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SUPER STRICTLY SINGULAR AND COSINGULAR OPERATORS AND RELATED CLASSES

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ABSTRACT. The notions of super strictly singular and cosingular operators are revisited and new characterizations given in terms of ultrapowers and operator local representability. The behaviour of the associated (Bernstein and Mityagin) s-numbers with respect to duality, ultraproducts and local representability is considered. We also give properties of these classes in the Banach lattice setting like a Dodds–Fremlin type domination result and introduce the class of super disjointly stricty singular operators. The equivalence between lattice strictly singular and disjointly strictly singular operators is considered, and we show that at the super level both classes coincide.

KEYWORDS: *Super striclty singular, cosingular operators.*

MSC (2000): 47B10, 47B65.

INTRODUCTION

Super strictly singular operators were introduced implicitly by Mityagin and Pełczyński in [34] and explicitly by Milman in [32] and [33]. This class of operators contains compact operators and it is contained in the class of strictly singular operators. Recall that a bounded operator is *strictly singular* (SS for short), or Kato [25] if its restriction to any infinite-dimensional subspace is never an isomorphism.

Given two Banach spaces X and Y, a bounded operator $T : X \to Y$ is *super strictly singular* (super-SS for short), or *finitely strictly singular*, if there does not exist a number $c > 0$ and a sequence of subspaces E_n of X, with $\dim E_n = n$, such that

$$\|Tx\| \geqslant c\|x\| \quad \text{for all } x \in \bigcup_n E_n.$$

In other words, T is super-SS if and only if the *Bernstein* numbers $b_n(T) \to 0$, as $n \to \infty$, where

$$b_n(T) = \sup \inf_{x \in S(E_n)} \|Tx\|,$$

and the supremum is taken among all n-dimensional subspaces E_n of X and $S(E_n)$ denotes the unit sphere of E_n. Clearly super-strict singularity implies strict singularity.

In [33] Milman proved that super-SS operators form a closed operator ideal in the sense of Pietsch [39].

Recently the dual point of view has been considered by Plichko in [41] where the class of *super strictly cosingular* operators (super-SC for short) is introduced, showing that both classes are in full duality. The precise definition of super-SC operators will be given in Section 2. Recall that an operator T between two Banach spaces X and Y is called *strictly cosingular* (SC for short) or Pełczyński [36] if for every infinite codimensional subspace E of Y the composition operator $\pi_E T$ is not surjective (here π_E denotes the quotient map from Y onto Y/E). This operator class is (partially) related by duality to that of strictly singular operators: If the conjugate operator T^* is SS (respectively SC), then T is SC (respectively SS), but the converses are not true in general ([36], [37]).

In this paper we consider the classes of super-SS and super-SC operators, and other closely related, with several purposes. One of these purposes is to establish that these classes constitute the super version of the classes of strictly singular (respectively cosingular) operators, in the following sense: an operator $T : X \to Y$ is super-SS (respectively super-SC) if and only if for every ultrafilter $\mathcal{U}$ the corresponding ultraoperator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is SS (respectively SC). It turns out that this is also equivalent to the fact that every operator locally representable in T (in the sense of Pietsch [40]) is SS (respectively SC) (see Proposition 1.5, Proposition 1.12 and Theorem 2.16). We also focus on reshaping somewhat the definition of super strict cosingularity in order to eliminate some inconsistencies resulting from the definition given in [41] (see Example 2.1 and Definition 2.10 below). It turns out that the super strict cosingularity is characterized by means of another class of s-numbers considered by Pietsch [38], namely Mityagin numbers. Recall that for an operator $T:X\to Y$ the *Mityagin* number $a_n(T)$ is defined as

$$a_n(T) = \sup\{q(\pi_E T);\ \operatorname{codim} E = n\}$$

where π_E denotes the quotient map $X \to Y/E$ and q is the modulus of surjectivity. Thus T is super strictly cosingular if the sequence $(a_n(T))$ of its Mityagin numbers converges to zero. With this modified definition of super strict cosingularity the full duality of super-SS and super-SC classes, stated by Plichko ([41], Theorem 4), is obtained as a direct consequence of a duality relation linking Bernstein and Mityagin numbers (see Proposition 2.6).

A third purpose is to study the problem of domination for these super operator classes in Banach lattices, motivated by the well-known result by Dodds and Fremlin concerning compact operators (cf. [2] and [46]) and the results obtained for strictly singular and cosingular operators in [14] and [15]. More precisely, let E and F be two Banach lattices and $0 \leqslant R \leqslant T : E \to F$ two positive operators and assume that T belongs to a given operator class; must R belong to that same class? The goal is to obtain positive answers by imposing mild conditions on the Banach lattices.

A final purpose is to obtain a deeper insight on the class of disjointly strictly singular operators. Recall that an operator T from a Banach lattice E to a Banach space Y is said to be *disjointly strictly singular* (DSS for short) if for every pairwise disjoint sequence $(x_n)_n$ in E the restriction of T to the (closed) span of $(x_n)_n$ is not an isomorphism. The class of DSS operators introduced in [22], strictly larger in general than the class of SS operators, has some remarkable applications ([14], [16]). Naturally close to this class is that of lattice strictly singular operators introduced in [12] and implicitly in [13]. Recall that T is *lattice strictly singular* (LSS for short) if the restriction of T to every infinite dimensional (closed) sublattice of E is never an isomorphism. Clearly every lattice strictly singular operator must be DSS, but the converse seems to be unknown. In [13] and [12] some results are given in this direction for regular operators. Here we formally introduce the classes of *super disjointly strictly singular* (super-DSS for short) and *super lattice strictly singular* (super-LSS for short) with the hope that some light might be shed on the question and in fact we show that these two super classes coincide (Theorem 6.3). From the techniques employed we obtain, as a consequence, new classes of spaces for which the equivalence between DSS and LSS operators holds (see Proposition 6.4).

The paper is organized in six sections. In sections one and two we give the announced modified definition of super-strict cosingularity, justify the role of super-SS and super-SC operators as the super classes of strictly singular and cosingular operators (see Proposition 1.5 and Theorem 2.16) and deduce the full duality between the two super classes (Proposition 2.13). The notion of operator *local representability* introduced by Pietsch gives a nice frame for treating these questions, and we use various results obtained by this author in [40]. The general idea of these sections is that "local representability decreases Bernstein and Mityagin numbers". More precisely, if an operator R is locally representable in T then $b_n(R) \leqslant b_n(T)$ and $a_n(R) \leqslant a_n(T)$ for every natural n.

In the third section we examine the question of preservation of theses super-properties by domination in the class of positive operators between Banach lattices.

In the fourth section we introduce the class of super-disjointly strictly singular operators, show that it is a "super" class and deduce domination results as consequences of known domination results for DSS operators.

The fifth section is devoted to the relations between the previously introduced classes of "singular" operators. We also consider rearrangement invariant function spaces E on $[0,1]$ showing that the canonical inclusions $L^\infty \hookrightarrow E$ and $E \hookrightarrow L^1$ are always super-SS and super-SC respectively (Proposition 5.7).

In the sixth section we introduce formally the class of super-lattice strictly singular operators and show that it actually coincides with the class of super-DSS operators (Theorem 6.3). We conclude by showing the equivalence between disjointly strictly singular and lattice strictly singular operators in the class of *stable* Banach lattices in the sense of Krivine and Maurey [27] (which contains for instance Orlicz and Lorentz spaces). This is the content of Pposition 6.4.

Our methods involve some ultrapower techniques, Pietsch's notion of operator local representability ([40]), the use of Krivine's theorem on finite representability of l_p ([26]), the spreading model construction by Brunel and Sucheston ([9]), and some results on domination in the classes of SS and DSS operators ([13], [14], [15]).

1. SUPER STRICTLY SINGULAR OPERATORS

Let us start recalling some basic definitions and fixing the notation.

Let $(X_i)_{i\in I}$ be a family of Banach spaces. We denote by $l^\infty((X_i);I)$ the Banach space

$$\Big\{x=(x_i)_{i\in I} : x_i \in X_i \text{ and } \|x\|_\infty = \sup_{i\in I}\|x_i\|_{X_i} < \infty\Big\}.$$

In the case that each X_i is a Banach lattice for every i then $l^\infty((X_i);I)$ is a Banach lattice under the pointwise lattice operations. If $\mathcal{U}$ denotes an ultrafilter on I, we consider the set

$$\mathcal{N}_{\mathcal{U}} = \Big\{x=(x_i)_{i\in I} \in l^\infty((X_i);I) : \lim_{i,\mathcal{U}}\|x_i\|_{X_i} = 0\Big\}.$$

It holds that $\mathcal{N}_{\mathcal{U}}$ is a closed vector subspace of $l^\infty((X_i);I)$; moreover, if X_i is a Banach lattice for every i, then $\mathcal{N}_{\mathcal{U}}$ is an order ideal in $l^\infty((X_i);I)$. If we consider the quotient space $l^\infty((X_i);I)/\mathcal{N}_{\mathcal{U}}$ we obtain a Banach space called the *ultraproduct* ([10], [20]) of the spaces $(X_i)_i$ with respect to the ultrafilter $\mathcal{U}$, which is denoted by $\prod_{\mathcal{U}} X_i$. If X_i is a Banach lattice for every i, then $\prod_{\mathcal{U}} X_i$ is a Banach lattice. For each $x=(x_i)_i \in l^\infty((X_i);I)$ we shall denote its equivalence class in $\prod_{\mathcal{U}} X_i$ by $[x]_{\mathcal{U}}$. Of particular interest is the situation in which all the Banach spaces X_i are the same. In this case we talk about the *ultrapower* of X with respect to the ultrafilter $\mathcal{U}$, and we denote it by $X_{\mathcal{U}}$. Note that there is a natural embedding of X into its ultrapower: the mapping $D_X : x \to [(x,x,\dots)]_{\mathcal{U}}$. This mapping is a linear isometry (which is also a lattice isometry if X is a Banach lattice); it is often called the *diagonal embedding*.

Analogously, given a family of bounded operators $T_i : X_i \to Y_i$ between Banach spaces X_i and Y_i with $i \in I$ and an ultrafilter $\mathcal{U}$ on I, the ultraproduct of the family $(T_i)_{i\in\mathcal{U}}$, denoted by $\prod_{\mathcal{U}} T_i : \prod_{\mathcal{U}} X_i \to \prod_{\mathcal{U}} Y_i$, is defined as $\prod_{\mathcal{U}} T_i([(x_i)]_{\mathcal{U}}) = [(T_i x_i)]_{\mathcal{U}}$. The operator norm of $\prod_{\mathcal{U}} T_i$ is $\|\prod_{\mathcal{U}} T_i\| = \lim_{i,\mathcal{U}}\|T_i\|$.

Also, if $T : X \to Y$ is a bounded operator, then for every ultrafilter $\mathcal{U}$ on I the mapping $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ defined for each $(x_i)_i \in l_\infty(X;I)$ by

$$T_{\mathcal{U}}([(x_i)]_{\mathcal{U}}) = [(Tx_i)]_{\mathcal{U}}$$

is a well defined bounded operator that extends T and satisfies $\|T_{\mathcal{U}}\| = \|T\|$. Moreover if X and Y are Banach lattices, then T is a positive operator (respectively lattice homomorphism) if and only if $T_{\mathcal{U}}$ is positive (respectively lattice homomorphism).

For the general terminology of Banach spaces we refer to [28], [6] and [11], while for that of Banach lattices we refer to [31], [2], [29] and [1].

Let us explicitly recall the well-known fact which shows in particular that every ultrapower $\widetilde{X}$ of a Banach space X is finitely representable in X. A proof can be found in p. 223 of [6].

PROPOSITION 1.1. *Let* $\widetilde{X} = \prod_{\mathcal{U}} X_i$ *be an ultraproduct of a family of Banach spaces* $(X_i)_{i\in I}$. *Let* $\widetilde{E}$ *be a m-dimensional subspace of* $\widetilde{X}$, $[\xi^{(1)}, \dots, \xi^{(m)}]$ *a basis of* $\widetilde{E}$, *and for every* $j = 1, \dots, m$ *let* $(x_i^{(j)})_{i\in I}$ *be a family representing* $\xi^{(j)}$. *Let* $E_i = \operatorname{span}[(x_i^{(j)}) : j = 1, \dots n]$ *and* $V_i : \widetilde{E} \to E_i$ *be the linear map which assigns* $x_i^{(j)}$ *to* $\xi^{(j)}$ *for every* $j = 1, \dots, m$. *Then for every* i *in some element* U *of the ultrafilter* $\mathcal{U}$, *the operator* V_i *is an isomorphism, and moreover* $\lim_{i,\mathcal{U}} \max(\|V_i\|, \|V_i^{-1}\|) = 1$.

DEFINITION 1.2. A bounded operator $T : X \to Y$ is *super strictly singular* (super-SS for short) if there does not exist a number $c > 0$ and a sequence of subspaces $E_n \subset X$, $\dim E_n = n$, such that

$$\|Tx\| \geqslant c\|x\| \quad \text{for all } x \in \bigcup_n E_n \tag{1.1}$$

The *Bernstein numbers* are considered for a given operator T:

$$b_n(T) = \sup \inf_{x\in S(E_n)} \|Tx\|,$$

where the supremum is taken over all n-dimensional subspaces E_n of X and $S(E_n)$ is the unit sphere of E_n. We have

$$\|T\| = b_1(T) \geqslant b_2(T) \geqslant \cdots \geqslant 0,$$

and it is seen that T is super-SS if and only if $b_n(T) \to 0$ as $n \to \infty$. The greatest constant c for which (1.1) is satisfied is equal to $\lim_{n\to\infty} b_n(T)$ ([32], [41]).

The class of super-SS operators is in between the classes of compact and SS operators; indeed the natural inclusion $l_p \hookrightarrow l_q, 1 \leqslant p < q \leqslant \infty$, is a (non compact) super-SS operator while the operator given in Example 5.1 is a (non super-SS) SS operator.

PROPOSITION 1.3. *Let* $(T_i : X_i \to Y_i)_{i\in I}$ *be a bounded family of bounded operators, and* $\mathcal{U}$ *an ultrafilter over the index set* I. *Let* $\widetilde{T} = \prod_{\mathcal{U}} T_i$ *be the corresponding ultraproduct of the family* $(T_i)_{i\in\mathcal{U}}$. *Then, for every* $n \geqslant 1$,

$$b_n(\widetilde{T}) = \lim_{i,\mathcal{U}} b_n(T_i).$$

Proof. Let c be a real number with $c < \lim_{i,\mathcal{U}} b_n(T_i)$, then there exists an element $U \in \mathcal{U}$ such that $c < \inf_{i\in U} b_n(T_i)$. For every $i \in U$, there is a n-dimensional subspace E_i of X_i such that $\|T_i x\| \geqslant c\|x\|$ for every $x \in E_i$. The ultraproduct $\widetilde{E} := \prod_{\mathcal{U}} E_i$ is an n-dimensional space which identifies with a subspace of the ultraproduct $\widetilde{X} := \prod_{\mathcal{U}} X_i$. For every $\xi = [x_i]_{\mathcal{U}} \in \widetilde{E}$ we clearly have $\|\widetilde{T}\xi\| = \lim_{i,\mathcal{U}} \|T_i x_i\| \geqslant c \lim_{i,\mathcal{U}} \|x_i\| = c\|\xi\|$ and thus $b_n(\widetilde{T}) \geqslant c$.

Conversely let $c < b_n(\widetilde{T})$ and let $\varepsilon > 0$ such that $c' := c(1+\varepsilon)^2 < b_n(\widetilde{T})$. Choose a subspace $F \subset \widetilde{X}$ of dimension n such that $\|\widetilde{T}\xi\| \geqslant c'\|\xi\|$ for every $\xi \in F$. Let $(\xi^{(1)}, \ldots, \xi^{(n)})$ be a basis of F: observe that $(\widetilde{T}\xi^{(1)}, \ldots, \widetilde{T}\xi^{(m)})$ is a basis of $\widetilde{T}F$. For $j = 1, \ldots, n$ let $(x_i^{(j)})_{i\in I}$ be a family representing $\xi^{(j)}$. We have

$$\Big\|\widetilde{T}\Big(\sum_{j=1}^{n} a_j \xi^{(j)}\Big)\Big\| = \lim_{i,\mathcal{U}} \Big\|\sum_{j=1}^{n} a_j T_i x_i^{(j)}\Big\|.$$

On the other hand, by Proposition 1.1 applied to both F and $\widetilde{T}F$, there is some $U \in \mathcal{U}$ such that for every $i \in U$ both operators $\phi_i : F \to [x_i^{(1)}, \ldots, x_i^{(n)}]$ and $\psi_i : \widetilde{T}F \to [T_i x_i^{(1)}, \ldots, T_i x_i^{(m)}]$) defined by $\phi_i(\xi^{(j)}) = x_i^{(j)}$ (respectively $\psi_i(\widetilde{T}\xi^{(j)}) = T_i x_i^{(j)}$) are isomorphisms satisfying $\max(\|\phi_i\|, \|\phi_i^{-1}\|) < 1+\varepsilon$ and $\max(\|\psi_i\|, \|\psi_i^{-1}\|) < 1+\varepsilon$ (here we use the fact that the intersection of two elements of $\mathcal{U}$ belongs to $\mathcal{U}$). Then we have the following chain of inequalities

$$\Big\|T_i\Big(\sum_{j=1}^{n} a_j x_i^{(j)}\Big)\Big\| \geqslant \frac{1}{(1+\varepsilon)}\Big\|\sum_{j=1}^{n} a_j \widetilde{T}\xi^{(j)}\Big\| \geqslant \frac{c'}{(1+\varepsilon)}\Big\|\sum_{j=1}^{n} a_j \xi^{(j)}\Big\| \geqslant \frac{c'}{(1+\varepsilon)^2}\Big\|\sum_{j=1}^{n} a_j x_i^{(j)}\Big\|,$$

showing that T_i is invertible on the n-dimensional subspace $E_i := [x_i^{(1)}, \ldots, x_i^{(n)}]$, with constant $\frac{c'}{(1+\varepsilon)^2} \geqslant c$. Thus $b_n(T_i) \geqslant c$ for every $i \in U$, and $\lim_{i,\mathcal{U}} b_n(T_i) \geqslant c$. ∎

COROLLARY 1.4. *If T is a bounded operator then any ultrapower $T_{\mathcal{U}}$ of T has the same Bernstein numbers as T: $\forall n,\ b_n(T_{\mathcal{U}}) = b_n(T)$.*

The following result was essentially given in [30]. We include its proof for the sake of completeness.

PROPOSITION 1.5. *Let $T : X \to Y$ be a bounded operator. The following assertions are equivalent:*

(i) *T is super strictly singular.*

(ii) *Every ultrapower of T is strictly singular.*

(iii) *Every ultrapower of T relative to a free ultrafilter on $\mathbb{N}$ is strictly singular.*

Proof. (i) $\Rightarrow$ (ii) Assume that there exists an ultrafilter $\mathcal{U}$ such that $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is not strictly singular. Then it is neither super strictly singular, thus $\inf_n b_n(T) = \inf_n b_n(T_{\mathcal{U}}) > 0$ by Corollary 1.4 and T is not super strictly singular.

(iii) $\Rightarrow$ (i) If T is not super strictly singular then there exists a sequence of subspaces $(B_n)_n$ in X such that $\dim(B_n) = n$ for every n, and a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in \bigcup_n E_n$. Consider a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and consider the ultraproduct $Z = (B_n)_{\mathcal{U}}$, of which an isometric copy, still denoted by Z, can be found in the ultrapower $X_{\mathcal{U}}$. Let us show that $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is invertible on $(B_n)_{\mathcal{U}}$. Indeed, for every element $[x]_{\mathcal{U}} = [(x_n)]_{\mathcal{U}}$ in $(B_n)_{\mathcal{U}}$ we have $T_{\mathcal{U}}([x]_{\mathcal{U}}) = [T(x_n)]_{\mathcal{U}}$ and

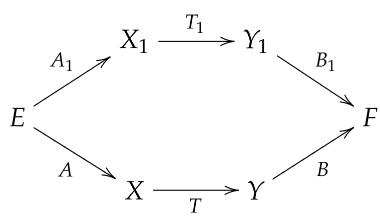
$$\|T_{\mathcal{U}}([x]_{\mathcal{U}})\| = \lim_{n,\mathcal{U}} \|Tx_n\| \geqslant \lim_{n,\mathcal{U}} C\|x_n\| = C\|[x]_{\mathcal{U}}\|.$$

Thus $T_{\mathcal{U}}$ is invertible on Z, hence $T_{\mathcal{U}}$ is not strictly singular, since Z has infinite dimension. ∎

REMARK 1.6. The collection of super strictly singular operators is an operator ideal in Pietsch sense [39]. This follows instantly from Proposition 1.5 and the fact that the collection of strictly singular operators is an operator ideal.

Let us now recall the notion of *local representability* of operators introduced by Pietsch in [40], that we shall use in the sequel.

DEFINITION 1.7. An operator $T_1 : X_1 \to Y_1$ is said to be *locally representable* in an operator $T : X \to Y$ (in short: T_1 l.r. T) if for every $\varepsilon > 0$, every finite-dimensional spaces E, F and every bounded linear operators $A_1 : E \to X_1$, $B_1 : Y_1 \to F$ there are bounded linear operators $A : E \to X$, $B : Y \to F$ such that $BTA = B_1 T_1 A_1$ and $\|B\|\|A\| \leqslant (1+\varepsilon)\|B_1\|\|A_1\|$.



Two operators T and T_1 are said to be *locally equivalent* if each of them is locally representable in the other one.

REMARK 1.8. An equivalent definition of Pietsch's local representability is the following: T_1 is locally representable in T if for every finite-dimensional subspace E_1 of X_1 and every finite-codimensional (closed) subspace N_1 in Y_1 there exist a finite dimensional subspace E in X, a finite-codimensional subspace N in Y, and bounded linear operators $V : E_1 \to E$ and $W : Y/N \to Y_1/N_1$ with $\|V\| \leqslant 1+\varepsilon$, $\|W\| \leqslant 1+\varepsilon$ such that the Diagramme 1.2 is commutative.

The following remark lists simple facts:

REMARK 1.9. (i) Local representability is transitive (i.e. T_2 l.r. T_1 and T_1 l.r. T imply T_2 l.r. T).

(ii) Let $U : X_1 \to X$ and $V : Y \to Y_1$ be norm one operators. Then for every $T : X \to Y$, the operator $VTU : X_1 \to Y_1$ is l.r. T.

(iii) Every operator $T : X \to Y$ is locally representable in $j_Y T$, where j_Y is the canonical injection of Y in its bidual.

(iv) Every operator T is locally representable in any of its ultrapowers $T_{\mathcal{U}}$.

(v) Every operator T is locally representable in its biconjugate T^{**}.

$$
(1.2) \qquad
\begin{array}{ccccccc}
E & \xrightarrow{i_E} & X & \xrightarrow{T} & Y & \xrightarrow{\pi_N} & Y/N \\
{\scriptstyle V}\big\uparrow & & & & & & \big\downarrow{\scriptstyle W} \\
E_1 & \xrightarrow{i_{E_1}} & X_1 & \xrightarrow{T_1} & Y_1 & \xrightarrow{\pi_{N_1}} & Y_1/N_1
\end{array}
$$

The connection between operator local representability and ultrapowers of operators is the following result of Pietsch ([40], Theorem 6.6):

THEOREM 1.10 (Pietsch). *Let $T : X \to Y$ and $T_1 : X_1 \to Y_1$ be bounded operators. Then $T_1 : X_1 \to Y_1$ is locally representable in T if and only if there exist an ultrapower $T_{\mathcal{U}}$ of T, and contractive operators $A : X_1 \to X_{\mathcal{U}}$ and $B : Y_{\mathcal{U}} \to Y_1^{**}$ such that $j_{Y_1} T_1 = B T_{\mathcal{U}} A$, where $j_{Y_1} : Y_1 \to Y_1^{**}$ is the canonical injection.*

Note that in particular every ultrapower $T_{\mathcal{U}}$ of T is locally representable in T. Similarly it results from the local reflexivity theorem that the biconjugate T^{**} is locally representable in T. Consequently we have ([40], Theorem 6.3):

PROPOSITION 1.11. *Every operator is locally equivalent to its biconjugate and to any of its ultrapowers.*

PROPOSITION 1.12. *An operator $T : X \to Y$ is super strictly singular if and only if every operator T_1 which is locally representable in T is strictly singular.*

Proof. If $T : X \to Y$ is super-SS and $T_1 : X_1 \to Y_1$ is l.r. T, then let $j_{Y_1} T_1 = B(T_{\mathcal{U}})A$ be a decomposition as in Theorem 1.10; since $T_{\mathcal{U}}$ is strictly singular by Proposition 1.5, so is $B(T_{\mathcal{U}})A$ by the ideal property and so is T_1 by the injectivity of the SS-operator ideal. The converse direction is clear since all ultrapowers of T are locally representable in T (Proposition 1.11). ∎

PROPOSITION 1.13. *The sequence of Bernstein numbers of an operator T dominates that of any operator T_1 locally representable in T: $\forall n,\ b_n(T_1) \leqslant b_n(T)$.*

Proof. Let $j_{Y_1} T_1 = B(T_{\mathcal{U}})A$, with $\|A\|\|B\| \leqslant 1$ be a decomposition as in Theorem 1.10. By injectivity of Bernstein numbers, s-number property ([38], pag. 202), and Corollary 1.4 we have

$$b_n(T_1) = b_n(j_{Y_1} T_1) = b_n(B T_{\mathcal{U}} A) \leqslant \|B\| b_n(T_{\mathcal{U}}) \|A\| \leqslant b_n(T_{\mathcal{U}}) = b_n(T). \quad ∎$$

In particular, from Propositions 1.11 and 1.13 we deduce the following statement which is implicit in p. 249–250 of [41], and could also be deduced directly from the local reflexivity theorem:

COROLLARY 1.14. *Any operator T and its biconjugate T^{**} have the same Bernstein numbers, i.e. $\forall n,\ b_n(T) = b_n(T^{**})$.*

2. SUPER STRICTLY COSINGULAR OPERATORS

We pass now to consider the class of super strictly cosingular operators introduced by Plichko in [41] as a dual version of the class of super strictly singular operators. There he introduces the following quantity numbers associated to a bounded operator $T : X \to Y$ between two Banach spaces:

$$a_n(T) = \sup_{E^n} \inf\{\|\widehat{Tx}\|_{Y/E^n} : x \in X\ ,\ \|\widehat{x}\|_{X/T^{-1}E^n} = 1\}\ ,$$

where supremum is taken over all n-codimensional subspaces E^n of Y and hats denote the corresponding quotient classes.

According to Definition 6 of [41], an operator $T : X \to Y$ is super strictly cosingular if the sequence $a_n(T) \to 0$ as n tends to infinity.

This class is shown to be in duality with that of super-SS operators ([41], Theorem 4); consequently this super class also contains the class of compact operators.

The definition of super strict cosingularity given in [41] may be inconsistent with the claimed duality results and with its relation with the class of compact operators. This is illustrated with the following:

EXAMPLE 2.1. Consider the rank-one operator $T : l_p \to l_p$ defined by $T((x_n)_n) = (x_1, 0, 0, \dots)$, and for every integer n the subspace $E_n = [e_{n+1}, e_{n+2}, \dots]$ spanned by the sequence $(e_{n+k})_{k=1}^{\infty}$. If P_n is the natural projection on l_p^n and π_{E_n} denotes the quotient linear map from l_p onto l_p/E_n, then it is clear that $\|P_n x\| = \|\pi_{E_n} x\|$ for all $x \in l_p$. Hence there is a linear isometry $\phi_n : l_p^n \to l_p/E_n$ such that $\pi_{E_n} = \phi_n P_n$.

If $\widehat{y}$ denotes the class of y in l_p/E_n, then $\|\widehat{Tx}\| = \|\widehat{x_1 e_1}\| = \|P_n(x_1 e_1)\| = |x_1|$. Also $T^{-1}E_n = [e_2, e_3, \dots]$ since $y \in T^{-1}E_n$ if and only if $y_1 = 0$. Therefore $l_p/T^{-1}E_n$ is isometric to l_p^1 and $\|\pi_{T^{-1}E_n} x\| = |x_1|$. Thus, $\|\widehat{Tx}\| = |x_1| = \|\pi_{T^{-1}E_n} x\|$, and hence

$$\inf\{\|\pi_{E_n} Tx\| : \|\pi_{T^{-1}E_n} x\| = 1\} = 1.$$

Since this is true for every integer n we obtain that $a_n(T) \geqslant 1$ for all n, in other words, T is not super strictly cosingular.

On the other hand, it is clear that the conjugate operator T^* has also rank one and sends every element $(x_n)_n$ in l_q to the sequence $(x_1, 0, 0, \dots)$ $(p^{-1} + q^{-1} = 1)$. Hence T^* is super-SS being compact and therefore T should be super strictly cosingular.

Let us show now how a simple modification of the definition of the numbers $a_n(T)$ will avoid the preceding trivial unconsistencies. To this end some preliminary preparation is needed.

DEFINITION 2.2. Let $T : X \to Y$ be a bounded linear operator. For every closed subspace E of Y set:

$$a_E(T) := \inf\{d(Tx, E)/d(x, T^{-1}(E)) : x \in X,\ Tx \notin E\}$$

if $TX \not\subset E$, and $a_E(T) = 0$ if $TX \subset E$. For every natural number $n \geqslant 1$ define the following, with the convention that $\sup \emptyset = 0$:

$$a_n(T) := \sup\{a_E(T) : E \subset Y,\ \operatorname{codim} E = n,\ Y \subset TX + E\}.$$

Note that if we denote by π_E the canonical surjection from Y onto Y/E, and if we assume that $TX \not\subset E$ we have:

$$a_E(T) = \inf\{\|\pi_E \circ Tx\|_{Y/E} : x \in X, \|\pi_{T^{-1}E}x\|_{X/T^{-1}E} = 1\}$$

consistently with Plichko's paper. As was said above our definition of $a_n(T)$ differs from that of Plichko insofar we consider in the supremum defining $a_n(T)$ only n-codimensional subspaces E such that $\pi_E \circ T$ *is surjective*. In particular if T has finite rank we have $a_n(T) = 0$ as soon as $n > \operatorname{rank} T$. Note also that this surjectivity condition, together with the fact that E is a proper subspace (in particular if E has nonzero codimension), implies that TX is not included in E and thus $\alpha_E(T)$ is defined by the first formula (or by Plichko's formula).

If X is a Banach space, we denote by B_X, respectively B_X^o, its unit ball, respectively its open unit ball. If $T : X \to Y$ is a bounded operator, we denote by $q(T)$ its *modulus of surjectivity* that is $q(T) = \sup\{\alpha \geqslant 0 : T(B_X) \supset \alpha B_Y\}$. In this definition balls can be replaced by open unit balls.

Note that if T is a Banach space isomorphism then $q(T) = \|T^{-1}\|^{-1}$.

LEMMA 2.3. *If $T : X \to Y$ is a bounded linear operator and $E \subsetneq Y$ is a proper subspace of finite codimension such that $Y \subset TX + E$ then $a_E(T) = \alpha_E(T)$, where*

$$\alpha_E(T) = \sup\{\alpha \geqslant 0 : \pi_E \circ T(B_X) \supset \alpha B_{Y/E}\} = q(\pi_E \circ T)$$

is the modulus of surjectivity of the operator $\pi_E \circ T$.

Proof. Let $T_E = \pi_E \circ T$ and consider the induced operator $\widehat{T}_E : \widehat{X} := X/\ker T_E \to Y/E$ (note that $\ker T_E = T^{-1}(E)$). Then clearly $a_E(T) = a_{\{0\}}(\widehat{T}_E)$ and $\widehat{T}_E(B^o_{X/\ker T_E}) = \pi_E \circ T(B^o_X)$ which implies $\alpha_E(T) = q(\widehat{T}_E)$. Now $\widehat{T}_E$ is injective by construction and surjective because $\pi_E \circ T$ is by hypothesis: so it is a linear isomorphism between $X/\ker T_E$ and Y/E; then:

$$\begin{aligned}\|\widehat{T}_E^{-1}\| &= \sup\{\|\widehat{T}_E^{-1}\eta\|/\|\eta\| : \eta \in Y/E, \eta \neq 0\} \\ &= \sup\{\|\xi\|/\|\widehat{T}_E\xi\| : \xi \in X/\ker T_E, \xi \neq 0\} = a_{\{0\}}(\widehat{T}_E)^{-1}.\end{aligned}$$

Passing to the inverses we obtain:

$$\alpha_E(T) = q(\widehat{T}_E) = \|\widehat{T}_E^{-1}\|^{-1} = a_{\{0\}}(\widehat{T}_E) = a_E(T). \quad \blacksquare$$

REMARK 2.4. Since $\alpha_E(T)=0$ when $\pi_E \circ T$ is not surjective, we have in fact, for every natural number $n \geqslant 1$:

$$a_n(T) = \sup\{\alpha_E(T) : E \subset Y,\ \operatorname{codim} E = n\}.$$

From this and the following lemma it will be clear that the $a_n(T)$ coincide with the *Mityagin numbers* defined in [38], pag. 210.

LEMMA 2.5. *The sequence $(a_n(T))_n$ is non increasing.*

Proof. It is sufficient to prove that if $F \subset E$ are subspaces of finite codimension of Y then $\alpha_F(T) \leqslant \alpha_E(T)$. Denote by π_E^F the canonical surjection $Y/F \twoheadrightarrow Y/E$. Then $\pi_E = \pi_E^F \circ \pi_F$ and π_E^F sends the unit ball of Y/F onto the unit ball of Y/E. Hence if $\alpha < \alpha_F(T)$:

$$\pi_E \circ T(B_X) = \pi_E^F(\pi_F \circ T(B_X)) \supset \pi_E^F(\alpha B_{Y/F}) \supset \alpha B_{Y/E}$$

thus $\alpha_E(T) \geqslant \alpha_F(T)$. $\blacksquare$

Mityagin and Bernstein s-numbers are related by duality as follows:

PROPOSITION 2.6. *For every operator $T : X \to Y$ we have for every $n \geqslant 1$:*

$$a_n(T) = b_n(T^*) \quad \text{and} \quad b_n(T) = a_n(T^*).$$

Proof. The lefthand equality was stated in Theorem 6.4 of [38], and proved there as a consequence of the equality

$$\alpha_E(T) = \inf\{\|T^* x'\| : x' \in E^\perp, \|x\| = 1\} \tag{2.1}$$

where $E \subset Y$, and $E^\perp \subset Y^*$ is the annihilator of E. Note that the map $E \mapsto E^\perp$ is one to one from the set $\mathcal{G}^n(Y)$ of all closed subspaces of codimension n in Y onto the set $\mathcal{G}_n(Y^*)$ of all subspaces of dimension n in Y^*, and the lefthand equality in the proposition is thus obtained by taking the suprema in (2.1) with respect to $E \in \mathcal{G}^n(Y)$. This reasoning does not work for the right hand equality in the proposition since the map $E \mapsto E^\perp$ is not onto from $\mathcal{G}_n(Y)$ onto $\mathcal{G}^n(Y^*)$ when Y is not reflexive.

However applying the left hand equality to the operator T^* and Corollary 1.14 we obtain:

$$a_n(T^*) = b_n(T^{**}) = b_n(T)$$

which is the right hand equality stated in the proposition (see Remark in [38], pag. 213). $\blacksquare$

COROLLARY 2.7. *If the operator R is locally representable in the operator T then the sequence of Mityagin numbers of T dominates that of R: $\forall n,\ a_n(R) \leqslant a_n(T)$.*

Proof. From the duality of Mityagin and Bernstein numbers (Proposition 2.6), the fact that R l.r. T is equivalent to R^* l.r. T^* ([40], Proposition 6.4), and Proposition 1.13 we have

$$a_n(R) = b_n(R^*) \leqslant b_n(T^*) = a_n(T). \quad \blacksquare$$

In particular by Proposition 1.11 we obtain:

COROLLARY 2.8. *Every operator T has the same Mityagin numbers as its biconjugate and any of its ultrapowers: $\forall n,\ a_n(T) = a_n(T^{**}) = a_n(T_{\mathcal{U}})$.*

The dual result to Proposition 1.3 holds true:

PROPOSITION 2.9. *Given a bounded family of operators $T_i : X_i \to Y_i$, $i \in I$ and an ultrafilter $\mathcal{U}$ on I, we have for every $n \geqslant 1$*

$$a_n\Big(\prod\nolimits_{\mathcal{U}} T_i\Big) = \lim_{i,\mathcal{U}} a_n(T_i).$$

Proof. By Propositions 2.6 and 1.3 is sufficient to prove that $b_n((\prod_{\mathcal{U}} T_i)^*) = b_n(\prod_{\mathcal{U}} T_i^*)$. Let $\iota_{(X_i)} : \prod_{\mathcal{U}} X_i^* \to (\prod_{\mathcal{U}} X_i)^*$ be the isometric linear injection defined by

$$\langle \iota_{(X_i)}([x_i^*]_{\mathcal{U}}), [x_i]_{\mathcal{U}} \rangle = \lim_{i,\mathcal{U}} \langle x_i^*, x_i \rangle$$

and $\iota_{(Y_i)}$ the similar one for $\prod_{\mathcal{U}} Y_i^*$, we have $\iota_{(X_i)} \circ \prod_{\mathcal{U}} T_i^* = (\prod_{\mathcal{U}} T_i)^* \circ \iota_{(Y_i)}$, thus

$$b_n(\prod\nolimits_{\mathcal{U}} T_i^*) = b_n(\iota_{(X_i)} \circ \prod\nolimits_{\mathcal{U}} T_i^*) \leqslant b_n((\prod\nolimits_{\mathcal{U}} T_i)^*)$$

by injectivity and s-number property of Bernstein numbers. For proving the converse inequality it is sufficient by Proposition 1.13 to prove that $(\prod_{\mathcal{U}} T_i)^*$ is locally representable in $\prod_{\mathcal{U}} T_i^*$. Indeed let $A : E \to (\prod_{\mathcal{U}} Y_i)^*$ and $B : (\prod_{\mathcal{U}} X_i)^* \to F$ where E, F are finite dimensional and $\|A\| \leqslant 1$, $\|B\| \leqslant 1$. Applying Proposition 5.6 of [40] to the operator $\prod_{\mathcal{U}} T_i$ we can assume that $B = C^*$ for some operator $C : F^* \to \prod_{\mathcal{U}} X_i$. Then applying the Kürsten-Stern "local duality theorem for ultraproducts" ([20], Theorem 7.3) we may find I_ε from the range of A to $(\prod_{\mathcal{U}} Y_i^*)$, with $\|I_\varepsilon\| \leqslant 1 + \varepsilon$, such that $\langle I_\varepsilon \phi, \eta \rangle = \langle \phi, \eta \rangle$ for all ϕ in the range of A and η in the range of $(\prod_{\mathcal{U}} T_i)C$. Setting $A_1 = I_\varepsilon A$ and $B_1 = C^* |_{\Pi_{\mathcal{U}} X_i^*}$ we obtain $B(\prod_{\mathcal{U}} T_i)^* A = B_1(\prod_{\mathcal{U}} T_i^*) A_1$, as we wanted. $\blacksquare$

Like Plichko, we define now:

DEFINITION 2.10. An operator $T : X \to Y$ is *super strictly cosingular* (super-SC) if $a_n(T) \to 0$ as $n \to \infty$.

EXAMPLE 2.11. We have seen that every finite rank operator is super-SC ($a_n(T) = 0$ for $n > \operatorname{rank} T$). More generally every compact operator T is super-SC: if not we can find a sequence $(E_n)_n$ of subspaces of Y with $\operatorname{codim} E_n \geqslant n$ and $\inf_n \alpha_{E_n}(T) = \alpha > 0$. Let $K = \overline{TB_X}$ which is compact. Then $\pi_{E_n}(K) \supset \alpha B_{Y/E_n}$ for every n. Since the normed space Y/E_n has dimension n, its unit ball contains a sequence of n points with mutual distances at least 1. Hence $\pi_{E_n}(K)$ contains at least n points with mutual distances at least α, and so does K since $\|\pi_{E_n}\| = 1$. Since this is true for every n, the set K cannot be compact, a contradiction.

REMARK 2.12. Super strict cosingularity implies strict cosingularity.

Indeed, consider a non-SC operator $T : X \to Y$; thus there exists a subspace E of Y of infinite codimension such that $\pi_E \circ T\colon\ X \to Y/E$ is surjective. By Banach's open mapping theorem the map $\pi_E \circ T$ is open, i.e. $\alpha_E(T) > 0$. For every subspace F of Y containing E we have $\alpha_F(T) \geqslant \alpha_E(T)$ (see the proof of Lemma 2.5); since E has infinite codimension, we may find for every $n \geqslant 1$ a subspace $F_n \supset E$ of codimension n: this shows that $a_n(T) \geqslant \alpha_E(T) > 0$ for every n, and T is not super-SC.

The following immediate consequence of Proposition 2.6 is stated as Theorem 4 in [41].

PROPOSITION 2.13. *Let $T : X \to Y$ be a bounded operator. Then T is super strictly cosingular if and only if T^* is super strictly singular; and T is super strictly singular if and only if T^* is super strictly cosingular.*

As an example we get that the canonical inclusions $L^p[0,1] \hookrightarrow L^1[0,1]$, $1 < p \leqslant \infty$, are super-SC operators since the conjugate operators $L^\infty[0,1] \hookrightarrow L^q[0,1]$, $1 \leqslant q < \infty$ are super-SS (cf. Theorem 5.2 of [43]). In Section 4 we extend these results to all rearrangement invariant function spaces (in particular to Lorentz and Orlicz spaces).

Our aim now is to prove that the class of super-SC operators is actually the "super" class of that of strictly cosingular operators. We first make a preliminary remark on ultraproducts of quotients.

LEMMA 2.14. *Let Z be a Banach space, $(H_i)_{i\in I}$ be a family of closed linear subspaces of Z, and $\mathcal{U}$ be an ultrafilter on the set I. Consider the ultraproduct $\widetilde{H} = \Pi_{\mathcal{U}} H_i$ as a subspace of the ultrapower $\widetilde{Z} = Z_{\mathcal{U}}$. Then the distance function to $\widetilde{H}$ is the ultraproduct of the distance functions to the subspaces H_i, that is:*

$$\forall \widetilde{x} = [x_i]_{\mathcal{U}} \in \widetilde{Z}, \quad d(\widetilde{x}, \widetilde{H}) = \lim_{i,\mathcal{U}} d(x_i, H_i).$$

Proof. Let $\varepsilon > 0$ be fixed; for every $i \in I$ chose $h_i \in H_i$ such that $d(x_i, h_i) \leqslant d(x_i, H_i) + \varepsilon$. Note that $\|h_i\| \leqslant \|x_i\| + \varepsilon$, so that the family $(h_i)_i$ is bounded and defines an element $\widetilde{h} = [h_i]_{\mathcal{U}}$ in $\widetilde{H}$. Then

$$d(\widetilde{x}, \widetilde{H}) \leqslant d(\widetilde{x}, \widetilde{h}) = \lim_{i,\mathcal{U}} d(x_i, h_i) \leqslant \lim_{i,\mathcal{U}} d(x_i, H_i) + \varepsilon.$$

Conversely let $\widetilde{h} = [h_i]_{\mathcal{U}} \in \widetilde{H}$ be such that $d(\widetilde{x}, \widetilde{h}) \leqslant d(\widetilde{x}, \widetilde{H}) + \varepsilon$, then $\lim_{i,\mathcal{U}} d(x_i, H_i) \leqslant \lim_{i,\mathcal{U}} d(x_i, h_i) = d(\widetilde{x}, \widetilde{h}) \leqslant d(\widetilde{x}, \widetilde{H}) + \varepsilon$. We conclude by letting $\varepsilon \to 0$. ∎

REMARK 2.15. By Lemma 2.14 there is a canonical identification of $\widetilde{Z}/\widetilde{H}$ with $\prod_{\mathcal{U}}(Z/H_i)$, namely the operator $J : \pi_{\widetilde{H}}\widetilde{x} \mapsto [\pi_{H_i}x_i]_{\mathcal{U}}$, which is well defined and isometric. Note that $J \circ \widetilde{\pi}_{\widetilde{H}} = \widetilde{\pi}$ where $\widetilde{\pi} = \prod_{\mathcal{U}} \pi_{H_i}$.

THEOREM 2.16. *Let $T : X \to Y$ be a bounded operator. The following assertions are equivalent:*

(i) *T is super strictly cosingular.*

(ii) *Every operator locally representable in T is strictly cosingular.*

(iii) *Every ultrapower of T is strictly cosingular.*

(iv) *Every ultrapower of T relative to a free ultrafilter on $\mathbb{N}$ is strictly cosingular.*

Proof. (i) $\Rightarrow$ (ii) This follows from Corollary 2.7 and Remark 2.12. (ii) $\Rightarrow$ (iii) results from Theorem 1.10. (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) Assume that T is not super-SC. Then there exist $\alpha > 0$ and for every $n \geqslant 1$ a subspace $E_n \subset Y$ with codim $E_n = n$, such that

$$\pi_{E_n} \circ T(B_X) \supset \alpha B_{Y/E_n}.$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and set $\widetilde{T} = T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$, and $\widetilde{E} = \prod_{\mathcal{U}} E_n$, which we consider as a closed subspace of $Y_{\mathcal{U}}$. Let $J : Y_{\mathcal{U}}/\widetilde{E} \to \widetilde{\Gamma} := \prod_{\mathcal{U}}(Y/E_n)$ be the isometric isomorphism of Remark 2.15, and set $\widetilde{\pi} = \prod_{\mathcal{U}} \pi_{E_n}$; then we have $\widetilde{\pi} = J \circ \pi_{\widetilde{E}}$. We claim that

$$\widetilde{\pi} \circ \widetilde{T}(B_{X_{\mathcal{U}}}) \supset \alpha B_{\widetilde{\Gamma}}$$

then it will follow that $\widetilde{\pi} \circ \widetilde{T}$ is surjective and so will be $\pi_{\widetilde{E}} \circ \widetilde{T}$ since $\widetilde{\pi} \circ \widetilde{T} = J \circ (\pi_{\widetilde{E}} \circ \widetilde{T})$; as $\widetilde{E}$ is infinite dimensional, this will show that $\widetilde{T}$ is not strictly cosingular. Now every $\widetilde{\eta} \in \alpha B_{\widetilde{\Gamma}}$ can be defined by a family $(\eta_n)_n$ with $\eta_n \in \alpha B_{Y/E_n}$ for every n. Thus for every n there is some $x_n \in B_X$ such that $\eta_n = \pi_{E_n} \circ T(x_n)$; setting $\widetilde{x} = [x_n]_{\mathcal{U}}$ we obtain the desired element $\widetilde{x} \in B_{X_{\mathcal{U}}}$ such that $\widetilde{\pi} \circ \widetilde{T}(\widetilde{x}) = \widetilde{\eta}$, and the claim is proved. ∎

REMARK 2.17. From Proposition 1.5 and Theorem 2.16 it follows immediately that for ultrapowers of a bounded operator with respect to a non-trivial ultrafilter on $\mathbb{N}$, strict (co) singularity implies super-strict (co) singularity. This can be extended to ultraproducts in the following way. Assume that the ultrafilter $\mathcal{U}$ on some set I is *countably incomplete*, that is, there exists a sequence (I_n) of members of $\mathcal{U}$ such that $\bigcap_{n \geqslant 1} I_n = \emptyset$. Then for any uniformly bounded family of operators $T_i : X_i \to Y_i$, $i \in I$, the ultraproduct $\widetilde{T} := \prod_{\mathcal{U}} T_i$ is super strictly singular (respectively cosingular), whenever it is strictly singular (respectively cosingular).

Indeed, if $\widetilde{T}$ is not, say, super strictly cosingular, then $\alpha := \inf_n a_n(\widetilde{T}) > 0$. By Proposition 2.9, for every n there is a set $U_n \in \mathcal{U}$ such that $a_n(T_i) > \alpha/2$ for every $i \in U_n$. By taking appropriate finite intersections we may assume that the sequence (U_n) is decreasing and that $U_n \subset I_n$ for every n, thus $\bigcap_n U_n = \emptyset$.

For $i \in U_n \setminus U_{n+1}$ we may choose a closed subspace E_i of codimension n such that $\pi_{E_i} \circ T_{E_i}(B_{Y_i}) \supset (\alpha/2)B_{Y_i/E_i}$. This defines E_i for all $i \in U_1$ (for $i \notin U_1$ set $E_i = (0)$). Set $\widetilde{E_i} = \prod_{\mathcal{U}} E_i$, which is an infinite-dimensional closed subspace of $\widetilde{Y} = \prod_{\mathcal{U}} Y_i$. Then as in the last part of the proof of Theorem 2.16 we see that $\pi_{\widetilde{E}} \circ \widetilde{T}$ is surjective, and thus $\widetilde{T}$ is not strictly cosingular.

REMARK 2.18. The preceding developments remain true if one replaces Pietsch's notion of operator local representability by the more ancient notion of "finite representability" due to Heinrich [19]. (Other notions of operator finite representability have been introduced by several authors ([5], [8], [3]) which seem to be less relevant for dealing with duality).

Recall that in Heinrich's definition of finite representability the arrows V, W in the diagramme (1.2) have to be $(1+\varepsilon)$-isomorphisms and the diagramme is only "ε-approximatively commutative", i.e. $\|W\pi_N T i_E V - \pi_{N_1} T_1 i_{E_1}\| \leqslant \varepsilon$. Moreover in Heinrich's analogue of Theorem 1.10, the operators A, B have to be an isometric injection, respectively a metric surjection. Thus Heinrich's notion of finite representability is strictly stronger than Pietsch's notion of local representability (see Remark 1.8).

3. DOMINATION BY SUPER-SS AND SUPER-SC OPERATORS

In this section we study the domination of super-SS and super-SC operators between Banach lattices. We start by showing that the domination problem is not trivial. We benefit from some examples given in [13] and [14].

EXAMPLE 3.1. There exist two operators $0 \leqslant R \leqslant T : l_1 \to L^\infty[0,1]$ such that T is super-SS but R is not.

Indeed, take $\widetilde{R} : l_1 \to L^\infty[0,1]$ the isometry which takes the n^{th} element, e_n, of the canonical basis of l_1 to the n^{th}-Rademacher function, r_n, on $[0,1]$ (cf. p. 203 of [11]). Consider also the positive operators $R_1, R_2 : l_1 \to L^\infty[0,1]$ defined by $R_1(e_n) = r_n^+$ and $R_2(e_n) = r_n^-$ respectively, where r_n^+ and r_n^- denote respectively the positive and negative part of r_n. Clearly $\widetilde{R} = R_1 - R_2$ and $0 \leqslant R_1, R_2 \leqslant T$, where T is the rank-one operator defined by $T(x) = \Big(\sum_{n=1}^{\infty} x_n\Big)\chi_{[0,1]}$. Note that the operator T is super-SS being compact, but from the equalities $T = R_1 + R_2$ and $\widetilde{R} = R_1 - R_2$ it results that if one of the operators R_1 or R_2 were SS then both would be, as well as $\widetilde{R}$; hence none of the operators R_1, R_2 can be SS.

EXAMPLE 3.2. There exist two operators $0 \leqslant Q \leqslant P : L^2[0,1] \to l_\infty$ such that P has rank one (hence is super-SS) and Q is not SS (hence non super-SS).

Indeed, consider the positive compact operator $T : L^1[0,1] \to l_\infty$ defined by the equality $T(f) = (\int f)(1,1,1,\dots)$. Consider also the standard isometry R from $L^1[0,1]$ into l_∞ given by $R(f) = (h_n'(f))_n$, where $(h_n)_n$ is a dense sequence in the unit ball of $L^1[0,1]$ and $(h'_n)_n$ is a sequence of norm-one functionals such that $h'_n(h_n) = \|h_n\|$ for all n (cf. Theorem 2.1.14 of [31]). Notice that the space of all bounded operators between $L^1[0,1]$ and l_∞ coincides with the space of all regular (in the lattice sense) operators between $L^1[0,1]$ and l_∞, and is a Dedekind complete vector lattice (cf. Theorem 1.5.11 of [31]). In particular the positive and negative parts R^+, R^- of R as well as the modulus $|R|$ exist; furthermore $0 \leqslant R^+, R^- \leqslant |R|$.

It can be seen that $0 \leqslant R^+, R^- \leqslant |R| \leqslant T : L^1[0,1] \to l_\infty$ (see [14] for details). If J denotes the natural inclusion of $L^2[0,1]$ into $L^1[0,1]$ then the operator $RJ : L^2[0,1] \to l_\infty$ is not SS since R is an isometry and the inclusion J is not SS. It follows that the operators $(RJ)^+$ and $(RJ)^-$ cannot simultaneously be SS. On the other hand the operator TJ is compact, hence SS, and clearly

$$0 \leqslant (RJ)^+, \quad (RJ)^- \leqslant |RJ| \leqslant |R|J \leqslant TJ.$$

Finally write $P = TJ$ and $Q = (RJ)^+$ to conclude the proof.

It is well known that for a Banach space X the property having type p (cotype q) is a super property. In other words, if X has type p (cotype q), then every ultrapower of X has type p (cotype q).

Recall that a Banach lattice E satisfies an *upper* (respectively *lower*) q-estimate if there exists a constant $M > 0$ such that $\Big\|\sum_{i=1}^{n} x_i\Big\| \leqslant M\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q}$ (respectively $\Big\|\sum_{i=1}^{n} x_i\Big\| \geqslant M\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q}$) for every disjoint sequence $(x_i)_{i=1}^n$ in E and every natural n. The lower (respectively upper) index of E is defined as $s(E) = \sup\{q \geqslant 1 : E \text{ satisfies an upper } q\text{-estimate}\}$ (respectively $\sigma(E) = \inf\{q \geqslant 1 : E \text{ satisfies a lower } q\text{-estimate}\}$). It is well known that $1 \leqslant s(E) \leqslant \sigma(E) \leqslant \infty$ and that $1/s(E) + 1/\sigma(E^*) = 1$ and $1/\sigma(E) + 1/s(E^*) = 1$ (cf. pag. 563 of [46]). Also well known is the fact that a Banach lattice E has type p (cotype q) for some $p > 1$ ($q < \infty$) if and only if $s(E) > 1$ ($\sigma(F) < \infty$) (cf. pag. 101 of [29]). From main results in [14] and [15] we have (see Theorem 1.4 of [15]):

THEOREM 3.3. *Let $0 \leqslant R \leqslant T : E \to F$ be two positive operators from a Banach lattice E to a Banach lattice F with order continuous norm. If $\sigma(E) < \infty$, then if T is* SS *so is R.*

We can give now a domination result for the super-SS operator class:

PROPOSITION 3.4. *Let $0 \leqslant R \leqslant T : E \to F$ be two positive operators from a Banach lattice E to a Banach lattice F. If $\sigma(E) < \infty$ and $\sigma(F) < \infty$, then if T is super-*SS *so is R. Moreover the order interval of operators $[0,T] := \{R : 0 \leqslant R \leqslant T\}$ is "uniformly super-*SS*" in the sense that, when $n \to \infty$,*

$$\sup_{R \in [0,T]} b_n(R) \to 0.$$

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $E_{\mathcal{U}}$, $F_{\mathcal{U}}$ be the ultrapower of E and F respect to $\mathcal{U}$. Consider the operators $0 \leqslant R_{\mathcal{U}} \leqslant T_{\mathcal{U}} : E_{\mathcal{U}} \to F_{\mathcal{U}}$ and note that, by Proposition 1.5, $T_{\mathcal{U}}$ is SS since T is super-SS. Also $\sigma(E), \sigma(F) < \infty$ implies $\sigma(E_{\mathcal{U}}), \sigma(F_{\mathcal{U}}) < \infty$ by the remark above, and $\sigma(F_{\mathcal{U}}) < \infty$ implies that $F_{\mathcal{U}}$ is order continuous. Thus $R_{\mathcal{U}}$ is SS by Theorem 3.3 and hence R is super-SS.

Note that the sequence $\beta_n := \sup\limits_{R \in [0,T]} b_n(R)$ is decreasing. Assume that $\inf\limits_n \beta_n > \beta > 0$, then for every n there exists an operator $R_n \in [0, T]$ with $b_n(R_n) > \beta$. Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$ and set $\widetilde{R} = \prod_{\mathcal{U}} R_n$. Then $0 \leqslant \widetilde{R} \leqslant T_{\mathcal{U}}$ and since $T_{\mathcal{U}}$ is super-SS, so is $\widetilde{R}$ by the preceding. On the other hand for every $k \geqslant 1$ we have by Proposition 1.3, the following wich is a contradiction:

$$b_k(\widetilde{R}) = \lim_{n,\mathcal{U}} b_k(R_n) \geqslant \lim_{n,\mathcal{U}} b_n(R_n) \geqslant \beta > 0. \quad ∎$$

The following example shows that the q-concavity condition in Proposition 3.4 above plays an important role:

EXAMPLE 3.5. Let $E = (\bigoplus_{n=1}^{\infty} l_1^n)_p$ and $F = (\bigoplus_{n=1}^{\infty} l_\infty^{2^n})_q$, be the usual direct sum spaces with the p-norm and q-norm respectively and $1 \leqslant p < q < \infty$. Consider for every n the operator $T_n : l_1^n \to l_\infty^{2^n}$ which sends an arbitrary finite sequence $(a_k)_1^n$ to the sequence $(\sum a_k)(1, 1, \ldots, 1)$ of $l_\infty^{2^n}$. Consider also for all n the isometry $R_n : l_1^n \to l_\infty^{2^n}$ represented by the $(2^n \times n)$ matrix with $\{1, -1\}$-entries defined as follows

$$R_n \equiv (x_{k,\ell}) = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & -1 \\ 1 & 1 & \ldots & -1 & 1 \\ \multicolumn{5}{c}{\dotfill} \\ -1 & -1 & \ldots & -1 & -1 \end{pmatrix}$$

(we set $x_{k,\ell} = \varepsilon_k(\ell)$, where ε_k, for $k = 1, \ldots, 2^n$ is an enumeration of $\{-1, 1\}^n$).

Consider now the operators $T = \bigoplus T_n$ and $R = \bigoplus R_n$ from E into F. The operator T is positive and factorizes through the natural injection $i : l_p \hookrightarrow l_q$. Indeed, the operator $\varphi : E \to l_p$ defined as $\varphi(x_n)_n = (\sigma(x_n))_n$, where $\sigma(x_n) = \sum\limits_{i=1}^{n} x_{n,i}$, is well defined and bounded. Consider next the bounded operator $\psi : l_q \to F$ defined as $\psi(a_n)_n = \bigoplus a_n 1_{2^n}$, where 1_{2^n} is the unit of $l_\infty^{2^n}$. Notice that $T = \psi\, i\, \varphi$. Since i is super-SS as observed by Plichko (cf. Corollary 1 of [41]), the operator T itself is super-SS.

On the other hand it is clear that the operator R cannot be super-SS since R_n is an isometry for every n. Standard facts show that R is regular and that the inequalities $0 \leqslant R^-, R^+ \leqslant |R| \leqslant T$ hold true. From this we obtain that R^+ and R^- cannot be simultanously super-SS since R is not. Notice that F is reflexive and E has the subsequence splitting property (compare with the situation for SS operators in Theorem 1.1 of [14]).

To obtain a result on domination in the class of super-SC operators we benefit from Proposition 2.13

PROPOSITION 3.6. *Let $0 \leqslant R \leqslant T : E \to F$ be positive operators defined on a Banach lattice E and taking values in a Banach lattice F. If $s(E) > 1$ and $s(F) > 1$, then T super-*SC *implies R is also super-*SC. *Moreover the order interval $[0,T]$ is uniformly super-*SC *in the sense that*

$$\sup_{R\in[0,T]} a_n(R) \to 0.$$

Proof. T super-SC implies that the conjugate operator $T^* : F^* \to E^*$ is super-SS by Proposition 2.13. Since $s(E), s(F) > 1$ imply $\sigma(E^*), \sigma(F^*) < \infty$, we see that R^* is super-SS by Proposition 3.4; hence R is super-SC by Proposition 2.13. Moreover, by Propositions 2.6 and 3.4,

$$\sup_{R\in[0,T]} a_n(R) = \sup_{R\in[0,T]} b_n(R^*) \leqslant \sup_{S\in[0,T^*]} b_n(S). \quad ∎$$

Like for super-SS operators the problem of domination in the super-SC class is not trivial. Indeed, if $T, R : L^1[0,1] \to l_\infty$ are the operators defined in Example 3.2 above, and J denotes the natural inclusion of $L^2[0,1]$ into $L^1[0,1]$, then the conjugate operator $(TJ)^* : (l_\infty)^* \to L^2[0,1]$ is compact and hence SC; on the other hand since the operator RJ is not SS, the operator $(RJ)^*$ cannot be SC (cf. [37]) and hence not super-SC. It follows that the operators $((RJ)^*)^+$ and $((RJ)^*)^-$ cannot simultaneously be super-SC. However $0 \leqslant ((RJ)^*)^+, ((RJ)^*)^- \leqslant (TJ)^*$. Similarly a counterexample is derived from Example 3.5 for reflexive spaces failing to have non-trivial concavity.

In the case of endomorphisms $0 \leqslant R \leqslant T : E \to E$ it has been studied when the compactness properties of T are inherited by some power of R under no assumption on the Banach lattice E (see for instance [2], [31], [46]). Using recent results in [15] we have the following

PROPOSITION 3.7. *Let $0 \leqslant R \leqslant T : E \to E$ be positive operators on a Banach lattice E. If T is super-*SS *(respectively super-*SC*) then R^4 is also super-*SS *(respectively super-*SC*). In fact the set $\{R^4 : 0 \leqslant R \leqslant T\}$ is uniformly super-*SS *(respectively uniformly super-*SC*).*

Proof. Since $0 \leqslant R_{\mathcal{U}} \leqslant T_{\mathcal{U}} : E_{\mathcal{U}} \to E_{\mathcal{U}}$ and $T_{\mathcal{U}}$ is SS we get by Theorem 1.5 of [15] that $(R_{\mathcal{U}})^4 = (R^4)_{\mathcal{U}}$ is SS. Hence R^4 is super-SS. When T is super-SC the proof follows now from Proposition 2.13. ∎

REMARK 3.8. The results of Proposition 3.4 could be uniformized not only with respect to $R \in [0,T]$, but also with $T : E \to F$ having a sequence of Bernstein numbers controlled by a given sequence $b = (b_n)$, with $b_n \to 0$ (and E, F having q-lower estimate constants bounded from below: $M_q(E), M_q(F) \geqslant M > 0$). In other words $\overline{\beta}_n := \sup\{b_n(R) : R \in [0,T]; b_k(T) \leqslant b_k \text{ for all } k\}$ converges to zero. The sequence $(\overline{\beta}_n)$ depends only on the sequence b and the constants q and M. It would be interesting to have an explicit estimation of this dependence. A similar question holds for the results of Propositions 3.6 and 3.7.

4. SUPER-DSS OPERATORS

In this section we introduce the analogous "super" version of the class of DSS operators and study some of its properties. We recall that a bounded operator T defined in a Banach lattice E and with values in a Banach space Y is *disjointly strictly singular* (DSS for short) if the restriction of T to the span of any infinite sequence of pairwise disjoint of vectors is never an isomorphism. This class, introduced in [22], has proven useful in comparing the lattice structure of function spaces ([16]) as well as in the study of the domination problem for SS operators ([14]).

We will say that a subspace of a Banach lattice E is *disjointly generated* if it has a basis consisting of pairwise disjoint vectors.

DEFINITION 4.1. Let E be a Banach lattice, Y be a Banach space and $T : E \to Y$ a bounded operator. We say that T is not super disjointly strictly singular if there is a sequence $(E_n)_n$ of subspaces of E, with each E_n being an n-dimensional disjointly generated subspace of E, and a constant $C > 0$ such that $\|Tx\| \geqslant C\|x\|$ for every $x \in \bigcup_{n=1}^{\infty} E_n$. We say that T is *super disjointly strictly singular* (super-DSS in short) otherwise.

Define

$$c_n(T) = \sup\Big\{ \inf_{x\in S(E_n)} \|Tx\| : E_n = [(x_i^n)_{i=1}^n],\ \|x_i^n\| = 1,\ x_i^n \perp x_j^n = 0,\ i \neq j\Big\},$$

where $S(E_n)$ is the unit sphere of E_n and E_n varies with each choice of n disjoint vectors $\{x_i\}_{i=1}^n$. Clearly $c_k(T) \geqslant c_{k+1}(T) \geqslant 0$ for all k.

PROPOSITION 4.2. *An operator $T : E \to Y$ is super-*DSS *if and only if $c_n(T) \to 0$ as $n \to \infty$.*

We also have the following characterization of super-DSS operators

PROPOSITION 4.3. *An operator T from a Banach lattice E to a Banach space Y is super-*DSS *if and only if for every sequence $(E_n)_n$ of disjointly generated finite dimensional subspaces of E, with $\dim E_n \to \infty$, there exists a sequence of disjointly generated finite dimensional subspaces $(F_n)_n$, with $\dim F_n \to \infty$ and $F_n \subset E_n$, such that $\|T|_{F_n}\| \to 0$ as $n \to \infty$.*

Proof. We only need to prove the non trivial implication (necessity). For this we essentially follow Theorem 1 of [41]. However we do not consider decomposition constants but use disjointness instead. Thus, let us assume that T is super-DSS, and let $(E_n)_n$ be a sequence of subspaces of E, each E_n having a basis $(x_i^n)_{i=1,\dots,n}$ consisting of pairwise disjoint vectors. For every $n \geqslant 1$ set $k(n) = \max\{k \geqslant 1 : k^2 \leqslant n\}$. Clearly $k(n) \to \infty$ as n tends to infinity. Now for every n write

$$E_n = \bigoplus_{j=1}^{k(n)} G_j^n \oplus W^n$$

where $G_j^n = [x^n_{(j-1)k(n)+1}, \dots, x^n_{jk(n)}]$ and $W^n = [x^n_{k(n)^2+1}, \dots, x^n_n]$.

Since $\dim G_j^n = k(n)$ for all $j = 1, \dots, k(n)$ we can find $z_j^n \in G_j^n$ of norm one such that $\|Tz_j^n\| \leqslant c_{k(n)}(T)$ (Proposition 4.2). Since $c_{k(n)}(T) \to 0$ we can find $l(n) \in \{1, \dots, k(n)\}$ such that $l(n) \underset{n}{\to} \infty$ and $l(n)c_{k(n)}(T) \underset{n}{\to} 0$.

Consider now the span $F_n = [(z_j^n)_{j=1}^{l(n)}]$ in $\bigoplus_{j=1}^{k(n)} G_j^n$. For an arbitrary norm-one vector $x = \sum_{j=1}^{l(n)} a_j^n z_j^n$ in F_n we have $|a_j^n| = \|a_j^n z_j^n\| \leqslant \left\| \sum_{s=1}^{l(n)} a_s^n z_s^n \right\| = 1$ since $z_j^n \wedge z_k^n = 0, j \neq k$. Then

$$\|Tx\| \leqslant \sum_{j=1}^{l(n)} |a_j^n| \, \|Tz_j^n\| \leqslant \sum_{j=1}^{l(n)} |a_j^n| \, c_{k(n)}(T) \leqslant l(n) \, c_{k(n)}(T) \underset{n}{\to} 0.$$

Hence $\|T|_{F_n}\| \to 0$ since x was arbitrarily taken. This concludes the proof. ∎

It is clear that every super-SS operator is super-DSS. However the converse is not true (see Example 5.3 below). It is clear that if T is super-DSS and R is a bounded operator then RT is super-DSS. Nevertheless super-DSS operators do not form an operator ideal since being super-DSS is a non-stable property by composition by the right. In fact, being DSS is not stable by composition by the right ([21]).

Like Bernstein numbers, the numbers c_n can be easily computed for an ultraproduct:

PROPOSITION 4.4. *Let $(T_i : E_i \to Y_i)_{i \in I}$ be a bounded family of bounded operators, each defined on a Banach lattice E_i, and $\mathcal{U}$ an ultrafilter over the index set I. Let $\widetilde{T} = \prod_{\mathcal{U}} T_i$ be the corresponding ultraproduct of the family (T_i). Then, for every $n \geqslant 1$,*

$$c_n(\widetilde{T}) = \lim_{i,\mathcal{U}} c_n(T_i).$$

Proof. The proof is similar to that of Proposition 1.3. Observe however that for proving the inequality $\lim_{i,\mathcal{U}} c_n(T_i) \geqslant c_n(\widetilde{T})$ we need to prove first that if $E_n = [(\xi_j)_{j=1}^n]$, $\xi_j \perp \xi_l = 0$, is a finite-dimensional subspace in $\widetilde{E} = \prod_{\mathcal{U}} E_i$ generated by disjoint elements then it is possible to find n families $(x_i^{(j)})_{i \in I}$, $j = 1, \dots, n$ in $\prod E_i$ whose corresponding classes $[x^{(j)}]_{\mathcal{U}}$ coincide with ξ_j for every j, and such that $x_i^{(j)} \perp x_i^{(l)}$ for every i and $j \neq l$. This fact is routine, we recall simply the proof for the sake of completeness. To avoid unnecessary notation we will restrict ourselves to the case $n = 2$. Since $|\xi_1| \wedge |\xi_2| = 0$, then for any families $(x_i^1), (x_i^2)$ representing ξ_1, respectively ξ_2 we have

$$0 = \lim_{i,\mathcal{U}} \| \, |x_i^1| \wedge |x_i^2| \, \| = \| \, |\xi_1| \wedge |\xi_2| \, \|.$$

For a fixed i consider the elements

$$y_k^1 = \operatorname{sign}(x_k^1)(|x_k^1| \wedge |x_k^2|) \quad \text{and} \quad y_k^2 = \operatorname{sign}(x_k^2)(|x_k^1| \wedge |x_k^2|),$$

and set $\widehat{x}_i^1 = x_i^1 - y_i^1$ and $\widehat{x}_i^2 = x_k^i - y_i^2$. Notice that $|\widehat{x}_k^1| \wedge |\widehat{x}_k^2| = 0$.

Since y_k^1 and y_k^2 have the same sign as x_k^1 and x_k^2 respectively, we have the equalities

$$|\widehat{x}_k^1| = | \, |x_k^1| - |y_k^1| \, | = |x_k^1| - (|x_k^1| \wedge |x_k^2|),$$

and similarly

$$|\widehat{x}_k^2| = |x_k^2| - (|x_k^1| \wedge |x_k^2|).$$

Since $\|y_k^1\| = \| |x_k^1| \wedge |x_k^2| \| = \|y_k^2\|$ tend to zero along $\mathcal{U}$ it follows that $[x_k^1]_{\mathcal{U}} = [\widehat{x}_k^1]_{\mathcal{U}}$ and $[x_k^2]_{\mathcal{U}} = [\widehat{x}_k^2]_{\mathcal{U}}$. Thus $(\widehat{x}_k^1)$, $(\widehat{x}_k^2)$ represent ξ_1, respectively ξ_2 as well. ∎

We can deduce now a characterization of super-DSS operators in terms of ultrapowers, analogous to that obtained for super-SS operators:

PROPOSITION 4.5. *An operator $T : E \to Y$ is super-*DSS *if and only if the ultrapower $T_{\mathcal{U}} : E_{\mathcal{U}} \to Y_{\mathcal{U}}$ is* DSS *for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (and then $T_{\mathcal{U}}$ is* DSS *for any ultrafilter $\mathcal{U}$ on any set).*

For the proof just mimic the proof of Proposition 1.5.

COROLLARY 4.6. *The class of super-*DSS *operators between E and Y forms a closed vector space.*

Regarding domination results we benefit from the next result given in [13]:

PROPOSITION 4.7. *Let $0 \leqslant R \leqslant T : E \to F$ be two positive operators between the Banach lattices E and F. Assume that F is order continous. If T is* DSS *then R is* DSS.

If we jointly use Proposition 4.7 and Proposition 4.5, and notice that for a Banach lattice F, the condition $\sigma(F) < \infty$ is equivalent to the order continuity of $F_{\mathcal{U}}$ for every ultrafilter $\mathcal{U}$ (cf. Proposition 4.6 of [18]), we obtain the following:

PROPOSITION 4.8. *Let $0 \leqslant R \leqslant T : E \to F$ be two positive operators between the Banach lattices E and F. Assume that $\sigma(F) < \infty$. If T is super-*DSS *then R is also super-*DSS.

In fact like in Proposition 3.4, but using now Proposition 4.4 one sees that the interval $[0, T]$ is uniformly super-DSS (i.e. $\sup\{c_n(R) : 0 \leqslant R \leqslant T\} \to 0$).

Notice that the operator R in Example 3.5 is actually not super-DSS since the subspaces l_1^n are sublattices of $E = (\bigoplus_{n=1}^{\infty} l_1^n)_p$; however the operator T is super-DSS being super-SS. Thus, in absence of non trivial concavity in F, Proposition 4.8 is no longer true.

REMARK 4.9. In the case of endomorphisms $0 \leqslant R \leqslant T : E \to E$ we benefit from Theorem 1.2 of [13] to obtain that if T is super-DSS then R^2 must also be super-DSS.

5. RELATIONS BETWEEN SINGULAR CLASSES

We pass to summarize the relations between the "singular" operator classes considered so far and provide examples for rearrangement invariant spaces.

EXAMPLE 5.1 ([32], [44]). Let $1 < p < q < \infty$. A bounded operator $T : l_p \to l_q$ which is SS and not super-SS.

Indeed, consider two isomorphisms $\varphi : l_p \to (\bigoplus_{n=1}^{\infty} l_2^n)_p$ and $\psi : (\bigoplus_{n=1}^{\infty} l_2^n)_q \to l_q$ (see for instance pag. 73 of [28]). Consider also the inclusion $J : (\bigoplus_{n=1}^{\infty} l_2^n)_p \to (\bigoplus_{n=1}^{\infty} l_2^n)_q$, and the composition $T = \psi J \varphi : l_p \to l_q$. For every n call $E_n = \varphi^{-1}(l_2^n)$ and $F_n = \psi(l_2^n)$; then $d(E_n, l_2^n) \leqslant \|\varphi\| \|\varphi^{-1}\|$ and $d(F_n, l_2^n) \leqslant \|\psi\| \|\psi^{-1}\|$ where $d(\cdot, \cdot)$ stands for the Banach–Mazur distance. The operator T uniformly preserves copies of l_2^n since $T(E_n) = F_n$ for all n; hence it is not super-SS. However T is SS as operator from l_p into l_q.

EXAMPLE 5.2. Let $1 < p < q < \infty$. A bounded operator $T : l_p \to l_q$ which is SC and yet not super-SC.

Consider the conjugate operator T^* in Example 5.1 and note that T^* is SC by Proposition 3 of [36] but not super-SC by Proposition 2.13.

EXAMPLE 5.3. The inclusion $i : L^2[0,1] \hookrightarrow L^1[0,1]$ is super-DSS but not super-SS.

Indeed, let $(E_n)_n$ be a sequence of subspaces of $L^2[0,1]$ where $E_n = [(x_i^n)_{i=1}^n]$ is generated by n nonzero disjoint elements. Then E_n is isometric to ℓ_2^n while $i(E_n)$ is isometric to ℓ_1^n. Since the Banach–Mazur distance of ℓ_2^n to ℓ_1^n is $\sqrt{n}$, it follows that the restrictions of i to the subspaces E_n's cannot have uniformly bounded isomorphism constants. This proves that i is super-DSS. On the other hand i is not SS (hence not super-SS) since by Khintchine inequality it preserves the span generated by the Rademacher functions which is isomorphic to l_2.

An example of a DSS operator which is not super-DSS is the natural inclusion $i_{p,q}$ from $E_p = (\bigoplus_{n=1}^{\infty} \ell_2^n)_p$ into $E_q = (\bigoplus_{n=1}^{\infty} \ell_2^n)_q$ (when $1 \leqslant p < q < \infty$). Indeed, this operator is not super-DSS since it preserves sublattices ℓ_2^n of arbitrary high dimension, but it is DSS, and even SS since every infinite dimensional subspace of E_p contains a copy of l_p, while every infinite dimensional subspace of E_q contains a copy of l_q (so E_p and E_q are "totally incomparable").

On the other hand all super properties imply the corresponding properties. We summarize the previous lines in the following diagram (no reverse arrows can be drawn):

$$\begin{array}{ccc} \text{super-SS} & \longrightarrow & \text{super-DSS} \\ \downarrow & & \downarrow \\ \text{SS} & \longrightarrow & \text{DSS} \end{array}$$

Given a Banach lattice E with a pairwise disjoint Schauder basis and a Banach space Y, it holds that a bounded operator $T : E \to Y$ is DSS if and only if it is SS ([21]). For super-DSS and super-SS operators the situation is different. Indeed, let $T : l_p \to l_q$, $1 < p < q$ be the non super-SS operator considered in Example 5.1. Like in Example 5.3, the operator T is super-DSS because the Banach–Mazur distance of ℓ_p^n and ℓ_q^n goes to infinity as $n \to \infty$.

Converses to domination results in the sense of obtaining necessary conditions on the Banach lattices involved from the (assumed) existence of a domination result for compact operators have been studied in [45]. In [13] and [14] it is shown that if there is a domination result in the classes of DSS or SS operators (F is assumed to be σ-Dedekind complete), then either E^* or F must be order continuous. The same proof works for super-SS and super-DSS operators. However one could conjecture that in the "super" situation the assumption of the existence of a domination result should yield stronger conditions like non-trivial convexity on E or concavity on F. This is not the case:

EXAMPLE 5.4. Consider $E = (\bigoplus_{n=1}^{\infty} l_1^n)_p$ and $F = (\bigoplus_{n=1}^{\infty} l_\infty^{2^n})_q$, for $1 < q < p < \infty$. An adaptation of a result by Pitt (cf. Proposition 2.c.3 of [28]) shows that every bounded operator from E to F must be compact. Hence the domination result trivially holds in the classes of super-SS and super-DSS operators from E to F. Yet the (reflexive) spaces E and F have trivial convexity and concavity respectively.

We pass now to consider *rearrangement invariant* function spaces E on the interval $[0, 1]$. We refer to [29] for definitions. We will show that the canonical inclusion $E \hookrightarrow L^1$ is always super-SC and that the canonical inclusion $L^\infty \hookrightarrow E$ is always super-SS. First we prove that both inclusions are super weakly compact.

Recall that an operator T from X to Y is *super weakly compact* if every ultrapower of T is weakly compact. This class is stable by duality: T is super weakly compact if and only if T^* is super weakly compact. Indeed, if $T : X \to Y$ is super-weakly compact, every ultrapower $T_{\mathcal{U}}$ is weakly compact; then $(T_{\mathcal{U}})^* : (Y_{\mathcal{U}})^* \to$

$(X_{\mathcal{U}})^*$ is weakly compact too. Let $\delta_{X,\mathcal{U}}$ (respectively $\delta_{Y,\mathcal{U}}$) be the canonical embedding of $(X^*)_{\mathcal{U}}$ into $(X_{\mathcal{U}})^*$ (respectively of $(Y^*)_{\mathcal{U}}$ into $(Y_{\mathcal{U}})^*$). Then the equality $\delta_{X,\mathcal{U}} \circ (T^*)_{\mathcal{U}} = (T_{\mathcal{U}})^* \circ \delta_{Y,\mathcal{U}}$ shows that $\delta_{X,\mathcal{U}} \circ (T^*)_{\mathcal{U}}$ is weakly compact. Since $\delta_{X,\mathcal{U}}$ is an isomorphic injection, $(T^*)_{\mathcal{U}}$ is weakly compact too, and T^* is super-weakly compact. Conversely if T^* is super-weakly compact then T^{**} is also by the preceding. Then $j_Y \circ T = T^{**} \circ j_X$ and T itself are super-weakly compacts too because the class of super-weakly compact operators form an injective operator ideal (these properties are trivially inherited from the class of weakly compact operators).

The following result is equivalent to a statement given as a remark without proof in p. 123 of [5].

PROPOSITION 5.5. *Let X be a Banach space. Every weakly compact operator T : $X \to L^1[0,1]$ is super weakly compact.*

Proof. Let $\mathcal{U}$ be an ultrafilter, the ultrapower $(L^1)_{\mathcal{U}}$ can be decomposed in two disjoint bands as $(L^1)_{\mathcal{U}} = C \oplus C^{\perp}$, where C is the band of elements represented by L^1-equi-integrable families which is lattice-isomorphically isometric to the space $L^1(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$, where the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is the ultrapower of the usual Lebesgue measure space $([0,1], \mathcal{A}, \mu)$ (see for instance [20]). Note that $T_{\mathcal{U}}(B_{X_{\mathcal{U}}})$ is a subset of $L^1(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$. Indeed, $T(B_X)$ is equi-integrable by Dunford–Pettis criterion, since it is relatively weakly compact. Every $\xi \in T_{\mathcal{U}}(B_{X_{\mathcal{U}}})$ is represented by a family $(Tx_i)_i$, with $x_i \in B_X$, which is equi-integrable; hence $T_{\mathcal{U}}(B_{X_{\mathcal{U}}}) \subset C$.

Let us show that $T_{\mathcal{U}}(B_{X_{\mathcal{U}}})$ is equi-integrable in $L^1(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$. Since $T(B_X)$ is equi-integrable there is a function $\delta : (0,\infty) \to (0,\infty)$ such that for all $\varepsilon > 0$:

$$x \in B_X,\ A \in \mathcal{A}, \quad \mu(A) < \delta(\varepsilon) \ \Rightarrow\ \|\chi_A Tx\| < \varepsilon.$$

If $\xi \in B_{X_{\mathcal{U}}}$ and $\widetilde{A} \in \widetilde{\mathcal{A}}$, with $\widetilde{\mu}(\widetilde{A}) < \delta(\varepsilon)$, we have $\chi_{\widetilde{A}} T_{\mathcal{U}} \xi = [\chi_{A_i} Tx_i]_{\mathcal{U}}$ where $x_i \in B_X$, $[x_i]_{\mathcal{U}} = \xi$, and $\forall i$, $A_i \in \mathcal{A}, \mu(A_i) < \varepsilon$. Then $\forall i \in I$ $\|\chi_{A_i} Tx_i\| < \varepsilon$ which implies $\|\chi_{\widetilde{A}} T_{\mathcal{U}} \xi\| < \varepsilon$. ∎

COROLLARY 5.6. *Let E be a rearrangement invariant space. If $E \neq L^1$ (respectively $E \neq L^\infty$), then the inclusion $i_E : E \hookrightarrow L^1$ (respectively $L^\infty \hookrightarrow E$) is super weakly compact.*

Proof. Since $E \neq L^1$ we have $\phi_E(t)/t \to \infty$ as $t \to 0$, where ϕ_E is the fundamental function of E (i.e. $\phi_E(t) = \|\chi_{[0,t]}\|_E$), and there exists an intermediate reflexive rearrangement invariant space between E and L^1 (cf. [23]): hence i_E is weakly compact. Apply Proposition 5.5 to conclude.

Also, when $E \neq L^\infty$ we have $\phi_E(t) \to 0$ as $t \to 0$, and there exists an intermediate reflexive rearrangement invariant space G between L^∞ and E. The conjugate space G^* is then a reflexive rearrangement invariant space, hence the inclusion $G^* \hookrightarrow L^1$ is weakly compact, and thus super weakly compact by Proposition 5.5. And, by duality, $L^\infty \hookrightarrow G$ is also super weakly compact, and so is the inclusion $L^\infty \hookrightarrow E$ by ideal property of this class of operators. ∎

PROPOSITION 5.7. *Let E be a rearrangement invariant space. If $E \neq L^\infty$ then the inclusion $i_E : L^\infty \hookrightarrow E$ is super*-SS. *If $E \neq L^1$ then the inclusion $E \hookrightarrow L^1$ is super*-SC.

Proof. Let $E \neq L^\infty$. The ultrapower inclusion $(i_E)_{\mathcal{U}} : (L^\infty)_{\mathcal{U}} \hookrightarrow E_{\mathcal{U}}$ is weakly compact by the previous corollary. Hence it is strictly singular since the AM-space $(L^\infty)_{\mathcal{U}}$ has the Dunford–Pettis property ([36]). Using Proposition 1.5 we conclude that $i_E : L^\infty \hookrightarrow E$ is super-SS.

The case $E \neq L^1$ is consequence of the first part and Proposition 2.13. ∎

Regarding disjoint strict singularity it is known that for every rearrangement invariant space $E \neq L^1$ the canonical inclusion $E \hookrightarrow L^1$ is DSS (cf. [35]); a stronger result is the following:

PROPOSITION 5.8. *Every weakly compact disjointness preserving operator T from a Banach lattice X to L^1 is super*-DSS.

Proof. The operator T is DSS, otherwise it would fix an isomorphic copy of ℓ_1. Using Proposition 5.5 we have that any ultrapower $T_{\mathcal{U}}$ of T is a weakly compact operator with values in some L^1-space and is also disjointness preserving (see proof of Proposition 4.4); hence $T_{\mathcal{U}}$ is DSS and thus T is super-DSS. ∎

In particular, if $E \neq L^1$ is a rearrangement invariant space then the inclusion $E \hookrightarrow L^1$ is super-DSS. This can also follow from the proof of Theorem 2.1 in [35].

6. COMPARING DSS AND LSS OPERATORS CLASSES

In [12], and implicitly in [13], the class of lattice strictly singular operators is formally introduced.

A bounded operator T from a Banach lattice E to a Banach space Y is said to be *lattice strictly singular* (LSS for short) if for every (closed) infinite-dimensional sublattice G of E the restriction $T : G \to Y$ is not an isomorphism onto its image. Equivalently T is LSS if for every pairwise disjoint sequence $(x_n)_n$ in E of *positive* vectors the restriction of T to the span $[(x_n)_n]$ is not an isomorphism onto its image. It seems unknown whether it has a vector space structure as well as if it actually constitutes a new class. Obviously every DSS operator is LSS. In [13] it is shown that for order continuous Banach lattices F, a positive operator $T : E \to F$ is LSS if and only if it is DSS. In [12] a few more cases are covered for regular operators.

In this section we formally introduce the class of super-LSS operators not for the sake of generalization but with the hope that its study will shed some light to the question above. Thus Theorem 6.3 shows that in the super setting both classes are the same. As a consequence it is seen that an operator which is an ultrapower of another operator is LSS if and only if it is DSS. Actually the proof of this result (which uses a well-known theorem of Krivine and the spreading-model construction by Brunel and Sucheston) essentially applies to the class of *stable* Banach lattices (in the sense of [27]) and, as a consequence, we obtain that LSS and DSS operators are the same in this case.

DEFINITION 6.1. A bounded operator T from a Banach lattice E to a Banach space Y is said to be *super-LSS* if it is not possible to find any sequence of sublattices $(E_n)_n$, where E_n is spanned by n pairwise disjoint positive vectors, and any $\alpha > 0$ such that $\|Tx\| \geqslant \alpha\|x\|$ for every $x \in \bigcup_n E_n$.

Of course every super-DSS operator is super-LSS. Notice also that there is a similar characterization to the one given in Proposition 4.5 for super-DSS operators: T is super LSS if and only if every ultrapower $T_{\mathcal{U}}$ is LSS for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (or equivalently for every ultrafilter).

The next proposition will be used in the proof of our equivalence result.

PROPOSITION 6.2. *Let $T : l_1 \to Y$ be a bounded operator with values in a Banach space Y. Then T is* LSS *if and only if T is* DSS.

Proof. If $T : l_1 \to Y$ is not DSS then there is a pairwise disjoint normalized sequence $(x_n)_n$ in l_1 such that the restriction of T to the span $[(x_n)_n]$ is an isomorphism. We may assume that $\|Tx_n^+\| \geqslant c$, $\|Tx_n^-\| \geqslant c$ for some $c > 0$. Indeed, if $\|Tx_n^+\|$ tends to zero, then the sequences $(Tx_n)_n$ and $(Tx_n^-)_n$ have equivalent basic subsequences by Proposition 1.a.9 of [28]. It follows that there is some subsequence $(x_{n_k})_k$ such that T is invertible on $[(x_{n_k}^-)_k]$ and we are done (a similar situation holds if $\|Tx_n^-\|$ tends to zero). By Rosenthal's dichotomy theorem (cf. Theorem 2.e.5 of [28]), there is a subsequence $(x_{n_k})_k$ such that either $(Tx_{n_k}^+)_k$ is equivalent to the unit basis of l_1 or $(Tx_{n_k}^+)_k$ is weakly Cauchy. Clearly we may disregard the first case and assume the second one. Hence, there is a further subsequence, still denoted by $(x_{n_k})_k$ such that either $(Tx_{n_k}^-)_k$ is equivalent to the unit basis of l_1 or $(Tx_{n_k}^-)_k$ is weakly Cauchy. Again the first case can be disregarded. Thus, both sequences $(Tx_{n_k}^+)_k$ and $(Tx_{n_k}^-)_k$ are weakly Cauchy and so is $(Tx_{n_k})_k$. Since T is an isomorphism on $[(x_n)_n]$, the sequence $(x_{n_k})_k$ is weakly Cauchy and in fact norm-convergent in l_1 by Schur lemma. Its limit must clearly be 0, which is a contradiction with being normalized. ∎

THEOREM 6.3. *Let E be a Banach lattice, Y a Banach space and $T : E \to Y$ a bounded operator. Then T is super-*LSS *if and only if T is super-*DSS.

Proof. Assume that T is not super-DSS; then according to Proposition 4.5 there exists some free ultrafilter $\mathcal{U}$ such that $T_{\mathcal{U}} : E_{\mathcal{U}} \to Y_{\mathcal{U}}$ is not DSS. Hence, by replacing T with $T_{\mathcal{U}}$, we can assume that there is a pairwise disjoint sequence of norm-one vectors $(x_n)_n$ in E such that T is invertible on the span $[(x_n)_n]$ (note that the vectors x_n may not be positive). From this point on we will make some reductions that eventually will help to prove the result. Notice that the ultrapower of an ultrapower of a Banach space (lattice) is an ultrapower of the initial space (lattice). This justifies all the "passing to some ultrapower" reductions used along the proof.

Step 1. We can assume that $(x_n)_n$ is equivalent to the l_p-canonical basis.

Indeed, by a well-known theorem of Krivine ([26]) l_p is finitely representable in the sequence $(x_n)_n$, that is, there exists some $p \in [1, \infty]$ such that for all integer N there is some finite sequence $(y_m^N)_{m=1,\dots,N}$ of normalized pairwise disjoint blocks of the sequence $(x_n)_n$ which is equivalent to the unit basis of l_p^N (with uniformly bounded constants for all N). Consider the ultrapower $E_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ and $T_{\mathcal{U}} : E_{\mathcal{U}} \to Y_{\mathcal{U}}$ (notice that E and Y were already assumed to be ultrapowers). If $\widetilde{y}_m$ denotes the class in $E_{\mathcal{U}}$ of the sequence $(y_m^N)_N$ (set $y_m^N = 0$ when $N < m$), then it is seen that the sequence $(\widetilde{y}_m)_m$ is equivalent to the unit basis of l_p, and $T_{\mathcal{U}}$ is an isomorphism onto its image when restricted to the span $[(\widetilde{y}_m)_m]$. To simplify notation let us replace from now on $T_{\mathcal{U}}$ by T, $E_{\mathcal{U}}$ by E and $Y_{\mathcal{U}}$ by Y.

Step 2. We can assume that the sequences $(x_n^+)_n$ and $(x_n^-)_n$ are equivalent to the canonical basis of l_p.

First notice that, by a standard perturbation argument, we can assume

$$\inf_n \{\|x_n^+\|, \|x_n^-\|\} \geqslant C > 0,$$

since otherwise we would obtain the invertibility of T on the span of a subsequence of either $(x_n^+)_n$ or of $(x_n^-)_n$, and the proof would be completed. Hence we can assume that both $(x_n^+)_n$ and $(x_n^-)_n$ are normalized unconditional basic sequences. We apply again Krivine's theorem to each sequence separately and obtain that there is some q and some r such that l_q is finitely representable in $(x_n^+)_n$ and l_r is finitely representable in $(x_n^-)_n$. In fact the unconditionality of the sequence $(x_n^+)_n$ allows to assume that the blocks obtained in Krivine theorem are positive by simply replacing the coefficients by their absolute value. Call the blocks $z_m^N = \sum_i \alpha_{m,i}^N x_i^+$ and call also $y_m^N = \sum_i \alpha_{m,i}^N x_i$. Clearly $\|z_m^N\| \leqslant \|y_m^N\|$. If we pass again to an ultrapower as in Step 1 we obtain a sequence $(\widetilde{y}_m')_m$ equivalent to the unit basis of l_p and a sequence $(\widetilde{z}_m')_m$ equivalent to the unit basis of l_q. Notice also that $\widetilde{z}_m' = \widetilde{y}_m'^{+}$. Again we may assume that $\inf_m \|\widetilde{z}_m'\| \geqslant c > 0$. Thus, so far the situation has been reduced to the case that $(x_n)_n$ is equivalent to the unit basis of l_p and $(x_n^+)_n$ is equivalent to the unit basis of l_q. Similarly we can assume that $(x_n^-)_n$ is isomorphic to the unit basis of l_r. Since we have the following, we necessarily obtain that $p \leqslant \min(q, r)$:

$$\max\left\{\Big\|\sum \alpha_n x_n^+\Big\|, \Big\|\sum \alpha_n x_n^-\Big\|\right\} \leqslant \Big\|\sum \alpha_n x_n\Big\|.$$

Assume that $q > p$. Then considering the blocks

$$y_m = (2m+1)^{-1/p} \sum_{n=m^2+1}^{(m+1)^2} x_n$$

we have that $(y_m)_m$ is still equivalent to the unit basis of l_p but $\|y_m^+\| = (2m+1)^{q^{-1}-p^{-1}}$ tends to zero. As above a perturbation argument shows that T is invertible on the span of some subsequence $(y_{m_k}^-)_k$ and we are done. Hence we can assume that $q = p$. Similarly we may assume that $r = p$.

Step 3. The sequences $(Tx_n^+)_n$ and $(Tx_n^-)_n$ can be assumed to be seminormalized and unconditional.

Indeed, once again the seminormalization stems from the perturbation argument above. Notice that we can disregard the case $p = 1$ according to Proposition 6.2.

Thus the sequences $(x_n^+)_n$ and $(x_n^-)_n$ are weakly convergent and so are the sequences $(Tx_n^+)_n$ and $(Tx_n^-)_n$. Since they are seminormalized we can assume, by passing to a subsequence, that both are basic sequences. To deal with the unconditionality we will apply Brunel–Sucheston's result ([9]) to the sequence $(Tx_n^+)_n$. Thus there exists a "good" subsequence, still denoted by (Tx_n^+), and a norm $|||\cdot|||$ on $\mathbb{R}^{(\mathbb{N})}$ such that for all natural M and all scalars $\alpha_1, \ldots, \alpha_M$ we have the following, where (e_i) are the unit vectors of $\mathbb{R}^{(\mathbb{N})}$,

$$\lim_{k_1\to\infty,\, k_1<k_2<\cdots<k_M} \Big\| \sum_1^M \alpha_i T x_i^+ \Big\| \to \Big|\Big|\Big| \sum_1^M \alpha_i e_i \Big|\Big|\Big|.$$

Note that for every N,

$$\Big|\Big|\Big| \sum_1^N \alpha_i e_i \Big|\Big|\Big| \leqslant \|T\| \Big(\sum_1^N |\alpha_i|^p \Big)^{1/p},$$

from which it follows that $N^{-1}\Big|\Big|\Big| \sum_1^N e_i \Big|\Big|\Big| \to 0$ as N tends to infinity. This implies that the fundamental sequence $(e_i)_i$ of the spreading model is unconditional ([7], Chapter I, Proposition 4). By passing to a further subsequence we may assume that the sequence $(Tx_n^-)_n$ is also "good" in Brunel–Sucheston's sense.

We pass again to the ultrapower along a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and consider the class $\widetilde{x_k} = [(x_{k+n})_n]_{\mathcal{U}}$ in $E_{\mathcal{U}}$ for every k. Then the equality $\Big\| \sum_1^M \alpha_k \widetilde{T x_k^+} \Big\| = \Big|\Big|\Big| \sum_1^M \alpha_k e_k \Big|\Big|\Big|$ easily holds true and from here the unconditionality of the sequence $(\widetilde{T x_k^+})_k$ follows.

Similarly we get that the sequence $(\widetilde{T x_k^-})_k$ is also unconditional.

Step 4. Both sequences $(Tx_k^+)_k$ and $(Tx_k^-)_k$ can be assumed to be equivalent to the unit basis of l_p.

Like in Step 1 there is some q and r such that l_q and l_r are finitely representable in $(Tx_k^+)_k$ and $(Tx_k^-)_k$ respectively. Notice that the blocks can be choosen to have positive coefficients (here the unconditionality is crucial). Since

$$\max\left\{\left\|\sum \alpha_k T x_k^+\right\|, \left\|\sum \alpha_k T x_k^-\right\|\right\} \leqslant C\|T\|\left(\sum |\alpha_k|^p\right)^{1/p},$$

we conclude as above that $q, r \geqslant p$. Now reasoning similarly as in Step 2 we can deduce that $p = q = r$. Hence T is invertible on $[(x_k^+)_k]$. The proof is completed. ∎

The proof of Theorem 6.3 easily adapts to the class of *stable* Banach spaces (of J.L. Krivine and B. Maurey [27]). Recall that a Banach space E is called *stable* if for every pair $(x_n)_n$, $(y_n)_n$ of bounded sequences the following equality holds

$$\lim_{n\to\infty}\lim_{m\to\infty}\|x_n + y_m\| = \lim_{m\to\infty}\lim_{n\to\infty}\|x_n + y_m\|,$$

whenever one of the iterated limits exists. Krivine-Maurey's theorem ([27]) states that every infinite-dimensional stable Banach space contains an almost isometric copy of some l_p, $1 \leqslant p < \infty$. In fact their proof shows that if $(x_n)_n$ is a sequence in X with no convergent subsequence, then a sequence of consecutive blocks $y_m = \sum_{N_m+1}^{N_{m+1}} \alpha_n x_n$ is asymptotically isometric to the unit basis of some l_p, that is

$$\left\|\sum_{j=m}^{\infty}\lambda_j y_j\right\| \overset{1+\varepsilon_m}{\sim} \left(\sum_{j=m}^{\infty}|\lambda_j|^p\right)^{1/p},$$

with $\varepsilon_m \to 0$ as m grows to infinity. Among stable spaces are L^p spaces for $1 \leqslant p < \infty$ ([27]), Orlicz spaces ([17]), Lorentz spaces ([42]) and the corresponding spaces of vector-valued functions with values in a stable Banach space ([4]).

Based on the ideas above we obtain the following result (which improves previous ones given in [13] and [12] for regular operators):

PROPOSITION 6.4. *Let E be a stable Banach lattice and let Y be a stable Banach space. Let $T : E \to Y$ be a bounded operator. Assume that Y has an unconditional finite dimensional decomposition. Then T is* LSS *if and only if T is* DSS.

Proof. We essentially mimic the proof of Theorem 6.3, replacing at each stage the finite sequences of blocks generating the l_p^n-copies with infinite sequences of blocks generating copies of l_p. Thus, if T is not DSS then there exists a disjoint sequence $(x_n)_n$ which generates l_p such that the restriction of T to the span $[(x_n)_n]$ is an isomorphism. As in the proof of Theorem 6.3 we can choose a sequence of consecutive disjoint blocks $(y_m)_m$ of $(x_n^+)_n$, respectively $(z_m)_m$ of $(x_n^-)_n$, with positive coefficients which are equivalent to the canonical basis of l_q, respectively of l_r. As in there it can be seen that $p = q = r$. Hence we can assume case that $(x_n)_n$, $(x_n^+)_n$ and $(x_n^-)_n$ are equivalent to the canonical basis of l_p. By applying Rosenthal's dichotomy theorem we may disregard the case $p = 1$. Then the sequences $(Tx_n^+)_n$ and $(Tx_n^-)_n$ are weakly null and in fact they can be assumed to be seminormalized. Furthermore they have subsequences $(Tx_{n_k}^+)_k$ and $(Tx_{n_k}^-)_k$ which are unconditional basic sequences by the assumption on the space Y. By applying Maurey–Krivine's result again to these two sequences we proceed as in Theorem 6.3 to conclude that T is invertible on $[(x_n^+)_n]$ and hence not LSS. ∎

REMARK 6.5. The previous result applies to $Y = L^p$ with $1 < p < \infty$. In contrast, Johnson, Maurey and Schechtmann produced a weakly null sequence in L^1 without unconditional subsequences ([24]). Thus, an attempt to adapt the proof of Proposition 6.4 to the case where $Y = L^1$ looks very unpromising.

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