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THE BAIRE PROPERTY AND THE DOMAIN OF ITERATES OF A PARACOMPLETE LINEAR RELATION

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ABSTRACT. The Baire category theorem for operators, concerning the denseness of the intersection of the ranges of certain sequences of continuous operators in Banach spaces, is generalized to multivalued linear operators. As an application we obtain a result which gives the denseness of the domains and ranges of certain sequences of paracomplete linear relations in Banach spaces. Results of Lennard and of Burlando about operators in Banach spaces are recovered.

KEYWORDS: *Paracomplete linear relation, Baire property, lower essential resolvent set.*

MSC (2000): 47A06.

INTRODUCTION

In 2.10 of [8], Burlando applies the Baire property for operators to obtain a result which deals with the iterates of a paracomplete operator having nonempty lower essential resolvent set. The purpose of this paper is to extend the results of the type mentioned above to linear relations.

Section 1 contains the auxiliary results that we will need to prove the main theorems of Section 2. We define and present some properties of the paracomplete linear relations in Banach spaces and we also establish several entirely algebraic results about linear relations in vector spaces.

In Section 2 we obtain the Baire category theorem for multivalued linear operators (Theorem 2.2 below) which is applied to generalize a result of Burlando ([8], 2.10) to paracomplete linear relations in Banach spaces (Theorem 2.4 below).

0.1. NOTATIONS. We recall some basic definitions from the theory of linear relations in vector spaces following the notation and terminology of the book [11].

Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation or multivalued linear operator in X is a mapping T from a subspace $D(T)$ of X, called the domain of T, into $P(X) \setminus \{\varnothing\}$ (the collection of nonempty subsets of X) such that $T(\alpha x_1 +$

$\beta x_2) = \alpha Tx_1 + \beta Tx_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. The class of such relations T is denoted by $LR(X)$. If T maps the points of its domain to singletons then T is said to be a single valued or simply an operator.

A linear relation T is uniquely determined by its graph, $G(T)$, defined by $G(T) := \{(x,y) : x \in D(T), y \in Tx\}$ which is a subspace of $X \times X$. Let $T, S \in LR(X)$. The sum $T + S$ is the linear relation whose graph is $G(T + S) := \{(x, y + z) : (x,y) \in G(T), (x,z) \in G(S)\}$. If $R(T) \cap D(S)$ is nonempty, then the composition or product ST is the linear relation defined by $G(ST) := \{(x,z) : \exists y \in X, (x,y) \in G(T), (y,z) \in G(S)\}$. The product of linear relations is clearly associative. Hence T^n, $n \in \mathbb{N}$, is defined as usual with $T^0 = I$ and $T^1 = T$. Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{K}$, with $a_n \neq 0$ if $n \geqslant 1$. If p denotes the polynomial of degree n defined by $p(\lambda) := \sum_{k=0}^{n} a_k \lambda^k$ for any $\lambda \in \mathbb{K}$, then the linear relation $p(T)$ is defined by $p(T) := \sum_{k=0}^{n} a_k T^k$. Notice that clearly $D(p(T)) = D(T^n)$.

Let $T \in LR(X)$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y,x) : (x,y) \in G(T)\}$. The null space of T, denoted $N(T)$, is the subspace $T^{-1}(0)$ and the range of T is the subspace $R(T) := T(D(T))$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if $N(T) = \{0\}$ and T is said to be surjective if its range coincides with the whole space X.

Assume now that X is a normed space. Let $Q^{X}_{\overline{T(0)}}$ (or simply Q_T), denote the natural quotient map from X onto $X/\overline{T(0)}$. Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and the norm of T is defined by $\|T\| := \|Q_T T\|$. We note that this quantity is not a true norm since $\|T\| = 0$ does not imply $T = 0$.

A linear relation $T \in LR(X)$ is said to be continuous if $\|T\| < \infty$, bounded, denoted $T \in BR(X)$, if T is continuous and $D(T) = X$, open if T^{-1} is continuous equivalently if $\gamma(T) > 0$ where $\gamma(T) := \sup\{\lambda \geqslant 0 : \lambda \mathrm{dist}(x, N(T)) \leqslant \|Tx\|, x \in D(T)\}$ and T is called closed if its graph is a closed subspace.

Throughout this paper X will denote an infinite dimensional Banach space and T will always denote an element of $LR(X)$ except where stated otherwise. We will denote the set of nonnegative integers by $\mathbb{N}$.

The importance of investigation of linear relations is demonstrated by the necessity of taking conjugates of operators with a non-dense domain used in applications to the theory of generalized differential equations ([9] and [25]) or by the need of considering the inverse of certain operators, used, for example, in the study of some Cauchy problems associated to parabolic type equations in Banach spaces [13]. The spectral theory of multivalued linear operators provides tools for studying important problems of operator theory as for example: the pseudoresolvent theory for operators, the spectral theory of ordered pair of operators and the study of linear bundles (see [7], [14] and the references therein).

Interesting works on multivalued linear operators include the treatise on partial differential relations by Gromov [20], the application of multivalued methods to solution of differential equations by Favini and Yagi [13], the development of fixed point theory for linear relations to the existence of mild solutions of quasilinear differential inclusions of evolution and also to many problems of fuzzy theory (see, for example [1], [18], [24] and [27]), the application of multivalued methods to Invariant Subspace problems ([19], [30]) and several papers on linear relations type semiFredholm in normed spaces and in Hilbert spaces (see [2], [4], [5], [11] and [15] among others).

Recall that a subspace M of X is called paracomplete or an operator range or an endomorphism range if it is the image of an injective bounded operator defined on some Banach space Z (see, for example, [10]).

We note that the injectivity condition is not too restrictive since if M is the image of a bounded operator S with domain a Banach space Y, we could consider the injective operator $\widehat{S}$ defined by $\widehat{S} : [y] \in Y/N(S) \to \widehat{S}[y] := Sy$. Therefore M is an operator range with $Z := Y/N(S)$.

Many normed spaces that appear in applications are paracomplete subspaces, like the space $C[0,1]$ with the norm of $L_2[0,1]$, some Sobolev spaces with suitable L_2-norms and the domain and range of a linear relation with complete graph are operator ranges. But not all subspaces of a Banach space are paracomplete; for example the null space of a discontinuous linear functional.

Operator ranges are natural objects of investigation because every operator range is the domain of a closed operator (see Lemma 1.6 below). There are many reasons for their investigation, for example, one is that the Burnside theorem on invariant subspaces of algebras of operators in finite dimensional spaces admits an adequate generalization to strongly closed algebras of operators in Hilbert spaces in terms of invariant operator ranges and other reason for their investigation is that every bounded operator defined on an operator range has a compact spectrum set [5].

Paracomplete subspaces in Banach spaces were studied in the papers [5], [6], [12], [17], [21], [26] and [31] among others.

1. BASIC ALGEBRAIC RESULTS AND PARACOMPLETE LINEAR RELATIONS

We begin this section giving some entirely algebraic results about linear relations in vector spaces which will be need in Section 2. In the second part of this section we define and obtain several properties of paracomplete linear relations in Banach spaces which we will use in Section 2.

The following result, together with its proof, can be found in [28].

LEMMA 1.1. *Let A be a linear relation in a vector space E and let $\lambda \in \mathbb{K}$. Then*
(i) *For all $n, m \in \mathbb{N}$*

$$D(A^{n+m}) \subset D(A^n), R(A^{n+m}) \subset R(A^n),$$
$$N(A^{n+m}) \supset N(A^n), A^{n+m}(0) \supset A^n(0),$$
$$N(A^n) \subset D(A^m), A^n(0) \subset R(A^m).$$

(ii) *For all* $n, m \in \mathbb{N}$

$$D((\lambda - A)^n) = D(A^n), (\lambda - A)^n(0) = A^n(0),$$
$$N((\lambda - A)^n) \subset D(A^m), A^n(0) \subset R((\lambda - A)^m).$$

We now establish the following four lemmata which were proved by Burlando ([8], 2.1, 2.2, 2.3 and 2.4) for the case when A is an operator.

LEMMA 1.2. *let A be a linear relation in a vector space E and let p be a nonzero polynomial with coefficients in $\mathbb{K}$. Then*

(i) $N(p(A)) \subset \bigcap_{m \in \mathbb{N}} D(A^m)$.

(ii) $N(p(A)) \subset D(q(A))$ *and* $q(A)N(p(A)) \subset N(p(A)) + q(A)(0)$ *for any polynomial q with coefficients in* $\mathbb{K}$

Proof. (i) Follows immediately from Lemma 1.1.

(ii) The first inclusion of this assertion is an obvious consequence of Lemma 1.1. Observe that clearly $p(A)$ and $q(A)$ commute, that is, $p(A)q(A) = q(A)p(A)$ in the sense of the product of linear relations. Therefore we have that $N(p(A)) \subset N(q(A)p(A))$ (Lemma 1.1) $= N(p(A)q(A))$ so that we have the following which completes the proof:

$$\begin{aligned} q(A)N(p(A)) &\subset q(A)N(p(A)q(A)) = q(A)q(A)^{-1}N(p(A)) \\ &= \{N(p(A)) \cap R(p(A))\} + q(A)(0) \quad ([11], \text{I.3.1}) \\ &\subset N(p(A)) + q(A)(0). \quad \blacksquare \end{aligned}$$

LEMMA 1.3. *Let A be a linear relation in a vector space E and let p, q be polynomials with coefficients in $\mathbb{K}$ without common roots in $\mathbb{C}$. Then*

(i) $N(p(A)) + q(A)(0) = q(A)N(p(A))$.

(ii) $R(p(A)) \cap R(q(A)) = R(p(A)q(A))$.

Proof. (i) According to Lemma 1.2, it remains to show that

$$N(p(A)) + q(A)(0) \subset q(A)N(p(A)).$$

Since p and q have no common roots in $\mathbb{K}$, there are polynomials u and v with coefficients in $\mathbb{K}$ such that $p(\lambda)u(\lambda) + q(\lambda)v(\lambda) = 1$ for every $\lambda \in \mathbb{K}$ and hence

$$p(A)u(A) + q(A)v(A) = I. \tag{1.1}$$

Let $x \in N(p(A)) + q(A)(0)$, so that $x = x_1 + x_2$ for some $x_1 \in N(p(A))$ and some $x_2 \in q(A)(0)$. Since $x_1 \in N(p(A))$, we have

$$\begin{aligned} &p(A)u(A)x_1 = p(A)u(A)(0) \quad \text{(by Lemma 1.1 and I.2.8 of [11]);} \\ &v(A)x_1 \subset N(p(A)) + v(A)(0) \quad \text{(by Lemma 1.2).} \end{aligned} \tag{1.2}$$

Combining (1.1) and (1.2) we obtain

$$\begin{aligned}x &\in p(A)u(A)x_1 + q(A)v(A)x_1 + q(A)(0)\\ &\subset p(A)u(A)(0) + q(A)N(p(A)) + q(A)v(A)(0) + q(A)(0)\\ &= q(A)N(p(A)) + q(A)(0) = q(A)N(p(A)).\end{aligned}$$

(ii) This statement is immediate from 3.3 of [28]. ∎

LEMMA 1.4. *Let A be a linear relation in a vector space E and let p be a polynomial of degree n, with coefficients in $\mathbb{K}$. Then for any $m \in \mathbb{N}$ we have that $p(A)D(A^{n+m}) = p(A)(0) + \{R(p(A)) \cap D(A^m)\}$.*

Proof. Since $D(p(A)) = D(A^n)$ we infer that $D(A^{n+m}) = D(p(A)A^m) = D(A^m p(A))$ and so from this equality combined with I.3.1 of [11] we conclude that $p(A)D(A^{n+m}) = p(A)(0) + \{R(p(A)) \cap D(A^m))\}$. ∎

LEMMA 1.5. *Let A be a linear relation in a vector space E. If for some $m \in \mathbb{N}$, $\dim(R(A^m)/R(A^{m+1})) < \infty$, then*

$$\dim(R(A^n)/R(A^{n+k})) < \infty \quad \textit{for any } n \geqslant m \textit{ and for any } k \in \mathbb{N}.$$

Proof. Since for each $i \in \mathbb{N}$ one has

$$\begin{aligned}(D(A^i) + R(A))/(N(A^i) + R(A)) &\cong D(A^i)/(D(A^i) \cap \{N(A^i) + R(A)\})\\ &\cong R(A^i)/R(A^{i+1}) \quad \text{(by 2.3 and 4.1 of [29])}\end{aligned}$$

and $D(A^{n+k}) \subset D(A^m)$ and $N(A^m) \subset N(A^{n+k})$ if $n \geqslant m$ (by Lemma 1.1). It now is easy to verify the desired statement. ∎

The following elementary lemma concerning paracomplete subspaces will be useful in the sequel.

LEMMA 1.6. *Let M be a subspace of a Banach space X. The following properties are equivalent:*

(i) *M is an operator range.*

(ii) *There exists a norm $\|\cdot\|_M$ on M such that $(M, \|\cdot\|_M)$ is a Banach space and $\|m\| \leqslant \|m\|_M$ for any $m \in M$.*

(iii) *M is the domain of a closed operator $S : D(S) \subset X \to Y$ where Y is a Banach space.*

Proof. (i) $\Rightarrow$ (ii) Let M be the image of a bounded operator defined in a Banach space $(M_1, \|\cdot\|_1)$ and we denote the operator mapping M_1 onto M by α_M and the inverse of α_M by β_M.

Renorm M with $\|m\|_M := \|m\| + \|\beta_M m\|_1$ for $m \in M$. It is easy to see that $\|\cdot\|_M$ is a complete norm on M.

(ii) $\Rightarrow$ (i) It is obvious, since the identity map from $(M, \|\cdot\|_M)$ onto $(M, \|\cdot\|)$ is bounded.

(i) $\Rightarrow$ (iii) If (i) holds then β_M satisfies (iii).

(iii) $\Rightarrow$ (i) Assume that M is the domain of a closed operator $S : D(S) \subset X \to Y$ where Y is a Banach space.

Renorm M as follows: $\|m\|_M := \|m\| + \|Sm\|$, $m \in M$. It is easy to prove that this quantity defines a complete norm on M, as desired. ∎

The notion of a paracomplete operator due to 2.1.2 of [8], can be naturally generalized to linear relations.

DEFINITION 1.7. T is said to be *paracomplete*, denoted $T \in PC(X)$, if its graph is a paracomplete subspace of $X \times X$.

Observe that a subclass of paracomplete linear relations in Hilbert spaces is given by the class of quasi-Fredholm relations introduced and studied in [22].

PROPOSITION 1.8. *Let $T \in LR(X)$ with $T(0)$ closed. Then T is paracomplete if and only if so is $Q_T T$.*

Proof. It is easy to prove that

$$\text{(1.3)} \qquad \textit{If } T(0) \textit{ is closed, then } (x,y) \in G(T) \Leftrightarrow (x, Q_T y) \in G(Q_T T).$$

Assume that T is paracomplete. Then $G(T)$ is the image of a bounded operator S defined in a Banach space Z. Now, if we consider the map $(I_X, Q_T) : (x,y) \in X \times X \to (x, Q_T y) \in X \times X/T(0)$ we obtain immediately from (1.3) that $G(Q_T T)$ is the image of the bounded operator $(I_X, Q_T)S$. Therefore, $G(Q_T T)$ is an operator range. Conversely, let $G(Q_T T)$ be a paracomplete subspace of $X \times X/T(0)$. Then by Lemma 1.6, there exist a closed operator U and a Banach space W such that $U : D(U) = G(Q_T T) \subset X \times X/T(0) \to W$ and applying again (1.3) we deduce that $G(T)$ is the domain of the closed operator $U(I_X, Q_T)$. ∎

Evidently, every closed linear relation is paracomplete. However there exists a paracomplete linear relation T in a Banach space such that $T(0)$ is not closed (and hence T is not closed by II.5.3 of [11]). Indeed, it follows from 1.1 of [8] that every infinite dimensional Banach space X has a non-closed paracomplete subspace M. Thus, let $T \in LR(X)$ be the linear relation whose graph is $X \times M$. Then it is clear that T is paracomplete and $T(0) = M$ is not closed.

PROPOSITION 1.9. *Let T be paracomplete. Then:*
(i) *$D(T)$ and $R(T)$ are paracomplete subspaces of X.*
(ii) *If* $\dim X/R(T) < \infty$, *then $R(T)$ is closed.*
(iii) *If $D(T) = X$ and $T(0)$ is closed, then T is bounded and closed.*

Proof. (i) Since the sum of two paracomplete subspaces of a Banach space is a paracomplete subspace ([21], 2.2) and $D(T) \times Y = G(T) + (\{0\} \times X), X \times R(T) = G(T) + (X \times \{0\})$ ([11], I. 3.1), it follows the desired assertion.

(ii) Combining (i) and 2.1.1 of [21].

(iii) By Proposition 1.8, $Q_T T$ is a paracomplete operator with $D(Q_T T) = D(T)$ and thus from 2.1.5 of [21], $Q_T T$ is bounded and since $\|T\| := \|Q_T T\|$ ([11], II.1.3), $T \in BR(X)$ and now the closedness of T follows from II.5.1 of [11]. ∎

PROPOSITION 1.10. *Let $S, T \in PC(X)$ such that $S(0) = \overline{S(0)} \subset T(0) = \overline{T(0)}$ and $D(T) \subset D(S)$. Then $T + S \in PC(X)$.*

Proof. Since $S(0) = \overline{S(0)} \subset T(0) = \overline{T(0)}$ we have that $T(0) = (T+S)(0)$ is closed, $Q_T = Q_{T+S}$ and $Q_T = Q^{X/S(0)}_{T(0)/S(0)} Q_S$ by virtue of IV.5.2 in [11]. These properties combined with Proposition 1.8 and the fact that the class of paracomplete operators is closed under sum and product ([21], 2.1.3) yield $Q_{T+S}(T+S) = Q_T T + Q_T S$ is a paracomplete operator and it follows again from Proposition 1.8 that $T + S$ is a paracomplete linear relation. ∎

Concerning product of paracomplete linear relations we will only need two properties which are established in the following propositions.

PROPOSITION 1.11. *Let $T, S \in PC(X)$. Then:*

(i) *$TS \in PC(X)$ if T is paracomplete and S is a bounded single valued.*

(ii) *$TS \in PC(X)$ if S is paracomplete and there is a bounded injective operator P such that $T = P^{-1}$.*

Proof. (i) It is clear that $G(TS) = \{(x,z) : (Sx, z) \in G(T)\}$. Now since $G(T)$ is paracomplete, $G(T)$ is the domain of a closed operator U (Lemma 1.6) and it is easy to show that then $G(TS)$ is the domain of the closed operator $U(S, I_X)$ and applying again Lemma 1.6 we obtain that TS is paracomplete.

(ii) By Lemma 1.6, there is a Banach space R and a closed operator $V : D(V) \subset X \times X \to R$ such that $D(V) = G(S)$. Moreover, by the injectivity of T we deduce that $G(TS) = \{(x, Ty) : (x,y) \in G(S)\}$ and now it is immediate to prove that $V(I_X, P)$ is a closed operator with domain $G(TS)$ and thus by Lemma 1.6, TS is a paracomplete linear relation, as desired. ∎

For a linear relation $A \in LR(E)$ where E is a vector space, Sandovici, Snoo and Winkler ([29], 3.2) introduced the concept of singular chain manifold by

$$R_c(A) := \Big(\bigcup_{n=1}^{\infty} N(A^n) \Big) \cap \Big(\bigcup_{n=1}^{\infty} A^n(0) \Big)$$

and they proved that many of the results concerning the relationship between ascent, descent, nullity and defect for the case operators remain valid in the context of linear relations only under the additional condition that the linear relation has a trivial singular chain manifold, that is, $R_c(A) = \{0\}$.

PROPOSITION 1.12. *Let $T \in PC(X)$ with $R_c(T) = \{0\}$. Then T^n is paracomplete for any $n \in \mathbb{N}$.*

Proof. Clearly, the condition $R_c(T) = \{0\}$ implies that if $(x,y) \in G(T^{n+m})$ with $n, m \in \mathbb{N}$ then there exists an unique $z \in X$ such that $(x,z) \in G(T^m)$ and $(z,y) \in G(T^n)$.

The proof is by induction. For $n = 1$ the result is trivial. Assume the assertion for n, and let $(x,y) \in G(T^{n+1})$. Then there is an unique $z \in X$ such that $(x,z) \in G(T)$ and $(z,y) \in G(T^n)$ and it is easy to see that $\|(x,y)\|^2_{G(T^{n+1})} := \|(x,z)\|^2_{G(T)} + \|(z,y)\|^2_{G(T^n)}$ defines a norm on $G(T^{n+1})$ and proceeding as for the single valued case ([21], 2.1.3), we conclude that $\|\cdot\|_{G(T^{n+1})}$ is a complete norm such that the canonical embedding of $(G(T^{n+1}), \|\cdot\|_{G(T^{n+1})})$ into $G(T^{n+1})$ is continuous, that is, $G(T^{n+1})$ is paracomplete. ∎

We do not know if, in general, the product of two paracomplete linear relations is a paracomplete linear relation.

2. THE BAIRE PROPERTY AND THE DOMAIN OF ITERATES OF A PARACOMPLETE LINEAR RELATION

Recall the following result due to Burlando ([8], 2.10) related to paracomplete operators T with nonempty lower essential spectrum, that is, such that the set $\rho_{PC_-}(T) := \{\lambda \in \mathbb{K} : \lambda - T \in PC(X), \dim X/R(\lambda - T) < \infty\}$ is nonempty.

THEOREM 2.1. *Let X be a complex Banach space and let T be an operator whose domain is a subspace of X, whose range is contained in X and its lower essential resolvent set is nonempty. If $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are scalar sequences such that $(\alpha_n T + \beta_n I_X)(D(T))$ is dense in X for any nonnegative integer n, then*

$$\bigcap_{n=1}^{\infty} (\alpha_0 T + \beta_0 I_X) \cdots (\alpha_{n-1} T + \beta_{n-1} I_X)(D(T^n)) \quad \text{is dense in } X.$$

It is the purpose of this section to extend the above theorem to multivalued linear operators. For this end, we shall use Lemmata 1.1, 1.2, 1.3, 1.4 and 1.5, the properties concerning paracomplete subspaces and paracomplete linear relations established in Section 1 and also the following Baire property for linear relations.

THEOREM 2.2 (The Baire property for linear relations). *Let $(X_n, \|\cdot\|_n)_{n=0}^{\infty}$ be a sequence of Banach spaces and let $(T_n)_{n=1}^{\infty}$ be a sequence of bounded linear relations such that each T_n maps X_n into X_{n-1} with dense range and $T_n(0)$ is closed for any $n \geqslant 1$. Then*

$$\bigcap_{n=1}^{\infty} T_1 T_2 \cdots T_n X_n \quad \text{is dense in } (X_0, \|\cdot\|_0).$$

Proof. For $m, n \in \mathbb{N}$ with $m \leqslant n$ we define

$$f_{m,n} := T_{m+1} T_{m+2} \cdots T_n \quad \text{if } m < n, \quad \text{and} \quad f_{m,n} := I_{X_m} \quad \text{if } m = n.$$

Let $x_o \in X_o$ and let $\varepsilon > 0$. For $n \geqslant 1$, select $x_n \in X_n$ such that

$$d(x_{n-1}, f_{n-1,n}(x_n)) < \frac{\varepsilon 2^{-(n-1)}}{\|T_1\|\|T_2\|\|T_{n-1}\|} \quad \text{if } n > 1, \tag{2.1}$$

where T_o is considered to be the identity in X_o.

For any $n \geqslant 1$, it is clear that $T_n(0) \subset R(T^n) \subset X_{n-1} = D(T_{n-1})$ with T_n and T_{n-1} continuous and thus by II.3.13 of [11], $\|T_{n-1}T_n\| \leqslant \|T_{n-1}\|\|T_n\|$.

Fixing $m \in \mathbb{N}$, it follows that for $n \geqslant m$

$$\|f_{m,n}\| \leqslant \|T_{m+1}\| \cdots \|T_n\| \quad \text{if } m < n, \quad \text{and} \quad \|f_{m,n}\| = 1 \quad \text{if } m = n. \tag{2.2}$$

Since $d(f_{m,n}(x_n), f_{m,n+1}(x_{n+1})) = d(f_{m,n}(x_n), f_{m,n}(f_{n,n+1}(x_{n+1})) \leqslant \|f_{m,n}\| d(x_n, f_{n,n+1}(x_{n+1})) \leqslant \varepsilon 2^{-n}/(\|T_1\| \cdots \|T_m\|)$ (by (2.1) and (2.2)) we have that

$$d(f_{m,n}(x_n), f_{m,n+1}(x_{n+1})) \leqslant \varepsilon \frac{2^{-n}}{(\|T_1\| \cdots \|T_m\|}. \tag{2.3}$$

Furthermore if $p \geqslant m$ and $q > 0$, by using (2.1), (2.2) and (2.3) we obtain

$$\begin{aligned}
&d(f_{m,p}(x_p), f_{m,p+q}(x_{p+q})) \\
&= d(f_{m,p}(x_p), f_{m,p}(f_{p,p+q}(x_{p+q})) \leqslant \|f_{m,p}\| d(x_p, f_{p,p+q}(x_{p+q})) \\
&\leqslant \|f_{m,p}\| [d(x_p, f_{p,p+1}(x_{p+1})) + d(f_{p,p+1}(x_{p+1}), f_{p+1,p+2}(x_{p+2})) \\
&\qquad\qquad + \cdots + d(f_{p,p+q-1}(x_{p+q-1}), f_{p,p+q}(x_{p+q}))] \\
&\leqslant \frac{\|f_{m,p}\| \varepsilon \sum_{i=p}^{i=p+q-1} 2^{-i}}{\|T_1\| \cdots \|T_p\|}.
\end{aligned}$$

In particular, if $p = m$ and $n > m$ we have

$$d(x_m, f_{m,n}(x_n)) < \frac{\varepsilon}{\|T_1\| \cdots \|T_m\|} \sum_{i=m}^{n} 2^{-i} := \delta_{m,n}. \tag{2.4}$$

For each $n > m$, choose $z_{m,n} \in f_{m,n}(x_n) = T_{m+1}T_{m+2} \cdots T_n(x_n) \subset R(T_{m+1}) \subset X_m$ such that $\|x_m - z_{m,n}\| < \delta_{m,n} + \varepsilon/(\|T_1\| \cdots \|T_m\|)$. Then, it follows from the above that $(z_{m,n})_{n>m}$ is a Cauchy sequence and since X_m is complete, $(z_{m,n})_{n>m}$ converges to some $z_m \in X_m$. Furthermore $\|x_m - z_m\| < (2\varepsilon)/(\|T_1\| \cdots \|T_m\|)$.

Now, for $n \geqslant m+1$ we have

$$\begin{aligned}
d(z_{m,n}, f_{m,m+1}(z_{m+1,n})) &\leqslant d(z_{m,n}, f_{m,n}(x_n)) + d(f_{m,n}(x_n), f_{m,m+1}(z_{m+1,n})) \\
&= d(f_{m,m+1}(f_{m+1,n}(x_n)), f_{m,m+1}(z_{m+1,n})) \\
&\leqslant \|f_{m,m+1}\| d(z_{m+1,n}, f_{m+1,n}(x_n)) = 0 \quad (\text{as } z_{m,n} \in f_{m,n}(x_n)))
\end{aligned}$$

that is

$$\text{for } n \geqslant m+1, \quad d(z_{m,n}, f_{m,m+1}(z_{m+1,n})) = 0. \tag{2.5}$$

Moreover, since $z_{m,n} \to z_m$ if $n \to \infty$ and $f_{m,m+1} = T_{m+1}$ is continuous it follows that

$$d(z_m, f_{m,m+1}(z_{m+1})) = 0 \tag{2.6}$$

that is, $z_m \in \overline{f_{m,m+1}(z_{m+1})}$ and thus

$$\begin{aligned} &Q_{f_{m,m+1}} z_m \\ &\in Q_{f_{m,m+1}} \overline{f_{m,m+1}(z_{m+1})} \subset \overline{Q_{f_{m,m+1}} f_{m,m+1}(z_{m+1})} \\ &= Q_{f_{m,m+1}} f_{m,m+1}(z_{m+1}) \quad (\text{since } Q_{f_{m,m+1}} f_{m,m+1} = Q_{T_{m,m+1}} T_{m+1} \text{ is single valued}) \\ &\Rightarrow z_m \in f_{m,m+1}(z_{m+1}) + N(Q_{f_{m,m+1}}) = f_{m,m+1}(z_{m+1}) + \overline{f_{m,m+1}}(0), \end{aligned}$$

and since $f_{m,m+1}(0) = T_{m+1}(0)$ is closed, we obtain that

$$z_m \in f_{m,m+1}(z_{m+1}). \tag{2.7}$$

Since $m \in \mathbb{N}$ was fixed arbitrarily, it follows that $z_o \in f_{0,1}(z_1) = T_1 z_1 \subset T_1 X_1, z_1 \in f_{1,2}(z_2) = T_2 z_2 \subset T_2 X_2$ and thus $z_o \in T_1 T_2 X_2$ and continuing in this way, we obtain that $z_o \in \bigcap_{n=1}^{\infty} T_1 T_2 \cdots T_n X_n$ and, since $\|x_o - z_o\| < 2\varepsilon$ with $\varepsilon > 0$ arbitrary, we conclude that $\bigcap_{n=1}^{\infty} T_1 T_2 \cdots T_n X_n$ is dense in X_o, as desired. ∎

The notion of lower essential resolvent set of an operator introduced in [8] is generalized to linear relations as follows.

DEFINITION 2.3. Let $T \in LR(X)$ where X is a complex Banach space. The *lower essential resolvent set* of T is defined by $\rho_{PC_-}(T) := \{\lambda \in \mathbb{C} : \lambda - T \in PC(X), \dim X/R(\lambda - T) < \infty\}$.

THEOREM 2.4. *Let X be a complex Banach space and let $T \in LR(X)$ such that $R_c(T) = \{0\}$, $T^n(0)$ is closed for any $n \in \mathbb{N}$ and $\rho_{PC_-}(T) \neq \emptyset$. If $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are scalar sequences such that $(\alpha_n T + \beta_n I_X)(D(T))$ is dense in X for any $n \in \mathbb{N}$, then*

$$\bigcap_{n=1}^{\infty} (\alpha_o T + \beta_o I_X) \cdots (\alpha_{n-1} T + \beta_{n-1} I_X)(D(T^n)) \quad \textit{is dense in } X.$$

In particular, it follows that

(i) $\bigcap_{n=1}^{\infty} D(T^n)$ *is dense in* X.

(ii) $\bigcap_{n=1}^{\infty} R(T^n)$ *is dense in* X *if* $R(T)$ *is dense in* X.

Proof. Since T has a lower essential resolvent set nonempty and $T(0)$ is closed we obtain from Proposition 1.10 that T is a paracomplete linear relation and since T has a trivial singular chain manifold we conclude from Proposition 1.12 that $T^n \in PC(X)$ for any $n \in \mathbb{N}$. This last fact and Proposition 1.9 imply that for each $n \in \mathbb{N}$, one has that $D(T^n)$ is a paracomplete subspace of X.

Now for $n \in \mathbb{N}$, we define $X_o := (X, \|\cdot\|_o)$ where $\|\cdot\|_o$ denotes the norm of X and if $n \geqslant 1$, $X_n := (D(T^n), \|\cdot\|_n)$ where $\|\cdot\|_n$ denote a complete norm on $D(T^n)$ such that the canonical injection Γ_n from X_n into X is continuous.

For any $n \in \mathbb{N}$, we define the linear relations $T_n, J_n : X_{n+1} \to X_n$ by

$$T_n x := Tx \quad \text{and} \quad J_n x := x \quad \text{for any } x \in X_{n+1}.$$

Let $n \in \mathbb{N}$. Clearly $T_n = \Gamma_n^{-1} T \Gamma_{n+1}$. Since T is paracomplete and Γ_{n+1} is a bounded operator, $T\Gamma_{n+1}$ is paracomplete by virtue of Proposition 1.11 and thus $\Gamma_n^{-1}(T\Gamma_{n+1})$ is also a paracomplete linear relation by Proposition 1.11. Hence

$$T_n \in PC(X_{n+1}, X_n). \tag{2.8}$$

Furthermore, by using I.1.2 and I.1.3 of [11] we have that

$$\begin{aligned} D(T_n) &:= D(\Gamma_n^{-1} T \Gamma_{n+1}) = (T\Gamma_{n+1})^{-1} D(\Gamma_n^{-1}) = \Gamma_{n+1}^{-1} T^{-1} D(\Gamma_n^{-1}) \\ &= \Gamma_{n+1}^{-1} T^{-1} R(\Gamma_n) = \Gamma_{n+1}^{-1} T^{-1} D(T^n) = \Gamma_{n+1}^{-1} D(T^{n+1}) = D(T^{n+1}). \end{aligned}$$

Consequently

$$D(T^n) = D(T^{n+1}). \tag{2.9}$$

We note that $D(T^2) = T^{-1}D(T) := \{x \in D(T) : Tx \cap D(T) \neq \emptyset\}$ ([11], I.1.2 and I.1.3), so that $T(0) \subset D(T)$ and continuing in this way, $T(0) \subset D(T^n)$ for all $n \in \mathbb{N}$. In this situation, we have that $T_n(0)$ is closed since

$$\Gamma_n \overline{T_n(0)} \subset \overline{\Gamma_n T_n(0)} = \overline{I_{R(\Gamma_n)} T(0)} = \overline{I_{D(T^n)} T(0)} = \overline{T(0)} = T(0)$$

which implies that

$$\overline{T_n(0)} \subset \Gamma_n^{-1} T(0) = T_n(0).$$

Now, it follows immediately from (2.8), (2.9) and Proposition 1.9 that

$$T_n \in BR(X_{n+1}, X_n) \quad \text{and} \quad \alpha T_n + \beta J_n \in BR(X_{n+1}, X_n) \quad \text{for all } \alpha, \beta \in \mathbb{C}. \tag{2.10}$$

Let $\alpha, \beta \in \mathbb{C}$ such that $(\alpha T + \beta I_X)(D(T))$ is dense in X. We shall prove that then

$$(\alpha T_n + \beta J_n)(D(T^{n+1}) \quad \text{is dense in } (D(T^n), \|\cdot\|_n) \text{ for any } n \in \mathbb{N}.$$

Let $\eta \in \rho_{PC_-}(T)$, so that $\eta - T \in PC(X)$ and $\dim X/R(\eta - T) < \infty$. Let $n \in \mathbb{N}$. Then arguing as in the single valued case (see the proof of 2.10 in [8]) by using the algebraic results obtained in Section 1 we obtain that

$$\begin{aligned} &N((\eta - T)^n) \subset R(\alpha T_n + \beta J_n) \quad \text{and} \\ &R((\eta - T)^n) \cap (\alpha T + \beta I_X)(D(T)) = (\eta - T)^n (R(\alpha T_n + \beta J_n)). \end{aligned} \tag{2.11}$$

Furthermore, it is known ([29], 7.1), that if A is a linear relation in a vector space and $\alpha \in \mathbb{K}$, then the singular chain manifold of A coincides with the singular chain manifold of $\alpha - T$. This fact combined with Proposition 1.12 yields $(\eta - T)^n \in PC(X)$ if $\eta \in \rho_{PC_-}(T)$. Now, from Proposition 1.11, $(\eta - T)^n \Gamma_n \in PC(X_n, X)$ and since $R(\Gamma_n) = D(T^n) = D((\eta - T)^n)$ and $T^n(0)$ is closed we deduce applying Proposition 1.9 that $(\eta - T)^n \Gamma_n \in BR(X_n, X)$ and it is clear that $R((\eta - T)^n \Gamma_n) = R((\eta - T)^n)$.

Furthermore, by virtue of Lemma 1.5 we have that $\dim X/R((\eta - T)^n) < \infty$ and therefore $R((\eta - T)^n \Gamma_n) = R((\eta - T)^n)$ is a closed subspace of X and thus it follows from Closed Graph Theorem for linear relations ([11], III.5.4) that $(\eta - T)^n \Gamma_n$ is open and proceeding exactly as in the proof of 2.10 of [8] we conclude that

$$R(\alpha T_n + \beta J_n) \quad \text{is dense in } X_n. \tag{2.12}$$

Now, by (2.11), (2.12) and the Baire property for linear relations (Theorem 2.2) we have that

$$\bigcap_{n=1}^{\infty} (\alpha_0 T + \beta_0 I_X)(\alpha_1 T + \beta_1 I_X) \cdots (\alpha_{n-1} T + \beta_{n-1} I_X)(D(T^n)) \quad \text{is dense in } X_0.$$

If $\alpha_n = 0$ and $\beta_n = 1$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} D(T^n)$ is dense in X. Similarly, (ii) follows if we take $\alpha_n = 1$ and $\beta_n = 0$ for all $n \in \mathbb{N}$. ∎

Theorem 2.4 has the following interesting consequence.

COROLLARY 2.5. *Let X be a complex Banach space and let $T \in LR(X)$ be closed and continuous such that $\overline{D(T)} = X$ and $\rho(T) \neq \emptyset$. If*

$$(\alpha_n T + \beta_n I_X)(D(T)) \quad \textit{is dense in } X \textit{ for any } n \in \mathbb{N},$$

then

$$\bigcap_{n=1}^{\infty} (\alpha_o T + \beta_o I_X) \cdots (\alpha_{n-1} T + \beta_{n-1} I_X(D(T^n)) \quad \textit{is dense in } X.$$

In particular, it follows that:

(i) $\bigcap_{n=1}^{\infty} D(T^n)$ *is dense in X.*

(ii) $\bigcap_{n=1}^{\infty} R(T^n)$ *is dense in X if $R(T)$ is dense in X.*

Proof. Let $T \in LR(X)$ be closed and continuous such that $\overline{D(T)} = X$ and $\rho(T) \neq \emptyset$. Then $T \in PC(X)$ (as T is closed). Assume that $\eta \in \rho(T) \subset \rho_{PC_-}(T)$. Then $\eta - T$ is closed and bijective and thus it follows from VI.5.2 of [11] that for each $n \in \mathbb{N}$ $(\eta - T)^n$ has the same properties; in particular $N(\eta - T) \cap R(\eta - T) = \{0\}$ which implies that $R_c(\eta - T) = \{0\}$ by virtue of 3.3 in [29] and $(\eta - T)^n(0)$ is closed. Hence the result follows by Theorem 2.4 upon noting that $R_c(T) = R_c(\alpha - T)$ for any $\alpha \in \mathbb{C}\backslash\{0\}$ by 7.1 of [29] and that for each $n \in \mathbb{N}$ one has $(\eta - T)^n(0) = T^n(0)$. ∎

We note that the result of Corollary 2.5 extends a result due to Lennard [23].

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