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PERTURBATIONS OF THE RIGHT AND LEFT SPECTRA FOR OPERATOR MATRICES

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ABSTRACT. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. For given $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $M_{(X,Y)}$ denotes an operator acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the form $M_{(X,Y)} = \left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)$, where $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. In this paper, a necessary and sufficient condition is given for $M_{(X,Y)}$ to be right invertible for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. In addition, it is shown that if $\dim \mathcal{H}_2 = \infty$ then $M_{(X,Y)}$ is left invertible for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$; if $\dim \mathcal{H}_2 < \infty$ then $M_{(X,Y)}$ is left invertible for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$ if and only if $\mathcal{R}(A)$ is closed and $\dim \mathcal{N}(A, C) \leqslant \dim \mathcal{H}_2$.

KEYWORDS: *Operator matrices, right (left) invertible operator, right (left) spectrum, perturbations of spectra.*

MSC (2000): 47A10, 47A55.

INTRODUCTION

The study of operator matrices arises naturally from the following fact: if $\mathcal{H}$ is a Hilbert space and we decompose $\mathcal{H}$ as a direct sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then each bounded operator $T : \mathcal{H} \to \mathcal{H}$ can be expressed as the operator matrix form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with respect to the space decomposition, where T_{ij} is an operator from $\mathcal{H}_j$ into $\mathcal{H}_i$, $i, j = 1, 2$. One way to study operators is to see them as entries of simpler operators. The operator matrices have been studied by numerous authors (see [1], [2], [3], [4], [5], [6], [7], [8], [11] and the references therein). This paper is concerned with the right and left spectra of 2×2 operator matrices.

In this paper, $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2$ we write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1)$. If $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we use $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* to denote the range space, the null space and the adjoint of T, respectively. For a linear subspace $\mathcal{M} \subset \mathcal{H}_1$, its closure and orthogonal complement are denoted by $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$. Write $P_{\overline{\mathcal{M}}}$ for the orthogonal projection onto $\overline{\mathcal{M}}$ along $\mathcal{M}^\perp$. If $T \in \mathcal{B}(\mathcal{H}_1)$, the right spectrum, $\sigma_{\mathrm{r}}(T)$, and the left spectrum, $\sigma_{\mathrm{l}}(T)$, of T are defined by

$$\sigma_{\mathrm{r}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right invertible}\};$$
$$\sigma_{\mathrm{l}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left invertible}\}.$$

For operators $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, we denote by $\mathcal{N}(A, C)$ the null space of $(A, C) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$.

When $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ are given, we denote by $M_{(X,Y)}$ an operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the form

$$\begin{pmatrix} A & C \\ X & Y \end{pmatrix}$$

for $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. In [2], a necessary and sufficient condition is given for $M_{(X,Y)}$ to be a square-zero operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$, and a necessary and sufficient condition is obtained for $M_{(X,Y)}$ to be an idempotent operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$ in [8]. In this paper, we investigate the right and left spectra of $M_{(X,Y)}$.

1. THE RIGHT SPECTRUM OF $M_{(X,Y)}$

The main result of this section follows.

THEOREM 1.1. *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ be given operators. Then $M_{(X,Y)}$ is a right invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$ if and only if $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$ and $\dim \mathcal{N}(A, C) \geqslant \dim \mathcal{H}_2$.*

Proof. Necessity: Suppose that $M_{(X,Y)}$ is a right invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. Then there exists a bounded linear operator

$$\begin{pmatrix} E & G \\ H & F \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

such that

$$\begin{pmatrix} A & C \\ X & Y \end{pmatrix} \begin{pmatrix} E & G \\ H & F \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}. \tag{1.1}$$

It follows that $AE + CH = I_{\mathcal{H}_1}$, which implies that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$. On the other hand, from (1.1) we have

$$XG + YF = I_{\mathcal{H}_2}, \quad AG + CF = 0.$$

Therefore

$$\dim \mathcal{R}(\begin{pmatrix} G \\ F \end{pmatrix}) \geqslant \dim \mathcal{H}_2 \text{ and } \mathcal{R}(\begin{pmatrix} G \\ F \end{pmatrix}) \subseteq \mathcal{N}(A,C).$$

Hence $\dim \mathcal{N}(A,C) \geqslant \dim \mathcal{H}_2$.

Sufficiency: Since $\mathcal{R}(A)+\mathcal{R}(C) = \mathcal{H}_1$, let $\binom{E}{H} : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the Moore–Penrose generalized inverse of $(A,C) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$. Then

$$AE+CH = I_{\mathcal{H}_1}, \mathcal{N}(A,C)^{\perp} = R(\begin{pmatrix} E \\ H \end{pmatrix}). \tag{1.2}$$

Since $\dim \mathcal{N}(A,C) \geqslant \dim \mathcal{H}_2$, there exists a left invertible operator $\binom{G}{F} : \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\mathcal{R}(\binom{G}{F}) \subseteq \mathcal{N}(A,C)$, which implies that

$$AG+CF = 0.$$

Let $(X,Y) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_2$ denote the Moore–Penrose generalized inverse of $\binom{G}{F}$. Then

$$XG+YF = I_{\mathcal{H}_2}, \mathcal{N}(X,Y) = R(\begin{pmatrix} G \\ F \end{pmatrix})^{\perp}. \tag{1.3}$$

Now we prove that $M_{(X,Y)}$ is a right invertible operator. Since $\mathcal{R}(\binom{G}{F}) \subseteq \mathcal{N}(A,C)$, it follows from (1.2) and (1.3) that $\mathcal{N}(X,Y) \supseteq \mathcal{R}(\binom{E}{H})$. Therefore

$$XE+YH = 0,$$

and so

$$\begin{pmatrix} A & C \\ X & Y \end{pmatrix} \begin{pmatrix} E & G \\ H & F \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}.$$

The proof is completed. ∎

From Theorem 1.1, we get the following results.

COROLLARY 1.2. *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$ be given operators. Assume that $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$. Then $M_{(X,Y)}$ is a right invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$ if and only if $\mathcal{R}(A)+\mathcal{R}(C) = \mathcal{H}_1$ and one of the following conditions holds:*

(i) *$\mathcal{N}(A)$ is infinite dimensional;*

(ii) *$\mathcal{N}(C)$ is infinite dimensional;*

(iii) *$\mathcal{R}(A) \cap \mathcal{R}(C)$ is infinite dimensional.*

Proof. The proof follows from Theorem 1.1 and

$$\mathcal{N}(A,C) = \begin{pmatrix} \mathcal{N}(A) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \mathcal{N}(C) \end{pmatrix}$$

$$\oplus \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : Ax+Cy = 0, x \in \mathcal{N}(A)^{\perp}, y \in \mathcal{N}(C)^{\perp} \right\}. \tag{1.4}$$ ∎

COROLLARY 1.3. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *be given operators. Assume that* $\dim \mathcal{H}_1 < \infty$. *Then* $M_{(X,Y)}$ *is a right invertible operator for some* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2)$ *if and only if* $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$.

Proof. If $\dim \mathcal{H}_2 < \infty$, it is easy to show that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$ implies $\dim \mathcal{N}(A, C) = \dim \mathcal{H}_2$. If $\dim \mathcal{H}_2 = \infty$, it follows from $\dim \mathcal{H}_1 < \infty$ that $\dim \mathcal{N}(C) = \infty = \dim \mathcal{H}_2$ and therefore $\dim \mathcal{N}(A, C) = \dim \mathcal{H}_2$. The proof of Corollary 1.3 is immediate from the discussion above and Theorem 1.1. ∎

COROLLARY 1.4. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *be given operators. Assume that* $\dim \mathcal{H}_2 < \dim \mathcal{H}_1 = \infty$. *Then* $M_{(X,Y)}$ *is a right invertible operator for some* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2)$ *if and only if* $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$ *and* $\dim \mathcal{N}(A) \geqslant \dim \mathcal{R}(A)^{\perp}$.

For the proof of Corollary 1.4, we need a lemma.

LEMMA 1.5 (See [9]). *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces,* $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ *and let* $\mathcal{F} \subset \mathcal{Y}$ *be a finite dimensional subspace. If* $\mathcal{R}(T) + \mathcal{F}$ *is closed, then* $\mathcal{R}(T)$ *is closed. Conversely, if* $\mathcal{R}(T)$ *is closed, then* $\mathcal{R}(T) + \mathcal{F}$ *is closed too.*

Proof of Corollary 1.4. Suppose that $M_{(X,Y)}$ is a right invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. By Theorem 1.1, $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$ and $\dim \mathcal{N}(A, C) \geqslant \dim \mathcal{H}_2$. From $\dim \mathcal{H}_2 < \infty$ it follows that $\mathcal{R}(A)$ is closed (by Lemma 1.5). On the other hand, it is not difficult to check that $\dim \mathcal{R}(A)^{\perp} + \dim \mathcal{R}(A) \cap \mathcal{R}(C) = \dim \mathcal{R}(C)$. Therefore

$$\dim \mathcal{N}(C) + \dim \mathcal{R}(A)^{\perp} + \dim \mathcal{R}(A) \cap \mathcal{R}(C) = \dim \mathcal{H}_2. \tag{1.5}$$

This, together with (1.4) and $\dim \mathcal{N}(A, C) \geqslant \dim \mathcal{H}_2$, shows that $\dim \mathcal{N}(A) \geqslant \dim \mathcal{R}(A)^{\perp}$.

Conversely, assume that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_1$ and $\dim \mathcal{N}(A) \geqslant \dim \mathcal{R}(A)^{\perp}$. Then, from (1.5) we have $\dim \mathcal{N}(A, C) \geqslant \dim \mathcal{H}_2$. By Theorem 1.1, $M_{(X,Y)}$ is a right invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. ∎

Immediately, from Theorem 1.1, Corollaries 1.2, 1.3 and 1.4 we get the following corollary, concerning perturbations of the right spectrum.

COROLLARY 1.6. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *be given operators.*

(i) *If* $\dim \mathcal{H}_1 < \infty$ *or* $\dim \mathcal{N}(C) = \infty$, *then*

$$\bigcap_{X,Y} \sigma_{\mathrm{r}}(M_{(X,Y)}) = \{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) + \mathcal{R}(C) \neq \mathcal{H}_1\}.$$

(ii) *If* $\dim \mathcal{H}_2 < \dim \mathcal{H}_1 = \infty$, *then*

$$\begin{aligned}\bigcap_{X,Y} \sigma_{\mathrm{r}}(M_{(X,Y)}) = &\{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) + \mathcal{R}(C) \neq \mathcal{H}_1\} \\ &\cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(A - \lambda I) < \dim \mathcal{R}(A - \lambda I)^{\perp}\}.\end{aligned}$$

(iii) *If* $\dim \mathcal{N}(C) < \dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$, *then*

$$\bigcap_{X,Y} \sigma_{\mathrm{r}}(M_{(X,Y)}) = \{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) + \mathcal{R}(C) \neq \mathcal{H}_1\} \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(A - \lambda I) < \infty, \dim \mathcal{R}(A - \lambda I) \cap \mathcal{R}(C) < \infty\}.$$

2. THE LEFT SPECTRUM OF $M_{(X,Y)}$

In this section, our main result is the following theorem.

THEOREM 2.1. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *be given operators.*

(i) *If* $\dim \mathcal{H}_2 = \infty$, *then* $M_{(X,Y)}$ *is a left invertible operator for some* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2)$.

(ii) *If* $\dim \mathcal{H}_2 < \infty$, *then* $M_{(X,Y)}$ *is a left invertible operator for some* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2)$ *if and only if* $\mathcal{R}(A)$ *is closed and* $\dim \mathcal{N}(A, C) \leqslant \dim \mathcal{H}_2$.

For the proof of Theorem 2.1, we need the following lemma.

LEMMA 2.2 (See [10]). *Let* $\mathcal{H}$ *be a Hilbert space. Assume that* $\mathcal{M} \subset \mathcal{H}$ *and* $\mathcal{N} \subset \mathcal{H}$ *are subspaces. If* $\dim \mathcal{M} > \dim \mathcal{N}$, *then* $\mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\}$.

Proof of Theorem 2.1. (i) Since $\dim \mathcal{H}_2 = \infty$, there exists a closed infinite dimensional subspace $\mathcal{M} \subset \mathcal{H}_2$ such that $\dim \mathcal{M}^{\perp} = \dim \mathcal{H}_1$. Thus there exist unitary operators $J_1 : \mathcal{H}_1 \to \mathcal{M}^{\perp}$ and $J_2 : \mathcal{H}_2 \to \mathcal{M}$. Define $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ by

$$X = \begin{pmatrix} 0 \\ J_1 \end{pmatrix} : \mathcal{H}_1 \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix}, \quad Y = \begin{pmatrix} J_2 \\ 0 \end{pmatrix} : \mathcal{H}_2 \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix}.$$

Then $M_{(X,Y)}$ is a left invertible operator. In fact, from the definition of X it follows that X is a left invertible operator. Similarly, we can show that Y is a left invertible operator. Let X^+ be the Moore–Penrose generalized inverse of X and Y^+ be the Moore–Penrose generalized inverse of Y. Then

$$X^+X = I_{\mathcal{H}_1}, \quad Y^+Y = I_{\mathcal{H}_2}, \quad X^+Y = 0 \quad \text{and} \quad Y^+X = 0.$$

Thus

$$\begin{pmatrix} 0 & X^+ \\ 0 & Y^+ \end{pmatrix} \begin{pmatrix} A & C \\ X & Y \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix},$$

which means that $M_{(X,Y)}$ is a left invertible operator.

(ii) Necessity: Suppose that $\dim \mathcal{H}_2 < \infty$ and $M_{(X,Y)}$ is a left invertible operator for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$. By the closeness of $\mathcal{R}(M_{(X,Y)})$,

$$\mathcal{R}(\left(\begin{smallmatrix} A^* & X^* \\ C^* & Y^* \end{smallmatrix}\right)) = \mathcal{R}(\left(\begin{smallmatrix} A^* & 0 \\ C^* & 0 \end{smallmatrix}\right)) + \mathcal{R}(\left(\begin{smallmatrix} 0 & X^* \\ 0 & Y^* \end{smallmatrix}\right))$$

is closed. It follows from Lemma 1.5 and $\dim \mathcal{H}_2 < \infty$ that $\mathcal{R}(\left(\begin{smallmatrix} A^* & 0 \\ C^* & 0 \end{smallmatrix}\right))$ is closed, which means that $\mathcal{R}(A) + \mathcal{R}(C)$ is closed. Applying again Lemma 1.5 we infer that $\mathcal{R}(A)$ is closed. On the other hand, from the injectivity of $M_{(X,Y)}$ we have that

$$\dim \mathcal{N}(A,C) \leqslant \dim \mathcal{N}(X,Y)^{\perp}. \tag{2.1}$$

Assume to contrary that $\dim \mathcal{N}(A,C) > \dim \mathcal{N}(X,Y)^{\perp}$. Then, by Lemma 2.2 we obtain

$$\mathcal{N}(A,C) \cap \mathcal{N}(X,Y) \neq \{0\},$$

which implies that $M_{(X,Y)}$ is not injective. This is in contradiction with the left invertibility of $M_{(X,Y)}$. Clearly, $\dim \mathcal{N}(X,Y)^{\perp} \leqslant \dim \mathcal{H}_2$. This, together with (2.1), shows that

$$\dim \mathcal{N}(A,C) \leqslant \dim \mathcal{H}_2.$$

Sufficiency: Let

$$\mathcal{K}_1 = \begin{pmatrix} \mathcal{N}(A) \\ 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 0 \\ \mathcal{N}(C) \end{pmatrix},$$
$$\mathcal{K}_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathcal{N}(A)^{\perp}, y \in \mathcal{N}(C)^{\perp}, Ax + Cy = 0 \right\}.$$

Then $\mathcal{N}(A,C) = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$. Since $\dim \mathcal{N}(A,C) \leqslant \dim \mathcal{H}_2 < \infty$, it follows that there exists a subspace $\mathcal{M} \subset \mathcal{H}_2$ such that $\dim \mathcal{M} = \dim \mathcal{K}_1$ and $\dim \mathcal{M}^{\perp} \geqslant \dim \mathcal{K}_2 + \dim \mathcal{K}_3 = \dim P_{\mathcal{H}_2}\mathcal{N}(A,C)$. Thus there exist a unitary $J_1 : \mathcal{N}(A) \to \mathcal{M}$ and a isometry $J_2 : P_{\mathcal{H}_2}\mathcal{N}(A,C) \to \mathcal{M}^{\perp}$. We define $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{B}(\mathcal{H}_2)$ as

$$X = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix} : \begin{pmatrix} (P_{\mathcal{H}_2}\mathcal{N}(A,C))^{\perp} \\ P_{\mathcal{H}_2}\mathcal{N}(A,C) \end{pmatrix} \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix}.$$

We shall prove that $M_{(X,Y)}$ is a left invertible operator. To see this, we will prove that $\mathcal{R}(M_{(X,Y)})$ is closed and $M_{(X,Y)}$ is injective.

From $\dim \mathcal{H}_2 < \infty$ and Lemma 1.5, we can infer that $\mathcal{R}(M_{(X,Y)})$ is closed. On the other hand, let

$$\begin{pmatrix} A & C \\ X & Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$Ax + Cy = 0 \quad \text{and} \quad Xx + Yy = 0.$$

Then clearly $y \in P_{\mathcal{H}_2}\mathcal{N}(A,C)$. By the definition of Y, $Yy \in \mathcal{M}^{\perp}$. From $Xx = -Yy$ and the definition of X we get $Xx = Yy = 0$, which implies that $y = 0$. Thus $Ax = 0$, and hence $x \in \mathcal{N}(A)$. Also, from $Xx = 0$ and the definition of X it follows that $x = 0$. This proves that $M_{(X,Y)}$ is injective. The proof is completed. ∎

As a corollary of Theorem 2.1, we get the following result.

COROLLARY 2.3. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *be given operators.*

(i) *If* $\dim \mathcal{H}_2 = \infty$, *then*

$$\bigcap_{X,Y} \sigma_l(M_{(X,Y)}) = \emptyset.$$

(ii) *If* $\dim \mathcal{H}_2 < \infty$, *then*

$$\bigcap_{X,Y} \sigma_l(M_{(X,Y)}) = \{\lambda \in \mathbb{C} : \mathcal{R}(A-\lambda I) \text{ is not closed}\}$$
$$\cup \{\lambda \in \mathbb{C} : \dim \mathcal{R}(A-\lambda I) \cap \mathcal{R}(C) + \dim \mathcal{N}(A-\lambda I) > \dim \mathcal{R}(C)\}.$$

Proof. The proof of Corollary 2.3 directly follows from Theorem 2.1, equality (1.4) and $\dim \mathcal{R}(C) + \dim \mathcal{N}(C) = \dim \mathcal{H}_2$. ∎

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