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WHEN STRICT SINGULARITY OF OPERATORS COINCIDES WITH WEAK COMPACTNESS

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ABSTRACT. We prove that the notions of finite strict singularity, strict singularity and weak compactness coincide for operators defined on various spaces: the disc algebra, subspaces of $C(K)$ with reflexive annihilator and subspaces of the Morse–Transue–Orlicz space $M^{\psi_q}(\Omega,\mu)$ with $q>2$.

KEYWORDS: *Weak compactness, strictly singular operators, finitely strictly singular operators.*

MSC (2000): 47B10, 47B38, 47B07.

1. INTRODUCTION

It is well known that every compact operator is strictly singular and that the converse is false in general. More precisely

$$\text{compact} \quad\Longrightarrow\quad \text{finitely strictly singular} \quad\Longrightarrow\quad \text{strictly singular}$$

and each reverse implication is false in general.

Let us recall that

DEFINITION 1.1. An operator from a Banach space X to a Banach space Y is *strictly singular* if it never induces an isomorphism on an infinite dimensional (closed) subspace of X: for every $\varepsilon>0$ and every infinite dimensional subspace E of X, there exists x in the unit sphere of E such that $\|T(x)\|\leqslant\varepsilon$.

See [11] or [10] for instance to know more on this notion.

DEFINITION 1.2. An operator from a Banach space X to a Banach space Y is *finitely strictly singular* if: for every $\varepsilon>0$, there exists $N_\varepsilon\geqslant 1$ such that for every subspace E of X with dimension greater than N_ε, there exists x in the unit sphere of E such that $\|T(x)\|\leqslant\varepsilon$.

In this paper, we are interested in the links between the strict singularity and the weak compactness of an operator: there is maybe a chance that, on particular Banach spaces, a weaker condition than compactness, such as weak compactness (or complete continuity), is maybe sufficient to ensure the finite strict singularity of an operator. On the other hand, as we stipulated before, neither the strict singularity nor the finite strict singularity implies the compactness on general spaces. But this sometimes implies the weak compactness (or complete continuity).

The question is not trivial in full generality: clearly there are some weakly compact operators which are not strictly singular (the identity on an infinite dimensional reflexive space).

Conversely, there are some finitely strictly singular operator which are not weakly compact: we shall use the James spaces J_p defined as follows:

$$J_p = \{x \in c_0 : \|x\|_{J_p} < \infty\}, \quad \text{where } \|x\|_{J_p}^p = \sup_s \sup_{j_1 < \cdots < j_s} \sum_{i=1}^{s} |x_{j_i} - x_{j_{i-1}}|^p.$$

Consider the formal identity from the James space J_2 to the James space J_3 for instance. It is shown in [1] that it defines a finitely strictly singular operator. It is not a weakly compact operator: denoting by $(e_n)_{n\in\mathbb{N}}$ the canonical basis: the sequence $x_m = \sum_1^m e_j$ is bounded in J_2 but admits no weakly convergent subsequence in J_3: else a cluster point would be $y \in J_3$ with $y_i = \lim e_i^*(x_m) = 1$ for every $i \in \mathbb{N}$, which is impossible.

We could also be interested in the comparison with the class of completely continuous operators. There is no relationship in general as well: the identity on ℓ^1 is completely continuous (thanks to the Schur property) but is not strictly singular. Conversely, the formal identity from ℓ^2 to ℓ^3 is finitely strictly singular but is not completely continuous (else it would be compact by the reflexivity of ℓ^2).

Once we have given these counterexamples, a natural question arises. If we assume that an operator T is both weakly compact and completely continuous, which is a (strictly in general) weaker condition than compactness: is this operator strictly singular, or merely finitely strictly singular ? First point out that T is necessarily strictly singular. Indeed, if T induces on a subspace E an isomorphism τ between E and $T(E)$, then τ is both weakly compact and completely continuous as well (by restriction). Now, the identity on E is equal to $\tau^{-1} \circ \tau$ and the ideal property of complete continuity and weak compactness implies that I_E shares the same properties, hence E is finite dimensional. Nevertheless, we cannot have finite strict singularity in general. We give a simple counterexample (strengthening some examples given above): consider the formal identity j from ℓ^1 to the reflexive space $R = \bigoplus_{\ell^2} \ell^1_n$. Since R is reflexive, j is weakly compact. The space ℓ^1 has the Schur property so that j is obviously completely continuous, but j is clearly not finitely strictly singular. Moreover, this operator j produces an explicit simple example of an operator which is strictly singular but not finitely strictly singular.

In Section 2, we prove that every absolutely continuous operator on a separable Banach space is finitely strictly singular. From this, we are able to deduce that on various specific subspaces of $C(K)$, the notions of weak compactness, strict singularity, finite strict singularity and complete continuity coincide. The counterexample (given by j) shows that these results are not trivial, in the sense that they are not a general consequence of both weak compactness and complete continuity. We also apply our technics in the framework of Orlicz spaces.

The following properties of Banach spaces will play a crucial role:

DEFINITION 1.3. A Banach space X has the *property (V)* of Pełczyński if, for every non relatively weakly compact bounded set $K \subset X^*$, there exists a weakly unconditionaly series $\sum x_n$ in X such that $\inf_n \sup\{|k(x_n)| : k \in K\} > 0$.

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is not weakly compact, there exists a subspace X_o of X isomorphic to c_o such that $T_{|X_o}$ is an isomorphic embedding.

Recall too

DEFINITION 1.4. A Banach space X has the *Dunford–Pettis property* if for every weakly null sequence (x_n) in X and every weakly null sequence (x_n^*) in X^*, then $x_n^*(x_n)$ tends to zero.

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is weakly compact, T is completely continuous i.e. maps a weakly Cauchy sequence in X into a norm Cauchy sequence.

The key tool in our work is the notion of absolutely continuous operator. Following the terminology of [3], p. 314, recall that

DEFINITION 1.5. An operator from a Banach space X to a Banach space Y is *absolutely continuous* if there exists a 2-summing operator j from X to a space Z such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (for every $x \in X$) with

$$\|T(x)\| \leqslant C_\varepsilon \|j(x)\|_Z + \varepsilon\|x\|.$$

Thanks to the Pietsch theorem, this is equivalent to asking that there exists a probability measure μ on the unit ball of X^* such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (for every $x \in X$) with

$$\|T(x)\| \leqslant C_\varepsilon \Big(\int_{B_{X^*}} |\xi(x)|^2 \,\mathrm{d}\mu(\xi) \Big)^{1/2} + \varepsilon\|x\|.$$

Moreover, when X is viewed as a (closed) subspace of a $C(K)$ space, the probability space can be chosen as a space (K, ν), where ν is a probability measure. We shall adopt this point of view in the sequel.

2. RESULTS

THEOREM 2.1. *Every absolutely continuous operator on a subspace of a $C(K)$ space is finitely strictly singular.*

Proof. We assume that T is not a finitely strictly singular operator then there exists some $\varepsilon_0 > 0$ such that for every $N \geqslant 1$, we can find some finite dimensional $E \subset X$ with dimension greater than N with the following property: for every $x \in E$, we have

$$\|T(x)\| \geqslant \varepsilon_0 \|x\|.$$

There exits a probability measure μ on K such that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ verifying: for every $f \in X$,

$$\|T(f)\| \leqslant C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\infty.$$

We apply this property with $\varepsilon = \varepsilon_0/2$ and we are given $K = (2/\varepsilon_0)C_{\varepsilon_0/2}$ which does not depend on the dimension N. Then we choose $N > K^2$. We are going to obtain a contradiction.

First point out, that the two preceding inequalities imply, for every $f \in E$,

$$\|f\|_\infty \leqslant K\|f\|_2.$$

Now, we consider N functions $f_1, \ldots, f_N$ in E, which are orthonormal for the euclidean structure of $L^2(\mu)$. We have for every family $a_1, \ldots, a_N \in \mathbb{C}^N$ and every $t \in K$:

$$\Big|\sum_{j=1}^N a_j f_j(t)\Big| \leqslant \Big\|\sum_{j=1}^N a_j f_j\Big\|_\infty \leqslant K\|a\|_{\ell^2}$$

and taking the supremum on the unit ball of ℓ^2_N, we have

$$\sum_{j=1}^N |f_j(t)|^2 \leqslant K^2.$$

Integrating over K, we have the following which is false:

$$N = \sum_{j=1}^N \|f_j\|_2^2 \leqslant K^2. \quad \blacksquare$$

REMARK 2.2. We could have presented the argument at the end of the proof in a slightly different manner, using the fact that the formal identity from $C(K)$ into any $L^2(\mu)$ space is 2-summing.

We have the following immediate corollaries.

COROLLARY 2.3. *Every absolutely continuous operator on a separable Banach space is finitely strictly singular.*

The following corollary was already proved by Plichko in [16] (using a different approach).

COROLLARY 2.4 ([16]). *Every absolutely summing operator is finitely strictly singular.*

We have another immediate corollary where we introduce the notion of ACW property: we say that a Banach space X has the *ACW property* if every weakly compact operator from X to any Banach space is absolutely continuous.

COROLLARY 2.5. *Every weakly compact operator on a space with the* ACW *property is finitely strictly singular.*

This is interesting when we know which spaces share this property: the case of $C(K)$ was proved by Niculescu (see [12] or [3], p. 314). The case of the disc algebra was proved by the author in Proposition 1.4. of [7]. We are going to settle the case of subspaces of $C(K)$ with reflexive annihilator:

PROPOSITION 2.6. *Every weakly compact operator on a subspace of $C(K)$ with reflexive annihilator is an absolutely continuous operator.*

Proof. This follows actually from Proposition 1.3 of [7] and the fact that any weakly compact subset of $M(K)/X^{\perp}$ can be lifted to a weakly compact subset of $M(K)$, since $X^{\perp}$ is reflexive (see [6]). ∎

Actually, there is a general principle. Assume that an operator T on X verifies an interpolation inequality with a L^2 norm (looking like the absolute continuity). If X does not admit an infinite dimensional complemented subspace which is isomorphic to a Hilbert space, then T is strictly singular. If X does not admit a complemented subspace (with uniformly bounded projection norm) of arbitrarily large dimension, isomorphic to a Hilbert space, then T is finitely strictly singular.

Let us concentrate on two examples:

First, one could try to extend Theorem 2.1 to the framework of C^*-algebras. Indeed, Jarchow [5] proved a criterion, which shows that every weakly compact operator T defined on a C^*-algebra $\mathcal{A}$, shares a property which looks like the absolute continuity: there exists a state $\varphi \in \mathcal{A}^*$ such that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ verifying: for every $a \in \mathcal{A}$,

$$\|T(a)\| \leqslant C_\varepsilon(\varphi(|a|^2))^{1/2} + \varepsilon\|a\|$$

where $|a| = ((xx^* + x^*x)/2)^{1/2}$.

Actually we could apply the principle in a C^*-algebra containing no (infinite dimensional) complemented subspace isomorphic to a Hilbert space and prove that T is strictly singular. Nevertheless, in full generality, it is false that a weakly compact operator on a C^*-algebra is strictly singular. Indeed, it is easy to see that $\mathcal{B}(\ell^2)$ contains a complemented subspace isomorphic to ℓ^2.

Point out that every strictly singular operator on a C^*-algebra is weakly compact. This follows from the fact that C^*-algebras share the property (V) (this is due to Pfitzner [14]).

Now, we consider another example, in the framework of Orlicz spaces. More precisely, we are going to obtain results in the framework of Morse–Transue spaces: when ψ is an Orlicz function, the Morse–Transue space M^ψ is the closure of L^∞ in the Orlicz space L^ψ (on a probability space). In our context, L^ψ is the bidual of M^ψ.

It is proved in [8] that every weakly compact operator T on a subspace X of the Morse–Transue space $M^\psi(\nu) \subset L^\psi(\nu)$ where ψ has the Δ^0 property and ν is a probability measure, verifies: for every $f \in X$ and every $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$\|T(f)\| \leqslant C_\varepsilon \|f\|_{L^2(\nu)} + \varepsilon \|f\|_\psi.$$

Recall that the Δ^0 property for ψ means that $\lim\limits_{t\to+\infty} \psi(ct)/\psi(t) = +\infty$ for some $c > 1$. The classical examples of Orlicz functions sharing this property are the functions $\psi_q(t) = \mathrm{e}^{t^q} - 1$, where $q \geqslant 1$.

Now consider for instance the gaussian case ($\psi = \psi_2$) and the formal identity from $M^{\psi_2}(\mathbb{T})$ to $L^2(\mathbb{T})$ (with the Haar measure on the torus): we obviously have a weakly compact operator which is not even strictly singular since on the subspace spanned by an infinite Sidon set (by the $e_n(t) = \exp(2\mathrm{i}\pi 3^n t)$ for example), the two norms $\|\cdot\|_{\psi_2}$ and $\|\cdot\|_2$ are equivalent: this is a reformulation of the classic theorem due to Rudin asserting that Sidon sets are $\Lambda(p)$ sets for every $p \geqslant 2$, with constant of order $\sqrt{p}$ (see [17]). We refer to the monograph [10] (Chapter 5.IV) for the terminology of lacunary sets used here.

Nevertheless when the Orlicz function grows very fast (even faster than in the Gaussian case), then the conclusion holds. We first need the following simple observation, where as usual $(a_1^*, \ldots, a_N^*)$ denotes the decreasing rearrangement of $(a_1, \ldots, a_N)$.

LEMMA 2.7. *Let (Ω, ν) be a probability space and $g_1, \ldots, g_N \in L^1(\Omega, \nu)$ be non negative functions. We assume that $\mathbb{E} g_j \geqslant 1$ and that there exists some $K > 0$ such that $\mathbb{E} g_j^2 \leqslant K$.*

Then for every $\delta \in (0, 1/K)$ and every $m \leqslant \delta N$, we have

$$\int_\Omega (g_m(\omega))^* \,\mathrm{d}\nu \geqslant \frac{1 - \sqrt{\delta K}}{1 - \delta}\cdot$$

Proof. We have $N = \sum\limits_{j=1}^N \int\limits_\Omega g_j(\omega)\,\mathrm{d}\nu = \int\limits_\Omega \sum\limits_{j=1}^N (g_j(\omega))^* \,\mathrm{d}\nu$.

Fixing any $m \in \{1, \ldots, N\}$, we have

$$N \leqslant \int_\Omega \sum_{j<m} (g_j(\omega))^* + \sum_{j=m}^N (g_j(\omega))^* \,\mathrm{d}\nu$$

$$\leqslant \int_\Omega \sqrt{m}\Big(\sum_{j=1}^{m}(g_j(\omega)^*)^2\Big)^{1/2} + \sum_{j=m}^{N}(g_m(\omega))^*\,\mathrm{d}\nu$$
$$\leqslant \int_\Omega \sqrt{m}\Big(\sum_{j=1}^{N} g_j^2(\omega)\Big)^{1/2}\mathrm{d}\nu + (N+1-m)\mathbb{E}(g_m(\omega))^*.$$

The result follows since $N \leqslant \sqrt{mNK} + (N+1-m)\int_\Omega (g_m(\omega))^*\,\mathrm{d}\nu$. ∎

PROPOSITION 2.8. *Let ν be a probability measure, $q > 2$ and $\psi_q(t) = \mathrm{e}^{t^q} - 1$. Every weakly compact operator on a subspace of a Morse–Transue space $M^{\psi_q}(\Omega,\nu)$ is finitely strictly singular.*

Proof. The beginning of the proof has the same structure as the one of Theorem 2.1:

If this was false, we could produce a constant $K > 0$ and some functions $f_1,\dots,f_N$, which are orthonormal for the euclidean structure of $L^2(\nu)$, such that for every family $a_1,\dots,a_N \in \mathbb{C}^N$:

$$\Big\|\sum_{j=1}^{N} a_j f_j\Big\|_{\psi_q} \leqslant K\|a\|_{\ell^2}.$$

Now we introduce the Rademacher functions (r_n), viewed as the characters on the Cantor group $Q = \{\pm 1\}^{\mathbb{N}}$. This is a Sidon set and we have for every $\theta \in Q$

$$\int_\Omega \psi_q\Big(\frac{1}{K\sqrt{N}}\Big|\sum_{j=1}^{N} r_j(\theta) f_j(\omega)\Big|\Big)\,\mathrm{d}\nu \leqslant 1.$$

Integrating over Q (and using Fubini), we have

$$\int_\Omega \mathbb{E}\,\psi_q\Big(\frac{1}{K\sqrt{N}}\Big|\sum_{j=1}^{N} r_j f_j(\omega)\Big|\Big)\,\mathrm{d}\nu \leqslant 1.$$

Let us point out that for any $h \in L^\psi$ (where ψ is an Orlicz function), we have $\|h\|_\psi \leqslant \sup\{1, \mathbb{E}\psi(|h|)\}$. This leads to

$$\int_\Omega \Big\|\frac{1}{K\sqrt{N}}\sum_{j=1}^{N} f_j(\omega) r_j\Big\|_{L^{\psi_q}(Q)}\,\mathrm{d}\nu \leqslant 1.$$

But, it is known by Proposition 2.2 of [15] that for every infinite Sidon set S, the space $L_S^{\psi_q}$ is isomorphic to the sequence Lorentz space $\ell^{q',\infty}$ (where q' is the conjugate exponent of q); hence, the Sidon property for the Rademacher implies that

$$\Big\|\sum_{j=1}^{N}\alpha_j r_j\Big\|_{L^{\psi_q}(Q)} \geqslant C\|\alpha\|_{\ell^{q',\infty}}$$

for some $C > 0$ and every $\alpha_1, \ldots, \alpha_N \in \mathbb{C}^N$, we would have for some $K' > 0$ and arbitrarily large values of N

$$K'\sqrt{N} \geqslant \int_{\Omega} \|(f_j(\omega))_{1\leqslant j\leqslant N}\|_{\ell^{q'},\infty}\,\mathrm{d}\nu.$$

We apply Lemma 2.7 with $g_j = k|f_j|$ with a suitable numerical constant k: indeed, we know that the L^2 norm and the L^{ψ_q} norm are equivalent on the vector space spanned by the functions f_j, hence using a standard Hölder argument, it is easy to show that there exists $\varepsilon_0 > 0$ such that $\mathbb{E}|f_j| \geqslant \varepsilon_0$, for every j. Moreover the L^4 norm is dominated (up to a constant) by the L^{ψ_q} norm.

We obtain a constant $c > 0$ such that for arbitrarily large values of N, where $m \approx N$,

$$K'\sqrt{N} \geqslant \int_{\Omega} m^{1/q'}|f_m(\omega)|^*\,\mathrm{d}\nu \geqslant cN^{1/q'}.$$

Since $q > 2$, N cannot be too large and the result follows. ∎

We summarize our results in the following theorem where X stands for one of the spaces listed below:

(1) $C(K)$;

(2) $A(\mathbb{D})$;

(3) X is a subspace of $C(K)$ with reflexive annihilator;

(4) X is a subspace of the Morse–Transue space $M^{\psi^q}(\Omega,\mu)$ with $q > 2$, on a probability space.

Of course, a $C(K)$ space is a particular case of (3).

THEOREM 2.9. *Let X be one of the space of the list above and T be any bounded operator from X to a Banach space Y. The following assertions are equivalent:*

(i) *T is a finitely strictly singular operator;*

(ii) *T is a strictly singular operator;*

(iii) *T is a weakly compact operator.*

Moreover for spaces (1), (2) *and* (3) *in the list above, these notions also coincide with the notion of complete continuity.*

On general Banach spaces, it is well known that the notions listed above are distinct.

Proof. The "moreover" part is already well known: this can be viewed as a consequence of the Dunford–Pettis property and the Pełczyński property for these spaces.

(i) ⇒ (ii) is always (obviously) true.

(ii) ⇒ (iii) It is a consequence of the Pełczyński property: if T were not weakly compact, there would exist some subspace isomorphic to c_0 on which T induces an isomorphism. This property is known for $C(K)$ (this due to Pełczyński itself [13]); for the disk algebra, this is due to Delbaen ([2]); for subspaces of $C(K)$

with reflexive annihilator, this is due to Godefroy-Saab [4]; and, at last, for subspaces of a Morse–Transue space M^{ψ}, whose associated conjugate function ϕ satisfies a growing condition Δ_2: $\phi(2x) \leqslant K\phi(x)$ for every x large enough ([8], [9]).

(iii) $\Rightarrow$ (i) was proved in Theorem 2.1, Proposition 2.8 and Corollary 2.5. ∎

Actually we can produce other examples of finitely strictly singular operators:

PROPOSITION 2.10. *Every weakly compact and weak* continuous operator from* H^{∞} *to a dual space is finitely strictly singular.*

We cannot expect the reverse implication in general. It is true that a finitely strictly singular operator on H^{∞} will be weakly compact (thanks to the property (V)), but the example of an arbitrary rank one operator shows that the weak* continuity does not hold in general.

Proof. The proof follows from Theorem 2.5: the absolute continuity itself being a consequence of Theorem 1.2 of [7]. ∎

REMARK 2.11. The natural question of the availibility of this result for L^{∞} holds. This actually follows from Theorem 2.1.

In the same spririt, we have

PROPOSITION 2.12. *Every biadjoint weakly compact operator from a subspace of* $L^{\psi_q}(\Omega, \mu)$, *with* $q > 2$ *is finitely strictly singular.*

An interesting particular case is given by the Hardy–Orlicz space $H^{\psi_q}(\Omega, \mu)$, where $q > 2$.

Proof. The proof still follows from Theorem 2.5 (the absolute continuity itself is established in Remark 13 of [8]). ∎

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