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# FERMI–MARKOV STATES

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*Dedicated to L. Accardi in occasion of his 60th birthday*

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ABSTRACT. We investigate the structure of the Markov states on general Fermi algebras. The situation treated in the present paper covers, beyond the $d$-Markov states on the CAR algebra on $\mathbb{Z}$ (i.e. when there are $d$ Fermionic annihilators and creators on each site), also the nonhomogeneous case (i.e. when the numbers of generators depends on the localization). The present analysis provides the first necessary step for the study of the general properties, and the construction of nontrivial examples of Fermi–Markov states on $\mathbb{Z}^\nu$, that is the Fermi–Markov fields. Natural connections with the KMS boundary condition and entropy of Fermi–Markov states are studied in detail. Apart from a class of Markov states quite similar to those arising in the tensor product algebras (called "strongly even" in the sequel), other interesting examples of Fermi–Markov states naturally appear. Contrarily to the strongly even examples, the latter are highly entangled and it is expected that they describe interactions which are not "commuting nearest neighbor". Therefore, the non-strongly even Markov states, in addition to the natural applications to quantum statistical mechanics, might be of interest for the quantum information theory as well.

KEYWORDS: *Applications of selfadjoint operator algebras to physics, quantum Markov processes, mathematical quantum statistical mechanics.*

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## 1. INTRODUCTION

The quantum analogues of Markov processes were first constructed in [1], where the notion of quantum Markov chain on infinite tensor product algebras was introduced.

Nowadays, the quantum Markov chains have become a standard computational tool in solid state physics, and several natural applications have emerged in quantum statistical mechanics and quantum information theory. On the other hand, the introduction in [8], [9], [10] of the notion of “product state” on CAR algebras motivated the analogue construction in [6] of quantum Markov chains on these algebras as local perturbation of product states. The Fermi extension of product and Markov states is nontrivial because, even if the Fermi algebra is isomorphic to an infinite tensor products of matrix algebras, this embedding does not preserve the natural localization which plays an essential role in the very definition of these states. Product states describe non interacting (free or independent) systems. Markov states describe nearest neighbor interactions. In both cases the notion of "localization" plays a crucial role. Typically a discrete system is identified to a point in a graph. If this graph is not isomorphic to an interval in $\mathbb{Z}$ (1-dimensional case), one speaks of a random field. The crucial role of the localization is at the root of the difficulties to construct nontrivial examples of Markov fields. As the "interacting degrees of freedom" localized in a finite volume, increase with the volume, the first step to achieve this result is to investigate the nonhomogeneous 1-dimensional case, one of the goals of the present paper. Our second goal has to do with the most important difference between tensor and Fermi–Markov chains, emerged from the analysis of [6], which in its turn is related to the difference between *quantum Markov chains* and *quantum Markov states*. The origin of this difference lies in the fact that, in the classical case, the simple structure of Markov states is equivalent to a single intrinsic condition: the Markov property. In the quantum case, while the Markov property can be formulated in terms of a localization property of the modular group of the state (see [4], [7]), there is a class of states which have a Markov like local structure but do not necessarily enjoy the Markov property. These states are called quantum Markov chains (see e.g. [6], Definition 2.2). This phenomenon is related to the fact that the natural probabilistic extension of the notion of conditional expectation to the quantum case in general is not a projection (cf. [2]). On the other hand, the results of [14] show that some of the most interesting physical applications involve precisely those Markov chains which are not Markov states. Such Markov chains can be explicitly constructed, but at the moment no intrinsic operator theoretic characterization is known. This distinction also appears in the Fermi case. However, the new phenomenon consists in the fact that, while in the tensor case all Markov states are convex combinations of states which are product states with respect to a new localization canonically associated to the original one (two block factors), this is not true in the Fermi case. The Fermi analogue of the convex combination of two block factors still appear. The last are called in the sequel *strongly even* Markov states. In addition, there is a completely new class of Fermi–Markov states. This new class of non-strongly even Markov states that appears in the classification theorem below (cf. Theorem 3.2) is likely to play in the Fermi case, the role played by the entangled states in the tensor product case.

One can define the notion of Fermi entanglement in analogy to the tensor product case. One can expect that the main problem of the entanglement theory, that is to find constructive and easily applicable criteria to discriminate entangled from non entangled states, will be in the Fermi case at least as difficult as in the tensor case. The first step to attack this problem is to have a full and detailed description of this new class of states. In the paper [6] only the simplest case was considered, that is when there is only one creator and annihilator in each site. The nonhomogeneous case, discussed in the present paper, includes as a particular case the translation invariant cases described in [6] and its natural translation invariant generalization when there are $d$ creators and annihilators localized in each site. This leads to a much larger class of non-strongly even Markov states, which can be completely described.

In the present paper, the program outlined below is carried out in the following steps. Section 2 contains the key result on the structure of the even transition expectations associated to Fermi–Markov states (Proposition 2.7). It is then possible to provide the full classification of all even Markov quasiconditional expectations. Section 3 is devoted to the study of the most general situation, including also the non homogeneous Fermi–Markov states. Even if the structure of the Markov states considered here is more complex than the one in [3], [6], we are still able to provide their decomposition as direct integrals of minimal ones (Theorem 3.2). Furthermore, the minimal Markov states are the building–blocks for the construction of all the Fermi–Markov states (Theorem 3.3). Section 4 deals with some general properties of the Markov states, such as the connection with the KMS boundary condition, and the entropy. Section 5 is devoted to the study of the detailed structure of the strongly even Markov states. They can be viewed as the Fermi analogue of the Ising type interactions. We show that a strongly even Markov state $\varphi$ on the Fermi algebra arises by a lifting of a classical Markov process on the spectrum of a maximal Abelian subalgebra, with respect to the same localization as $\varphi$. In addition, we establish the equality between the Connes–Narnhofer–Thirring dynamical entropy $h_\varphi(\alpha)$ with respect to the shift and the mean entropy $s(\varphi)$. Section 6 provides the full list of translation invariant Fermi–Markov states for low dimensional single site local algebras. The same method is applicable to higher dimensions. Finally, by using Moriya criterion, we show (cf. Proposition 6.1) that the Fermi–Markov states which are not strongly even are indeed entangled, that is they provide a wide class of examples of entangled states on the CAR algebra which can be directly constructed and investigated in detail.

## 2. PRELIMINARIES

For the convenience of the reader, we collect some preliminary facts needed in the sequel.

2.1. UMEGAKI CONDITIONAL EXPECTATIONS. By a (Umegaki) *conditional expectation* $E : \mathfrak{A} \to \mathfrak{B} \subset \mathfrak{A}$ we mean a norm one projection of the $C^*$-algebra $\mathfrak{A}$ onto the $C^*$-subalgebra (with the same identity $\mathbf{1}$) $\mathfrak{B}$. The case of interest for us is when $\mathfrak{A}$ is a full matrix algebra. Consider a set $\{P_i\}$ of central orthogonal projections of the range $\mathfrak{B}$ of $E$, summing up to the identity. We have

$$E(x) = \sum_i E(P_i x P_i) P_i \,. \tag{2.1}$$

Then $E$ is uniquely determined by its values on the reduced algebras $\mathfrak{A}_{P_i} := P_i \mathfrak{A} P_i$. When the above set $\{P_i\}$ consists of minimal projections, we get $\mathfrak{A}_{P_i} = N_i \otimes \overline{N}_i$, and there exist states $\phi_i$ on $\overline{N}_i$ such that

$$E(P_i(a \otimes \overline{a})P_i) = \phi_i(\overline{a}) P_i (a \otimes \mathbf{1}) P_i \,. \tag{2.2}$$

The reader is referred to [17] for further details.

2.2. QUASICONDITIONAL EXPECTATIONS. Consider a triplet $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$ of unital $C^*$-algebras. A *quasiconditional expectation* with respect to the given triplet, is a completely positive, identity preserving linear map $E : \mathfrak{A} \to \mathfrak{B}$ such that

$$E(ca) = cE(a) \,, \quad a \in \mathfrak{A} \,, \, c \in \mathfrak{C} \,.$$

Notice that, as the quasiconditional expectation $E$ is a real map, we have

$$E(ac) = E(a)c \,, \quad a \in \mathfrak{A} \,, \, c \in \mathfrak{C} \,.$$

If $\varphi$ is a normal faithful state on the $W^*$-algebra $\mathfrak{A}$, the $\varphi$-expectation $E^\varphi : \mathfrak{A} \to \mathfrak{B}$ by Accardi and Cecchini preserving the restriction of $\varphi$ to the $W^*$-subalgebra $\mathfrak{B}$, provides an example of quasiconditional expectation. Namely, it is enough to choose for $\mathfrak{C}$ any unital $C^*$-subalgebra of $\mathfrak{B}$ contained in the $E^\varphi$-fixed point algebra. $E^\varphi$ is a conditional expectation if and only if the modular group of $\varphi$ leaves globally stable the subalgebra $\mathfrak{B}$, see [2].

2.3. THE CAR ALGEBRA. Denote $[a,b] := ab - ba$, $\{a,b\} := ab + ba$ the commutator and anticommutator between elements $a$, $b$ of an algebra, respectively.

Let $J$ be a set. The *Canonical Anticommutation Relations* (CAR for short) algebra over $J$ is the $C^*$-algebra $\mathfrak{A}_J$ with the identity $\mathbf{1}$ generated by the set $\{a_j, a_j^\dagger\}_{j \in I}$, and the relations

$$(a_j)^* = a_j^\dagger \,, \; \{a_j^\dagger, a_k\} = \delta_{jk}\mathbf{1} \,, \; \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \,, \quad j,k \in J \,.$$

When there is no matter of confusion, we denote $\mathfrak{A}_J$ simply as $\mathfrak{A}$. The parity automorphism $\Theta$, of $\mathfrak{A}$ acts on the generators as

$$\Theta(a_j) = -a_j \,, \; \Theta(a_j^\dagger) = -a_j^\dagger \,, \quad j \in J \,,$$

and induces on $\mathfrak{A}$ the $\mathbb{Z}_2$-grading $\mathfrak{A} = \mathfrak{A}_+ \oplus \mathfrak{A}_-$ where

$$\mathfrak{A}_+ := \{a \in \mathfrak{A} : \Theta(a) = a\} \,, \quad \mathfrak{A}_- := \{a \in \mathfrak{A} : \Theta(a) = -a\} \,.$$

Elements in $\mathfrak{A}_+$ (respectively $\mathfrak{A}_-$) are called *even* (respectively *odd*).

A map $T : \mathfrak{A}^1 \to \mathfrak{A}^2$ between the $\mathbb{Z}_2$-graded algebras $\mathfrak{A}^1$, $\mathfrak{A}^2$ with $\mathbb{Z}_2$-gradings $\Theta_1$, $\Theta_2$ is said to be *even* if it is grading-equivariant:

$$T \circ \Theta_1 = \Theta_2 \circ T \,.$$

The previous definition applied to states $\varphi \in \mathcal{S}(\mathfrak{A})$ leads to $\varphi \circ \Theta = \varphi$, that is $\varphi$ is even if it is $\Theta$-invariant.

When the index set $J$ is countable, the CAR algebra is isomorphic to the $C^*$-infinite tensor product of $J$-copies of $\mathbb{M}_2(\mathbb{C})$:

$$\mathfrak{A}_J \sim \overline{\bigotimes_J \mathbb{M}_2(\mathbb{C})}^{C^*} . \tag{2.3}$$

For the convenience of the reader, we report the Jordan–Klein–Wigner transformation establishing the mentioned isomorphism. Fix any enumeration $j = 1, 2, \ldots$ of the set $J$. Let $U_j := a_j a_j^\dagger - a_j^\dagger a_j$, $j = 1, 2, \ldots$ . Put $V_0 := \mathbf{1}$, $V_j := \prod_{n=1}^{j} U_n$, and denote

$$\begin{aligned} &e_{11}(j) := a_j a_j^\dagger , \quad e_{12}(j) := V_{j-1} a_j , \\ &e_{21}(j) := V_{j-1} a_j^\dagger , \quad e_{22}(j) := a_j^\dagger a_j . \end{aligned} \tag{2.4}$$

$\{e_{kl}(j) : k, l = 1, 2\}_{j \in I}$ provides a system of commuting $2 \times 2$ matrix units realizing the mentioned isomorphism.

The CAR algebra $\mathfrak{A}_J$ has a unique tracial state $\tau$ as the extension of the unique tracial state on $\mathfrak{A}_I$, $|I| < +\infty$. Let $J_1 \subset J$ be a finite set and $\varphi \in \mathcal{S}(\mathfrak{A})$. Then there exists a unique positive element $T$ such that $\varphi\lceil_{\mathfrak{A}_{J_1}} = \tau\lceil_{\mathfrak{A}_{J_1}}(\,\cdot\, T)$. The element $T$ is called the *adjusted matrix* of $\varphi\lceil_{\mathfrak{A}_{J_1}}$. (For the standard applications to quantum statistical mechanics, we also use the density matrix with respect to the unnormalized trace, see Section 5.) The state $\varphi\lceil_{\mathfrak{A}_{J_1}}$ is even (faithful) if and only if its adjusted matrix is even (invertible). The reader is referred to Section XIV.1 of [23], and [9] for further details.

We end the present subsection by recalling the description of product state (cf. [9]), and the definition of entanglement (cf. Section 2 of [18]). Let $J_1, J_2 \subset I$ with $J_1 \cap J_2 = \emptyset$. Fix $\varphi_1 \in \mathcal{S}(\mathfrak{A}_{J_1})$, $\varphi_2 \in \mathcal{S}(\mathfrak{A}_{J_2})$. If at least one among them is even, then according to Theorem 11.2 of [9], the product state extension (called *product state* for short) $\varphi \in \mathcal{S}(\mathfrak{A}_{J_1 \cup J_2})$ is uniquely defined. We write with an abuse of notation, $\varphi = \varphi_1 \varphi_2$. Suppose that $J_1, J_2$ are finite sets. Let $T_1 \in \mathfrak{A}_{J_1}$, $T_2 \in \mathfrak{A}_{J_2}$ be the adjusted densities relative to $\varphi_1 \in \mathcal{S}(\mathfrak{A}_{J_1})$, $\varphi_2 \in \mathcal{S}(\mathfrak{A}_{J_2})$, respectively. If at least one among $T_1$ and $T_2$ is even, then $[T_1, T_2] = 0$ and $T := T_1 T_2$ is a well defined positive element of $\mathfrak{A}_{J_1 \cup J_2}$ which is precisely the density matrix of $\varphi = \varphi_1 \varphi_2$. $\varphi \in \mathcal{S}(\mathfrak{A}_{J_1 \cup J_2})$ is even if and only if $\varphi_1$ and $\varphi_2$ are both even.

A state $\varphi \in \mathcal{S}(\mathfrak{A}_{J_1 \cup J_2})$ is called *separable* (with respect to to the decomposition $\mathfrak{A}_{J_1 \cup J_2} = \overline{\mathfrak{A}_{J_1} \vee \mathfrak{A}_{J_2}}$) if it is in the closed convex hull of all the product states over $\mathfrak{A}_{J_1 \cup J_2}$. Otherwise it is called *entangled*.

2.4. PRELIMINARIES ON FERMI–MARKOV STATES. Let us start as in [3], with a totally ordered countable set $I$ containing, possibly a smallest element $j_-$ and/or a greatest element $j_+$. If $I$ contains neither $j_-$, nor $j_+$, then $I \sim \mathbb{Z}$. If only $j_+ \in$

$I$, then $I \sim \mathbb{Z}_-$, and if only $j_- \in I$, then $I \sim \mathbb{Z}_+$. Finally, if both $j_-$ and $j_+$ belong to $I$, then $I$ is a finite set and the analysis becomes easier. If $I$ is order isomorphic to $\mathbb{Z}$, $\mathbb{Z}_-$ or $\mathbb{Z}_+$, we put simbolically $j_-$ and/or $j_+$ equal to $-\infty$ and/or $+\infty$ respectively. In such a way, the objects with indices $j_-$ and $j_+$ will be missing in the computations.

Let $\mathfrak{A}_j$ be the CAR algebra generated by $d_j$ creators and annihilators

$$\{a_{j,1}, a^{\dagger}_{j,1}, a_{j,2}, a^{\dagger}_{j,2}, \dots, a_{j,d_j}, a^{\dagger}_{j,d_j}\}$$

localized on the site $j \in I$. The numbers of the $2d_j$ generators of $\mathfrak{A}_j$ may depend on $j$. We call the following the *Fermi algebra*:

$$\mathfrak{A} := \overline{\bigvee_{j\in I} \mathfrak{A}_j}^{C^*}. \tag{2.5}$$

Let $J := \bigcup_{j\in I}\{1,2,\dots,d_j\}$ be the disjoint union of the sets $\{1,2,\dots,d_j\}$, $j \in I$. Then the Fermi algebra $\mathfrak{A}$ given in (2.5) is nothing but the CAR algebra over the set $J$ previously described.

Now we pass to describe the local structure of the Fermi algebra $\mathfrak{A}$. For each $\Lambda \subset I$, the local algebra $\mathfrak{A}_\Lambda \subset \mathfrak{A}$ is defined as $\mathfrak{A}_\Lambda := \overline{\bigvee_{j\in\Lambda} \mathfrak{A}_j}$. According to this notation, $\mathfrak{A}_{\{j\}} = \mathfrak{A}_j$ and $\mathfrak{A}_I = \mathfrak{A}$. Then $\Lambda \subset I \mapsto \mathfrak{A}_\Lambda \subset \mathfrak{A}$ describes the local structure of the Fermi algebra. The $C^*$-algebra $\mathfrak{A}$ has the structure of a *quasi local algebra*, $\bigcup\{\mathfrak{A}_\Lambda : \Lambda \subset I\,,\ \text{finite}\}$ being the set of the *localized elements*. Particular subsets of $I$ are

$$[k,n] := \{l \in I : k \leqslant l \leqslant n\}\,, \quad n] := \{l \in I : l \leqslant n\}\,.$$

We put for $\Lambda \subset I$, $S_\Lambda := S\lceil_{\mathfrak{A}_\Lambda}$, $S$ being any map defined on $\mathfrak{A}$. The reader is referred to Section 2.6 of [11] and Section 4 of [9], for further details.

A state $\varphi \in \mathfrak{A}$ is said to be *locally faithful* if $\varphi_\Lambda$ is faithful whenever $\Lambda \subset I$ is finite.

If the number $d_j$ of the local generators depends on $j$ we refer to this situation as the *nonhomogeneous case*. Conversely, when $I = \mathbb{Z}$ and $d_j = d$, $j \in \mathbb{Z}$, the shift $j \to j+1$ acts in a natural way as an automorphism $\alpha$ of $\mathfrak{A}$. A state $\varphi \in \mathcal{S}(\mathfrak{A})$ is *translation invariant* if $\varphi \circ \alpha = \varphi$. If a state is translation invariant, then it is automatically even, see e.g. Example 5.2.21 of [11].

We specialize the definition of Markov states which parallels Definition 4.1 of [6].

DEFINITION 2.1. An even state $\varphi$ on $\mathfrak{A}$ is called a *Markov state* if, for each $n < j_+$, there exists an even quasi conditional expectation $E_n$ with respect to the triplet $\mathfrak{A}_{n-1]} \subset \mathfrak{A}_{n]} \subset \mathfrak{A}_{n+1]}$ satisfying

$$\varphi_{n]} \circ E_n = \varphi_{n+1]}\,, \quad E_n(\mathfrak{A}_{[n,n+1]}) \subset \mathfrak{A}_{\{n\}}\,. \tag{2.6}$$

Notice that the local structure $\Lambda \mapsto \mathfrak{A}_\Lambda$, $\Lambda$ finite subset of $I$, plays a crucial role in defining the Markov property. In fact, the isomorphism in (2.3) does not preserve neither the grading, nor the natural localization. (The algebra on the right hande side of (2.3) is naturally equipped with the trivial parity automorphism. Thus, its $\mathbb{Z}_2$-grading is trivial.) Hence, it does not intertwine the corresponding Markov states.

When the numbers $d_j$ of the generators of $\mathfrak{A}_j$ depend of the site, we call a Markov state $\varphi$ (or equally well a Markov measure in the Abelian case) a *non-homogeneous Markov state*. If $d_j = d$ for each $j$ we refer to the *d-Markov property*. Thus, *homogeneity* means $d$-Markov property for some $d$. For the applications to quantum statistical mechanics, $d_j$ is nothing but the "range of interaction" on the chain which might depend on the site. When $d = 1$ we speak of *nearest neighbor interaction*. The reader is referred to [3], [4], [20] and the literature cited therein, for the connection between the Markov property and the statistical mechanics, and for further details.

Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a locally faithful Markov state. Then the restriction $e_n := E_n\lceil_{\mathfrak{A}_{[n,n+1]}}$ is a completely positive identity preserving linear map $e_n : \mathfrak{A}_{[n,n+1]} \to \mathfrak{A}_{\{n\}} \subset \mathfrak{A}_{[n,n+1]}$ leaving invariant the faithful state $\varphi_{[n,n+1]}$. It is a quite standard fact (see e.g. [2]) that the ergodic average

$$\varepsilon_n := \lim_k \frac{1}{k} \sum_{h=0}^{k-1} (e_n)^h$$

exists and defines a conditional expectation

$$\varepsilon_n : \mathfrak{A}_{[n,n+1]} \to \mathcal{R}(\varepsilon_n) \subset \mathfrak{A}_{\{n\}}$$

projecting onto the fixed point algebra of $e_n$, the last coinciding with the range $\mathcal{R}(\varepsilon_n)$ of $\varepsilon_n$. The sequence $\{\varepsilon_n\}_{n<j_+}$ of two point conditional expectations is called in the sequel the sequence of *transition expectations* associated to the locally faithful Markov state $\varphi$. They uniquely determine, and are determined by the conditional expectations $\mathcal{E}_n : \mathfrak{A}_{n+1]} \to \mathfrak{A}_{n]}$, given for $x \in \mathfrak{A}_{n-1]}$, $y \in \mathfrak{A}_{[n,n+1]}$ by

$$\mathcal{E}_n(xy) = x\varepsilon_n(y)\,. \tag{2.7}$$

In addition, it is quite standard to verify (cf. Proposition 4.2 of [6]) that we can freely replace the quasiconditional expectation $E_n$ in Definition 2.1 with its ergodic average $\mathcal{E}_n$. For the convenience of the reader we report Proposition 4.3 of [6].

PROPOSITION 2.2. *Let $f : \mathfrak{A}_{[n,n+1]} \to \mathcal{R}(f) \subset \mathfrak{A}_{\{n\}}$ be an even conditional expectation. The formula*

$$\mathcal{F}(xy) := xf(y)\,, \quad x \in \mathfrak{A}_{n-1]}\,, \, y \in \mathfrak{A}_{[n,n+1]}$$

*uniquely defines an even conditional expectation*

$$\mathcal{F} : \mathfrak{A}_{n+1]} \to \mathfrak{A}_{n-1]} \vee \mathcal{R}(f) \subset \mathfrak{A}_{n]}\,.$$

From now on, we deal without further mention with even (quasi) conditional expectations. In addition, all the Markov states we deal with are even and locally faithful if it is not otherwise specified.

LEMMA 2.3. *Let $\mathcal{E} : \mathfrak{A}_{[k,l+1]} \to \mathcal{R}(\mathcal{E}) \subset \mathfrak{A}_{[k,l]}$ be a conditional expectation with $\mathfrak{A}_{[k,l-1]} \subset \mathcal{R}(\mathcal{E})$. Then $\mathcal{E}$ is faithful provided that $\mathcal{E}\lceil_{\mathfrak{A}_{[l,l+1]}}$ is faithful.*

*Proof.* Let $\varphi_1$, $\psi$ be faithful even states on $\mathfrak{A}_{[k,l-1]}$, $\mathfrak{A}_{[l,l+1]}$ respectively. The state $\varphi_2 := \psi \circ (\mathcal{E}\lceil_{\mathfrak{A}_{[l,l+1]}})$ on $\mathfrak{A}_{[l,l+1]}$ is even and faithful. Then the product state $\varphi := \varphi_1\varphi_2$ is a faithful state on $\mathfrak{A}_{[k,l+1]}$ left invariant by $\mathcal{E}$. Fix $a \in \mathfrak{A}_{[k,l+1]}$ with $\mathcal{E}(a^*a) = 0$. Then $\varphi(a^*a) = \varphi(\mathcal{E}(a^*a)) = 0$ which implies that $a = 0$ as $\varphi$ is faithful. Namely, $\mathcal{E}$ is faithful. ∎

We then pass to study the structure of the even conditional expectations

$$\varepsilon_n : \mathfrak{A}_{[n,n+1]} \to \mathcal{R}(\varepsilon_n) \subset \mathfrak{A}_{\{n\}}\,.$$

To shorten the notations, it is enough to consider the case when $n = 0$. After putting $\varepsilon := \varepsilon_0$, let us start with the finite set $\{P_j\}$ of the minimal projections of the centre $\mathcal{Z}(\mathcal{R}(\varepsilon))$ of $\mathcal{R}(\varepsilon)$.

LEMMA 2.4. *The parity automorphism $\Theta$ acts on $\mathcal{Z}(\mathcal{R}(\varepsilon))$, and the orbits of minimal projections consist of one or two elements.*

*Proof.* Let $P_j$ be a minimal projection of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. As $\varepsilon$ is even and $\Theta^2 = \mathrm{id}$, we have that $\Theta(P_j)$ is a minimal projection of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. This means that either $\Theta(P_j) = P_j$, or $\Theta(P_j)$ is orthogonal to $P_j$. The latter means that the orbit of $\Theta(P_j)$ consists of two elements. ∎

We showed in [6] that there are interesting examples with $\Theta(P_j) \neq P_j$. Let $\varepsilon$ be as above. Some useful properties of the pieces $\varepsilon(PxP)P$, $P$ being an even projection of the centre of $\mathcal{R}(\varepsilon)$, minimal among the invariant ones, are described below.

LEMMA 2.5. *Let $M = M_+ \oplus M_-$ be a $\mathbb{Z}_2$-graded full matrix algebra. If $x \in M_-$ commutes with $M_+$, then $x = 0$.*

*Proof.* Let the $\mathbb{Z}_2$-grading be implemented by the automorphism $\Theta$. As $\Theta$ is inner, there exists an even selfadjoint unitary $V \in M$, uniquely determined up to a sign, implementing $\Theta$ on $M$, see Corollary 8.11 of [22]. This means that $M_+ = A'$, $A$ being the Abelian algebra generated by $V$, and the commutant is taken in the full matrix algebra $M$. As $x \in (M_+)'$, $x \in A'' \equiv A$. As $x$ is odd, we have $VxV = -x$. Collecting together, we obtain $x = 0$. ∎

LEMMA 2.6. *Let $M = M_+ \oplus M_-$ be a $\mathbb{Z}_2$-graded full matrix algebra. For every $\Theta$-invariant full matrix subalgebra $N \subset M$, there exists a unique $\Theta$-invariant full matrix subalgebra $\overline{N} \subset M$ such that $M = N \vee \overline{N}$, and*

$$x\overline{x} + \sigma(x,\overline{x})\overline{x}x = 0\,, \quad x \in N_\pm\,, \overline{x} \in \overline{N}_\pm \tag{2.8}$$

*where $\sigma(x,\overline{x})$ is* 1 *if both $x$, $\overline{x}$ are odd, and* $-1$ *in the remaining cases. Moreover, we have* $N \wedge \overline{N} = \mathbb{C}\mathbf{1}$.

*Proof.* Let $\widetilde{N} := N' \wedge M$ which is a $\Theta$-invariant full matrix subalgebra of $M$ as well. Fix a (even) unitary $V \in N$ uniquely determined up to a sign, implementing $\Theta$ on $N$. Define for $x = x_+ + x_- \in \widetilde{N}$,

$$\beta(x) := x_+ + Vx_- \,. \tag{2.9}$$

It is easy to see that $\beta$ defines a $*$-algebra isomorphism between $\widetilde{N}$ and $\beta(\widetilde{N})$. Thus, the full matrix algebra $\overline{N} := \beta(\widetilde{N})$ is the algebra we are looking for.

For the uniqueness, let $\overline{R} \subset M$ be a $\Theta$-invariant full matrix algebra fulfilling the commutation relations in (2.8) whenever $x \in N_\pm$, $\overline{x} \in \overline{R}_\pm$, such that $N \vee \overline{R} = M$. Then it is easy to verify that $\widetilde{R} := \{x_+ + Vx_- \ : \ x \in \overline{R}\}$ is a full matrix subalgebra of $\widetilde{N}$. Since $M = N \vee \widetilde{N} \sim N \otimes \widetilde{N}$, we get that $\widetilde{R}$ must coincide with $\widetilde{N}$ which implies $\overline{R} = \overline{N}$. ∎

We call the algebra

$$\overline{N} := (N' \wedge M)_+ + V(N' \wedge M)_- \tag{2.10}$$

obtained in Lemma 2.6, the *Fermion complement* of $N$ in $M$.

PROPOSITION 2.7. *Let $\mathfrak{A} := \bigvee_{j\in I} \mathfrak{A}_j$ be the Fermi $C^*$-algebra with $I = \{0,1\}$.*

(i) *Let $P \in \mathfrak{A}_{\{0\}}$ be a $\Theta$-invariant projection. Then there is a one–to–one correspondence between:*

(a) *$\varepsilon : P\mathfrak{A}P \to P\mathfrak{A}P$ an even conditional expectation such that $\mathcal{R}(\varepsilon)$ is a full matrix subalgebra of $P\mathfrak{A}_{\{0\}}P$;*

(b) *$N \subset P\mathfrak{A}_{\{0\}}P$ a $\Theta$-invariant full matrix subalgebra and $\Phi$ an even state on $\overline{N} \vee P\mathfrak{A}_{\{1\}}P$.*

*The correspondence is given for $x \in N$, $y \in \overline{N} \vee P\mathfrak{A}_{\{1\}}P$ by*

$$\varepsilon(xy) = \Phi(y)x \tag{2.11}$$

*where $\overline{N}$ is the Fermion complement of $N$ in $P\mathfrak{A}_{\{0\}}P$ given in* (2.10). *In particular, $\mathcal{R}(\varepsilon) = N$.*

(ii) *Let $P_1, P_2 \in \mathfrak{A}_{\{0\}}$ such that $\Theta(P_1) = P_2$, $P_1P_2 = 0$. Then there is a one-to-one correspondence between:*

(a) *$\varepsilon : (P_1 + P_2)\mathfrak{A}(P_1 + P_2) \to (P_1 + P_2)\mathfrak{A}(P_1 + P_2)$ an even conditional expectation such that $\mathcal{R}(\varepsilon) \subset \mathfrak{A}_{\{0\}}$ and $\mathcal{Z}(\mathcal{R}(\varepsilon)) = \mathbb{C}P_1 \oplus \mathbb{C}P_2$;*

(b) *$N_1 \subset P_1\mathfrak{A}_{\{0\}}P_1$ full matrix algebra and $\Phi$ a state on $M_1 := N_1' \wedge P_1\mathfrak{A}P_1$.*

*The correspondence is given for $x_i \in N_i$, $y_i \in M_i$, $i = 1,2$,*

$$\varepsilon(x_1y_1 + x_2y_2) = \Phi(y_1)x_1 + \Phi(\Theta(y_2))x_2 \tag{2.12}$$

*where $N_2 := \Theta(N_1)$, $M_2 := \Theta(M_1)$. In particular, $\mathcal{R}(\varepsilon) = N_1 \oplus \Theta(N_1)$.*

*In addition, if $z \in \mathfrak{A}_{\{1\}}$ is even, then*

$$\varepsilon((P_1+P_2)z)=\Phi(P_1zP_1)(P_1+P_2)\,. \tag{2.13}$$

*Proof.* (i) Let $N:=\mathcal{R}(\varepsilon)$. As $\varepsilon$ is even, $N$ is a $\Theta$-invariant full matrix algebra of $P\mathfrak{A}_{\{0\}}P$. Let $\overline{N}$ be the Fermion complement of $N$ in $P\mathfrak{A}_{\{0\}}P$, and $y\in\overline{N}\vee P\mathfrak{A}_{\{1\}}P$ an odd element. Then $\varepsilon(y)\in N$ is odd too, and by the bimodule property of $\varepsilon$, $[\varepsilon(y),N_+]=0$. By Lemma 2.5, $\varepsilon(y)=0$. If $x\in N$, $y\in\overline{N}\vee\mathfrak{A}_{\{1\}}$, we have

$$x\varepsilon(y)=x\varepsilon(y_+)=\varepsilon(xy_+)=\varepsilon(y_+x)=\varepsilon(y_+)x=\varepsilon(y)x\,.$$

This means that $\varepsilon(y)\in\mathcal{Z}(N)\equiv\mathbb{C}P$, that is $\varepsilon(xy)=\Phi(y)x$ for a uniquely determined even state $\Phi$ on $\overline{N}\vee P\mathfrak{A}_{\{1\}}P$.

Fix now an invariant full matrix subalgebra $N$ of $P\mathfrak{A}_{\{0\}}P$. By uniqueness, the Fermion complement of $N$ in $P\mathfrak{A}P$ is all of $\overline{N}\vee P\mathfrak{A}_{\{1\}}P$. Thus, in order to shorten the notations, we can suppose that $\overline{N}$ is the Fermion complement of $N$ in $P\mathfrak{A}P$. Thus,

$$P\mathfrak{A}P=N\vee\beta^{-1}(\overline{N})\sim N\otimes\beta^{-1}(\overline{N})\,,$$

where $\beta:\widetilde{N}\to\overline{N}$ is the isomorphism given in (2.9). Define $\varepsilon:=E_N^{\Phi\circ\beta}$ as the Fubini mapping given in 9.8.4 of [21]. Let now $x\in N$, $y\in\overline{N}$. We get

$$\begin{aligned}\varepsilon(xy)&=\varepsilon(x(y_++y_-))=\varepsilon(xy_+)+\varepsilon[(xV)(Vy_-)]\\&=\Phi(y_+)x+\Phi(\beta(Vy_-))xV=\Phi(y_+)x+\Phi(y_-)xV\\&=\Phi(y_+)x=\Phi(y_+)x+\Phi(y_-)x=\Phi(y)x\end{aligned}$$

as, being $\Phi$ even, it is zero on the odd part of $\overline{N}$.
(ii) Take $N_i:=P_i\mathcal{R}(\varepsilon)P_i$, $M_i:=P_i(\mathcal{R}(\varepsilon)'\wedge\mathfrak{A})P_i$, $i=1,2$. As $\varepsilon$ is even, we have

$$\Theta(N_1)=N_2\,,\quad\Theta(M_1)=M_2\,,\quad\Theta(P_1\mathfrak{A}P_1)=P_2\mathfrak{A}P_2\,,$$

and

$$P_1\mathfrak{A}P_1+P_2\mathfrak{A}P_2=N_1\vee M_1+N_2\vee M_2\sim N_1\otimes M_1\oplus N_2\otimes M_2\,.$$

As $\varepsilon$ is uniquely determined by the restriction on the reduced algebras $\mathfrak{A}_{P_i}$, $i=1,2$, according to (2.1) and (2.2), there exist uniquely determined states $\varphi_i$ on $M_i$, such that

$$\varepsilon(x_1y_1+x_2y_2)=\varphi_1(y_1)x_1+\varphi_2(y_2)x_2$$

whenever $x_i\in N_i$, $y_i\in M_i$, $i=1,2$. Thus, it is enough to show that $\varphi_2=\varphi_1\circ\Theta$. We compute

$$\varepsilon(\Theta(x_1y_1+x_2y_2))=\varphi_1(\Theta(y_2))\Theta(x_2)+\varphi_2(\Theta(y_1))\Theta(x_1)\,,$$

and

$$\Theta(\varepsilon(x_1y_1+x_2y_2))=\varphi_2(y_2)\Theta(x_2)+\varphi_1(y_1)\Theta(x_1)\,.$$

Thanks to the $\Theta$-equivariance of $\varepsilon$, we conclude that $\varphi_2=\varphi_1\circ\Theta$ and vice versa.

Finally, if $z\in\mathfrak{A}_{\{1\}}$ is even, then

$$P_izP_i\equiv P_iz\in P_i(\mathcal{R}(\varepsilon)'\wedge\mathfrak{A}_{[0,1]})P_i=M_i\,,\quad i=1,2\,.$$

By the first part, we get

$$\begin{aligned}\varepsilon((P_1+P_2)z) &= \Phi(P_1zP_1)P_1+\Phi(\Theta(P_2zP_2))P_2\\ &= \Phi(P_1zP_1)P_1+\Phi(P_1zP_1)P_2=\Phi(P_1zP_1)(P_1+P_2)\,. \qquad \blacksquare\end{aligned}$$

The previous results relative to the action of the grading automorphism on the centers of the transition expectations allow us to provide the definition of the *strongly even* and *minimal* Markov states.

We start by noticing that Definition 2.1 can be slightly generalized by simply requiring that the local subalgebras $\mathfrak{A}_j$ appearing in (2.5) are full matrix $C^*$-algebras such that the grading automorphism $\Theta$ leaves each algebra $\mathfrak{A}_j$ globally stable. In this case, the Markov property is still described by the transition expectations $\varepsilon_n$ previously described, and Lemma 2.4 still works. Thus, the following definition is meaningful.

DEFINITION 2.8. Let $\varphi \in \mathcal{S}(\mathfrak{B})$ be a Markov state on the $\mathbb{Z}_2$-graded quasi local $C^*$-algebra

$$\mathfrak{B} := \overline{\bigvee_{j\in I}\mathfrak{B}_j}^{C^*}\,.$$

It is called *strongly even* (respectively *minimal*) if the parity automorphism $\Theta$ acts trivially (respectively transitively) on each $\mathcal{Z}(\mathcal{R}(\varepsilon_n))$, $\{\varepsilon_n\}_{n<j_+}$ being the transition expectations canonically associated to $\varphi$ through Proposition 2.2.

For some interesting applications (see e.g. Corollary 4.3), it is enough to consider a Markov state as strongly even if $\Theta$ acts trivially on the centers of the transition expectations, infinitely often. Then a Markov state $\varphi$ will be *non-strongly even* if there exists $k\in\mathbb{N}$ such that the action of $\Theta$ on $\mathcal{Z}(\mathcal{R}(\varepsilon_n))$ is nontrivial either for each $n>k$, or for each $n<-k$.

## 3. THE STRUCTURE OF GENERAL FERMI–MARKOV STATES

In the present section we investigate the structure of Fermi–Markov states. We follow Section 3 of [3], where we dealt with the quasi local algebra

$$\mathfrak{A} = \overline{\bigotimes_{j\in I}\mathbb{M}_{d_j}(\mathbb{C})}^{C^*}\,,$$

equipped with the local structure $\mathfrak{A}_\Lambda = \bigotimes_{j\in\Lambda}\mathbb{M}_{d_j}(\mathbb{C})$, $\Lambda\subset I$ finite, and trivial $\mathbb{Z}_2$-grading. The forthcoming analysis also represents the extension to the most general Fermi algebra of the results in Section 5 of [6], where only the homogeneous situation $\mathfrak{A} := \overline{\bigvee_I \mathbb{M}_2(\mathbb{C})}^{C^*}$, and only the strongly even Markov states were considered.

The program in [3] cannot be directly implemented in this situation. In fact the parity automorphism $\Theta$ does not act trivially on the centres of $\mathcal{Z}(\mathcal{R}(\varepsilon_j))$ in general. Thus, the minimal projections of the centers $\mathcal{Z}(\mathcal{R}(\varepsilon_j))$ of the ranges $\mathcal{R}(\varepsilon_j)$ does not generate an Abelian algebra. Yet, we are able to decompose non-homogeneous Markov states on the Fermi algebras into minimal ones (cf. Definition 2.8).

Let $\varphi$ be a Markov state, together with the sequence $\{\varepsilon_j\}_{j<j_+}$ of transition expectations canonically associated to $\varphi$ as previously explained. We start by considering the centre $Z_j$ of $\mathcal{R}(\varepsilon_j)$, together with the generating family $\{P^j_{\gamma_j}\}_{\gamma_j\in\Gamma_j}$ of minimal projections. Define $\Omega_j := \Gamma_j/\sim$ where "$\sim$" stands for the equivalence relation induced by $\Theta$ on the spectrum $\Gamma_j$ of $Z_j$. Let $p_j : \Gamma_j \to \Omega_j$ be the corresponding canonical projection. Put

$$Q^j_{\omega_j} := \bigvee_{\gamma_j=p_j^{-1}(\{\omega_j\})} P^j_{\gamma_j}\,. \tag{3.1}$$

For $j < j_+$, denote $C_j \subset Z_j$ the subalgebra generated by $\{Q^j_{\omega_j} : \omega_j \in \Omega_j\}$. Notice that $\mathrm{spec}(C_j) = \Omega_j$. The $C_j$ generate an Abelian subalgebra of $\mathfrak{A}$ whose spectrum is precisely

$$\Omega := \prod_{j\in I}\Omega_j \equiv \prod_{j\in I}\mathrm{spec}(C_j)\,, \tag{3.2}$$

where the product in (3.2) stands for the topological product of the finite sets $\Omega_j$. In order to simplify the notations, we define $\varepsilon_{j_+} := \mathrm{id}_{\mathfrak{A}_{j_+}}$. This means $\Omega_{j_+} := \{j_+\}$, $Q_{j_+} \equiv P_{j_+} := \mathbf{1}$, and finally, for $N_{j_+}$, $\overline{N}_{j_+}$ given in Proposition 2.7, $N_{j_+} := \mathfrak{A}_{\{j_+\}}$, $\overline{N}_{j_+} := \mathbb{C}\mathbf{1}$ with an obvious meaning. Put

$$B_j := \bigoplus_{\omega_j\in\Omega_j} Q^j_{\omega_j}\mathfrak{A}_{\{j\}}Q^j_{\omega_j} \subset \mathfrak{A}_{\{j\}}\,, \quad \text{and} \quad \mathfrak{B} := \overline{\bigvee_{j\in I} B_j} \subset \mathfrak{A}\,.$$

The next step is to construct a conditional expectation of $\mathfrak{A}$ onto $\mathfrak{B}$. Thanks to the fact that the $Q^j_{\omega_j}$ are even and thus mutually commuting, we have for each $x \in \mathfrak{A}_{[k,l]}$,

$$\begin{aligned}\sum_{\omega_{k-1},\omega_k,\dots,\omega_l,\omega_{l+1}} &(Q^{k-1}_{\omega_{k-1}}Q^k_{\omega_k}\cdots Q^l_{\omega_l}Q^{l+1}_{\omega_{l+1}})x(Q^{k-1}_{\omega_{k-1}}Q^k_{\omega_k}\cdots Q^l_{\omega_l}Q^{l+1}_{\omega_{l+1}})\\
&=\sum_{\omega_{k-1}}Q^{k-1}_{\omega_{k-1}}\sum_{\omega_{l+1}}Q^k_{\omega_{l+1}}\sum_{\omega_k,\dots,\omega_l}(Q^k_{\omega_k}\cdots Q^l_{\omega_l})x(Q^k_{\omega_k}\cdots Q^l_{\omega_l})\\
&=\sum_{\omega_k,\dots,\omega_l}(Q^k_{\omega_k}\cdots Q^l_{\omega_l})x(Q^k_{\omega_k}\cdots Q^l_{\omega_l})\,.\end{aligned}$$

Moreover, if $x = x_k x_{k+1} \cdots x_l$, then

$$\sum_{\omega_k,\omega_{k+1},\dots,\omega_l,\omega_l} (Q^k_{\omega_k} Q^{k+1}_{\omega_{k+1}} \cdots Q^l_{\omega_l}) x (Q^k_{\omega_k} Q^{k+1}_{\omega_{k+1}} \cdots Q^l_{\omega_l})$$
$$= \sum_{\omega_k,\omega_{k+1},\dots,\omega_l} (Q^k_{\omega_k} x_k Q^k_{\omega_k})(Q^{k+1}_{\omega_{k+1}} x_{k+1} Q^{k+1}_{\omega_{k+1}}) \cdots (Q^l_{\omega_l} x_l Q^l_{\omega_l}) .$$

Thus, on the dense subalgebra $\mathcal{A} := \bigcup_{\Lambda \subset I} \mathfrak{A}_\Lambda$, $\Lambda$ finite, we get a norm one projection $E : \mathcal{A} \to \mathfrak{B}$, given on the algebraic generators of $\mathcal{A}$ by

$$E(x_{j_1} \cdots x_{j_n}) = \sum_{\omega_{j_1},\dots,\omega_{j_n}} (Q^{j_1}_{\omega_{j_1}} x_{j_1} Q^{j_1}_{\omega_{j_1}}) \cdots (Q^{j_n}_{\omega_{j_n}} x_{j_n} Q^{j_n}_{\omega_{j_n}}) \tag{3.3}$$

which uniquely extends to a conditional expectation (denoted again by $E$ by an abuse of notations) $E : \mathfrak{A} \to \mathfrak{B}$ of $\mathfrak{A}$ onto $\mathfrak{B}$. It is also a quite standard fact to see that

$$\varphi = \varphi \circ E \equiv \varphi\lceil_{\mathfrak{B}} \circ E .$$

By taking into account the previous considerations we can investigate the structure of Fermi–Markov states following the lines in [3]. We recover the following objects canonically associated to the Markov state $\varphi$ under consideration.

(a) A classical Markov process on the compact space $\Omega$ given in (3.2), whose law $\mu$ is uniquely determined by the sequences of compatible distributions and transition probabilities at the place $j$ given respectively by

$$\pi^j_{\omega_j} := \varphi(Q^j_{\omega_j}) , \quad j < j_+; \quad \pi^j_{\omega_j,\omega_{j+1}} := \frac{\varphi(\varepsilon_j(Q^j_{\omega_j} Q^{j+1}_{\omega_{j+1}}))}{\varphi(Q^j_{\omega_j})} , \quad j < j_+ . \tag{3.4}$$

(b) For each trajectory $\omega \equiv (\dots, \omega_{j-1}, \omega_j, \omega_{j+1}, \dots) \in \Omega$, the $C^*$-algebra $\mathfrak{B}^\omega$ given by

$$\mathfrak{B}^\omega := \overline{\bigvee_{j\in I} Q^j_{\omega_j} \mathfrak{A}_{\{j\}} Q^j_{\omega_j}}^{C^*} . \tag{3.5}$$

Notice that, in the non trivial cases (i.e. when $I$ is infinite), $\mathfrak{B}^\omega$ cannot be viewed in a canonical way as a subalgebra of $\mathfrak{A}$. Yet, whenever $\Lambda \subset I$ is finite,

$$\mathfrak{B}^\omega_\Lambda := \bigvee_{j\in\Lambda} Q^j_{\omega_j} \mathfrak{A}_{\{j\}} Q^j_{\omega_j}$$

is a subalgebra of $\mathfrak{A}_\Lambda$ with the identity the projection $\bigvee_{j\in\Lambda} Q^j_{\omega_j}$. Namely, $\mathfrak{B}^\omega$ is equipped with a canonical localization $\{\mathfrak{B}^\omega_\Lambda : \Lambda \text{ finite subset of } I\}$, and a $\mathbb{Z}_2$-grading implemented by the automorphism $\Theta^\omega$ arising from the restrictions $\Theta\lceil_{\mathfrak{A}_\Lambda}$.

(c) A completely positive identity preserving map $E^\omega : \mathfrak{A} \to \mathfrak{B}^\omega$, which is uniquely determined as in (3.3) by

$$x_{j_1}x_{j_2}\cdots x_{j_n} \in \mathfrak{A} \mapsto (Q^{j_1}_{\omega_{j_1}}x_{j_1}Q^{j_1}_{\omega_{j_1}})(Q^{j_2}_{\omega_{j_2}}x_{j_2}Q^{j_2}_{\omega_{j_2}})\cdots(Q^{j_n}_{\omega_{j_n}}x_{j_n}Q^{j_n}_{\omega_{j_n}})\,. \tag{3.6}$$

The above map satisfies $E^\omega \circ E = E^\omega$.

(d) A sequence $\{\mathcal{E}^\omega_j\}_{j\in I}$ of maps

$$\mathcal{E}^\omega_j : \mathfrak{B}^\omega_{j+1]} \to \mathfrak{B}^\omega_{j]}$$

given, for $x \in \mathfrak{B}^\omega_{j-1]}$, $y \in \mathfrak{B}^\omega_{[j,j+1]}$ by

$$\mathcal{E}^\omega_j(xy) := \frac{x\varepsilon_j(y)}{\pi^j_{\omega_j,\omega_{j+1}}}\,.$$

PROPOSITION 3.1. *The maps $\mathcal{E}^\omega_j$ are even conditional expectations.*

*Proof.* As for each $k \leqslant j$,

$$\mathfrak{B}^\omega_{[k,j+1]} = \Big(\prod_{l=k}^{j+1} Q^l_{\omega_l}\Big)\mathfrak{A}_{[k,j+1]}\Big(\prod_{l=k}^{j+1} Q^l_{\omega_l}\Big) \subset \mathfrak{A}_{[k,j+1]}\,,$$

and

$$\mathcal{E}^\omega_j\lceil_{\mathfrak{B}^\omega_{[k,j+1]}} = \frac{\mathcal{E}_j\lceil_{\mathfrak{B}^\omega_{[k,j+1]}}}{\pi^j_{\omega_j,\omega_{j+1}}}\,.$$

Thanks to Proposition 2.2, $\mathcal{E}^\omega_j$ is an even conditional expectation, provided that it is identity preserving. This means that we must check $\mathcal{E}^\omega_j(Q^j_{\omega_j}Q^{j+1}_{\omega_{j+1}}) = Q^j_{\omega_j}$. But, we have by (2.13) that $\varepsilon_j(Q^j_{\omega_j}Q^{j+1}_{\omega_{j+1}}) = cQ^j_{\omega_j}$. The proof follows as the number $c$ is precisely $\pi^j_{\omega_j,\omega_{j+1}}$. ∎

(e) The state $\psi^\omega \in \mathcal{S}(\mathfrak{B}^\omega)$, uniquely determined on localized elements by

$$\psi^\omega := \lim_{k\downarrow j_-,\, l\uparrow j_+} \frac{\varphi\lceil_{Q^k_{\omega_k}\mathfrak{A}_{\{k\}}Q^k_{\omega_k}} \circ \mathcal{E}^\omega_k \circ \cdots \circ \mathcal{E}^\omega_l}{\pi^k_{\omega_k}}\,. \tag{3.7}$$

It is straightforward to check that the state $\psi^\omega$ is a minimal Markov state on $\mathfrak{B}^\omega$ with respect to the conditional expectations

$$\widetilde{\mathcal{E}}^\omega_j := \mathcal{E}^\omega_j \circ \mathcal{E}^\omega_{j+1}\,.$$

In addition, the following field is $\sigma(\mathfrak{A}^*,\mathfrak{A})$-measurable:

$$\omega \in \Omega \mapsto \psi^\omega \circ E^\omega \in \mathcal{S}(\mathfrak{A}).$$

THEOREM 3.2. *Let $\varphi$ be a Markov state on the Fermi algebra $\mathfrak{A}$ with respect to the associated sequence $\{\mathcal{E}_j\}_{j_-\leqslant j<j_+}$ of conditional expectations given in* (2.7). *Define the compact set $\Omega$ by* (3.2)*, the probability measure $\mu$ on $\Omega$ by* (3.4)*, the quasi local algebra $\mathfrak{B}^\omega$ by* (3.5)*, the map $E^\omega$ by* (3.6)*, the state $\psi^\omega$ on $\mathfrak{B}^\omega$ by* (3.7).

*Then $\varphi$ admits the integral decomposition*

$$\varphi(A) = \int_\Omega^\oplus \psi^\omega \circ E^\omega(A)\mu(\mathrm{d}\omega)\,, \quad A \in \mathfrak{A}\,. \tag{3.8}$$

*Proof.* We outline the proof which is similar to that of Theorem 3.2 of [3] after writing down the corresponding objects relative to the Fermi case.

Consider the Abelian $C^*$-subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ given by

$$\mathfrak{C} := \overline{\bigvee_{j\in I} C^j} \sim \overline{\bigotimes_{j\in I} C^j}^{C^*}\,,$$

together its spectrum $\operatorname{spec}(\mathfrak{C}) = \Omega$. By restricting $\varphi$ to $\mathfrak{C}$, we obtain a possibly nonhomogeneous Markov random process on $\Omega$ with law $\mu$ described above. Let $\pi$ be the GNS representation of $\mathfrak{B}$ relative to $\varphi\lceil_{\mathfrak{B}}$. Then $L^\infty(\Omega,\mu) \sim \pi(\mathfrak{C})'' \subset \pi(\mathfrak{B})' \cap \pi(\mathfrak{B})''$. Thus, we have for $\pi$ the direct integral decomposition

$$\pi = \int_\Omega^\oplus \pi_\omega \mu(\mathrm{d}\omega)\,,$$

where $\omega \mapsto \pi_\omega$ is a measurable field of representations of $\mathfrak{B}$. This leads to the direct integral decomposition of $\varphi\lceil_{\mathfrak{B}}$, and then the decomposition of $\varphi \equiv \varphi\lceil_{\mathfrak{B}} \circ E$ as $\varphi = \int_\Omega \varphi_\omega \mu(\mathrm{d}\omega)$, see e.g. Section IV of [23]. It is then straightforward to see that

$$\varphi_\omega(A) = \psi^\omega(E^\omega(A))\,,$$

almost everywhere on $\Omega$ for each $A \in \mathfrak{A}$. ∎

The constructive part of Proposition 2.7 allows us to provide the following reconstruction theorem for the class of Fermi–Markov states considered in the sequel. It parallels the analogous one (cf. Theorem 3.3 of [3]).

Let $\mathfrak{A}$ be a Fermi algebra. Take for every $j < j_+$, a $\Theta$-invariant commutative subalgebra $Z_j$ of $\mathfrak{A}_{\{j\}}$ with spectrum $\Gamma_j$ and generators $\{P^j_{\gamma_j}\}_{\gamma_j\in\Gamma_j}$. Put $Z_{j_+} := \mathbb{C}\mathbf{1}$. Let "$\sim$" be the equivalence relation on the $\Gamma_j$ induced by the action of $\Theta$, and $p_j$ the corresponding projection map. Set $\Omega_j := \Gamma_j/\sim$, and define $Q^j_{\omega_j}$ as in (3.1). Choose a full matrix subalgebra $N^j_{\gamma_j} \subset P^j_{\gamma_j}\mathfrak{A}_{\{j\}}P^j_{\gamma_j}$ which is $\Theta$-invariant whenever $P^j_{\gamma_j}$ is a fixed point of $\Theta$. (Notice that $\overline{N}^j_{\gamma_j}$ given in Proposition 2.7 is also left globally invariant under the parity.) Form for $j < j_+$, the two point even, faithful conditional expectations

$$\varepsilon^{j}_{\omega_j,\omega_{j+1}} : Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j} \vee Q^{j+1}_{\omega_{j+1}}\mathfrak{A}_{\{j+1\}}Q^{j+1}_{\omega_{j+1}} \to \bigoplus_{\gamma_j=p_j^{-1}(\{\omega_j\})} N^{j}_{\gamma_j} \subset Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}\,,$$

$$\varepsilon^{j_+-1}_{\omega_{j_+-1},j_+} : Q^{j_+-1}_{\omega_{j_+-1}}\mathfrak{A}_{\{j_+-1\}}Q^{j_+-1}_{\omega_{j_+-1}} \vee \mathfrak{A}_{\{j_+\}} \to \bigoplus_{\gamma_{j_+-1}=p_j^{-1}(\{\omega_{j_+-1}\})} N^{j_+-1}_{\gamma_{j_+-1}} \subset Q^{j_+-1}_{\omega_{j_+-1}}\mathfrak{A}_{\{j_+-1\}}Q^{j}_{\omega_{j_+-1}}\,,$$

according to Proposition 2.7, by taking for the states in (2.11), (2.12), faithful ones. Define $\mathfrak{B}^{\omega}$, $E^{\omega}$ as in (3.5), (3.6) respectively. For the trajectory

$$\omega = (\dots,\omega_{j-1},\omega_j,\omega_{j+1},\dots)\,,$$

and $j < j_+$, define the map $\mathcal{E}^{\omega}_j$ as

$$\mathcal{E}^{\omega}_j(xy) := x\varepsilon^{j}_{\omega_j,\omega_{j+1}}(y)\,,$$

which is an even faithful conditional expectation according to Proposition 2.2, and Lemma 2.3. Take, for $j < j_+$, a compatible sequence of even faithful states $\varphi^{\omega}_j$ on $Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}$. (It can be shown by a standard compactness property (cf. Proposition 5.1 of [5]), that the set of sequences of even compatible states $\varphi^{\omega}_j$, that is such that $\varphi^{\omega}_{j+1} = \varphi^{\omega}_j \circ \mathcal{E}^{\omega}_j\lceil_{\mathfrak{B}^{\omega}_{\{j+1\}}}$, is nonvoid.) Form the state $\psi^{\omega} \in \mathcal{S}(\mathfrak{B}_{\omega})$ as in (3.7) by taking as initial distributions the $\varphi^{\omega}_j$. Finally, fix a Markov process on the product space $\Omega := \prod_{j\in I}\Omega_j$ with law $\mu$ determined, for $\omega_j \in \Omega_j$, $\omega_{j+1} \in \Omega_{j+1}$, by the marginal distributions $\pi^{j}_{\omega_j} > 0$, and transition probabilities $\pi^{j}_{\omega_j,\omega_{j+1}} > 0$.

THEOREM 3.3. *In the above notations, the state $\varphi$ on $\mathfrak{A}$ given by*

$$\varphi := \int_{\Omega} \psi^{\omega} \circ E^{\omega}\mu(\mathrm{d}\omega)$$

*is a Markov state with respect to the sequence $\{\mathcal{E}_j\}_{j<j_+}$ of conditional expectations uniquely determined (with the convention $\mathfrak{A}_{\{j_--1\}} = \mathbb{C}\mathbf{1}$) for $a \in \mathfrak{A}_{j-1]}$, $x \in \mathfrak{A}_{\{j\}}$, $y \in \mathfrak{A}_{\{j+1\}}$ by*

$$\mathcal{E}_j(axy) = a\sum_{\omega_j,\omega_{j+1},\omega_{j+2}} \pi^{j}_{\omega_j}\pi^{j}_{\omega_j,\omega_{j+1}}\pi^{j+1}_{\omega_{j+1},\omega_{j+2}}\mathcal{E}^{\omega}_j(Q^{j}_{\omega_j}x\mathcal{E}^{\omega}_{j+1}(Q^{j+1}_{\omega_{j+1}}yQ^{j+1}_{\omega_{j+1}}Q^{j+2}_{\omega_{j+2}})Q^{j}_{\omega_j})\,,\qquad j \leqslant j_+ - 2\,;$$

$$\mathcal{E}_j(axy) = a\sum_{\omega_j}\pi^{j}_{\omega_j}\mathcal{E}^{\omega}_j(Q^{j}_{\omega_j}xQ^{j}_{\omega_j}y)\,,\quad j = j_+ - 1\,.$$

*Proof.* A straightforward computation shows that, for all generators of the form $x_k\cdots x_l \in \mathfrak{A}_{[k,l]}$, $\varphi$ satisfies (2.6), for the sequence of conditional expectations constructed as above (cf. Theorem 4.1 of [3]). The proof follows as the state $\varphi$ is locally faithful, by taking into account Lemma 2.3. ∎

## 4. GENERAL PROPERTIES OF FERMI–MARKOV STATES

Let $\varphi \in \mathcal{S}(\mathfrak{A})$, and $D_{[k,l]}$ be the adjusted density matrix of the restriction $\varphi_{[k,l]}$. For $k < n < j_+$, define the unitary $w_{k,n}(t) \in \mathfrak{A}_{[k,n+1]}$ as

$$w_{k,n}(t) := D^{\mathrm{i}t}_{[k,n+1]} D^{-\mathrm{i}t}_{[k,n]}\,, \quad t \in \mathbb{R}\,.$$

The unitaries $\{w_{k,n}(t)\}_{t\in\mathbb{R}}$ give rise to a cocycle called *transition cocycle* when $\varphi$ is a Markov state (cf. [7]). Denote $S(\cdot)$ the von Neumann entropy (see e.g. [19]).

The following theorem collects some properties of the Fermi–Markov states, which parallel the analogous ones relative to Markov states on tensor product algebras (cf. [3], [7], [19]). For the natural applications of the properties described below to the variational principle in quantum statistical mechanics, the reader is referred to [9], [19].

THEOREM 4.1. *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a locally faithful even state. Then the following assertions are equivalent:*

(i) *$\varphi \in \mathcal{S}(\mathfrak{A})$ is a Markov state;*

(ii) *for each $t \in \mathbb{R}$ and $k < n < j_+$, $w_{k,n}(t) \in \mathfrak{A}_{[n,n+1],+}$.*

*Moreover, if $I = \mathbb{Z}$, $\mathfrak{A}_{\{n\}} = \mathbb{M}_{2^d}(\mathbb{C})$ for each $n \in \mathbb{Z}$, and $\varphi$ is translation invariant, the previous assertions are also equivalent to*

(iii) $S(\varphi_{[0,n+1]}) - S(\varphi_{[0,n]}) = S(\varphi_{[0,1]}) - S(\varphi_{\{0\}})$, $n \geqslant 1$.

*Proof.* (i) $\Rightarrow$ (ii) Thanks to Lemma 4.1 of [7], if $\varphi$ is a Markov state, then there exists an unitary $u_t \in \mathfrak{A}'_{[k,n-1]} \wedge \mathfrak{A}_{[k,n+1]}$ such that, for each $x \in \mathfrak{A}_{[k,n-1]}$,

$$w_{k,n}(t)xw_{k,n}(t)^* \equiv \sigma^{\varphi_{[k,n+1]}}_{-t}(\sigma^{\varphi_{[k,n]}}_{t}(x)) = u_t x u_t^* \equiv x\,,$$

$\sigma^{\varphi}_t$ denoting the modular group of a faithful state $\varphi$ on a von Neumann algebra, see e.g. [22]. As $w_{k,n}(t)$ is even, we have

$$w_{k,n}(t) \in \mathfrak{A}'_{[k,n-1]} \wedge \mathfrak{A}_{[k,n+1]} \wedge \mathfrak{A}_+ = \mathfrak{A}_{[n,n+1],+}\,,$$

see Lemma 11.1 and Theorem 4.17 of [9].

(ii) $\Rightarrow$ (i) The Accardi–Cecchini $\varphi$-expectation $E_{k,n}$ of $\varphi_{[k,n+1]}$ with respect to the inclusion $\mathfrak{A}_{[k,n]} \subset \mathfrak{A}_{[k,n+1]}$ (cf. [2]) is written as

$$E_{k,n}(x) = \mathcal{E}^0_{[k,n]}(w_{k,n}(-\mathrm{i}/2)^* x w_{k,n}(-\mathrm{i}/2))$$

where $w_{k,n}(-\mathrm{i}/2)$ is the analytic continuation at $-\mathrm{i}/2$ of $w_{k,n}(t)$, and $\mathcal{E}^0_{[k,n]}$ is the conditional expectation of $\mathfrak{A}_{[k,n+1]}$ onto $\mathfrak{A}_{[k,n]}$ preserving the normalized trace. If the $w_{k,n}(t)$ satisfy all the properties listed above, the Accardi–Cecchini expectation $E_{k,l}$ is a $\varphi_{[k,n+1]}$-preserving quasiconditional expectation with respect to the triplet $\mathfrak{A}_{[k,n-1]} \subset \mathfrak{A}_{[k,n]} \subset \mathfrak{A}_{[k,n+1]}$. By taking for each fixed $n$ the pointwise limit

$$\varepsilon_n := \lim_{k\downarrow j_-}\Big(\lim_{L}\frac{1}{L}\sum_{l=0}^{L-1}(E_{k,n}\lceil_{\mathfrak{A}_{[n,n+1]}})^l\Big)\,,$$

we obtain by $\mathcal{E}_n(xy) := x\varepsilon_n(y)$, $x \in \mathfrak{A}_{[n-1]}$, $y \in \mathfrak{A}_{[n,n+1]}$, a conditional expectation (cf. Proposition 2.2) fulfilling all the properties listed in Definition 2.1.

(ii) $\Leftrightarrow$ (iii) We have

$$w_{0,n}(t) = [D\varphi_{[0,n+1]} : D(\varphi_{[0,n+1]} \circ \mathcal{E}^0_{[0,n]})]_t\,,$$

the last being the Connes–Radon–Nikodym cocycle of $\varphi_{[0,n+1]}$ with respect to $\varphi_{[0,n+1]} \circ \mathcal{E}^0_{[0,n]}$ (cf. [22]). The assertion follows from the fact that (iii) is equivalent to the fact that $\mathfrak{A}_{[n,n+1]}$ is a sufficient subalgebra for both the mentioned states. It turns out to be equivalent to (ii) by translation invariance, see Proposition 11.5 and Proposition 9.3 of [19]. ∎

COROLLARY 4.2. *Suppose that $j_- \in I$. If $\varphi \in \mathcal{S}(\mathfrak{A})$ is a Markov state, then its support in $\mathfrak{A}^{**}$ is central. In addition, $\varphi$ is faithful.*

*Proof.* By Theorem 4.1, the pointwise norm limit

$$\lim_{n\uparrow j_+} D^{-\mathrm{i}t}_{[j_-,n]} x D^{\mathrm{i}t}_{[j_-,n]}$$

exists as it is asymptotically constant in $n$, on localized elements. Thus, it defines a one parameter group of automorphisms $t \mapsto \sigma_t$ of $\mathfrak{A}$ which admits, by construction, $\varphi$ as a KMS state. This means that $\pi_\varphi(\mathfrak{A})'\xi_\varphi$ is dense in $\mathcal{H}_\varphi$, $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ being the GNS triplet of $\varphi$. Furthermore, $\varphi$ is faithful as $\mathfrak{A}$ is a simple $C^*$-algebra, see Proposition 2.6.17 of [11]. ∎

COROLLARY 4.3. *Suppose that, for each $n \in I$, there exists a $k(n) \in I$ with $k(n) \leqslant n$, such that $\Theta$ acts trivially on $\mathcal{Z}(\mathcal{R}(\varepsilon_{k(n)}))$, $\varepsilon_j$ being the transition expectations associated to the Markov state $\varphi$. Then the assertions in Corollary* 4.2 *hold true as well.*

*Proof.* By regrouping the local algebras, we can suppose that there exists a $j_0 \in I$ such that, for $j < j_0$, $\Theta$ acts trivially on $\mathcal{Z}(\mathcal{R}(\varepsilon_j))$. Consider for $k < j_0$, $l > j_0$ the local algebras

$$\mathfrak{M}_{[k,l]} := \mathfrak{R}^c_k \vee \mathfrak{A}_{[k+1,l]}\,,$$

with $\mathfrak{R}^c_k$ given in (5.4). The last assertion follows as in Corollary 4.2, by looking at the transition cocycles of $\varphi$ relative to the new localization $\{\mathfrak{M}_{[k,l]}\}_{k<j_0<j}$. ∎

Let $\varphi$ be a translation invariant locally faithful state on the Fermion algebra $\mathfrak{A} \equiv \mathfrak{A}_{\mathbb{Z}}$. The mean entropy $s(\varphi)$ of $\varphi$ (see e.g. [19]) is defined as

$$s(\varphi) := \lim_n \frac{1}{n+1} S(\varphi_{[0,n]})\,,$$

$S(\varphi_{[0,n]})$ being the von Neumann entropy of $\varphi_{[0,n]}$.

COROLLARY 4.4. *We have for the translation invariant Markov state $\varphi$,*

$$s(\varphi) = S(\varphi_{[0,1]}) - S(\varphi_{\{0\}})\,.$$

*Proof.* It immediately follows by (iii) of Theorem 4.1. ∎

## 5. STRONGLY EVEN MARKOV STATES

In the present section we investigate the structure of strongly even Markov states (cf. Definition 2.8). By taking into account the structure of the local densities (or equally well the local Hamiltonians by passing to the logaritm) described in (5.3), the strongly even Markov states can be viewed as the Fermi analogue of the Ising type interactions. In addition, they enjoy a kind of local entanglement effect, see Section 4 of [15] for further details.

Notice that, as explained in the previous section, the forthcoming analysis extends to the situation when there exists a subsequence $\{n_j\} \subset I$ such that $\Theta$ acts trivially on all the $\mathcal{Z}(\mathcal{R}(\varepsilon_{n_j}))$.

We start with the following lemma which is known to the experts.

LEMMA 5.1. *Let $\mathfrak{C}_n \subset \mathfrak{B}_n$, $n \in \mathbb{N}$, be an increasing sequence of inclusions of unital $C^*$-subalgebras of $\mathfrak{B} := \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n}$ satisfying $(\mathfrak{C}_k)' \cap \mathfrak{B}_n = \mathfrak{C}_n$, $k \geqslant n$. Then $\mathfrak{C} := \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{C}_n}$ is a maximal Abelian $C^*$-subalgebra of $\mathfrak{B}$.*

*Proof.* We have for the commutant $\mathfrak{C}'$ in the ambient algebra $\mathfrak{B}$,

$$\begin{aligned}\mathfrak{C}' &= \overline{\bigcup_{n\in\mathbb{N}} (\mathfrak{C}' \cap \mathfrak{B}_n)} = \overline{\bigcup_{n\in\mathbb{N}} \Big( \Big( \bigcap_{k\in\mathbb{N}} (\mathfrak{C}_k)' \Big) \cap \mathfrak{B}_n \Big)} \\ &= \overline{\bigcup_{n\in\mathbb{N}} \Big( \bigcap_{k\in\mathbb{N}} \Big( (\mathfrak{C}_k)' \cap \mathfrak{B}_n \Big) \Big)} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{C}_n} = \mathfrak{C}. \quad \blacksquare\end{aligned}$$

Let $\omega = (\dots, \omega_{j-1}, \omega_j, \omega_{j+1}, \dots) \in \Omega$ be a trajectory. Thanks to part (i) of Proposition 2.7,

$$\begin{aligned}\mathfrak{B}^\omega &\equiv \overline{\Big( \bigvee_{j<j_+} Q^j_{\omega_j} \mathfrak{A}_{\{j\}} Q^j_{\omega_j} \Big) \vee \mathfrak{A}_{\{j_+\}}}^{C^*} \\ &= \overline{N^{j_-}_{\omega_{j_-}} \bigvee \Big( \bigvee_{j<j_+-1} (\overline{N}^j_{\omega_j} \vee N^{j+1}_{\omega_{j+1}}) \Big) \vee (\overline{N}^{j_+-1}_{\omega_{j_+-1}} \vee \mathfrak{A}_{\{j_+\}})}^{C^*},\end{aligned} \tag{5.1}$$

$N^j_{\omega_j}$ $\overline{N}^j_{\omega_j}$ providing the (Fermi) decompositions of the $Q^j_{\omega_j} \mathfrak{A}_{\{j\}} Q^j_{\omega_j}$ described by (2.10) in Proposition 2.7. This decomposition is quite similar to the analogous one described in Theorem 3.2 of [3], and generalize the situation treated in Section 5 of [6].

LEMMA 5.2. *Any maximal Abelian subalgebra of $(\overline{N}^j_{\omega_j} \vee N^{j+1}_{\omega_{j+1}})_+$ is maximal Abelian in $\overline{N}^j_{\omega_j} \vee N^{j+1}_{\omega_{j+1}}$ as well.*

*Proof.* Let $V \in \overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}}$ be any selfadjoint unitary implementing $\Theta$. Then

$$(\overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}})_+ \equiv (\mathbb{C}E_1 \oplus \mathbb{C}E_{-1})' = E_1(\overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}})E_1 \oplus E_{-1}(\overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}})E_{-1}\,,$$

$V = E_1 - E_{-1}$ being the resolution of $V$. ∎

LEMMA 5.3. *The unnormalized trace of*

$$R := N^{k}_{\omega_k} \vee \overline{N}^{k}_{\omega_k} \vee \cdots \vee N^{l}_{\omega_l} \vee \overline{N}^{l}_{\omega_l}$$

*is the product of the unnormalized traces of the $N^{j}_{\omega_j}$ and $\overline{N}^{j}_{\omega_j}$, $k \leqslant j \leqslant l$.*

*Proof.* Put $R = (Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}) \vee \cdots \vee (Q^{j+1}_{\omega_{j+1}}\mathfrak{A}_{\{j+1\}}Q^{j+1}_{\omega_{j+1}})$. By the product property of $\mathrm{Tr}_{\mathfrak{A}_{[k,l]}}$, we get

$$\mathrm{Tr}_R = \prod_{j=k}^{l} \mathrm{Tr}_{Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}}\,.$$

Thus, we reduce the situation to the algebra $N^{j}_{\omega_j} \vee \overline{N}^{j}_{\omega_j} \equiv Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}$. Notice that

$$Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j} = N^{j}_{\omega_j} \vee \widetilde{N}^{j}_{\omega_j} \sim N^{j}_{\omega_j} \otimes \widetilde{N}^{j}_{\omega_j}\,,$$

$N^{j}_{\omega_j}$, $\widetilde{N}^{j}_{\omega_j}$ are both globally stable under the action of $\Theta$, $\overline{N}^{j}_{\omega_j} = \widetilde{N}^{j}_{\omega_j,+} + V\widetilde{N}^{j}_{\omega_j,-}$, $V$ being any unitary of $N^{j}_{\omega_j}$ implementing $\Theta$ on itself, see Proposition 2.7. As the traces are invariant under any automorphism, we get

$$\begin{aligned}\mathrm{Tr}_{Q^{j}_{\omega_j}\mathfrak{A}_{\{j\}}Q^{j}_{\omega_j}} &= \mathrm{Tr}_{N^{j}_{\omega_j}}\,\mathrm{Tr}_{\widetilde{N}^{j}_{\omega_j}} = \Big(\mathrm{Tr}_{N^{j}_{\omega_j,+}} \circ \frac{\mathrm{id}+\Theta}{2}\Big)\Big(\mathrm{Tr}_{\widetilde{N}^{j}_{\omega_j,+}} \circ \frac{\mathrm{id}+\Theta}{2}\Big)\\ &= \Big(\mathrm{Tr}_{N^{j}_{\omega_j,+}} \circ \frac{\mathrm{id}+\Theta}{2}\Big)\Big(\mathrm{Tr}_{\overline{N}^{j}_{\omega_j,+}} \circ \frac{\mathrm{id}+\Theta}{2}\Big) = \mathrm{Tr}_{N^{j}_{\omega_j}}\,\mathrm{Tr}_{\overline{N}^{j}_{\omega_j}}\,. \quad ∎\end{aligned}$$

Let the initial distributions $\eta^{j_-}_{\omega_{j_-}} \in \mathcal{S}(N^{j_-}_{\omega_{j_-}})$, the states $\eta^{j}_{\omega_j,\omega_{j+1}}\mathcal{S}(\overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}})$ be recovered by $\varphi$ according to (2.11). (If $j_-$ and/or $j_+$ do not belong to $I$, they do not appear in the formulae, the last having an obvious meaning. In addition, as $\Omega_{j_+} \equiv \{j_+\}$, we use the symbology $\eta^{j}_{\omega_j,\omega_{j+1}}$ also for the final distributions $\eta^{j_+-1}_{\omega_{j_+-1},j_+}$.) Consider the even densities $T^{(j)}_{\omega_j}$, $\widehat{T}^{(j)}_{\omega_j}$, $T^{(j)}_{\omega_j,\omega_{j+1}}$ localized in $N^{j}_{\omega_j}$, $\overline{N}^{j}_{\omega_j}$, $\overline{N}^{j}_{\omega_j} \vee N^{j+1}_{\omega_{j+1}}$, and associated to $\eta^{j_-}_{\omega_{j_-}}$ or $\eta^{j}_{\omega_j,\omega_{j+1}}\lceil_{N^{j+1}_{\omega_{j+1}}}$, $\eta^{j}_{\omega_j,\omega_{j+1}}\lceil_{\overline{N}^{j}_{\omega_j}}$, $\eta^{j}_{\omega_j,\omega_{j+1}}$ respectively.

PROPOSITION 5.4. *The states $\eta^{j_-}_{\omega_{j_-}}$, $\eta^{j}_{\omega_j,\omega_{j+1}}$ uniquely define a product state on $\mathfrak{B}^{\omega}$, coinciding with $\psi^{\omega}$ in* (3.8), *which is symbolically written as*

$$\psi^{\omega} = \eta^{j_-}_{\omega_{j_-}} \prod_{j \leqslant j_+-1} \eta^{j}_{\omega_j,\omega_{j+1}}\,. \tag{5.2}$$

*Proof.* Consider on $\mathfrak{B}^\omega$ the localization

$$\mathfrak{B}^\omega = \overline{\bigvee_{j\in I}\mathfrak{N}^j_{\omega_j}}$$

suggested by (5.1). Here, $\mathfrak{N}^{j_-}_{\omega_{j_-}} := N^{j_-}_{\omega_{j_-}}$, $\mathfrak{N}^j_{\omega_j} := \overline{N}^{j-1}_{\omega_{j-1}} \vee N^{j+1}_{\omega_j}$, $j_- < j < j_+$, and finally $\mathfrak{N}^{j_+}_{\omega_{j_+}} := \overline{N}^{j_+-1}_{\omega_{j_+-1}} \vee \mathfrak{A}_{\{j_+\}}$. As the above densities commute each other, for each $k < l$, the product of local densities

$$T^{(k-1)}_{\omega_{k-1},\omega_k} \times \cdots \times T^{(l-1)}_{\omega_{l-1},\omega_l}$$

is a well defined positive even operator on $\bigvee_{k\leqslant j\leqslant l}\mathfrak{N}^j_{\omega_j}$ which by Lemma 5.3, is the density of $\psi^\omega\lceil_{\bigvee_{k\leqslant j\leqslant l}\mathfrak{N}^j_{\omega_j}}$ with respect to the unnormalized trace of $\bigvee_{k\leqslant j\leqslant l}\mathfrak{N}^j_{\omega_j}$. As explained in Section 2.3, this means that $\psi^\omega$ is the product states of $\eta^{j_-}_{\omega_{j_-}}$ with the $\eta^j_{\omega_j,\omega_{j+1}}$ as explained in (5.2) (see Theorem 11.2 of [9] for a similar situation). ∎

As all the states appearing in (5.2) are even, we can explicitely write the local densities associated to the strongly even Markov state. Namely, consider the Radon–Nikodym derivatives (i.e. the densities) $T_{\mathfrak{A}_{[k,l]}}$ with respect to the unnormalized trace of $\mathfrak{A}_{[k,l]}$,

$$\varphi_{[k,l]} = \mathrm{Tr}_{\mathfrak{A}_{[k,l]}}(T_{\mathfrak{A}_{[k,l]}}\cdot)\,.$$

Then $T_{\mathfrak{A}_{[k,l]}}$ has the nice decomposition

$$T_{\mathfrak{A}_{[k,l]}} = \bigoplus_{\omega_k,\dots,\omega_l} T^{(k)}_{\omega_k}T^{(k)}_{\omega_k,\omega_{k+1}} \times \cdots \times T^{(l-1)}_{\omega_{l-1},\omega_l}\widehat{T}^{(l)}_{\omega_l}\,. \tag{5.3}$$

By Corollary 4.2, any strongly even Markov state is a KMS state for the one parameter group of automomorphisms $\sigma_t$ given, for $x\in\mathfrak{A}$, by

$$\sigma_t(x) := \lim_{k\downarrow j_-,\, l\uparrow j_+} T^{-\mathrm{i}t}_{\mathfrak{A}_{[k,l]}}xT^{\mathrm{i}t}_{\mathfrak{A}_{[k,l]}}\,.$$

In addition, each strongly even Markov state is faithful.

We now show that each strongly even Markov state is a lifting of a classical Markov process. This result parallels the analogous one relative to the tensor product algebra, obtained first in [16] for some particular cases, and then in [15] for the general situation. Such property was called *diagonalizability* in [16]. After adapting the situation relative to the tensor product case to the strongly even Fermi–Markov states, we can follow the same line of the proof of Theorem 3.2 of [15].

We start by defining increasing subalgebras of the Fermi algebra $\mathfrak{A}$ equipped with a natural local structure inherited from that of the original algebra. Let $\mathfrak{R}_j := \mathcal{R}(\varepsilon_j)$, with relative commutant

$$\mathfrak{R}^c_j := \mathfrak{R}'_j \wedge \mathfrak{A}_{\{j\}}\,. \tag{5.4}$$

Define

$$\begin{aligned}\mathfrak{N}_{\{k\}} &:= \mathcal{Z}(\mathfrak{R}_k)\,, \quad \mathfrak{N}_{[k,k+1]} := \mathfrak{R}_k^c \vee \mathfrak{R}_{k+1}\,,\\ \mathfrak{N}_{[k,l]} &:= \mathfrak{R}_k^c \vee \mathfrak{A}_{[k+1,l-1]} \vee \mathfrak{R}_l\,, \quad k < l+1\,.\end{aligned} \tag{5.5}$$

Thanks to Lemma 5.2, for each $k \leqslant j < l$ and $\omega_j \in \Omega_j$, we can choose an even maximal Abelian subalgebra $D^j_{\omega_j,\omega_{j+1}}$ of $\overline{N}^j_{\omega_j} \vee N^{j+1}_{\omega_{j+1}}$ containing $T^{(j)}_{\omega_j,\omega_{j+1}}$. Put

$$\begin{aligned}\mathfrak{D}_{\{k\}} :=& \mathfrak{N}_{\{k\}} \equiv \mathcal{Z}(\mathfrak{R}_k)\,,\\ \mathfrak{D}_{[k,l]} :=& \bigoplus_{\omega_k,\dots,\omega_l} (D^k_{\omega_k,\omega_{k+1}} \vee \cdots \vee D^{l-1}_{\omega_{l-1},\omega_l})\,, \quad k < l\,,\\ \mathfrak{D} :=& \overline{\Big(\bigcup_{[k,l]\subset I} \mathfrak{D}_{[k,l]}\Big)}\,.\end{aligned} \tag{5.6}$$

THEOREM 5.5. *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a strongly even Markov state. Then there exists an even maximal Abelian $C^*$-subalgebra $\mathfrak{D} \subset \mathfrak{A}$, and a conditional expectation $\mathfrak{E} : \mathfrak{A} \to \mathfrak{D}$ such that $\varphi = \varphi\lceil_{\mathfrak{D}} \circ \mathfrak{E}$. In addition, the measure $\mu$ on* spec$(\mathfrak{D})$ *associated to $\varphi\lceil_{\mathfrak{D}}$ is a Markov measure with respect to the natural localization of $\mathfrak{D}$ given in* (5.6).

*Proof.* Let $[m_k, n_k]$ be an increasing sequence of intervals such that $[m_k, n_k] \uparrow I$. Then

$$\mathfrak{A} = \overline{\Big(\varinjlim_{[m_k,n_k]\uparrow I} \mathfrak{N}_{[m_k,n_k]}\Big)}^{C^*}\,.$$

As $\mathfrak{D}_{[m,n]}$ is an even maximal Abelian subalgebra of $\mathfrak{N}_{[m,n]}$, the increasing sequence $\mathfrak{D}_{[m_k,n_k]} \subset \mathfrak{N}_{[m_k,n_k]}$ satisfies the hypotheses of Lemma 5.1. Thus, $\mathfrak{D}$ is an even maximal Abelian $C^*$-subalgebra of $\mathfrak{A}$. According to (5.3), we have

$$T_{\mathfrak{N}_{[m,n]}} = \bigoplus_{\omega_m,\dots,\omega_n} T^{(m)}_{\omega_m,\omega_{m+1}} \times \cdots \times T^{(n-1)}_{\omega_{n-1},\omega_n}\,,$$

that is, $\{T_{\mathfrak{N}_{[m,n]}}\}_{m<n} \subset \mathfrak{D}$. Let $E^0_{m,n} : \mathfrak{N}_{[m,n]} \to \mathfrak{D}_{[m,n]}$ be the canonical conditional expectation of $\mathfrak{N}_{[m,n]}$ onto the maximal abelian subalgebra $\mathfrak{D}_{[m,n]}$ (cf. Footnote 4 of [15]). We have

$$\begin{aligned}\varphi\lceil_{\mathfrak{N}_{[m,n]}} \equiv& \mathrm{Tr}_{\mathfrak{N}_{[m,n]}}(T_{\mathfrak{N}_{[m,n]}}\,\cdot) = \mathrm{Tr}_{\mathfrak{N}_{[m,n]}}(E^0_{m,n}(T_{\mathfrak{N}_{[m,n]}}\,\cdot))\\ =& \mathrm{Tr}_{\mathfrak{N}_{[m,n]}}(T_{\mathfrak{N}_{[m,n]}} E^0_{m,n}(\cdot)) \equiv \varphi\lceil_{\mathfrak{N}_{[m,n]}} \circ E^0_{m,n}\,.\end{aligned} \tag{5.7}$$

As the sequence $\{E^0_{m,n}\}_{m<n}$ is projective, the direct limit $\varinjlim_{[m,n]\uparrow I} E^0_{m,n}$ uniquely defines a conditional expectation $\mathfrak{E} : \mathfrak{A} \to \mathfrak{D}$ fulfilling by (5.7), $\varphi = \varphi\lceil_{\mathfrak{D}} \circ \mathfrak{E}$. The measure $\mu$ on spec$(\mathfrak{D})$ associated to $\varphi\lceil_{\mathfrak{D}}$ is a Markov measure with respect to the natural localization of $\mathfrak{D}$ previously described. This follows as in Section 6 of [15], after noticing that $D^m_{\omega_m,\omega_{m+1}} \vee \cdots \vee D^{n-1}_{\omega_{n-1},\omega_n}$ in (5.6) generates a tensor product, and the restriction $\varphi\lceil_{D^m_{\omega_m,\omega_{m+1}} \vee \cdots \vee D^{n-1}_{\omega_{n-1},\omega_n}}$ defines a product measure on spec$(D^m_{\omega_m,\omega_{m+1}}) \times \cdots \times$ spec$(D^n_{\omega_n,\omega_{n+1}})$. ∎

Now we pass to the dynamical entropy $h_\varphi(\alpha)$ with respect to the right shift $\alpha$ for translation invariant strongly even Markov states. The reader is referred to [12], [13], [19] for the definition and technical details on the dynamical entropy.

The definition of the dynamical entropy $h_\varphi(\alpha)$ is based on the multiple subalgebra entropy $H_\varphi(N_1,\dots,N_k)$, with $N_1,\dots,N_k\subset M$. We start by pointing out that, if the subalgebras $N_1,\dots,N_k$ are the range of $\varphi$-preserving conditional expectations and are contained in different factors of a tensor product algebra, then

$$H_\varphi(N_1,\dots,N_k)=S(\varphi\lceil_N)\,, \tag{5.8}$$

with $N:=N_1\vee\cdots\vee N_k$. (Fix a faithful trace on $M$. Let $T_1,\cdots,T_k$, $T$ be the corresponding densities of $N_1,\dots,N_k$, $M$ respectively. Choose maximal Abelian subalgebras $A_j$ of $N_j$ containing $T_j$, $j=1,\dots,k$. As the $N_j$ are expected, we have for $a\in A_j$,

$$T^{-\mathrm{i}t}aT^{\mathrm{i}t}=T_j^{-\mathrm{i}t}aT_j^{\mathrm{i}t}=a\,,$$

that is $A_j\subset M_\varphi$, $M_\varphi$ being the centralizer of the faithful state $\varphi$. As the $A_j$ are contained in different factors of a tensor product, $A_1\vee\cdots\vee A_k$ is maximal Abelian in $N$. Thus, (5.8) follows by Corollary VIII.8 of [13].)

THEOREM 5.6. *Let $\varphi\in\mathcal{S}(\mathfrak{A})$ be a translation invariant strongly even Markov state. Then $h_\varphi(\alpha)=s(\varphi)$.*

*Proof.* The proof follows the same lines of the tensor product case. We keep into account some boundary effects which cannot be neglected in proving the result. Fix $n$, and consider $\mathfrak{N}_{[0,n+1]}$ given in (5.5). We have $\mathfrak{A}_{[1,n]}\subset\mathfrak{N}_{[0,n]}\subset\mathfrak{A}_{[0,n+1]}$, and $\mathfrak{N}_{[0,n]}$ is expected. We compute,

$$\begin{aligned}H(k):=&H_\varphi(\mathfrak{N}_{[0,n]},\alpha(\mathfrak{N}_{[0,n]}),\dots,\alpha^{k(n+2)}(\mathfrak{N}_{[0,n]}))\\ \geqslant&H_\varphi(\mathfrak{N}_{[0,n]},\alpha^{n+2}(\mathfrak{N}_{[0,n]}),\dots,\alpha^{k(n+2)}(\mathfrak{N}_{[0,n]}))\\ \geqslant&H_\varphi(\mathfrak{N}_{[0,n],+},\alpha^{n+2}(\mathfrak{N}_{[0,n],+}),\dots,\alpha^{k(n+2)}(\mathfrak{N}_{[0,n],+}))\,,\end{aligned}$$

Now, $\mathfrak{N}_{[0,n],+},\alpha^{n+2}(\mathfrak{N}_{[0,n],+},),\dots,\alpha^{k(n+2)}(\mathfrak{N}_{[0,n],+})$ are all expected, and generate a tensor product. Then

$$H(k)\geqslant S(\varphi\lceil_{M_k})=-S(\varphi\lceil_{M_k},\tau\lceil_{M_k})+k\ln d\,.$$

Here, $M_k:=\mathfrak{N}_{[0,n],+}\vee\alpha^{n+2}(\mathfrak{N}_{[0,n],+})\vee\cdots\vee\alpha^{k(n+2)}(\mathfrak{N}_{[0,n],+})$, $d$ is the tracial dimension of $\mathfrak{N}_{[0,n],+}$, $\tau$ the normalized trace on $\mathfrak{A}$, and finally $S(\cdot,\cdot)$ the relative entropy (see e.g. [19]). As $\mathfrak{A}_{[1,m],+}\subset\mathfrak{N}_{[0,m],+}$, and the tracial dimension of $\mathfrak{A}_{[1,m],+}$ coincides with that of $\mathfrak{A}_{[1,m]}$ (cf. Lemma 5.2), we obtain by the monotonicity of the relative entropy,

$$\begin{aligned}H(k)&\geqslant-S(\varphi\lceil_{\mathfrak{A}_{[1,(n+2)(k+1)]}},\tau\lceil_{\mathfrak{A}_{[1,(n+2)(k+1)]}})+k\ln d\\ &=S(\varphi\lceil_{\mathfrak{A}_{[1,(n+2)(k+1)]}})+[kn-(n+2)(k+1)]\ln l\,,\end{aligned}$$

$l$ being the tracial dimension of $\mathfrak{A}_{\{0\}}$. Finally, we get

$$h_\varphi(\alpha) \geqslant \lim_k \frac{H(k)}{(n+2)k} \geqslant \lim_k \Big[\frac{k+1}{k}s(\varphi) + \frac{kn-(n+2)(k+1)}{(n+2)k}\ln l\Big] = s(\varphi) - \frac{2\ln l}{n+2}\,.$$

Since $h_\varphi(\alpha) \leqslant s(\varphi)$ and $n$ is arbitrary, the assertion follows. ∎

## 6. EXAMPLES OF TRANSLATION INVARIANT FERMI–MARKOV STATES

In the present section we exhibit examples of Fermi–Markov states. It exhausts all the low dimensional translation invariant cases. The present construction can be extended to the most general situation when the dimensions $\dim(\mathfrak{A}_j)$, $j \in I$, are arbitrary and/or the Markov state under consideration is not translation invariant. It furnishes the direct application of Theorem 3.3, or equally well Proposition 2.7.

Thanks to translation invariance, it is enough to construct a two point even transition expectation $\varepsilon : \mathfrak{A}_{[0,1]} \to \mathfrak{A}_{\{0\}}$, and compute the stationary even distributions by solving $\rho = \rho \circ \varepsilon \circ \alpha\lceil_{\mathfrak{A}_{\{0\}}}$, $\rho$ running into the even states of $\mathfrak{A}_{\{0\}}$. A translation invariant Markov state $\varphi$ is then recovered by the marginals

$$\varphi(x_k \cdots x_l) = \rho(\varepsilon_k(x_k\varepsilon_{k+1}(x_{k+1}\cdots\varepsilon_{l-1}(x_{l-1}\varepsilon_l(x_l))\cdots)))\,. \tag{6.1}$$

### 6.1. CASE 1: $\mathfrak{A}_{\{n\}} \sim \mathbb{M}_2(\mathbb{C})$, $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^2$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))} = 1$.

We start with the pivotal example in Subsection 6.4 of [6] by showing that it provides examples of Fermi–Markov states which are entangled. Define, for a fixed $\chi$ in the unit circle $\mathbb{T}$,

$$q_\chi := \frac{1}{2}(\mathbf{1} + \chi a_0 + \overline{\chi}a_0^\dagger)\,.$$

Choose a faithful state $\eta \in \mathcal{S}(q_\chi \mathfrak{A}_{[0,1]} q_\chi)$. Put

$$\varepsilon(x) = \eta(q_\chi x q_\chi)q_\chi + \eta(q_\chi\Theta(x)q_\chi)q_{-\chi}\,, \quad x \in \mathfrak{A}_{[0,1]}\,. \tag{6.2}$$

With $\tau$ the normalized trace on $\mathbb{M}_2(\mathbb{C})$, $\varepsilon_n := \varepsilon \circ \alpha^{-n}$, and $x_k \in \mathfrak{A}_{\{k\}}, \dots, x_l \in \mathfrak{A}_{\{l\}}$, the marginals (6.1) with $\rho = \tau$, uniquely determine a translation invariant locally faithful Markov state $\varphi$ on the Fermion algebra $\mathfrak{A} := \mathfrak{A}_{\mathbb{Z}}$ satisfying the required properties. We show that $\varphi$ is entangled for particular choices of $\eta$. Thanks to shift invariance, it suffices to consider $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$.

Let $\xi_\chi$, $\xi_\chi^\perp$ be the (uniquely determined up to a phase) normalized eigenvectors of $q_\chi$, $q_{-\chi} = q_\chi^\perp$ acting on $\mathbb{C}^2$, corresponding to the eigenvalues 1, respectively. Put

$$V := \langle\,\cdot\,,\xi_\chi\rangle\xi_\chi^\perp\,.$$

As $V \in \mathbb{M}_2(\mathbb{C}) = \mathfrak{A}_{\{0\}} \subset \mathfrak{A}_{[0,1]}$, $V$ is also in $\mathfrak{A}_{[0,1]}$. Put $\delta := \eta(V(\chi a_1 + \overline{\chi}a_1^\dagger)q_\chi)$. We have

$$\varphi(xy) = \langle x_+\xi_\chi,\xi_\chi\rangle\langle\alpha^{-1}(y_+)\xi_\chi,\xi_\chi\rangle + \delta\langle x_-\xi_\chi,\xi_\chi^\perp\rangle\langle\alpha^{-1}(y_-)\xi_\chi,\xi_\chi\rangle\,.$$

Now we show that there exists a faithful state $\eta$ as above, such that $\eta(X) \neq 0$, where

$$X := V\alpha(q_{\chi,-})q_\chi \equiv \frac{1}{2}V(\chi a_1 + \overline{\chi}a_1^\dagger)q_\chi) \neq 0\,. \tag{6.3}$$

Pick a functional which is different from zero on $X$, hence a state $\eta_0$ which is nonnull on $X$. Let $p \in q_\chi \mathfrak{A}_{[0,1]} q_\chi$ be the support of $\eta_0$. Choose a state $\eta_1$ with support $q_\chi - p$. Then $\eta := \beta\eta_0 + (1-\beta)\eta_1$ is a faithful state on $q_\chi \mathfrak{A}_{[0,1]} q_\chi$ which is nonnull on $X$ for an appropriate choice of $\beta \in [0,1]$. (The last claim easily follows as $\eta(X) = 0$ means $\eta_0(X) \neq \eta_1(X)$, and $\beta = \frac{\eta_1(X)}{\eta_1(X)-\eta_0(X)}$.) We then have the following

PROPOSITION 6.1. *Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, $\Lambda_1 \cup \Lambda_2 = \mathbb{Z}$. Suppose that $\eta(X) \neq 0$, where $\eta$ is the state in* (6.2) *and $X$ is given in* (6.3). *Then the state $\varphi$ described above is entangled with respect to the decomposition $\mathfrak{A} = \overline{\mathfrak{A}_{\Lambda_1} \vee \mathfrak{A}_{\Lambda_2}}$.*

*Proof.* Let $\mathfrak{A}_{\{n\}} \subset \mathfrak{A}_{\Lambda_1}$, $\mathfrak{A}_{\{n+1\}} \subset \mathfrak{A}_{\Lambda_2}$ for some $n \in \mathbb{Z}$ (which is always the case after a possible renumbering of $\Lambda_1$, $\Lambda_2$). Under the above assumption, $\varphi(x_- y_-)$ cannot be identically zero for each $x \in \mathfrak{A}_{\Lambda_1}$, $y \in \mathfrak{A}_{\Lambda_2}$ due to the shift invariance. The proof now follows by applying the Moriya criterion established in Proposition 1 of [18]. ∎

By extending the previous computations to more general cases, it is then possible to construct many examples of entangled translation invariant Fermi–Markov states for the situation when $\mathfrak{A}_{\{0\}} = \mathbb{M}_{2^d}$, $d > 1$. We are going to describe a sample of pivotal examples.

We now consider the successive step $\mathfrak{A}_{\{k\}} \sim \mathbb{M}_4(\mathbb{C})$. We exhibit examples for each possible structure of the Abelian algebra $\mathcal{Z}(\mathcal{R}(\varepsilon))$, and for the action of $\Theta$ on it. Let $\{a_i, a_i^\dagger : i = 1,2\}$ be the creators and annihilators generating $\mathfrak{A}_{\{0\}}$. Consider the system $\{e_{kl}(j) : j,k,l = 1,2\}$ of commuting $2 \times 2$ matrix units obtained via the Jordan–Klein–Wigner transformation (2.4). Putting $e_{(i,j)(k,l)} := e_{ik}(1)e_{jl}(2)$, we obtain a system of matrix units for $\mathfrak{A}_{\{0\}}$ which realizes the isomorphism $\mathfrak{A}_{\{0\}} \sim \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$.

## 6.2. CASE 2. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^4$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))} = 4$.

Choose $\{e_{(i,j)(i,j)} : i,j = 1,2\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. In this situation, there exist even states $\varphi_{ij}$, $i,j = 1,2$ on $\mathfrak{A}_{\{1\}}$ such that for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\varepsilon(xy) = \sum_{i,j=1}^{2} \mathrm{Tr}(xe_{(i,j)(i,j)})\varphi_{ij}(y)e_{(i,j)(i,j)}\,.$$

This is nothing but the example in Subsection 6.2 of [6]. Thus, $\varphi$ is strongly clustering with respect to the shift on the chain, and the von Neumann algebra $\pi_\varphi(\mathfrak{A})''$ generated by the GNS representation $\pi_\varphi$ of $\varphi$ is a type $\mathrm{III}_\lambda$ factor for some $\lambda \in (0,1]$, see [15].

6.3. CASE 3. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^4$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))} = 3$.

For a fixed $\chi$ in the unit circle $\mathbb{T}$, define

$$Q_\chi := \frac{1}{2}(\mathbf{1} + \chi a_2 + \overline{\chi} a_2^\dagger)\,.$$

Choose $\{e_{(1,j)(1,j)}, e_{22}(1)Q_{\pm\chi} : j = 1,2\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. In this situation, there exist even states $\varphi_j$, $j = 1,2$ on $\mathfrak{A}_{\{1\}}$, and a state $\varphi$ on $e_{22}(1)Q_\chi\mathfrak{A}_{[0,1]}$ $e_{22}(1)Q_\chi$ such that, for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\begin{aligned}\varepsilon(xy) = \sum_{j=1}^{2}\operatorname{Tr}(xe_{(1,j)(1,j)})\varphi_j(y)e_{(1,j)(1,j)} + \varphi(e_{22}(1)Q_\chi xye_{22}(1)Q_\chi)e_{22}(1)Q_\chi&\\ + \varphi(e_{22}(1)Q_\chi\Theta(xy)e_{22}(1)Q_\chi)e_{22}(1)Q_{-\chi}\,.&\end{aligned}$$

6.4. CASE 4. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^4$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))} = 2$.

First choose $\{e_{ii}(1)Q_{\pm\chi} : i = 1,2\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. In this situation, there exist states $\varphi_i$, on $e_{ii}(1)Q_\chi\mathfrak{A}_{[0,1]}e_{ii}(1)Q_\chi$, $i = 1,2$ such that, for $x \in \mathfrak{A}_{[0,1]}$,

$$\varepsilon(x) = \sum_{i=1}^{2}(\varphi_i(e_{ii}(1)Q_\chi xe_{ii}(1)Q_\chi)e_{ii}(1)Q_\chi + \varphi_i(e_{ii}(1)Q_\chi\Theta(x)e_{ii}(1)Q_\chi)e_{ii}(1)Q_{-\chi})\,.$$

Next, for fixed $(\chi,\eta) \in \mathbb{T}^2$, define with $V := a_1^\dagger a_1 - a_1 a_1^\dagger$,

$$P_{\chi,\eta} := \frac{1}{4}(\mathbf{1} + \chi a_1 + \overline{\chi} a_1^\dagger)(\mathbf{1} + \eta V a_2 + \overline{\eta} V a_2^\dagger)\,.$$

Choose $\{P_{\pm\chi,\pm\eta}\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. In this situation, there exist states $\varphi_\pm$ on $P_{\pm\chi,\eta}\mathfrak{A}_{[0,1]}P_{\pm\chi,\eta}$ respectively, such that for $x \in \mathfrak{A}_{[0,1]}$,

$$\begin{aligned}\varepsilon(x) = \varphi_+(P_{\chi,\eta}xP_{\chi,\eta})P_{\chi,\eta} + \varphi_+(P_{\chi,\eta}\Theta(x)P_{\chi,\eta})P_{-\chi,-\eta}&\\ + \varphi_-(P_{-\chi,\eta}xP_{-\chi,\eta})P_{-\chi,\eta} + \varphi_-(P_{-\chi,\eta}\Theta(x)P_{-\chi,\eta})P_{\chi,-\eta}\,.&\end{aligned}$$

6.5. CASE 5. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^3$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))} = 3$.

First choose $\{e_{11}(1)e_{jj}(2), e_{22}(1) : j = 1,2\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. We have two possibilities. Namely, there exist even states $\varphi_j$, on $\mathfrak{A}_{\{1\}}$, $i = 1,2$, and an even state $\varphi$ either on $\mathfrak{A}_{\{1\}}$, or on $(e_{22}(1)\mathfrak{A}_{\{0\}}e_{22}(1)) \vee \mathfrak{A}_{\{1\}}$ such that, for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\varepsilon(xy) = \sum_{j=1}^{2}\operatorname{Tr}(xe_{11}(1)e_{jj}(2))\varphi_j(y)e_{11}(1)e_{jj}(2) + \varphi(y)e_{22}(1)xe_{22}(1)\,,$$

respectively

$$\varepsilon(xy) = \sum_{j=1}^{2}\operatorname{Tr}(xe_{11}(1)e_{jj}(2))\varphi_j(y)e_{11}(1)e_{jj}(2) + \varphi(e_{22}(1)xe_{22}(1)y)e_{22}(1)\,.$$

Next, put $P := e_{(1,2)(1,2)} + e_{(2,1)(2,1)}$ and choose $\{e_{(i,i)(i,i)}, P : i = 1,2\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. Again, we have two possibilities. Namely, there exist even states $\varphi_j$, on $\mathfrak{A}_{\{1\}}$, $i = 1,2$, and an even state $\varphi$ either on $\mathfrak{A}_{\{1\}}$, or on $(P\mathfrak{A}_{\{0\}}P) \vee \mathfrak{A}_{\{1\}}$ such that, for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\varepsilon(xy) = \sum_{j=1}^{2} \operatorname{Tr}(xe_{11}(1)e_{jj}(2))\varphi_j(y)e_{11}(1)e_{jj}(2) + \varphi(y)PxP\,,$$

respectively

$$\varepsilon(xy) = \sum_{j=1}^{2} \operatorname{Tr}(xe_{11}(1)e_{jj}(2))\varphi_j(y)e_{11}(1)e_{jj}(2) + \varphi(PxPy)P\,.$$

Notice that the last possibilities correspond to nontrivial cases with $\mathcal{R}(\varepsilon) \subset \mathfrak{A}_+$.

6.6. CASE 6. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^3$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))}= 2$.

For $\chi \in \mathbb{T}$, choose $\{e_{11}(1)Q_{\pm\chi}, e_{22}(1)\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. We have two possibilities. Namely, choose a state $\varphi$ on $e_{11}(1)Q_\chi\mathfrak{A}_{[0,1]}e_{11}(1)Q_\chi$, and an even state $\psi$ either on $\mathfrak{A}_{\{1\}}$, or on $(e_{22}(1)\mathfrak{A}_{\{0\}}e_{22}(1)) \vee \mathfrak{A}_{\{1\}}$ such that, for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\begin{aligned}\varepsilon(xy)=\varphi(e_{11}(1)Q_\chi xye_{11}(1)Q_\chi)e_{11}(1)Q_\chi+\varphi(e_{11}(1)Q_\chi\Theta(xy)e_{11}(1)Q_\chi)e_{11}(1)Q_{-\chi}\\ +\psi(y)e_{22}(1)xe_{22}(1)\,,\end{aligned}$$

respectively

$$\begin{aligned}\varepsilon(xy)=\varphi(e_{11}(1)Q_\chi xye_{11}(1)Q_\chi)e_{11}(1)Q_\chi+\varphi(e_{11}(1)Q_\chi\Theta(xy)e_{11}(1)Q_\chi)e_{11}(1)Q_{-\chi}\\ +\psi(e_{22}(1)xe_{22}(1)y)e_{22}(1)\,.\end{aligned}$$

6.7. CASE 7. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^2$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))}= 2$.

We treat only the following cases, the remaining ones follow analogously. Choose $p = e_{(1,1)(1,1)}$, $p^\perp$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. We have two possibilities. Namely, there exists an even state $\varphi$ on $\mathfrak{A}_{\{1\}}$, and an even state $\psi$ either on $\mathfrak{A}_{\{1\}}$, or on $(p^\perp\mathfrak{A}_{\{0\}}p^\perp) \vee \mathfrak{A}_{\{1\}}$ such that, for $x \in \mathfrak{A}_{\{0\}}$, $y \in \mathfrak{A}_{\{1\}}$,

$$\varepsilon(xy) = \operatorname{Tr}(xp)\varphi(y)p + \psi(y)p^\perp xp^\perp\,,$$

respectively

$$\varepsilon(xy) = \operatorname{Tr}(xp)\varphi(y)p + \psi(p^\perp xp^\perp y)p^\perp\,.$$

6.8. CASE 8. $\mathcal{Z}(\mathcal{R}(\varepsilon)) \sim \mathbb{C}^2$, # of orbits of $\Theta\lceil_{\mathcal{Z}(\mathcal{R}(\varepsilon))}= 1$.

Choose $\{Q_{\pm\chi}\}$ as the generators of $\mathcal{Z}(\mathcal{R}(\varepsilon))$. We have two possibilities. First

$$\varepsilon(x) = \varphi(Q_\chi xQ_\chi)Q_\chi + \varphi(Q_\chi\Theta(x)Q_\chi)Q_{-\chi}\,,\quad x \in \mathfrak{A}_{[0,1]}\,,$$

$\varphi$ being a state on $Q_\chi\mathfrak{A}_{[0,1]}Q_\chi$. Second, let $\mathfrak{B} \subset \mathfrak{A}_{[0,1]}$ be the tensor completion of $\mathfrak{A}_{\{0\}}$ in $\mathfrak{A}_{[0,1]}$. (According to (2.4), the subalgebra $\mathfrak{B}$ is obtained by constructing a systems $\{e_{kl}(j), f_{kl}(j) : j,k,l = 1,2\}$ of four mutually commuting $2\times 2$ matrix units for $\mathfrak{A}_{[0,1]}$. Notice that $\mathfrak{B}$ is localized in the whole $\mathfrak{A}_{[0,1]}$, and is $\Theta$-invariant.) Then there exists a state $\varphi$ on $\mathfrak{B}$ such that for $x\in\mathfrak{A}_{\{0\}}$, $y\in\mathfrak{B}$,

$$\varepsilon(xy)=\varphi(y)Q_\chi xQ_\chi+\varphi(\Theta(y))Q_{-\chi}xQ_{-\chi}\,.$$

6.9. CASE 9. $\mathcal{Z}(\mathcal{R}(\varepsilon))\sim\mathbb{C}$.

We treat only one possibility, the two remaining ones generating one step product states (see e.g. Subsection 6.1 of [6]). Let $N$, $\overline{N}$ be the algebra generated by $a_1,a_1^\dagger$, $a_2,a_2^\dagger$ respectively. Then there exists an even state $\varphi$ on $\overline{N}\vee\mathfrak{A}_{\{1\}}$ such that for $x\in N$, $y\in\overline{N}\vee\mathfrak{A}_{\{1\}}$,

$$\varepsilon(xy)=\varphi(y)x\,.$$

Notice that this example is nothing but that the two block factor treated in Subsection 6.3 of [6]. This is easily seen by passing in [6], to the two point regrouped algebra.

6.10. CASE 10. Two examples with $\mathfrak{A}_{\{n\}}\sim\mathbb{M}_{2^3}(\mathbb{C})$.

We describe two examples relative to more complicated situations than the previous ones. Let $\{a_i,a_i^\dagger\,:\,i=1,2,3\}$, $\{b_i,b_i^\dagger\,:\,i=1,2,3\}$ be the generators of $\mathfrak{A}_{\{0\}}$, $\mathfrak{A}_{\{1\}}$ respectively. Let $\{e_{kl}(j)\,,f_{kl}(j)\,:\,j,k,l=1,2\}$ of commuting $2\times2$ matrix units obtained according to (2.4), and realizing the isomorphism $\mathfrak{A}_{[0,1]}\sim\underbrace{\mathbb{M}_2(\mathbb{C})\otimes\cdots\otimes\mathbb{M}_2(\mathbb{C})}_{6\text{–times}}$. Put for $\chi\in\mathbb{T}$,

$$P_\chi:=\frac{1}{2}(\mathbf{1}+\chi a_1+\overline{\chi}a_1^\dagger)\,.$$

First define $N_i$, $\overline{N}_i$ as the algebras generated by $\{e_{ii}(1)a_2\,,e_{ii}(1)a_2^\dagger\}$, $\{e_{ii}(1)a_3\,,e_{ii}(1)a_3^\dagger\}$, $i=1,2$ respectively. Choose even states $\varphi_i$ on $\overline{N}_i\vee\mathfrak{A}_{\{1\}}$. Then for $x_i\in N_i$, $y\in\overline{N}_i\vee\mathfrak{A}_{\{1\}}$,

$$\varepsilon\Big(\sum_{i=1}^2 x_iy_i\Big)=\sum_{i=1}^2\varphi_i(y_i)x_i\,.$$

Second define $N_\chi$, $M_\chi$ as the algebras generated by $\{P_\chi e_{ij}(2):i,j=1,2\}$, $\{P_\chi e_{ij}(3)f_{kl}(n):i,j,k,l=1,2\,,n=1,2,3\}$ respectively. Choose a state $\varphi$ on $M_\chi$. Then for $x_{\pm\chi}\in N_{\pm\chi}$, $y_{\pm\chi}\in M_{\pm\chi}$,

$$\varepsilon(x_\chi y_\chi+x_{-\chi}y_{-\chi})=\varphi(y_\chi)x_\chi+\varphi(\Theta(y_{-\chi}))x_{-\chi}\,.$$

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## REFERENCES


[1] L. ACCARDI, On noncommutative Markov property, *Funct. Anal. Appl.* **8**(1975), 1–8.

[2] L. ACCARDI, C. CECCHINI, Conditional expectations in von Neumann algebras and a theorem of Takesaki, *J. Funct. Anal.* **45**(1982), 245–273.

[3] L. ACCARDI, F. FIDALEO, nonhomogeneous quantum Markov states and quantum Markov fields, *J. Funct. Anal.* **200**(2003), 324–347.

[4] L. ACCARDI, F. FIDALEO, Quantum Markov fields, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6**(2003), 123–138.

[5] L. ACCARDI, F. FIDALEO, Entangled Markov chains, *Ann. Mat. Pura Appl.* **184**(2005), 327–346

[6] L. ACCARDI, F. FIDALEO, F. MUKHAMEDOV, Markov states and chains on the CAR algebra, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10**(2007), 165–183.

[7] L. ACCARDI, A. FRIGERIO, Markovian cocycles, *Proc. Royal Irish Acad.* **83**(1983), 251–263.

[8] H. ARAKI, Conditional expectations relative to a product state and the corresponding standard potentials, *Comm. Math. Phys.* **246**(2004), 113–132.

[9] H. ARAKI, H. MORIYA, Equilibrium statistical mechanics of Fermi lattice systems, *Rev. Math. Phys.* **15**(2003), 93–198.

[10] H. ARAKI, H. MORIYA, Joint extension of states of subsystems, *Comm. Math. Phys.* **237**(2003), 105–122.

[11] O. BRATTELI, D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics.* vol. I, II, Springer, Berlin–Heidelberg–New York 1987.

[12] C. CECCHINI, D. PETZ, On the fixed point algebras for $\varphi$-conditional expectations in von Neumann algebras, *Studia Sci. Math. Hungar.* **24**(1989), 133–157.

[13] A. CONNES, H. NARNHOFER, W. THIRRING, Dynamical entropy of $C^*$-algebras and von Neumann algebras, *Comm. Math. Phys.* **112**(1987), 691–719.

[14] M. FANNES, B. NACHTERGAELE, R.F. WERNER, Finitely correlated states of quantum spin chains, *Comm. Math. Phys.* **144**(1992), 443–490.

[15] F. FIDALEO, F. MUKHAMEDOV, Diagonalizability of nonhomogeneous quantum Markov states and associated von Neumann algebras, *Probab. Math. Statist.* **24**(2004), 401–418.

[16] YA.V. GOLODETS, G.N. ZHOLTKEVICH, Markovian KMS states [Russian], *Teoret. Mat. Phys.* **56**(1983), 686–690.

[17] J.F. HAVET, Espérance conditionelle minimale, *J. Operator Theory* **24**(1990), 35–55.

[18] H. MORIYA, On separable states for composite systems of distinguishable fermions, *J. Phys. A* **39**(2006), 3753–3762.

[19] M. OHYA, D. PETZ, *Quantum Entropy and its Use*, Springer, Berlin–Heidelberg–New York 1993.

[20] C.J. PRESTON, *Gibbs States on Countable Sets*, Cambrige Univ. Press, Cambrige 1974.

[21] S. STRĂTILĂ, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells, Kent 1981.

[22] S. STRĂTILĂ, L. ZSIDÓ, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells, Kent 1979.

[23] M. TAKESAKI, *Theory of Operator Algebras*, vol. I, III, Springer, Berlin–Heidelberg–New York 2003.

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