, \varepsilon_2, \varepsilon' > 0$ there is a $C > 0$ such that the following holds. For any $h, p \in A_{\mathrm{sa}}$ such that $0 \leqslant h \leqslant 1$, $0 \leqslant p \leqslant 1$, and $\|hp - p\| < \delta$ the spectral projections F of h and G of p corresponding to $[1-\varepsilon_1, 1]$ and $[1-\varepsilon_2, 1]$, respectively, satisfy $\|FG - G\| < \varepsilon' + C\delta$.*

The proof is similar to that of Lemma 2.4.

LEMMA 2.12. *Fix $t_0 \in (0,1/2)$ and $\varepsilon \in (0,1/6)$. Define $g \in C_0(\mathbb{R})$ by $g(s) = 0$ for $|t| > 1+\varepsilon$ and $g(s) = 1$ for $t \in [-1,1]$ and by linearity elsewhere. Define $f \in C_0(\mathbb{R})$ by $f = \alpha_{-\varepsilon}(g) \vee \alpha_{t_0+\varepsilon}(g)$ where α is the translation flow of Example* 2.10. *(Note that $f(s) = 1$ for $s \in [-1-\varepsilon, 1+t_0+\varepsilon]$ and $f\alpha_t(g) = \alpha_t(g)$ for $t \in [0,t_0]$.) Define*

$h \in C_0(\mathbb{R})$ by $h(t) = 0$ for $t < -1 + t_0$ and $t > 1$, $h(t) = 1$ for $t \in [-1 + t_0 + \varepsilon, 1 - \varepsilon]$ and by linearity elsewhere. (Note that $f \cdot \bigvee\{\alpha_s(h) : |s| \leqslant t_0 + 3\varepsilon\} = f$.)

Let $\delta \in (0, 1)$ and suppose that α is pseudo-diagonal. Then for any $\varepsilon' > 0$ there is a unital CP *map ϕ of $C_0(\mathbb{R}^+)$ into M_n and a flow β on M_n satisfying the following assertion. If F, G, and H are spectral projections of $\phi(f)$, $\phi(g)$, and $\phi(h)$, respectively, corresponding to $[1 - \delta, 1]$, then $\|F\beta_t(G) - \beta_t(G)\| < \varepsilon'$ for $t \in [0, t_0]$ and* $\dim F \leqslant 3 \dim H$.

Proof. The estimate $\|FG - G\| < \varepsilon'$ follows from Lemma 2.11 by assuming $\|\phi(f)\phi(g) - \phi(g)\| \approx 0$. If $\phi(\alpha_t(g)) \approx \beta_t(\phi(g))$ sufficiently closely then $\beta_t(G)$ is almost dominated by the spectral projection G_t of $\phi\alpha_t(g)$ corresponding to $[1 - 3\delta/2, 1]$ (and almost dominates the spectral projection corresponding to $[1 - \delta/2, 1]$). (See Lemma 2.2 of [3].) If $\phi(f)\phi\alpha_t(g) \approx \phi\alpha_t(g)$ sufficiently closely then $FG_t \approx G_t$. Thus, assuming $\|\phi(f)\phi\alpha_t(g) - \phi\alpha_t(g)\| \approx 0$, it follows that $F\beta_t(G) \approx \beta_t(G)$ for $t \in [0, t_0]$.

Let $t_1 = -t_0 - 3\varepsilon$, $t_2 = 2 - 2t_0 - 6\varepsilon$, and $t_3 = 4 - 3t_0 - 9\varepsilon$ and note that $\alpha_{t_1}(h) \vee \alpha_{t_2}(h) \vee \alpha_{t_3}(h) \cdot f = f$ and that there are non-negative $f_1, f_2, f_3 \in C_0(\mathbb{R})$ such that $f = f_1 + f_2 + f_3$ and $\alpha_{t_i}(h) f_i = f_i$. Suppose that $\dim F > 3 \dim H$ however we choose ϕ and β. Then, since $\dim F > \dim(\beta_{t_1}(H) \vee \beta_{t_2}(H) \vee \beta_{t_3}(H))$, there is a state φ on M_n such that $\varphi(F) = 1$ and $\varphi(\beta_{t_i}(H)) = 0$ for $i = 1, 2, 3$. By assuming that $\|\phi(f_i)\phi\alpha_{t_i}(h) - \phi(f_i)\| \approx 0$ etc. we would have $\varphi(\phi(f_i)) \approx 0$ for $i = 1, 2, 3$, which implies that $\varphi(\phi(f)) \approx 0$. But since $\varphi(\phi(f)) \geqslant 1 - \delta$ due to $\varphi(F) = 1$ this is a contradiction. Hence $\dim F \leqslant 3 \dim H$ follows if $\phi(f_i)\phi\alpha_{t_i}(h) \approx \phi(f_i)$. ∎

EXAMPLE 2.13. Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| \leqslant 1\}$. Define a flow α of homeomorphisms of $\mathbb{D}$ by $\alpha_t(z) = z\mathrm{e}^{\mathrm{i}t}$. Then the induced flow on $C(\mathbb{D})$ is quasi-diagonal. More generally let v be a continuous function on $[0, 1]$ of finite variation and define a flow α' on $\mathbb{D}$ by $\alpha'_t(z) = \mathrm{e}^{\mathrm{i}tv(|z|)}z$. Then the induced flow on $C(\mathbb{D})$ is quasi-diagonal. Note that the origin is a fixed point which is neither absorbing nor repelling.

We shall prove the first assertion here. The second one will not be proved but follows from the proof of Proposition 2.15 given later.

Let α denote the induced flow on $C(\mathbb{D})$ and β the rotation flow on $C(\mathbb{T})$, i.e. $\beta_t(x)(z) = x(z\mathrm{e}^{-\mathrm{i}t})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We regard $C(\mathbb{T})$ as acting on $L^2(\mathbb{T})$. Then β is implemented by the unitary flow U defined by $U_t\xi(z) = \xi(z\mathrm{e}^{-\mathrm{i}t})$. Note that $U_t = \sum\limits_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kt} P_k$ where P_k is a rank-one projection. For $r \in [0, 1]$ let π_r be the restriction map of $C(\mathbb{D})$ onto $C(\mathbb{T})$: $\pi_r(x)(z) = x(rz)$. For $n \in \mathbb{N}$, we define a covariant representation ρ_n of $C(\mathbb{D})$ by $\rho_n = \bigoplus\limits_{k=0}^{n} \pi_{k/n}$ with the unitary flow $U^{(n)}$ defined by $U_t^{(n)} = \bigoplus\limits_{k=0}^{n} U_t$.

Let $\mathcal{F}$ be a finite subset of $C(\mathbb{D})$ and $\varepsilon > 0$. For any $\delta > 0$ there is an $n \in \mathbb{N}$ such that $\|\rho_n(x)\| \geqslant (1-\delta)\|x\|$ for all $x \in \mathcal{F}$ and $\|\pi_r(x) - \pi_s(x)\| \leqslant \delta\|x\|$ if $x \in \mathcal{F}$ and $\|r - s\| \leqslant 1/n$. We find a decreasing sequence

$$T_0 = F_0, G_0, T_1, F_1, G_1, \ldots, T_n, F_n, G_n = 0$$

of non-negative operators in the convex hull of the P_k such that all F_k and G_k are projections, $\|(F_k - G_k)\pi_{k/n}(x)(F_k - G_k)\| \geqslant (1-\delta)\|\pi_{k/n}(x)\|$ for $x \in \mathcal{F}$ and $\|[T_k, \pi_{k/n}(x)]\| \leqslant \delta\|x\|$ for $x \in \mathcal{F}$. We construct the sequence in the reverse order.

After choosing G_k, since $\pi_{k/r}(x)$ is not compact, one can choose $F_k \geqslant G_k$ to satisfy the condition $\|(F_k - G_k)\pi_{k/n}(x)(F_k - G_k)\| \geqslant (1-\delta)\|\pi_{k/n}(x)\|$. By the general theory of quasi-central approximate units, (see [1] or [20]), one can choose $T_k \geqslant F_k$. If T_k is chosen we set G_{k-1} to be the support projection of T_k. After repeating this process a finite number of times we construct F_0. Since the condition $\|[T_k, \pi_{k/n}(x)]\| \leqslant \delta\|x\|$ is void for $k = 0$ we can set $T_0 = F_0$.

We define a finite-rank projection $E = S^*S$ on $\bigoplus_{k=0}^{n} L^2(\mathbb{T})$ with $S = ((T_0 - T_1)^{1/2}, (T_1 - T_2)^{1/2}, \ldots, (T_n - T_{n+1})^{1/2})$ with $T_{n+1} = 0$. Since $SS^* = T_0$, E is indeed a finite-rank projection. Since all the T_k commute with U, it follows that $[E, U_t^{(n)}] = 0$. Since E is tri-diagonal, $[E, \rho_n(x)]$ is expressed as the sum of the diagonal part $\bigoplus_{k=0}^{n} [T_k - T_{k+1}, \pi_{k/n}(x)]$, the upper off-diagonal part and the lower off-diagonal part, respectively,

$$\bigoplus_{k=0}^{n-1}\{E_{k,k+1}\pi_{(k+1)/n}(x) - \pi_{k/n}(x)E_{k,k+1}\}, \quad \bigoplus_{k=0}^{n-1}\{E_{k+1,k}\pi_{k/n}(x) - \pi_{(k+1)/n}(x)E_{k+1,k}\},$$

where $E_{k,k+1} = E_{k+1,k} = (T_k - T_{k+1})^{1/2}(T_{k+1} - T_{k+2})^{1/2} = (T_{k+1} - T_{k+1}^2)^{1/2}$. Thus one can conclude that $\|[E, \rho_n(x)]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$ for a sufficiently small δ. The diagonal part of $E\rho_n(x)E$ is given by the direct sum of

$$(T_k - T_{k+1})\pi_{k/n}(x)(T_k - T_{k+1}) + E_{k,k+1}\pi_{(k+1)/n}(x)E_{k+1,k} + E_{k,k-1}\pi_{(k-1)/n}(x)E_{k-1,k},$$

over $k = 0, 1, \ldots, n$, where $E_{k,k+1}$ etc. are given above and the term should be omitted if $k+1 = n+1$ or $k-1 = -1$. Hence the norm of this is greater than or equal to

$$\bigoplus_{k=0}^{n}(F_k - G_k)\pi_{k/n}(x)(F_k - G_k).$$

Thus we obtain $\|E\rho_n(x)E\| \geqslant (1-\varepsilon)\|x\|$ for $x \in \mathcal{F}$ for a small δ.

Since $\pi_0(x) = x(0)$, π_0 is an α-invariant character. Hence we could choose $t \mapsto 1$ for a unitary flow implementing α instead of the U which has spectrum $2\pi\mathbb{Z}$, but then the above proof would fail.

The above proof was taken from the proof of Proposition 3 of [24]. It is appropriate to indicate how to prove Theorem 1.7 at this point. First we establish the following analogue of Proposition 3 of [24].

PROPOSITION 2.14. *Let α (respectively β) be a flow on a C^*-algebra A (respectively B). Let $\{\phi_s : s \in [0,1]\}$ be a homotopy of homomorphisms of A into B such that $\phi_s\alpha_t = \beta_t\phi_s$ and $\bigcap_s \mathrm{Ker}(\phi_s) = \{0\}$. If $\beta|_{\phi_1(A)}$ is quasi-diagonal then α is quasi-diagonal.*

Proof. Let (ρ, V) be a covariant representation of (B, β) such that $\rho \times V$ is faithful and contains no non-zero compact operators in its range. Then, by the assumption, $\bigoplus_s \rho\phi_s$ is faithful and $(\rho\phi_1(A), V)$ is quasi-diagonal. Let H denote the self-adjoint generator of V. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ and $\varepsilon > 0$. There is a self-adjoint compact operator K on $\mathcal{H}_\rho$ such that $\|K\| < \varepsilon/2$ and $H_1 = H + K$ is diagonal. For any small constant $\delta > 0$, there is an $n \in \mathbb{N}$ such that if $|s_1 - s_2| \leqslant 1/n$ and $x \in \mathcal{F}$ then $\|\phi_{s_1}(x) - \phi_{s_2}(x)\| \leqslant \delta\|x\|$ and if $x \in \mathcal{F}$ then $\max_k \|\pi_{k/n}(x)\| \geqslant (1-\delta)\|x\|$. There is a finite increasing sequence $G_0 = 0, F_0, T_0, G_1, F_1, T_1, G_2, \ldots, G_n, F_n = T_n$ of non-negative compact operators in the maximal commutative von Neumann algebra generated by a family of minimal projections commuting with H_1 such that all G_k, F_k are projections, $\|(F_k - G_k)\pi_{k/n}(x)(F_k - G_k)\| \geqslant (1-\delta)\|\pi_{k/n}(x)\|$ for $x \in \mathcal{F}$ and $\|[T_k, \phi_{k/n}(x)]\| < \delta\|x\|$ for $x \in \mathcal{F}$. Let $\pi = \bigoplus_{k=0}^{n} \rho\phi_{k/n}$ and $U = \bigoplus_{n=0}^{n} V$. In this covariant representation space $\bigoplus_{k=0}^{n} \mathcal{H}_\rho$ we define a finite-rank projection E as $S^*S \in M_{n+1} \otimes \mathcal{B}(\mathcal{H}_\rho)$, where S is the row vector $(T_0, (T_1 - T_0)^{1/2}, \ldots, (T_n - T_{n-1})^{1/2})$. If δ is sufficiently small, one can show that (π, U) and E satisfy condition (iii) of Theorem 1.5 for $(\mathcal{F}, \varepsilon)$. (See the proof of Proposition 3 of [24] for more details.) ∎

Theorem 1.7 follows from Proposition 2.14 exactly as in the proof of Theorem 5 of [24]. Let us reproduce the proof here. We have two flows α on A and β on B such that (B, β) dominates (A, α), i.e. there are intertwining homomorphisms $\phi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi\phi$ is homotopic to the identity in the endomorphisms of A commuting with α. The assumption that β is quasi-diagonal implies that $\beta|_{\phi(A)}$ is quasi-diagonal and hence that the flow $\dot{\alpha}$ on $A/\mathrm{Ker}(\phi)$ induced from α is quasi-diagonal. Let $D = A \oplus A/\mathrm{Ker}(\phi)$ with the flow $\alpha \oplus \dot{\alpha}$ and consider two intertwining homomorphisms $\psi\phi \oplus \pi$ and $\mathrm{id} \oplus \pi$ from A into D, where π is the quotient map of A onto $A/\mathrm{Ker}(\phi)$. Then, since $\mathrm{Ran}(\psi\phi \oplus \pi)$ is isomorphic to $A/\mathrm{Ker}(\phi)$, we conclude that $\alpha \oplus \dot{\alpha}|_{\mathrm{Ran}(\psi\phi\oplus\pi)}$ is quasi-diagonal. Since $\psi\phi \oplus \pi$ is homotopic to $\mathrm{id} \oplus \pi$ in the intertwining homomorphisms and $\mathrm{id} \oplus \pi$ is injective Proposition 2.14 implies that α is quasi-diagonal. This concludes the proof of 1.7. (See [24] for another formulation.)

We can also show the following variant of Proposition 3 of [24].

PROPOSITION 2.15. *Let α be a flow on a quasi-diagonal C^*-algebra A and define a flow β on $B = A \otimes C[0,1]$ by $\beta_t(x)(s) = \alpha_{st}(x(s))$. Then β is quasi-diagonal.*

Proof. Let (π, U) be a covariant representation of (A, α) such that $\pi \times U$ is faithful. For $s \in [0,1]$ we define a map ϕ_s of B onto A by $\phi_s(x) = x(s)$. For each $n \in \mathbb{N}$ let $L_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ and let $s_k = L_n^{-1} \sum_{m=1}^{k} 1/m$ for $k = 1, 2, \ldots, n$ with $s_0 = 0$. We define a representation π_n of B by $\pi_n = \bigoplus_{k=0}^{n} \pi\phi_{1-s_k}$ and a unitary flow $U_t^{(n)} = \bigoplus_{k=0}^{n} U_{(1-s_k)t}$ which implements β. Given $\mathcal{F}$ and $\varepsilon > 0$ we construct the required finite-rank projection in the representation space $\mathcal{H}_n = \bigoplus_{k=0}^{n} \mathcal{H}_\pi$ of π_n for some large n. The finite-rank projection $E = (E_{k\ell})$ is defined just as before as the tri-diagonal matrix S^*S by choosing a finite increasing sequence $T_0, T_1, \ldots, T_n$ of finite-rank non-negative operators as above; $E_{kk} = T_k - T_{k-1}$ and $E_{k,k+1} = E_{k+1,k} = (T_k - T_{k-1})^{1/2}(T_{k+1} - T_k)^{1/2} = (T_k - T_k^2)^{1/2}$. Since $U^{(n)}$ is not the direct sum of $n+1$ copies of the same flow it is not sufficient to assume T_k almost commutes with $U_{t(1-s_k)}$. To achieve $\operatorname{Ad} U_t^{(n)}(E) \approx E$ we must also have

$$U_{(1-s_k)t} E_{k,k+1} U^*_{(1-s_{k+1})t} \approx E_{k,k+1}.$$

Since $E_{k,k+1}$ almost commutes with U, this amounts to $E_{k,k+1} U_{t(k+1)^{-1}L_n^{-1}} \approx E_{k,k+1}$, i.e. $(U_t - 1)(T_k - T_k^2) \approx 0$ if $|t| \leqslant (k+1)^{-1} L_n^{-1}$.

Let X be the linear subspace of $C_0(\mathbb{R}^+)$ consisting of non-increasing C^∞-functions f on $\mathbb{R}$ such that $f(t) = 1$ for all small t and $f(t) = 0$ for all large t. We regard H as the generator of $t \mapsto \lambda_t$ in the multiplier algebra $M(A \times_\alpha \mathbb{R})$ and write $f(H) = \int \widehat{f}(t) \lambda_t \mathrm{d}t$. Since $f(H)$ with $f \in X$ is a subspace in $M(A \times_\alpha \mathbb{R})$ and contains the identity in its closure with respect to the strict topology, there is an $f_- \in X$ such that $\|[f_-(H), x]\| \leqslant (\varepsilon/4)\|x\|$ for all $x \in \mathcal{F}$, where $\mathcal{F}$ is the given finite subset of A. Since this inequality is left invariant under the dual flow we may assume that $f_-(t) = 1$ for $t \leqslant 0$ and $f_-(t) < 1$ for $t > 0$. Let $L = \min\{t : f_-(t) = 0\}$ and define f_+ by $f_+(t) = 1 - f_-(t + L)$. On the other hand there is an $L' > 0$ such that

$$\|\chi_{(-L',L')}(H) x \chi_{(-L',L')}(H)\| \geqslant \Big(1 - \frac{\varepsilon}{2}\Big)\|x\|$$

for $x \in \mathcal{F}$ where $\chi_{(-L',L')}$ is the characteristic function of the interval $(-L', L')$ (and hence $\chi_{(-L',L')}(H)$ is an open projection in the second dual of $A \times_\alpha \mathbb{R}$). We define $f_k \in C_0(\mathbb{R})$ by

$$f_k(t) = f_+(t + kL + (2k+1)L') f_-(t - kL - (2k+1)L')$$

for $k = 0, 1, 2, \ldots$. We note that $\operatorname{supp}(f_k) = [-(k+1)L - (2k+1)L', (k+1)L + (2k+1)L']$, $f_k(t) = 1$ for $t \in [-kL - (2k+1)L', kL + (2k+1)L']$, $f_k f_{k+1} = f_k$, and $\|[f_k(H), x]\| \leqslant (\varepsilon/2)\|x\|$ for $x \in \mathcal{F}$.

Let $P_0 = \chi_{(-L',L')}(H)$ where H now denotes the generator of U. We choose a finite-rank non-negative operator T_0 on $\mathcal{H}_\pi$ such that $T_0 \leqslant f_0(H)$, $\|[T_0, U_t]\| < \varepsilon$ for $t \in [-1, 1]$, and $\|[T_0, \pi(x)]\| \leqslant (\varepsilon/2)\|x\|$ and $\|P_0 T_0 \pi(x) T_0 P_0\| \geqslant (1-\varepsilon)\|x\|$ for $x \in \mathcal{F}$. This is possible since the set of finite-rank operators T satisfying $0 \leqslant T \leqslant f_0(H)$ forms a convex set invariant under $\operatorname{Ad} U_t$ and contains $f_0(H)$ in its closure with the strict topology on $\mathcal{K}(\mathcal{H}_\pi)$ and $f_0(H)$ satisfies all the conditions required for T_0 with stricter coefficients. Let G_1 be the support projection of T_0 and let $P_1 = \chi_{(-L-3L',-L-L')}(H) + \chi_{(L+L',L+3L')}(H)$. We will find a finite-rank non-negative operator T_1 on $\mathcal{H}_\pi$ such that $G_1 \leqslant T_1 \leqslant f_1(H)$, $\|[T_1, U_t]\| < \varepsilon$ for $t \in [-1,1]$, and $\|[T_1, \pi(x)]\| \leqslant (\varepsilon/2)\|x\|$ and $\|P_1 T_1 \pi(x) T_1 P_1\| \geqslant (1-\varepsilon)\|x\|$ for $x \in \mathcal{F}$. For this purpose let N be a large number and let F be a finite-rank projection such that $F \leqslant \chi_{(-L-L',L+L')}(H)$ and $\|FU_tG_1U_t^* - U_tG_1U_t^*\| = \|U_t^*FU_tG_1 - G_1\| \approx 0$ for all $t \in [-N,N]$. We then choose a finite-rank T such that $F \leqslant T \leqslant f_1(H)$, $\|[T, P_1]\| < \varepsilon$, $\|[T, \pi\alpha_t(x)]\| \leqslant (2\varepsilon/3)\|x\|$ and $\|P_1 T \pi\alpha_t(x) T P_1\| \geqslant (1-2\varepsilon/3)\|x\|$ for $x \in \mathcal{F}$ and $t \in [-N,N]$. Then by taking $\int h(t) U_t T U_t^* \mathrm{d}t$ instead of T with an appropriate function $h \geqslant 0$ and assuming that N is large enough we can see that all the conditions are satisfied with $TG_1 \approx G_1$ instead of $G_1 \leqslant T$. We then modify T slightly to obtain a T_1 which satisfies all the conditions. Note that $T_1T_0 = T_0$ and $\|[T_1 - T_0, \pi(x)]\| \leqslant \varepsilon\|x\|$.

By repeating this process we obtain $T_0, T_1, \ldots, T_{n-1}$ such that $T_k \leqslant f_k(H)$, $T_kT_{k-1} = T_{k-1}$, $\|[T_k, U_t]\| < \varepsilon$ for $t \in [-1,1]$, $\|[T_k - T_{k-1}, \pi(x)]\| \leqslant \varepsilon\|x\|$ and $\|P_kT_k\pi(x)T_kP_k\| \geqslant (1-\varepsilon)\|x\|$ for $x \in \mathcal{F}$, where

$$P_k = \chi_{(-kL-(2k+1)L',-kL-(2k-1)L')}(H) + \chi_{(kL+(2k-1)L',kL+(2k+1)L')}(H).$$

Let $P_n' = \chi_{(-\infty,-nL-(2n+1)L')}(H) + \chi_{(2nL+(2n+1)L',\infty)}(H)$. By using that A is quasi-diagonal, we then choose a finite-rank projection T_n such that $T_n \geqslant T_{n-1}$ and $\|P_n'T_n\pi(x)T_nP_n'\| \geqslant (1-\varepsilon)\|x\|$ and $\|[T_n, \pi(x)\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$. If $|t| \leqslant (k+1)^{-1}L_n^{-1}$, then $\|(U_t - 1)(T_k - T_k^2)\| \leqslant |L(k+1) + L'(2k+1)|(k+1)^{-1}L_n^{-1} \leqslant (L + 2L')/L_n$. Since $L_n \to \infty$ as $n \to \infty$ we obtain the desired sequence $T_0, T_1, \ldots, T_n$ for some n. ∎

EXAMPLE 2.16. Let A_θ denote the irrational rotation C^*-algebra generated by two unitaries u, v satisfying $uv = \mathrm{e}^{\mathrm{i}2\pi\theta}vu$ with $\theta \in (0,1)$ irrational. (Then A_θ is a unital simple AT-algebra with a unique tracial state.) Let α be a flow on A_θ such that $\alpha_t(u) = \mathrm{e}^{\mathrm{i}t}u$ and $\alpha_t(v) = \mathrm{e}^{\mathrm{i}pt}v$ with some $p \in \mathbb{R}$. Then α is not pseudo-diagonal. This follows because if ω is a KMS state for the inverse temperature $\beta \neq 0$ then one must have $\mathrm{e}^{-\beta} = \omega(u^*\alpha_{\mathrm{i}\beta}(u)) = \omega(uu^*) = 1$ which is a contradiction. Thus α has no KMS states and α is not pseudo-diagonal.

PROPOSITION 2.17. *Let A be a quasi-diagonal C^*-algebra and α an approximately inner flow on A. Then α is pseudo-diagonal.*

Proof. Suppose A acts non-degenerately on a Hilbert space $\mathcal{H}$ such that A is a quasi-diagonal set of $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{F}$ be a finite subset of A and $\varepsilon > 0$. By assumption there is an $h = h^* \in A$ such that $\|\alpha_t(x) - \mathrm{Ad}\,\mathrm{e}^{\mathrm{i}th}(x)\| \leqslant \varepsilon/3\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1,1]$. There is a finite-rank projection E on $\mathcal{H}$ such that $\|ExE\| \geqslant (1-\varepsilon)\|x\|$ and $\|[E,x]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$, and $\|[E,h]\| < \varepsilon/3$. Since $\|E\mathrm{e}^{\mathrm{i}th}E - \mathrm{e}^{\mathrm{i}tEhE}E\| < \varepsilon/6$ for $t \in [-1,1]$, it follows that

$$\|E\alpha_t(x)E - \operatorname{Ad} e^{itEhE}(ExE)\| \leqslant \varepsilon\|x\|$$

for all $x \in \mathcal{F}$. Note also that $\|ExEyE - ExyE\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$. By setting $B = \mathcal{B}(E\mathcal{H})$, $\beta_t = \operatorname{Ad} e^{itEhE}$ and $\phi(x) = ExE$ we obtain condition (ii) of Theorem 1.6. ∎

If α is a pseudo-diagonal flow on A and B is an α-invariant C^*-subalgebra of A then $\alpha|_B$ is pseudo-diagonal, i.e. pseudo-diagonality is preserved under passing to an invariant C^*-subalgebra. But it is not evident that this property holds for approximate innerness.

PROPOSITION 2.18. *If α is an* AF *flow then α is quasi-diagonal.*

Proof. By assumption the C^*-algebra A is an AF algebra and has an increasing sequence (A_n) of finite-dimensional C^*-subalgebras such that $\bigcup_n A_n$ is dense in A and $\alpha_t(A_n) = A_n$. We choose a maximal abelian C^*-subalgebra D_n of $A_n \cap A'_{n-1}$ (with $A_0 = \mathbb{C}1$) such that α is trivial on D_n and let D be the C^*-subalgebra of A generated by all D_n. Let (ϕ_n) be a dense sequence in the characters of D. Each ϕ_n uniquely extends to a pure α-invariant state of A which we also denote by ϕ_n. Note that $\bigoplus_n \pi_{\phi_n}$ is a faithful representation of A.

In the GNS representation π_ϕ of A for $\phi = \phi_n$ we define a unitary flow U by $U_t\pi_\phi(x)\Omega_\phi = \pi_\phi\alpha_t(x)\Omega_\phi$, $x \in A$. It follows that (π_ϕ, U) is a covariant representation of (A, α). Denote by E_n the finite-rank projection onto the subspace $\pi_\phi(A_n)\Omega_n$. Then $[E_n, \pi_\phi(x)] = 0$ for $x \in A_n$, $[E, U_t] = 0$ and $\lim_n E_n = 1$. This shows that $(\pi_\phi(A), U)$ is quasi-diagonal. Denoting U by U^ϕ we conclude that $((\bigoplus_n \pi_{\phi_n})(A), \bigoplus_n U^{\phi_n})$ is quasi-diagonal. ∎

From the above proof one can construct a projection of norm one ϕ_n of A onto A_n such that $\alpha_t|_{A_n} \circ \phi_n = \phi_n \circ \alpha_t$, from which follows a stronger form of condition (ii) of Theorem 1.5. By using this fact one can show that all the KMS states are locally approximable for an AF flow (see 4.6.1 of [21] and the comment after Proposition 2.8). This remark also applies to approximate AF flows [16].

A quasi-diagonal flow on an AF algebra is not expected to be simply a cocycle perturbation of an AF flow because there is a more general type of AF flow (see [21] for commutative derivations which generate such flows). Specifically, there is a flow α on a unital simple AF algebra A, which is not a perturbation of an AF flow, such that A has an increasing sequence (B_n) of α-invariant C^*-subalgebras of A with dense union satisfying $\alpha|_{B_n}$ is uniformly continuous and $B_n \cong A_n \otimes C[0,1]$ with A_n finite-dimensional [15]. We take an α-invariant pure state f such that $f|_{B_n}$ reduces to an evaluation on $C[0,1]$. Then, in the GNS representation associated with f, the subspace $\pi_f(B_n)\Omega_f$ is finite-dimensional, which gives the desired finite-rank projections.

EXAMPLE 2.19. Let (A_n) be an increasing sequence of finite-dimensional C^*-algebras and A the closure of the union $\bigcup_n A_n$. Let $B_n = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. We define an embedding of B_n into B_{n+1} as follows: if $k < n$ then A_k of B_n is identified with A_k of B_{n+1} and A_n is mapped into $A_n \oplus A_{n+1}$ by duplication. Let B be the closure of the union $\bigcup_n B_n$ (which is also defined as the C^*-algebra of bounded sequences (x_n) with $x_n \in A_n$ such that $\lim_n x_n$ converges in A). We note that B has many finite-dimensional quotients. Let I_n be the ideal generated by all $A_i \subset B_m$ with $i \neq n \leqslant m$. Then B/I_n is isomorphic to A_n. Since $\bigcap I_n = \{0\}$, one concludes that B is quasi-diagonal. This shows that any flow β on B is quasi-diagonal (since β fixes I_n).

Let α be an approximately inner flow on A and choose an $h_n = h_n^* \in A_n$ such that $\operatorname{Ad} e^{ith_n}(x) \to \alpha_t(x)$ for $x \in A$. We can define a flow β on B as follows. Let $\beta^{(n)}$ denote the flow on B_n implemented by $\bigoplus_{k \leqslant n} h_k$. One shows that $\beta_t^{(n)}(x)$ converges for $x \in B$ and defines β as the limit.

Conversely, if a flow β is given on B then we have an $h_n = h_n^* \in A_n$ such that the induced flow on B/I_n is given by $\operatorname{Ad} e^{ith_n}$ and can argue that $\operatorname{Ad} e^{ith_n}$ converges to a flow on A. Hence one concludes that β is defined just as in the previous paragraph.

EXAMPLE 2.20. Let A be a residually finite-dimensional C^*-algebra and α a flow on A which fixes each ideal of A. Then α is quasi-diagonal. This follows because A has a separating family of finite-dimensional representations which must be covariant under α. Thus the direct sum of these representations gives the required faithful representation of A.

EXAMPLE 2.21. Let γ denote the periodic flow on the UHF algebra $\bigotimes_{n=1}^{\infty} M_2$ of type 2^∞ given by $\gamma_t = \bigotimes_n \operatorname{Ad}(1 \oplus e^{2\pi i t})$ and let A be the fixed point algebra of γ. The dimension group of A is isomorphic to $\mathbb{Z}[t]$ with the positive cone of strictly positive functions on the open interval $(0,1)$. There is a decreasing sequence $I_1, I_2, \ldots$ of ideals of A such that $A/I_1 \cong \mathbb{C}1$, $\bigcap_n I_n = \{0\}$, and $I_n/I_{n+1} \cong \mathcal{K}$ for $n \geqslant 1$, where $\mathcal{K}$ is the compact operators on a separable infinite-dimensional Hilbert space. It follows from the next lemma that any flow on A/I_n is quasi-diagonal and this then implies that any flow on A is quasi-diagonal.

LEMMA 2.22. *If A is a type* I AF *algebra then any flow on A is quasi-diagonal.*

Proof. There is a strictly increasing family $\{I_\mu\}$ of (closed) ideals of A indexed by μ in a segment $[0, \nu]$ of ordinals such that $I_0 = \{0\}$, $I_\nu = A$, $\bigcup_{\mu<\gamma} I_\mu$ is dense in I_γ for any limit ordinal γ, and $I_{\mu+1}/I_\mu$ is generated as an ideal by a minimal projection for any $\mu < \nu$, i.e. $I_{\mu+1}/I_\mu \cong \mathcal{K}$ or otherwise M_m for some $m = 1, 2, \ldots$ (see, e.g. [20] for type I C^*-algebras). In the following we allow

$I_{\mu+1}/I_\mu \cong 0$ and call this a composite series for A. Note that any flow α on A fixes each ideal (because the ideal is generated by projections). We shall prove the statement that any flow on A (with a composite series indexed by $[0,\nu]$) is quasi-diagonal by induction on ν. If $\nu = 1$ this is obvious since $A \cong \mathcal{K}$ or M_m for some m and any flow on A is inner. (If $A \cong \mathcal{K}$ we use the Weyl–von Neumann theorem.)

Suppose that this is shown for any $\nu < \sigma$. Let A be a type I AF algebra with a composite series $\{I_\mu\}$ with $\mu \in [0,\sigma]$. If σ is a limit ordinal, then $\bigcup_{\nu<\sigma} I_\nu$ is dense in A. Since I_ν has a composite series indexed by $[0,\nu]$ the induction hypothesis implies that any flow on I_ν is quasi-diagonal which in turn implies that any flow on A is quasi-diagonal. If σ is not a limit ordinal (and $I_{\sigma-1} \neq A$) then there is a minimal projection in $A/I_{\sigma-1}$ which is an image of a projection e of A. The existence of such a projection e follows since A is AF. Let $J(e)$ denote the ideal of A generated by e and note that $A = J(e) + I_{\sigma-1} = J(e) + (1-e)I_{\sigma-1}(1-e)$. Since $(1-e)I_{\sigma-1}(1-e)$ is an ideal of $(1-e)A(1-e)$ and is generated by an increasing sequence (p_n) of projections it follows that the sequence $J(e+p_n)$ is increasing and $\bigcup_n J(e+p_n)$ is dense in A. Set $e_n = e + p_n$.

Let α be a flow on A. Then there is an α-cocycle u in A (or $A + \mathbb{C}1$ if A is not unital) such that $\operatorname{Ad} u_t\alpha_t(e_n) = e_n$. To prove that α is quasi-diagonal on $J(e_n)$ it suffices to show, by Proposition 2.2, that $\operatorname{Ad} u\alpha$ is quasi-diagonal on $B = e_nAe_n$. Note that B has a composite series $\{J_\mu\}$ with $J_\mu = e_nI_\mu e_n \subset I_\mu$ and $\mu \in [0,\sigma]$. Since $B/J_{\sigma-1} \cong \mathbb{C}1$ and any flow on $J_{\sigma-1}$ is quasi-diagonal one can conclude that any flow on B is quasi-diagonal. Thus we conclude that $\alpha|_{J(e_n)}$ is quasi-diagonal for any n, which implies that α is quasi-diagonal. ∎

REMARK 2.23. If A is a type I AF algebra then any flow on A is an approximate AF flow (or a cocycle perturbation of an AF flow [16]). The proof is quite similar to the above but using Corollary 1.6 of [17] in place of Proposition 2.7. Thus the above lemma also follows from this fact and Proposition 2.18.

REMARK 2.24. Let A be a type I C^*-algebra and α a flow on A which fixes each ideal. Then α is universally weakly inner and is approximately inner. (N.B. A need not be quasi-diagonal.) This follows from [13], [8].

It looks that the quasi-diagonal condition is strong for flows on AF algebras. But we could not decide if they are approximately inner or not.

Summarizing Example 2.21 gives the following.

PROPOSITION 2.25. *Let A be an* AF *algebra and suppose that there is a sequence $\{I_n\}$ of ideals of A such that $\bigcap_n I_n = \{0\}$ and A/I_n is of type* I *for all n. Then any flow on A is quasi-diagonal.*

PROPOSITION 2.26. *Let A be a unital* AF *algebra and let α be a flow on A. Suppose that there is a covariant irreducible representation (π, U) such that $(\pi(A), U)$ is* *quasi-diagonal. Then there is an increasing sequence (E_n) of finite-rank projections on $\mathcal{H}_\pi$ and an α-cocyle u in A such that $[E_n, \pi(u_t)U_t] = 0$ and $\bigcup_n B_n$ is dense in A where*

$$B_n = \{x \in A : [E_k, \pi(x)] = 0,\ k \geqslant n\}.$$

Proof. Since A is an AF algebra there is an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that the union is dense in A. We omit π in the arguments below.

Given a finite-rank projection E on $\mathcal{H}$ and $\varepsilon > 0$ there is a finite-rank projection F on $\mathcal{H}$ such that $A_1 E\mathcal{H} \subset F\mathcal{H}$, $\|[F, x]\| \leqslant \varepsilon\|x\|$ for $x \in A_1$ and $\|[F, U_t]\| < \varepsilon$ for $t \in [-1, 1]$. Since A_1 is finite-dimensional the average of vFv^* over v in the unitary group of A_1 is in a small vicinity of F (depending on ε and $\dim A_1$) which yields a projection F' with $\|F - F'\|$ small by functional calculus. We note that $E \leqslant F'$, $F' \in A_1'$, and $\|[F', U_t]\| < \varepsilon' + 2\|F' - F\|$ for $t \in [-1, 1]$. Since $\|F - F'\|$ can be made arbitrarily small we now suppose that the finite-rank projection F satisfies $E \leqslant F$, $F \in A_1'$ and $\|[F, U_t]\| < \varepsilon$ for $t \in [-1, 1]$. By Proposition 1.3 one obtains a projection F' in a small vicinity of F such that $\|[F', H]\|$ is small where H is the self-adjoint operator with $U_t = \mathrm{e}^{\mathrm{i}tH}$. Using the irreducibility of A and Kadison's transitivity theorem we obtain an $h = h^* \in A$ such that $[h, F'] = [H, F']$ and $\|h\| = \|[H, F']\|$. We define V to be the unitary part of the polar decomposition of $X = F'F + (1 - F')(1 - F)$. Since $\|X - 1\| \leqslant 2\|F - F'\|$ and $XF = F'X$ we conclude that $\|V - 1\|$ is small and that $VFV^* = F'$. A second application of Kadison's transitivity theorem gives a unitary $v \in A$ such that $vF' = VF'$ and $\|v - 1\| \leqslant \|V - 1\|$. Note that $v\mathrm{e}^{\mathrm{i}t(H-h)}v^*$ commutes with F. Let $u_t = v\mathrm{e}^{\mathrm{i}t(H-h)}v^*\mathrm{e}^{-\mathrm{i}tH} \in A$. This is an α-cocycle satisfying $u_t U_t F = F u_t U_t$ and $\|u_t - 1\|$ is small for $t \in [-1, 1]$ because $\|h\|$ and $\|v - 1\|$ are small. Note that if E satisfies that $[E, U_t] = 0$ then we may suppose that $u_t E = E$.

We apply the foregoing procedure repeatedly and each time make a perturbation by selecting a cocycle.

Let (ξ_n) be an orthonormal basis for $\mathcal{H}$. We construct a sequence (E_n) of finite-rank projections and a sequence $(u^{(n)})$ of cocycles such that $E_n\xi_n = \xi_n$, $E_{n-1} \leqslant E_n$, $E_n \in A_n'$, $u_t^{(n)}E_{n-1} = E_{n-1}$, $\|u_t^{(n)} - 1\| < 2^{-n}$ for $t \in [-1, 1]$, and $u^{(n)}$ is $\alpha^{(n-1)}$-cocycle, where $\alpha^{(0)} = \alpha$ and $\alpha^{(n)} = \operatorname{Ad} u^{(n)}\alpha^{(n-1)}$. Then $\alpha^{(n)}$ converges to a flow $\alpha^{(\infty)}$ which is a cocycle perturbation of α with the cocycle u_t obtained as the limit of $u_t^{(n)}u_t^{(n-1)} \cdots u_t^{(1)}$. Since $B_n \supset A_n$, the union $\bigcup_n B_n$ is dense in A. The other requirements follow easily. ∎

Let α be a flow on a unital simple AF algebra A and consider the following conditions:

(i) There is a covariant irreducible representation (π, U) of A such that the pair $(\pi(A), U)$ is quasi-diagonal.

(ii) There exist an α-cocycle u and an increasing sequence (B_n) of residually finite-dimensional C^*-subalgebras of A such that $\bigcup_n B_n$ is dense in A and each B_n is left invariant under $\operatorname{Ad} u\alpha$ and has a faithful family of covariant finite-dimensional representations.

(iii) α is quasi-diagonal.

The above proposition shows that (i) implies (ii). It is immediate that (ii) implies (iii) since $\operatorname{Ad} u\alpha|_{B_n}$ is quasi-diagonal. But to show that (iii) implies (i) we would need to extract more information on (A, α) in addition to the conclusion of Proposition 2.8.

3. VOICULESCU'S WEYL–VON NEUMANN THEOREM

Our aim is to prove a version of Voiculescu's non-commutative Weyl–von Neumann theorem [22]. A new feature is that we deal with a C^*-algebra together with a derivation implemented by an unbounded self-adjoint operator.

THEOREM 3.1. *Let α be a flow on a separable C^*-algebra A and (π, U) (respectively (ρ, V)) a covariant representation of (A, α) on a separable Hilbert space such that the range of $\rho \times V$ does not contain a non-zero compact operator. If* $\operatorname{Ker}(\rho \times V) \subset \operatorname{Ker}(\pi \times U)$*, then there is a sequence $W_1, W_2, \dots$ of isometries from $\mathcal{H}_\pi$ into $\mathcal{H}_\rho$ such that $\pi(x) - W_n^* \rho(x) W_n \in \mathcal{K}(\mathcal{H}_\pi)$ and $\|\pi(x) - W_n^* \rho(x) W_n\| \to 0$ for $x \in A$. In addition $H_U - W_n^* H_V W_n \in \mathcal{K}(\mathcal{H}_\pi)$ and $\|H_U - W_n^* H_V W_n\| \to 0$ where H_U (respectively H_V) is the self-adjoint generator of U (respectively V). Furthermore if* $\operatorname{Ker}(\rho \times V) = \operatorname{Ker}(\pi \times U)$ *and the range of $\pi \times U$ (as well as of $\rho \times V$) does not contain a non-zero compact operator then the W_n can be assumed to be unitary.*

This theorem immediately allows a proof of Theorem 1.4.

Poof of Theorem 1.4. Let (ρ, V) be a covariant representation of (A, α) such that $\operatorname{Ker}(\rho \times V) = \{0\}$ and $\operatorname{Ran}(\rho \times V) \cap \mathcal{K}(\mathcal{H}_\rho) = \{0\}$. Let $\mathcal{F}$ be a finite subset of A, ω a finite subset of $\mathcal{H}_\rho$ and $\varepsilon > 0$. Let B be the α-invariant C^*-subalgebra of A generated by $\mathcal{F}$ and let $\mathcal{H}'$ be a separable closed subspace of $\mathcal{H}_\rho$ which is invariant under $\rho(B)$ and V such that $\mathcal{H}' \supset \omega$ and the representation $\rho|_B \times V$ on $\mathcal{H}'$ is faithful. We set $\rho' = \rho|_B$ on $\mathcal{H}'$ and $V' = V|_{\mathcal{H}'}$. Note that B is separable and $\alpha|_B$ is quasi-diagonal. Let (π, U) be a covariant representation of (B, α_B) on a separable Hilbert space $\mathcal{H}_\pi$ such that $(\pi(B), U)$ is quasi-diagonal, $\operatorname{Ker}(\pi \times U) = \{0\}$, and $\operatorname{Ran}(\pi \times U) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$. We now apply Theorem 3.1 to (π, U) and (ρ', V'). For any $\varepsilon > 0$ there is a unitary W from $\mathcal{H}'$ onto $\mathcal{H}_\pi$ such that $\|\rho'(x) - W^* \pi(x) W\| \leqslant (\varepsilon/4)\|x\|$ for $x \in \mathcal{F}$ and $\|H_{V'} - W^* H_U W\| \leqslant \varepsilon/2$. Let E be a finite rank projection on $\mathcal{H}_\pi$ such that $\|[E, \pi(x)]\| \leqslant (\varepsilon/2)\|x\|$ for $x \in \mathcal{F}$, $\|(1-E)W\xi\| \leqslant \varepsilon\|\xi\|$ for $\xi \in \omega$ and $\|[E, H_U]\| < \varepsilon/2$. We set $F = W^* E W$, which is a finite-rank projection on $\mathcal{H}' \subset \mathcal{H}_\rho$. Then $\|[F, \rho(x)]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$, $\|(1-F)\xi\| = \|(1-E)W\xi\| \leqslant \varepsilon\|\xi\|$ for $\xi \in \omega$, and $\|[F, H_V]\| < \varepsilon/2 + \|[E, H_U]\| <$

ε. This completes the proof in the quasi-diagonal case. A similar proof can be given in the pseudo-diagonal case. ∎

Now we turn to the proof of Theorem 3.1.

Let σ be a completely positive map (or CP map) of $A \times_\alpha \mathbb{R}$ into $B(\mathcal{H})$. Let (e_ν) be an approximate identity in $A \times_\alpha \mathbb{R}$. Then $\sigma(e_\nu)$ is increasing and bounded in $B(\mathcal{H})$ and hence converges in the strong operator topology. Denote the limit by I and remark that I is the supremum of $\{\sigma(x) : x \in A \times_\alpha \mathbb{R},\ 0 \leqslant x \leqslant 1\}$ in $\mathcal{B}(\mathcal{H})$.

More generally we extend σ to a CP map from the multiplier algebra $M(A \times_\alpha \mathbb{R})$ into $B(\mathcal{H})$. For $x \in M(A \times_\alpha \mathbb{R})$ one shows that $\sigma(e_\nu x)$ converges in the weak operator topology since $|\langle \xi, \sigma((e_\mu - e_\nu)x)\eta\rangle| \leqslant \langle \xi, \sigma(e_\mu - e_\nu)\xi\rangle^{1/2} \langle \eta, \sigma(x^*(e_\mu - e_\nu)x)\eta\rangle^{1/2}$ for $\mu \geqslant \nu$. We denote the limit by $\sigma(x)$. It is also the limit of $\sigma(e_\nu x e_\nu)$ since $\sigma(e_\nu x(e_\mu - e_\nu))$ converges to zero for all pairs μ, ν with $\mu > \nu$ as $\nu \to \infty$. Note that $\sigma(\lambda(f))$ for $f \in L^1(\mathbb{R})$ and $\sigma(\lambda_t)$ for $t \in \mathbb{R}$ are all well-defined where λ denotes the unitary group implementing α on A and $\lambda(f) = \int f(t)\lambda_t \mathrm{d}t \in M(A \times_\alpha \mathbb{R})$. From the definition we have that $\sigma(1) = I$.

We next define the α-spectrum of σ. Set

$$J = \{f \in L^1(\mathbb{R}) : \lambda(f) \geqslant 0,\ \sigma(\lambda(f)) = 0\}.$$

Then J is a hereditary closed cone in $L^1(\mathbb{R})_+$ and the α-spectrum of σ, denoted by $\mathrm{Sp}_\alpha(\sigma)$, is defined by $\bigcap\{\mathrm{Ker}\widehat{f} : f \in J\}$. When $\mathrm{Sp}_\alpha(\sigma)$ is compact and $f \in L^1(\mathbb{R})_+$ satisfies $\widehat{f} = 1$ on $\mathrm{Sp}_\alpha(\sigma)$ then it follows that $\sigma(\lambda(f)) = \sigma(1)$.

Let $\mathcal{D}$ denote the set of $\xi \in \mathcal{H}$ such that $\sigma(\lambda_t - 1)\xi/it$ converges strongly as $t \to 0$. If $\mathcal{D}$ is a dense linear subspace then the operator H' defined on $\mathcal{D}$ as the limit of $\sigma(\lambda_t - 1)/it$ is symmetric. If the closure of H', which we will denote by $\sigma(H)$, is self-adjoint we will say that σ is α-differentiable. If $\mathrm{Sp}_\alpha(\sigma)$ is compact then σ is α-differentiable and $\sigma(H)$ is bounded (because if $f \in L^1(\mathbb{R})$ satisfies $\widehat{f} = 1$ on $\mathrm{Sp}_\alpha(\sigma)$ and $\mathrm{supp}(\widehat{f})$ is compact then it follows that $\sigma(\lambda_t) = \sigma(\lambda(f)\lambda_t)$ and $t \mapsto \lambda(f)\lambda_t$ is differentiable in t). If σ is a homomorphism then $t \mapsto \sigma(\lambda_t)$ is a unitary flow and thus σ is α-differentiable.

From now on we always assume that the C^*-algebra A is separable.

DEFINITION 3.2. Let $\sigma\colon A \times_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H})$ and $\sigma'\colon A \times_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H}')$ be α-differentiable CP maps. For two bounded (or unbounded self-adjoint) operators T and T' (with a common domain) we write $T \sim T'$ if the difference $T - T'$ is (or extends to) a compact operator.

We write $\sigma \sim \sigma'$ if there is a unitary $V\colon \mathcal{H} \to \mathcal{H}'$ such that $\sigma(x) \sim V^*\sigma'(x)V$ for $x \in A \cup A \times_\alpha \mathbb{R}$ and $\sigma(H) \sim V^*\sigma'(H)V$ and $\sigma \lesssim \sigma'$ if there is an isometry $W\colon \mathcal{H} \to \mathcal{H}'$ such that $\sigma(x) \sim W^*\sigma'(x)W$ for $x \in A \cup A \times_\alpha \mathbb{R}$ and $\sigma(H) \sim W^*\sigma'(H)W$ where $W^*\mathcal{D}(\sigma'(H)) = \mathcal{D}(\sigma(H))$ and $WW^*\mathcal{D}(\sigma'(H)) \subset \mathcal{D}(\sigma'(H))$.

We write $\sigma \approx \sigma'$ if there is a sequence of unitaries $V_n\colon \mathcal{H} \to \mathcal{H}'$ such that the V_n satisfy the above conditions for V and $\|\sigma(x) - V_n^*\sigma'(x)V_n\| \to 0$ for $x \in$

$A \cup A \times_\alpha \mathbb{R}$ and $\|\sigma(H) - V_n^*\sigma'(H)V_n\| \to 0$, and $\sigma \lessapprox \sigma'$ if there is a sequence of isometries $W_n \colon \mathcal{H} \to \mathcal{H}'$ such that the W_n satisfy the above conditions for W, $W_nW_n^* \to 0$, $\|\sigma(x) - W_n^*\sigma'(x)W_n\| \to 0$ for $x \in A \cup A \times_\alpha \mathbb{R}$ and $\|\sigma(H) - W_n^*\sigma'(H)W_n\| \to 0$.

Note that $\lessapprox$ is transitive: if $\sigma \lessapprox \sigma'$ and $\sigma' \lessapprox \sigma''$ then $\sigma \lessapprox \sigma''$ as easily follows. We are now able to state the following version of the Theorem 3.1.

THEOREM 3.3. *Let (π, U) be a covariant representation of (A, α) on a separable Hilbert space such that the range of $\pi \times U$ does not contain a non-zero compact operator. If ρ is an α-differentiable* CP *map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H}_\rho)$ such that $Q\rho$ is a homomorphism with* $\mathrm{Ker}(Q\rho) \subset \mathrm{Ker}(\pi \times U)$*, where Q is the quotient map of $\mathcal{B}(\mathcal{H}_\rho)$ onto $\mathcal{B}(\mathcal{H}_\rho)/\mathcal{K}(\mathcal{H}_\rho)$, then $\pi \times U \lessapprox \rho$.*

The following lemma is an adaptation of 3.5.5 of [12].

LEMMA 3.4. *Let σ be a homomorphism of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space. Then there exists a sequence of* CP *maps $\sigma_n \colon A \times_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H}_n)$ such that* $\mathrm{Sp}_\alpha(\sigma_n)$ *is compact,* $\dim \mathcal{H}_n$ *is finite, $\sigma_n(1) = 1$ and $\sigma \lessapprox \bigoplus_{n=1}^{\infty} \sigma_n$.*

Proof. Let P be the spectral measure of the generator $\sigma(H)$ of $t \mapsto \sigma(\lambda_t)$. On each spectral subspace $P(n, n+1]\mathcal{H}$ we find a compact operator K_n such that $H_n' = \sigma(H)P(n, n+1] + K_n$ is diagonal with eigenvalues in $(n, n+1]$ and $\|K_n\| < 1/(|n|+1)$. Then $H' = \sum_n H_n'$ is diagonal and is given by $\sigma(H) + K$ where $K = \sum_n K_n$ is compact. Let (E_n) be an approximate unit for $\mathcal{K}(\mathcal{H})$ consisting of projections such that $[E_n, H'] = 0$. Note that each E_n commutes with $P(k, k+1]$, is dominated by $P[-m, m]$ for some m and satisfies $\|[E_n, H]\| = \|[E_n, K]\| \to 0$ as $n \to \infty$. In the convex hull of (E_n) there is an approximate unit (F_n) for $\mathcal{K}(\mathcal{H})$ such that (F_n) satisfies $\|[F_n, \sigma(x)]\| \to 0$ for $x \in A \cup A \times_\alpha \mathbb{R}$ in addition to the conditions on H. Note that F_n is of finite rank.

Let $(\mathcal{F}_k)$ be an increasing sequence of finite subsets of $A \cup A \times_\alpha \mathbb{R}$ such that $\bigcup_k \mathcal{F}_k$ is dense in $A \cup A \times_\alpha \mathbb{R}$ and $\varepsilon > 0$. By assuming $\|[F_n, \sigma(x)]\|$ and $\|[F_n, \sigma(H)]\|$ are sufficiently small we may suppose that $D_n = (F_n - F_{n-1})^{1/2}$ with $F_0 = 0$ satisfies

$$\|D_n\sigma(x) - \sigma(x)D_n\| < \varepsilon 2^{-n}, \quad x \in \mathcal{F}_n, \quad \text{and} \quad \|D_n\sigma(H) - \sigma(H)D_n\| < \varepsilon 2^{-n}.$$

The former follows from $\|[F_n - F_{n-1}, \sigma(x)]\| \approx 0$ as D_n is just a continuous function of $F_n - F_{n-1}$ which can be approximated by polynomials. For the latter, where $\sigma(H)$ is unbounded in general, we use the fact that both D_n and $\sigma(H)$ commute with $P(k, k+1]$. Thus we have

$$[D_n, \sigma(H)] = \sum_k [D_nP(k, k+1], (\sigma(H) - k)P(k, k+1]],$$

where each commutator is between elements of norm one or less. Since $D_nP(k,k+1] = (F_nP(k,k+1] - F_{n-1}P(k,k+1])^{1/2}$ the latter inequality then follows just as the former.

Introduce the finite-dimensional subspace $\mathcal{H}_n$ of $\mathcal{H}$ by $\mathcal{H}_n = D_n\mathcal{H}$. Let $\mathcal{H}' = \bigoplus_n \mathcal{H}_n$ and define a linear map V from $\mathcal{H}$ into $\mathcal{H}'$ by $V\xi = \bigoplus_n D_n\xi$. This is an isometry since $\sum_n D_n^2 = 1$. Let Q_n be the projection onto $\mathcal{H}_n$ in $\mathcal{H}$ and let $\sigma_n(x) = Q_n\sigma(x)Q_n$ for $x \in A \times_\alpha \mathbb{R}$. Since $\mathcal{H}_n$ is finite-dimensional and $D_n = D_nP[-m,m]$ for some m one has $\sigma_n(1) = 1$ and $\mathrm{Sp}_\alpha(\sigma_n)$ is compact with $\sigma_n(H) = Q_n\sigma(H)Q_n = Q_n\sigma(H)P[-m,m]Q_n$. Define $\sigma' = \bigoplus_n \sigma_n$. Then $\sigma'(H)$ is well-defined and is equal to $\bigoplus_n \sigma_n(H)$.

We will show that $V\mathcal{D}(\sigma(H)) \subset \mathcal{D}(\sigma'(H))$ and $\mathcal{D}(\sigma(H)) \supset V^*\mathcal{D}(\sigma'(H))$. If $\xi \in \mathcal{D}(\sigma(H))$ then

$$\begin{aligned}\sum_n \|\sigma_n(H)D_n\xi\|^2 &\leqslant \sum_n \|[\sigma_n(H), D_n]\xi + D_n\sigma(H)\xi\|^2 \\ &\leqslant 2\sum_n \|[\sigma(H), D_n]\xi\|^2 + 2\|\sigma(H)\xi\|^2,\end{aligned}$$

which implies that $V\xi \in \mathcal{D}(\sigma'(H))$. If $\eta \in \mathcal{D}(\sigma'(H))$ and P_N denotes the projection onto the first N direct summands in $\mathcal{H}'$ then

$$\begin{aligned}\|\sigma(H)V^*P_N\eta\| &= \Big\| \sum_{n=1}^N [\sigma_n(H), D_n]\eta_n + D_n\sigma_n(H)\eta_n \Big\| \\ &\leqslant \Big\| \sum_{n=1}^N [\sigma_n(H), D_n]\eta_n \Big\| + \Big\| \sum_{n=1}^N D_n\sigma_n(H)\eta_n \Big\| \leqslant \varepsilon\|\eta\| + \|V^*P_N\sigma'(H)\eta\|,\end{aligned}$$

where we have used $\|[\sigma(H), D_n]\| < \varepsilon 2^{-n}$. From this kind of computation we can conclude that $(\sigma(H)V^*P_N\eta)_N$ is a Cauchy sequence and so $V^*\eta \in \mathcal{D}(\sigma(H))$.

From the two inequalities above it follows that $\mathcal{D}(\sigma(H)) = V^*\mathcal{D}(\sigma'(H))$ and $V\mathcal{D}(\sigma(H)) = VV^*\mathcal{D}(\sigma'(H)) \subset \mathcal{D}(\sigma'(H))$. Thus $\mathcal{D}(V\sigma(H)) = \mathcal{D}(\sigma(H))$ and $\mathcal{D}(\sigma'(H)V) = V^*\mathcal{D}(\sigma'(H))$ are equal and on this common domain

$$V\sigma(H)\xi - \sigma'(H)V\xi = (D_n\sigma(H)\xi - \sigma(H)D_n\xi)_n.$$

Since $\|[D_n, \sigma(H)]\| < \varepsilon 2^{-n}$, the closure of $V\sigma(H) - \sigma'(H)V$ on $\mathcal{D}(\sigma(H))$ is a compact operator with norm less than ε. Since the closure of $V\sigma(H)V^* - \sigma'(H)VV^*$ on $\mathcal{D}(\sigma'(H))$ is compact it follows that the closure of $V\sigma(H)V^* - VV^*\sigma'(H)$ on $\mathcal{D}(\sigma'(H))$ is also compact. Thus we can conclude that the closure of $\sigma'(H)VV^* - VV^*\sigma'(H)$ on $\mathcal{D}(\sigma'(H))$ is compact. In the same way one concludes that $V\sigma(x) - \sigma'(x)V$ is a compact operator for $x \in \bigcup \mathcal{F}_k$ and so for all $x \in A$ and all $x \in A \times_\alpha \mathbb{R}$. Note that $\|V\sigma(x) - \sigma'(x)V\| < \varepsilon$ for $x \in \mathcal{F}_1$. This concludes the proof of $\sigma \lesssim \sigma' = \bigoplus_n \sigma_n$.

It also follows from the foregoing construction of σ' and V that one has bounds $\|V\sigma(H) - \sigma'(H)V\| < \varepsilon$ and $\|V\sigma(x) - \sigma'(x)V\| < \varepsilon$ for $x \in \mathcal{F}_1$ where

$\mathcal{F}_1$ is a prescribed finite subset of $A \cup A \times_\alpha \mathbb{R}$. One can then obtain a sequence of such (σ'_k, V_k) such that $\|V_k\sigma(H) - \sigma'_k(H)V_k\| \to 0$ and $\|V_k\sigma(x) - \sigma'_k(x)V_k\| \to 0$ for all $x \in A \cup A \times_\alpha \mathbb{R}$. Since the direct sum $\bigoplus \sigma'_k$ is of the form $\bigoplus \sigma_n$ described in the statement this concludes the proof. ∎

LEMMA 3.5. *Let ρ be a homomorphism of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H})$ such that* $\mathrm{Ran}(\rho) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ *and let σ be a* CP *map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathbb{C}^n)$ such that* $\sigma(1) = 1$, $\mathrm{Sp}_\alpha(\sigma)$ *is compact, and* $\ker\sigma \supset \ker\rho$. *Then it follows that $\sigma \lessapprox \rho$.*

More generally let ρ be an α-differentiable CP *map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H})$ such that $Q\rho$ is a homomorphism satifying* $\mathrm{Ker}(\sigma) \supset \mathrm{Ker}(Q\rho)$ *where Q is the quotient map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then the same conclusion follows.*

Proof. We may assume that A is unital. There is a C^∞-function $f \in L^1(\mathbb{R})$ such that the support of $\widehat{f}$ is compact, $0 \leqslant \lambda(f) \leqslant 1$ and $\widehat{f} = 1$ on $\mathrm{Sp}_\alpha(\sigma)$. Then it follows that $\sigma(\lambda(f)) = 1$. We denote by ρ_n the representation $\mathrm{id} \otimes \rho$ of $M_n(A \times_\alpha \mathbb{R})$ on $\mathcal{H}_n = \mathbb{C}^n \otimes \mathcal{H}$. We define a state ϕ on $M_n(A \times_\alpha \mathbb{R})$ by

$$\phi([x_{ij}]) = \frac{1}{n}\sum_{i,j=1}^{n} \langle \xi_i, \sigma(x_{ij})\xi_j \rangle,$$

where (ξ_i) is the standard orthonormal basis for $\mathbb{C}^n$. Since

$$\phi|_{\mathrm{Ker}(\sigma_n)} = 0, \quad \mathrm{Ker}(\rho_n) \subset \mathrm{Ker}(\sigma_n), \quad \phi(\sigma_n(1 \otimes \lambda(f))) = 1$$

and $\mathrm{Ran}(\rho_n) \cap \mathcal{K}(\mathbb{C}^n \otimes \mathcal{H}) = \{0\}$ there is a sequence (η_k) of vectors in $\mathbb{C}^n \otimes \mathcal{H}$ such that $\langle \eta_k, \rho_n(x)\eta_k \rangle \to \phi(x)$ for $x \in M_n(A \times_\alpha \mathbb{R})$, $\langle \eta_k, \rho_n(1 \otimes \lambda(f))\eta_k \rangle = 1$ and η_k converges to zero weakly. Since $\phi(e_{ij} \otimes \lambda(f)) = \delta_{ij}/n$ we may assume

$$\langle \eta_k, \rho_n(e_{ij} \otimes \lambda(f))\eta_k \rangle = \frac{\delta_{ij}}{n}.$$

If $\eta_k = (\eta_{k1}, \ldots, \eta_{kn}) \in \mathbb{C}^n \otimes \mathcal{H} \cong \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ then we define an isometry $V_k \colon \mathbb{C}^n \to \mathcal{H}$ by $V_k\xi_i = \sqrt{n}\eta_{ki}$. Then one can conclude that $\|\sigma(x) - V_k^*\rho(x)V_k\| \to 0$ for $x \in A \times_\alpha \mathbb{R}$. (See the proof of 3.6.7 of [12] for details.) Since $\rho(\lambda(f))V_k = V_k$, $\sigma(H) = \mathrm{i}\lambda(f')$ and $\rho(H\lambda(f)) = \mathrm{i}\rho(\lambda(f'))$ it also follows that

$$\|\sigma(x) - V_k\rho(x)V_k\| \to 0, \quad x \in A$$

and $\|\sigma(H) - V_k\rho(H)V_k\| \to 0$.

The condition required for the existence of the foregoing (η_k) is precisely the content of the additional statement. Let B be the C^*-algebra generated by the range of ρ and the compact operators. Then σQ is a CP map of B into $\mathcal{B}(\mathbb{C}^n)$ vanishing on $\mathcal{K}(\mathcal{H})$. Then the above arguments show that $\sigma Q \lessapprox \mathrm{id}_B$ for maps from B. Composing with the homomorphism ρ of A into B one arrives at the conclusion. ∎

The following is an adaptation of 3.5.2 of [12].

LEMMA 3.6. *Let ρ be an α-differentiable* CP *map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H})$ such that $Q\rho$ is a homomorphism where Q is the quotient map of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $\sigma_n \colon A \times_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H}_n)$ be a sequence of* CP *maps such that* $\dim \mathcal{H}_n < \infty$, *$\sigma_n(1) = 1$ and* $\mathrm{Sp}_\alpha(\sigma_n)$ *is compact. If $\sigma_n \lessapprox \rho$ for all n then $\sigma \equiv \bigoplus_n \sigma_n \lessapprox \rho$.*

Proof. For each n there is a C^∞-function $f_n \in L^1(\mathbb{R})$ such that the support of $\widehat{f}$ is compact, $0 \leqslant \lambda(f_n) \leqslant 1$ and $\widehat{f_n} = 1$ on $\mathrm{Sp}_\alpha(\sigma_n)$. By $\sigma_n(\lambda(f_n)) = 1$ and the assumption $\sigma_n \lessapprox \rho$ there is a sequence (V_k) of isometries of $\mathcal{H}_n$ into $\mathcal{H}$ such that $\rho(\lambda(f_n))V_k = V_k$, $V_k\eta$ converges to zero weakly for $\eta \in \mathcal{H}_n$ and $\|\sigma_n(x) - V_k^*\rho(x)V_k\| \to 0$ for $x \in A \cup A \times_\alpha \mathbb{R}$ and $x = H$.

Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of $A \cup A \times_\alpha \mathbb{R}$ such that $\bigcup_n \mathcal{F}_n$ is dense in $A \cup A \times_\alpha \mathbb{R}$ and let $\varepsilon > 0$. We construct inductively a sequence $V_n \colon \mathcal{H}_n \to \mathcal{H}$ of isometries such that $\rho(\lambda(f_n))V_n = V_n$, $\|\sigma_n(H) - V_n^*\rho(H)V_n\| < \varepsilon 2^{-n}$ and $\|\sigma_n(x) - V_n^*\rho(x)V_n\| < \varepsilon 2^{-n}$ for all $x \in \mathcal{F}_n$, and moreover $V_n\mathcal{H}_n$ with $n > 1$ is orthogonal to the finite-dimensional subspace spanned by $V_m\xi, \rho(H)V_m\xi, \rho(x)V_m\xi, \rho(x^*)V_m\xi$ with $\xi \in \mathcal{H}_m$, $x \in \mathcal{F}_n$ and $m < n$. (This last condition may require a slight modification of V_n, retaining $\rho(\lambda(f_n))V_n = V_n$, which will not affect the condition $\|\sigma_n(H) - V_n^*\rho(H)V_n\| < \varepsilon 2^{-n}$ since $\rho(H)$ may be replaced by bounded $\rho(H)\rho(\lambda(f_n))$.) We define $V = \bigoplus_n V_n$, which is an isometry from $\bigoplus_n \mathcal{H}_n$ into $\mathcal{H}$. Since for $x \in \mathcal{F}_m$

$$V^*\rho(x)V = (V_i^*\rho(x)V_j)_{i,j<m} \oplus \bigoplus_{n \geqslant m} V_n\rho(x)V_n,$$

one has

$$\sigma(x) - V^*\rho(x)V = (\sigma_i(x)\delta_{ij} - V_i^*\rho(x)V_j)_{i,j<m} \oplus \bigoplus_{n \geqslant m} (\sigma_n(x) - V_n^*\rho(x)V_n),$$

where if $m = 1$ we ignore the first direct summand, otherwise it is an operator on $\bigoplus_{i<m} \mathcal{H}_i$, which is finite-dimensional. Then one concludes that the displayed operator is compact and that $\|\sigma(x) - V^*\rho(x)V\| < \varepsilon$ for $x \in \mathcal{F}_1$. Similarly one has

$$\sigma(H) - V^*\rho(H)V = \bigoplus_n (\sigma_n(H) - V_n^*\rho(H)V_n),$$

which is compact with norm less than ε. ∎

Proof of Theorem 3.3. Let (π, U) be a covariant representation of (A, α) and ρ an α-differentiable CP map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(\mathcal{H}_\rho)$ as in the theorem. We may assume that A is unital.

Let $\sigma = \pi \times U$. By Lemma 3.4 we find a sequence $\sigma_n \colon A \times_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H}_n)$ of CP maps such that $\dim \mathcal{H}_n < \infty$, $\sigma_n(1) = 1$, $\mathrm{Sp}_\alpha(\sigma_n)$ is compact and $\sigma \lessapprox \bigoplus_n \sigma_n$. Since $\mathrm{Ker}(\sigma_n) \supset \mathrm{Ker}(\sigma) \supset \mathrm{Ker}(\rho)$ Lemma 3.5 shows that $\sigma_n \lessapprox \rho$. Since $\bigoplus_n \sigma_n \lessapprox \rho$ by Lemma 3.6 one concludes that $\sigma \lessapprox \rho$. ∎

Proof of Theorem 3.1. The first part of the theorem is a special case of Theorem 3.3.

Let $\overline{\pi} = \pi \times U$ and $\overline{\rho} = \rho \times V$ and suppose that $\mathrm{Ker}(\overline{\rho}) = \mathrm{Ker}(\overline{\pi})$. Let $\overline{\rho}^{\infty}$ denote the direct sum of infinite copies of $\overline{\rho}$. Then applying the first part of the theorem to $\overline{\rho}^{\infty}$ and $\overline{\pi}$ we deduce that $\overline{\rho}^{\infty}$ can be approximated by a direct summand $\overline{\pi}_1$ of $\overline{\pi}$ through a unitary. Here $\overline{\pi}_1$ is obtained as $P\overline{\pi}(\,\cdot\,)P$ where P is a projection such that $P\mathcal{D}(\overline{\pi}(H)) \subset \mathcal{D}(\overline{\pi}(H))$ and $\|[P, \overline{\pi}(H)]\|$ is small depending on the approximation. Thus $\overline{\pi}_1$ is an α-differentiable unital CP map and this situation will simply be written as $\overline{\rho}^{\infty} \sim \overline{\pi}_1$. Writing $\overline{\pi}_2 = (1-P)\overline{\pi}(\,\cdot\,)(1-P)$ one obtains that $\overline{\pi} \sim \overline{\pi}_1 \oplus \overline{\pi}_2$ and that $\overline{\rho}^{\infty} \oplus \overline{\pi}_2 \sim \overline{\pi}$. Since $\overline{\rho} \oplus \overline{\rho}^{\infty} \sim \overline{\rho}^{\infty}$, one calculates that $\overline{\rho} \oplus \overline{\pi} \sim \overline{\rho} \oplus \overline{\rho}^{\infty} \oplus \overline{\pi}_2 \sim \overline{\rho}^{\infty} \oplus \overline{\pi}_2 \sim \overline{\pi}$. By changing the roles of $\overline{\rho}$ and $\overline{\pi}$ we conclude that $\overline{\rho} \sim \overline{\pi}$. Since this is true for any degree of approximation one obtains the conclusion (see the arguments on page 340 of [1] for more details). ∎

4. ABSTRACT CHARACTERIZATIONS

Voiculescu [24] gave conditions for C^*-algebras to be quasi-diagonal. By mimicking his proof we shall establish Theorems 1.5 and 1.6.

Proof of Theorem 1.5. (i) $\Rightarrow$ (ii). Suppose α is quasi-diagonal. Then there is a faithful representation π of A and a unitary flow U on $\mathcal{H}_\pi$ such that $(\pi(A), U)$ is quasi-diagonal. For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there is a finite subset ω of unit vectors in $\mathcal{H}_\pi$ such that $\max\{|\langle \xi, \pi(x)\eta\rangle| : \xi, \eta \in \omega\} \geqslant (1-\varepsilon/3)\|x\|$ for all $x \in \mathcal{F}$. Then there is a finite-rank projection E on $\mathcal{H}_\pi$ such that $\|[E, \pi(x)]\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{F}$, $\|(1-E)\xi\| \leqslant \varepsilon/3$ for $\xi \in \omega$ and $\|[E, H]\| < \varepsilon$, where H is the self-adjoint generator of U. Set $B = \mathcal{B}(E\mathcal{H}_\pi)$, $\beta_t = \mathrm{Ad}\, \mathrm{e}^{\mathrm{i}tEHE}$ and $\phi(x) = E\pi(x)E$ for $x \in A$. Then it follows that $\|\phi(x)\| \geqslant (1-\varepsilon)\|x\|$ for $x \in \mathcal{F}$, $\|\phi(x)\phi(y) - \phi(xy)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$ and $\|\beta_t\phi - \phi\alpha_t\| < \varepsilon$ for $t \in [-1, 1]$. The triple B, β, ϕ satisfies the required conditions.

(ii) $\Rightarrow$ (iii). Let $\mathcal{F}$ be a finite subset of A and $\varepsilon > 0$. Suppose there is a CP map ϕ of A into a finite-dimensional C^*-algebra B with a flow β such that $\|\phi\| \leqslant 1$, $\|\phi(x)\| \geqslant (1-\varepsilon)\|x\|$ and $\|\phi(xy) - \phi(x)\phi(y)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{F} \cup \mathcal{F}^*$ and $\|\beta_t\phi - \phi\alpha_t\| < \varepsilon$ for $t \in [-1, 1]$. We may assume that ϕ is unital. (If A is not unital, we may assume this by extending ϕ as such; if A is unital we modify ϕ using the fact that $\phi(1)$ is close to a projection.) By Stinespring's theorem there is a representation π of A and a finite-rank projection E on $\mathcal{H}_\pi$ such that ϕ identifies with $E\pi(\,\cdot\,)E$. It follows that

$$\|\phi(x^*x) - \phi(x^*)\phi(x)\| = \|E\pi(x^*)(1-E)\phi(x)E\| = \|(1-E)\pi(x)E\|^2.$$

Since $\|[E, \pi(x)]\| = \max\{\|E\pi(x)(1-E)\|, E\pi(x^*)(1-E)\|\}$ we obtain (iii) except for the condition concerned with the flow.

To prove (iii) fully we have to modify ϕ and go back to the proof of Stinespring's theorem. For a small $\gamma > 0$ we replace ϕ by

$$\varphi = \frac{\gamma}{2} \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma|t|} \beta_{-t} \phi \alpha_t \mathrm{d}t.$$

Then $\varphi(1) = 1$, $\beta_{-t}\varphi\alpha_t \leqslant \mathrm{e}^{\gamma|t|}\varphi$ and $\|\beta_t\varphi - \varphi\alpha_t\| < \varepsilon$ for $t \in [-1,1]$. (This method is used in [18].) Noting that $\|\beta_{-t}\phi\alpha_t - \phi\| < \varepsilon(1+|t|)$ we have $\|\varphi - \phi\| < \varepsilon(1+1/\gamma)$, which is an arbitrarily small constant if ε is chosen after γ. Hence we may assume that ϕ satisfies $\beta_{-t}\phi\alpha_t \leqslant \mathrm{e}^{\varepsilon|t|}\phi$ in addition to the conditions in (ii).

The above representation π is constructed as follows. Assuming B acts on a finite-dimensional Hilbert space $\mathcal{H}$ we define an inner product on the algebraic tensor product $A \otimes \mathcal{H}$ by

$$\Big\langle \sum_i x_i \otimes \xi_i, \sum_j y_j \otimes \eta_j \Big\rangle = \sum_{i,j} \langle \xi_i, \phi(x_i^* y_j)\eta_j \rangle.$$

We obtain a Hilbert space $\mathcal{H}_\pi$ by the standard process of dividing $A \otimes \mathcal{H}$ out by the null space followed by completion and then a representation π of A on $\mathcal{H}_\pi$ from the multiplication of A on A, the first factor of $A \otimes \mathcal{H}$. Using a flow V on $\mathcal{H}$ such that $\beta_t(x) = V_t x V_t^*$ for $x \in B$ we define a one-parameter group of operators W_t on $\mathcal{H}_\pi$ by

$$W_t \sum_i x_i \otimes \xi_i = \sum_i \alpha_t(x_i) \otimes V_t \xi_i.$$

Since

$$\Big\| W_t \sum_i x_i \otimes \xi_i \Big\|^2 = \sum_{i,j} \langle \xi_i, \beta_{-t}\phi\alpha_t(x_i^* y_j)\eta_j \rangle,$$

the group W is well-defined and we obtain estimates $\mathrm{e}^{-\varepsilon t}1 \leqslant W_t^* W_t \leqslant \mathrm{e}^{\varepsilon t}1$. Denoting the generator of W by $\mathrm{i}K$ one concludes that $\mathcal{D}(K^*) = \mathcal{D}(K)$ and $-\varepsilon \leqslant -\mathrm{i}K^* + \mathrm{i}K \leqslant \varepsilon 1$. Then the closure k of $K - K^*$ has norm less than or equal to ε. Since $W_t\pi(x)W_{-t} = \pi\alpha_t(x)$ we conclude that $W_t^*W_t \in \pi(A)'$ and so $k \in \pi(A)'$. Set $U_t = \mathrm{e}^{\mathrm{i}t(K-k/2)}$ for $t \in \mathbb{R}$. The U is a unitary flow such that $U_t\pi(x)U_t^* = \pi\alpha_t(x)$ for $x \in A$.

We denote by $\mathcal{H}'$ the subspace of $\mathcal{H}_\pi$ generated by $1 \otimes \xi$ with $\xi \in \mathcal{H}$. Let E denote the projection onto $\mathcal{H}'$. Then it follows that $\phi(x) = E\pi(x)E$ for $x \in A$. Since $W_tEW_{-t^*} = E$ one obtains $\|U_tEU_t^* - E\| \leqslant \|k\||t|$. Thus condition (iii) follows.

(iii) $\Rightarrow$ (i). We prove this by following the argument given in [24]. We may suppose that A is separable.

Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of A with dense union. For $(\mathcal{F}_n, n^{-1})$ we choose a covariant representation (π_n, U^n) and a finite-rank projection E_n on the representation space $\mathcal{H}_n$ satisfying the conditions described in (iii). Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, $\pi = \bigoplus_{n=1}^{\infty} \pi_n$, $U = \bigoplus_{n=1}^{\infty} U^n$ and $E = \bigoplus_{n=1}^{\infty} E_n$. For each

$k \in \mathbb{N}$ let P'_k denote the projection in $\mathcal{H}$ onto the first k direct summands and let $P_k = (1 - P'_k)E = E(1 - P'_k)$. If $\pi_n(H)$ (respectively $\pi(H)$) denotes the self-adjoint generator of U^n (respectively U) then $\pi(H) = \bigoplus_{n=1}^{\infty} \pi_n(H)$. Since $\|[E_n, \pi_n(x)]\| \leqslant n^{-1}\|x\|$ for $x \in \mathcal{F}_n$ and $\|[E_n, \pi_n(H)]\| < n^{-1}$ one has $[P_k, \pi(x)] \in \mathcal{K}$ for $x \in A$ and $[P_k, \pi(H)] \in \mathcal{K}$ where $\mathcal{K}$ denotes the compact operators on $\mathcal{H}$. If we denote by $\pi \times U$ the covariant representation of $A \times_\alpha \mathbb{R}$, then it follows that $\sigma_k = P_k(\pi \times U)P_k$ is an α-differentiable CP map of $A \times_\alpha \mathbb{R}$ into $\mathcal{B}(P_k\mathcal{H})$ such that the composition $Q \circ \sigma_k$ is an isomorphism and

$$\sigma_k(H) = (1 - P'_k) \cdot \bigoplus_{n=1}^{\infty} E_n \pi_n(H) E_n \cdot (1 - P'_k),$$

where Q is the quotient map of $\mathcal{B}(P_k\mathcal{H})$ into $\mathcal{B}(P_k\mathcal{H})/\mathcal{K}$.

Let (ρ, V) be a covariant representation on a separable Hilbert space X such that $\rho \times V$ is faithful and $\mathrm{Ran}(\rho \times V) \cap \mathcal{K} = \{0\}$. We set $X_\infty = X \oplus X \oplus \cdots$, $\rho^\infty = \rho \oplus \rho \oplus \cdots$, and $V^\infty = V \oplus V \oplus \cdots$. Let G_k denote the projection onto the direct sum of the first k copies of X in X_∞. We will show that (ρ^∞, V^∞) is quasi-diagonal as required in (i).

Fix $k \in \mathbb{N}$. By Theorem 3.3 applied to $(\rho^\infty, V^\infty)|_{G_k X_\infty}$ and $\sigma_n = P_n(\pi \times U)P_n$ we find partial isometries $S_n \colon X_\infty \to \mathcal{H}$ such that $S_n^* S_n = G_k$, $\mathrm{Ran}(S_n) \subset P_n\mathcal{H}$ and $\|S_n \rho^\infty(x) - \pi(x) S_n\| \to 0$ for $x \in A$ and $\|S_n \rho^\infty(H) - \pi(H) S_n\| \to 0$. Similarly from the pair of (π, U) and (ρ, V) we also find isometries $T_n \colon \mathcal{H} \to X$ such that $\|T_n \pi(x) - \rho(x) T_n\| \to 0$ for $x \in A$ and $\|T_n \pi(H) - \rho(H) T_n\| \to 0$. We define an isometry W_n of $\mathcal{H}$ into X_∞ by

$$W_n \xi = S_n^* \xi \oplus T_n(1 - S_n S_n^*)\xi \oplus 0 \oplus 0 \oplus \cdots,$$

which satisfies that $G_k X_\infty \subset W_n \mathcal{H} \subset G_{k+1} X_\infty$ and $\|W_n \pi(x) - \rho^\infty(x) W_n\| \to 0$ for $x \in A$ and $\|W_n \pi(H) - \rho^\infty(H) W_n\| \to 0$ as $n \to \infty$. We note that $G_k = W_n P_n W_n^*$.

Now $F_m = P'_m E$ is a finite-rank projection such that $[F_m, P_n] = 0$ and $F_m P_n = (P'_m - P'_n)E \to P_n$ as $m \to \infty$. Thus choosing $m_n > n$ for each n sufficiently large we may suppose that $W_n F_{m_n} P_n W_n^* \xi \to \xi$ for each $\xi \in G_k X_\infty$. Since $F_{m_n} P_n$ commutes with the range of σ_n, $W_n F_{m_n} P_n W_n^*$ will serve as the required finite-rank projection on X_∞ for a large n. ∎

Proof of Theorem 1.6. (i) ⇒ (ii). This is easy.

(ii) ⇒ (iii). Given $(\mathcal{F}, \varepsilon)$ let $\mathcal{G} = \{x, x^*, xy : x, y \in \mathcal{F}\}$. By condition (ii) there is a flow β on a finite-dimensional C^*-subalgebra B and a CP map ϕ of A into B such that $\|\phi\| \leqslant 1$, $\|\phi(x)\| \geqslant (1 - \varepsilon)\|x\|$ and $\|\phi(x)\phi(y) - \phi(xy)\| \leqslant \varepsilon\|x\|\|y\|$ for $x, y \in \mathcal{G}$, and $\|\beta_t \phi(x) - \phi\alpha_t(x)\| \leqslant \varepsilon\|x\|$ for $x \in \mathcal{G}$ and $t \in [-1, 1]$. We may assume that A and ϕ are unital. For $\gamma = -\log\varepsilon > 0$ we replace ϕ by

$$\varphi = \frac{\gamma}{2} \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma|t|} \beta_{-t} \phi \alpha_t \mathrm{d}t.$$

Since $\|\varphi(x) - \phi(x)\| \leqslant (\varepsilon + e^{-\gamma})\|x\| = 2\varepsilon\|x\|$ for $x \in \mathcal{G}$, we may suppose, starting with $\varepsilon/7$ instead of ε, that ϕ satisfies the above properties as well as the covariance $\beta_t\phi\alpha_t \leqslant e^{\gamma|t|}\phi$ for $\gamma \approx -\log\varepsilon$. We suppose that B acts on a finite-dimensional Hilbert space $\mathcal{H}$ and choose a unitary flow V on $\mathcal{H}$ such that $\beta_t = \mathrm{Ad}\, V_t|_B$. Then, by Stinespring's construction for ϕ as in the proof of Theorem 1.5, we obtain a representation π of A, a (non-unitary) flow W and a finite-rank projection E such that $\phi(x) = E\pi(x)E$ for $x \in A$, under the identification of $E\mathcal{H}$ with $\mathcal{H}$, $W_t\pi(x)W_{-t} = \pi\alpha_t(x)$ for $x \in A$ and $W_tE = EW_tE = V_tE$. By a perturbation of W we obtain a unitary flow U such that $\pi\alpha_t(x) = \mathrm{Ad}\, U_t\pi(x)$, $x \in A$. Then we conclude that (π, U, E, V) satisfies the required properties.

(iii) $\Rightarrow$ (i). The proof is similar to the proof of the corresponding implication in Theorem 1.5. ∎

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ADDED IN PROOFS. The two notions, quasi-diagonality and pseudo-diagonality, for flows on C^*-algebras are in fact equivalent, which will be shown in a forthcoming paper.