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TRACE FORMULAS AND p-ESSENTIALLY NORMAL PROPERTIES OF QUOTIENT MODULES ON THE BIDISK

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ABSTRACT. Let M be an invariant subspace of the multiplication operators M_z and M_w on the Hardy or Bergman space on $D^2 = \{(z,w) : |z|, |w| < 1\}$, and $S_f = P_{M^\perp} M_f P_{M^\perp}$ be the compressions on the quotient module $M^\perp$ of the multiplication operators M_f. We study the Schatten–von Neumann, in particular trace and weak trace class, properties of commutators $[S_f^*, S_f]$, and we prove the trace formulas for the commutators. Similar trace formulas for Hankel type operators are also obtained.

KEYWORDS: *Hilbert module, quotient module, essentially normal quotient, trace class, Hilbert–Schmidt class, Dirichlet norm.*

MSC (2000): 47A13, 47A20, 46H25, 46C99.

1. INTRODUCTION

Trace formulas for commutators of Toeplitz operators are of much interest as they are closely related to various subjects such as index theory and complex analysis. There is a well-known formula for Toeplitz operators on the unit disk. Consider the Bergman or Hardy space on the unit disk D and the Toeplitz operator T_f with symbol f. Then for holomorphic function f, the trace of the commutator $[T_f^*, T_f]$ is given by the Dirichlet norm of f,

$$\operatorname{tr}[T_f^*, T_f] = \int_D |f'(z)|^2 \mathrm{d}m(z);$$

see e.g. [19]. Actually, it is proved in [4] that the same formula holds with the Bergman space on the unit disk replaced by the Bergman space on any complex domain Ω defined by a measure μ (under certain mild assumptions). Some further generalization of the trace formula for the commutator $[S^*, S]$ of a subnormal operator S was given in [2]. There is a similar trace formula for the anticommutator of the $2n$-tuple (T^*, T), where $T = (T_1, \dots, T_n)$ is an n-tuple of commuting operators, see [19], [12], [24] and [26]. In this paper we will prove trace and Dixmier trace formulas for certain Hankel and Toeplitz type operators acting on function modules over the bidisk.

Let H be the Hardy space or a weighted Bergman space of holomorphic functions on the unit disk. We consider the Hardy or Bergman space $H \otimes H$ on the bidisk D^2. For an invariant subspace M generated by homogeneous polynomials, we will study the Schatten–von Neumann $\mathcal{L}^p$ properties of the quotient module $M^\perp$, namely the membership in $\mathcal{L}^p$ of the operators $[S_f^*, S_f]$ for any polynomial symbol $f(z,w)$, where S_f is the compression of M_f on $M^\perp$. The classification of those quotients $M^\perp$ with compact properties has been done in [11], [8], [20], [17]. We will prove that $M^\perp$ is in $\mathcal{L}^1$, i.e., in the trace class, precisely when $M = [p]$ with p being one of the polynomials $(z - \alpha w)^{n+1}$, $(z - \beta w)$, $(w - \gamma z)$ and $(z - \beta w)(w - \gamma z)$, for some $|\alpha| = 1$, $|\beta|$, $|\gamma| < 1$. Moreover, it is proved that the trace $[S_f^*, S_f]$ is given by the Dirichlet norm of the restriction of f on the zero set of the polynomial p. Note that the trace formula in [4] is applicable to our case only when $p = (z - \alpha w)$ for $|\alpha| \leqslant 1$, since for other cases the operator S_z is not unitarily equivalent to any Toeplitz operator T_f (or its dual) as in [4] and is even not hyponormal (or co-hyponormal). We will also study the Hankel type operator H_z from $M^\perp$ to M. The square of its modulus is $|H_z|^2 = H_z^* H_z = P_{M^\perp}[M_z^*, M_z]P_{M^\perp} - [S_z^*, S_z]$ and thus measures the discrepancy between the compression of the commutator and the commutator of the compressions on $M^\perp$. It turns out that there is a subtle difference between the Hardy case $\nu = 1$ and the weighted Bergman case $\nu > 1$. The operator $H_z^* H_z$ is in the weak trace class $\mathcal{L}^{1,\infty}$ but not the trace class for $\nu = 1$. It is in the trace class for $\nu > 1$. We prove then that the Dixmier trace of $H_f^* H_f$ is also given by the Dirichlet norm. The proof of the $\mathcal{L}^p$-properties involves some rather delicate estimates of eigenvalues of related operators. For the computation of the trace and Dixmier trace we use certain Möbius invariance which might be somewhat ad hoc. Indeed some direct computations instead of invariant arguments are also possible, and they might provide more insights for the study of general non-homogeneous modules; see Remark 5.12 for a concrete question.

It is worthwhile to mention that there are several related interesting problems on submodules of the Hardy space on the unit ball B^d generated by homogeneous polynomials. In [6] (see also [5]) Arveson conjectures that the operator $[S_i^*, S_j]$ on the quotient module is always in $\mathcal{L}^p$ for $p > n$. This conjecture has recently been proved to be true for $d = 2, 3$ by Guo and Wang [18], [16]; roughly speaking the Toeplitz operators on the quotient modules behave as they are on the unit ball. Thus there would be no trace formula for a single commutator. However we may still consider the question of trace class property of the anti-commutators of several operators as in [19]. There is also a formula for the Dixmier trace of the product of commutators of Toeplitz operators on the unit ball [13] and the same question makes also sense for the quotients. However the function theory on the bidisk or polydisks is quite different from that on the unit ball, in particular the Toeplitz operators on the bidisk are not essentially commuting, and the above conjecture does not hold generally. Our results rise a natural question of characterizing those quotient modules of the polydisk which are 1-essentially normal, namely classifying quotient modules with all the commutators being in the trace class $\mathcal{L}^1$.

2. QUOTIENT MODULES $[(z-w)^{N+1}]^\perp$ AND THEIR REALIZATIONS

Consider the functional Hilbert space $H = H_\nu$ on the unit disk D with the reproducing kernel $K_w(z) = \frac{1}{(1-z\overline{w})^\nu}$ for $\nu \geqslant 1$. It is the Hardy space $H^2(T)$ ($\nu = 1$) or the weighted Bergman space $L_a^2(D, \mathrm{d}\mu_{\nu-2})$ ($\nu > 1$); here T is the unit circle and $\mathrm{d}\mu_\alpha = c_\alpha(1-|z|^2)^\alpha \mathrm{d}m(z)$ is the normalized measure on D.

The space $H \otimes H$ is then the Hilbert space $H^2(T^2)$ or $L_a^2(D^2, \mathrm{d}\mu_{\nu-2} \times \mathrm{d}\mu_{\nu-2})$ on the bidisk D^2. Let M_f be the multiplication operator on $H \otimes H$ for $f \in H^\infty(D^2)$. For an invariant subspace M of the multiplication operators M_z, M_w on $H \otimes H$, we denote

$$S_f = P_{M^\perp} M_f P_{M^\perp}, \quad H_f = P_M M_f P_{M^\perp}$$

the compression of M_f to the quotient module $M^\perp$ and Hankel type operator, respectively, where P_M and $P_{M^\perp}$ are projections from H onto M and $M^\perp$. When M is generated by homogenous polynomials, the essentially normal properties (see Section 3 for the definition) of (S_z, S_w) have been studied in [17]. It is proved that the problem can be reduced to the special class of modules M generated by $(z-w)^j$. We consider this case first. The compression S_f on the quotient $[(z-w)^j]^\perp$ can be realized as certain block matrix acting on direct sum of usual weighted Bergman spaces. Let us recall briefly this realization; see [15], [23].

For any $j \geqslant 0$, let M_j be the invariant subspace of the tuple (M_z, M_w) generated by $(z-w)^j$. We will fix $N \geqslant 0$ in the sequel and consider the submodule

$$M := M_{N+1} = [(z-w)^{N+1}].$$

Equivalently, it is the subspace of holomorphic functions in $H \otimes H$ which are vanishing along the diagonal of D^2 of degree $N+1$. Using the filtration

$$M_{N+1} \subset M_N \subset \cdots \subset M_1 \subset M_0 = H \otimes H, \tag{2.1}$$

we find

$$M_{N+1}^\perp = \bigoplus_{j=0}^{N} (M_j \ominus M_{j+1}).$$

Under this decomposition the operator $S = S_z$ on the quotient is a lower triangular $(N+1) \times (N+1)$-matrix $S = (S_{ij})$ with

$$S_{ij} = P_i S_z P_j, \quad 0 \leqslant j, i \leqslant N,$$

where P_j is the projection from $H \otimes H$ onto $M_j \ominus M_{j+1}$ for $0 \leqslant j \leqslant N$. The spaces $M_j \ominus M_{j+1}$ as well as the multiplication operators on $M^\perp$ have certain Möbius group invariance, which we shall also need.

Let

$$SU(1,1) = \left\{ g = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

be the Möbius group acting on the unit disk D by

$$g : z \to g \cdot z = \widetilde{\phi}_g(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

It induces a unitary action of $g \in SU(1,1)$ on H via

$$\pi_\nu(g) : f(z) \mapsto f(g^{-1}z)(\widetilde{\phi}_g^{-1})'(z)^{\nu/2}. \tag{2.2}$$

(The power $(\widetilde{\phi}_g^{-1})'(z)^{\nu/2}$ can be properly defined for non-integral values of $\frac{\nu}{2}$ so that $g \mapsto \pi_\nu(g)$ forms a projective representation.) Its action on $H \otimes H$ is

$$(\pi_\nu \otimes \pi_\nu)(g) : F(z,w) \mapsto F(\widetilde{\phi}_g^{-1}z, g^{-1}w)(\widetilde{\phi}_g^{-1})'(z)^{\nu/2}(\widetilde{\phi}_g^{-1})'(w)^{\nu/2}. \tag{2.3}$$

Observing that

$$(gz - gw)^j = (z-w)^j(\widetilde{\phi}_g'(z))^{j/2}(\widetilde{\phi}_g'(w))^{j/2},$$

we see that the filtration (2.1) is invariant under the action (2.3). In particular, the subspaces $M_k \ominus M_{k+1}$ are also invariant. As a representation of $SU(1,1)$, it is equivalent to the space $H_{2\nu+2k}$ with the action $\pi_{2\nu+2k}$. We will need a concrete intertwining operator. Let T_k be the following operator from the space holomorphic functions of two variables into that of one variable,

$$(T_kF)(z) = C_k \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\partial_z^j \partial_w^{k-j} F(z,z)}{(\nu)_j(\nu)_{k-j}}, \quad C_k = \frac{(\nu)_k}{\sqrt{k!(2\nu-1+k)_k}}, \tag{2.4}$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ is the generalized Pochhammer symbol. This operator has been well-studied in classical invariant theory and representation theory (see e.g. [21], [23], [15] and references therein). We recall two known results; see e.g. Theorem 1.2 of [15] and references therein.

LEMMA 2.1. *The operator T_k is a unitary operator from $M_k \ominus M_{k+1}$ onto the Bergman space $H_{2\nu+2k}$ and intertwines the action $\pi_\nu \otimes \pi_\nu$ with $\pi_{2\nu+2k}$ on $H_{2\nu+2k}$.*

An elementary computation shows that the adjoint T_k^* of T_k is given by

$$(T_k^* f)(z,w) = C_k(z-w)^k \int_D f(\xi) \frac{1}{(1-z\overline{\xi})^{\nu+k}(1-w\overline{\xi})^{\nu+k}} \mathrm{d}\mu_{2\nu+2k-2}(\xi). \tag{2.5}$$

It follows that T_k^* maps the standard orthonormal basis

$$E_m = \sqrt{\frac{(2\nu+2k)_m}{m!}} z^m \tag{2.6}$$

of $H_{2\nu+2k}$ onto the orthonormal basis, of $M_k \ominus M_{k+1}$:

$$e_m^k = C_k \sqrt{\frac{m!}{(2\nu+2k)_m}} (z-w)^k \sum_{l=0}^{m} \frac{(\nu+k)_l}{l!} \frac{(\nu+k)_{m-l}}{(m-l)!} z^l w^{m-l}. \tag{2.7}$$

LEMMA 2.2. *The map*

$$\bigoplus_{k=0}^{N} T_k : H \otimes H \to \bigoplus_{k=0}^{N} H_{2\nu+2k}$$

induces a Möbius invariant unitary operator

$$M^\perp = \bigoplus_{k=0}^{N} (M_k \ominus M_{k+1}) \to \bigoplus_{k=0}^{N} H_{2\nu+2k}.$$

Under this unitary equivalence, the diagonal components S_{kk} are then the Bergman multiplication on $H_{2\nu+2k}$.

We shall also need the Gauss summation formula for the hypergeometric series

$$\sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} := F(a,b;c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.8}$$

and in particular its special case

$$\sum_{j=0}^{k} \frac{(\alpha)_j (\beta)_{k-j}}{j!(k-j)!} = \frac{(\alpha+\beta)_k}{k!}, \tag{2.9}$$

which can also be easily proved by binomial expansions.

3. TRACE FORMULAS

Recall that the Schatten–von Neumann class $\mathcal{L}^p$, $p > 0$, consists of compact operators T such that the eigenvalues $\{\mu_n(|T|)\}$, $\mu_1(|T|) \geqslant \mu_2(|T|) \geqslant \cdots$, of $|T| = (T^*T)^{1/2}$ are in l^p. In particular, $\mathcal{L}^2$ is the Hilbert–Schmidt class, $\mathcal{L}^1$ the trace class and $\mathcal{L}^\infty$ compact operators. We shall also need the Macaev class $\mathcal{L}^{p,\infty}$, or the weak $\mathcal{L}^p$ class, (see e.g. Example 2.2 of [22]) which consists of all compact operators T satisfying

$$\mu_n(|T|) = O(n^{-1/p}) \quad \text{if } p > 1; \quad \sum_{i=1}^{n} \mu_i(|T|) = O(\log n), \quad \text{if } p = 1.$$

One may also define the Macaev class $\mathcal{L}^{p,q}$ by using the interpolation between $\mathcal{L}^\infty$ and $\mathcal{L}^{1,\infty}$; see e.g. [22] and Chapter IV of [9].

For a submodule M of $H \otimes H$, we say that $M^\perp$ is (p,q)*-essentially normal* or simply $M^\perp$ is $\mathcal{L}^{(p,q)}$, if all the cross commutators of the operators $\{S_z^*, S_w^*, S_z, S_w\}$

are in $\mathcal{L}^{p,q}$ (see e.g. [5], [6] for the case of unit ball). We abbreviate (∞,∞)-essentially normal as essentially normal or compact.

We observe that the commutators $[S_z, S_w] = 0$ and $[S_z^*, S_w^*] = 0$, and the definition is only about the $\mathcal{L}^{(p,q)}$ property of $[S_z^*, S_z]$, $[S_z^*, S_w]$, $[S_w^*, S_w]$.

In this section we will show the quotient module $M^\perp = [(z-w)^{N+1}]^\perp$ is $\mathcal{L}^1$ and we shall compute the trace of the commutators. Let us recall first the following result in Proposition 6 of [15].

LEMMA 3.1. *The operator S_{ji}, for $j > i$, realized as the operator $T_j M_z T_i^*$ from $H_{2\nu+2i} \to H_{2\nu+2j}$ is a differentiation operator of degree $j-i-1$,*

$$(T_j M_z T_i^* f)(z) = \frac{C_i}{C_j} \frac{(\nu+i)_{j-i}}{(2\nu+2i)_{2j-2i-1}} \Big(-\frac{\partial}{\partial z}\Big)^{j-i-1} f(z), \tag{3.1}$$

where $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ and $C_k = \frac{(\nu)_k}{\sqrt{k!(2\nu-1+k)_k}}$.

We shall need to understand the $\mathcal{L}^{(p,q)}$ property of the above differential operators.

LEMMA 3.2. *The k-th differentiation*

$$f \mapsto \Big(\frac{\partial}{\partial z}\Big)^k f,$$

from H_ν to H_σ belongs to the Schatten class $\mathcal{L}^p$ for $p > 1$ if and only if $p(\frac{\sigma-\nu}{2} - k) > 1$. It belongs to the weak trace class $\mathcal{L}^{1,\infty}$ if and only if $\frac{\sigma-\nu}{2} - k \geqslant 1$.

Proof. The functions $e_m = \sqrt{\frac{(\nu)_m}{m!}} z^m$ form an orthonormal basis of H_ν. The differentiation maps the basis $\{e_m\}$ to a system of orthogonal vectors. In fact, writing $T = (\frac{\partial}{\partial z})^k$, we have

$$Te_m = \begin{cases} 0 & \text{if } m < k, \\ m(m-1)\cdots(m-k+1)\sqrt{\frac{(\nu)_m}{(\sigma)_{m-k}}} f_{m-k} & \text{if } m \geqslant k, \end{cases}$$

where $f_m = \sqrt{\frac{(\sigma)_m}{m!}} z^m$ is the orthonormal basis of H_σ. Therefore, T belongs to $\mathcal{L}^p$ and $\mathcal{L}^{1,\infty}$ if and only if

$$\sum_m \|Te_m\|^p < \infty \quad \text{and respectively} \quad \sum_{m \leqslant n} \|Te_m\| = O(\log n).$$

A direct calculation shows the following, leading to desired results:

$$\|Te_m\| \approx m^k m^{(1/2)(\nu-\sigma)} = m^{-((1/2)(\sigma-\nu)-k)}. \qquad \blacksquare$$

As a consequence, we see that

$$\|S_{ij} e_m^j\| \leqslant C\frac{1}{m}, \quad 1 \leqslant j < i \leqslant N,\ m = 0, 1, 2, \ldots \tag{3.2}$$

for some C independent of i, j, N. Therefore, the operator S_{ij} is in the weak trace class $\mathcal{L}^{1,\infty}$, and in particular in any $\mathcal{L}^p$ for $p > 1$.

THEOREM 3.3. *The commutators $[S_z^*, S_z]$, $[S_w^*, S_w]$ and $[S_z^*, S_w]$ are of trace class. Thus, the quotient module $M^\perp$ is in $\mathcal{L}^1$.*

Proof. We prove first that $[S_z^*, S_z]$ is of trace class. Writing $S = S_z$ as a block lower triangular $(N+1) \times (N+1)$-matrix $S = (S_{ij})$ with $S_{ij} = P_i S P_j$, where P_i is the projection from $H \otimes H$ onto $M_j \ominus M_{j+1}$, we have $S_{ij} = 0$ for $i < j$. The (ij)-entry of the self-adjoint operator $[S^*, S]$ is

$$[S^*, S]_{ij} = \sum_{k=0}^{N} (S_{ki}^* S_{kj} - S_{ik} S_{jk}^*). \tag{3.3}$$

If $i = j$, all terms except possibly the term $S_{ii}^* S_{ii} - S_{ii} S_{ii}^*$ in the sum in (3.3) are trace class since S_{ki} is in $\mathcal{L}^p$ for any $p > 1$; but S_{ii} is unitarily equivalent to the Bergman multiplication operator on $H_{2\nu+2i}$ and consequently the commutator $S_{ii}^* S_{ii} - S_{ii} S_{ii}^*$ is also trace class.

Suppose $i > j$. Again all terms in the sum in (3.3) are trace class for $S_{ji} = 0$ and $S_{ki} \in \mathcal{L}^{1,\infty}$ if $k \neq i$, except possibly the terms with $k = i, j$. In the latter case the sum is

$$W := S_{ii}^* S_{ij} - S_{ii} S_{ji}^* + S_{ji}^* S_{jj} - S_{ij} S_{jj}^* = S_{ii}^* S_{ij} - S_{ij} S_{jj}^*.$$

We now compute its action on the orthonormal basis $E_m = \frac{z^m}{\|z^m\|_{2\nu+2j}}$. We write $(\frac{\partial}{\partial z})^{i-j-1} z^m = p(m) z^{m-(i-j-1)}$, where $p(m)$ is a polynomial in m of degree $i - j - 1$ with leading term m^{i-j-1}. By direct computations we have

$$W(z^m) = c(m) z^{m-1-(i-j-1)}$$

with

$$c(m) = p(m) \frac{m - (i-j-1)}{2\nu + 2i + m - (i-j-1) - 1} - p(m-1) \frac{m}{2\nu + 2j + m - 1}.$$

As a rational function of m, it is clear that the leading term m^{i-j-1} cancels each other, and $c(m)$ is of lower order m^{i-j-2},

$$|c(m)| \approx m^{i-j-2}.$$

Observing that $\|z^m\|_\sigma \approx m^{(1-\sigma)/2}$ we get

$$\|W(E_m)\|_{2\nu+2i} \approx m^{-2},$$

proving that W is of trace class.

Since the operator S_w is also a lower triangular matrix and the (ij) entries of it differ only by a factor of $(-1)^{i-j}$, the same proof works also for $[S_w^*, S_w]$ and $[S_z^*, S_w]$. ∎

For any polynomials $F(z,w)$ and $G(z,w)$ the commutator $[S_F^*, S_G]$ is then also trace class, as it can be seen by using

$$[AB, C] = A[B, C] + [A, C]B.$$

We prove below a trace formula for $[S_F^*, S_F]$.

THEOREM 3.4. *Let $F(z,w)$ be a polynomial in (z,w) and $f(z) = F(z,z)$ be its restriction to the diagonal. Then*

$$\mathrm{Tr}[S_F^*, S_F] = (N+1)\int_D |f'(z)|^2 \mathrm{d}m(z).$$

We divide the proof into some elementary lemmas.

LEMMA 3.5. *Let $G(z,w) = (z-w)g(z,w)$ for some polynomial g. Then the operators $[S_G^*, S_f]$ are of trace class for any polynomial $f(z,w)$ and* $\mathrm{Tr}[S_G^*, S_f] = 0$.

Proof. The multiplication by $G(z,w) = (z-w)g(z,w)$ maps M_i into M_{i+1}, thus S_G is a lower triangular matrix with diagonal entries being 0, with the (ij)-entry T_{ij} being Hilbert–Schmidt, by Proposition 3.3. Denoting $S_f = (S_{jk})$, the (ii)-entry of $[S_G^*, S_f]$ is

$$\sum_{j>i} T_{ji}^* S_{ji} - \sum_{j<i} S_{ij} T_{ij}^*,$$

where each term is of trace class since both S_{ij} and T_{ij} are Hilbert–Schmidt. Taking trace and summing over i we see that it is zero due to the anti-symmetry of the sum. ∎

The following lemma is elementary and known as the uniqueness of Möbius invariant spaces; see [3]. It can also be proved by elementary computations using the skew-adjointness of the Lie algebra elements on group-invariant pre-Hilbert spaces. (A much general form is known as Schur's lemma [10] for irreducible representations of semisimple Lie algebra).

LEMMA 3.6. *Let $\|\cdot\|$ be a pre-Hilbert norm on a space of analytic functions, which includes all polynomials. If $\|\cdot\|$ is invariant under the action of the Lie group of $SU(1,1)$ via change of variables, that is*

$$\|f(z)\| = \|f(g\cdot z)\|, \quad \text{for } g \text{ in } SU(1,1),\ f \text{ polynomial},$$

then it is the Dirichlet norm,

$$\|f\|^2 = c\int_D |f'(z)|^2 \mathrm{d}m(z), \quad \text{for } f \text{ polynomial},$$

for some constant $c \geqslant 0$.

Now we prove Theorem 3.4.

Proof. Writing $f(z) = F(z,z)$, we claim

$$\mathrm{Tr}[S_F^*, S_F] = \mathrm{Tr}[S_f^*, S_f].$$

Indeed $F(z,w) = f(z) + G(z,w)$, with $G(z,w) = (z-w)g(z,w)$ for some polynomials $g(z,w)$. By Lemma 3.5,

$$\mathrm{Tr}[S_F^*, S_F] = \mathrm{Tr}[S_f^*, S_f] + \mathrm{Tr}[S_f^*, S_G] + \mathrm{Tr}[S_G^*, S_f] + \mathrm{Tr}[S_G^*, S_G] = \mathrm{Tr}[S_f^*, S_f].$$

It follows from the proof of Theorem 3.3 and the invariance of $\mathrm{Tr}[S_f^*, S_f]$ under rotations $f(z) \to f(\mathrm{e}^{\mathrm{i}\theta} z)$ that the trace $\mathrm{Tr}[S_f^*, S_f]$ is sum of $\mathrm{Tr}[S_{z^n}^*, S_{z^n}]$ with nonnegative coefficients, and each $\mathrm{Tr}[S_{z^n}^*, S_{z^n}]$ is nonnegative. Thus the trace defines an pre-Hilbert norm on the space of all polynomials $f(z)$. Moreover, for any Möbius transformation ϕ and $g(z) = f(\phi(z))$, we have that S_f and S_g are unitarily equivalent. Thus the trace $\mathrm{Tr}[S_f^*, S_f]$ is Möbius invariant whenever it exists. Therefore, according to the previous lemma, $[S_f^*, S_f]$ is a constant multiple of the Dirichlet norm for polynomial f, and the constant can be evaluated by taking $F(z,w) = z$. ∎

4. THE TRACE AND DIXMIER TRACE OF HANKEL TYPE OPERATORS

We recall very briefly the Dixmier trace on the weak trace class $\mathcal{L}^{1,\infty}$. There exist ([9], Chapter IV) linear functionals $\mathrm{tr}_\omega : \mathcal{L}^{1,\infty} \to \mathbb{C}$, depending on certain functionals ω on the space of bounded continuous functions over the half line $[1,\infty)$, called Dixmier traces, which are similar to the usual trace. In particular for a positive operator T with eigenvalues μ_n,

$$\mathrm{tr}_\omega(T) = \lim_{n\to\infty} \frac{\sum_{k=1}^n \mu_k}{\log n} \tag{4.1}$$

whenever the limit exists. It satisfies $\mathrm{tr}_\omega(AB) = \mathrm{tr}_\omega(BA)$, and $\mathrm{tr}_\omega(T) = 0$ if T is of trace class.

In this section we will prove that the operator $H_z^* H_z$ is in the weak trace class $\mathcal{L}^{1,\infty}$ in the case of the Hardy space ($\nu = 1$), and we shall compute the Dixmier trace $\mathrm{tr}_\omega H_f^* H_f$ and show it is independent of the linear functional ω.

THEOREM 4.1. *Suppose $\nu > 1$. For any polynomial $f(z)$ we have*

$$\mathrm{Tr}\, H_f^* H_f = c\frac{1}{\pi} \int_D |f'(z)|^2 \mathrm{d}m(z)$$

where $c = \sum_{i=0}^{N} c_i$ and $c_i = \sum_{m=0}^{\infty} \|P_M M_z e_m^i\|^2$.

We consider the case $f(z) = z$ first.

LEMMA 4.2. *Suppose $\nu > 1$. The operator $H_z^* H_z$ is a trace class operator.*

Proof. Since $H_z^* H_z$ is a positive operator, we need only to prove that each $P_i H_z^* H_z P_i$ is a trace class operator, which is equivalent to that the series

$$c_i := \sum_{m=0}^{\infty} \|P_M M_z e_m^i\|^2 = \sum_{m=0}^{\infty} \sum_{j \geqslant i+1} \|P_j M_z e_m^i\|^2 \tag{4.2}$$

is convergent. In terms of the basis $E_m = \sqrt{\frac{(2\nu+2i)_m}{m!}} z^m$ of $H_{2\nu+2i}$, we have by Lemma 3.1

$$\begin{aligned}\|P_j M_z e_m^i\|^2 &= \|T_j M_z T_i^* E_m\|^2 \\ &= \frac{C_i^2}{C_j^2} \Big(\frac{(\nu+i)_{j-i}}{(2\nu+2i)_{2j-2i-1}} \Big)^2 \frac{(2\nu+2i)_m}{m!} \Big\| \Big(-\frac{\partial}{\partial z} \Big)^{j-i-1} z^m \Big\|_{2\nu+2j}^2,\end{aligned}$$

with (the norm being computed in $H_{2\nu+2j}$)

$$\begin{aligned}\Big\| \Big(-\frac{\partial}{\partial z} \Big)^{j-i-1} z^m \Big\|_{2\nu+2j}^2 &= (m(m-1)\cdots(m-j+i+2))^2 \frac{(m-j+i+1)!}{(2\nu+2j)_{m-j+i+1}} \\ &= \frac{(m!)^2}{(m-(j-i-1))!(2\nu+2j)_{m-(j-i-1)}},\end{aligned}$$

which is nonzero only for $m \geqslant j-i-1$. Writing all terms using Gamma function we find that $\|P_j M_z e_m^i\|^2$ is, apart from the constants independent of the summation index j and m, equal to

$$\frac{j!(2\nu-1+j)_j(\nu+i)_{j-i}^2 m!(2\nu+2i)_m}{(\nu)_j^2(2\nu+2i)_{2j-2i-1}^2(m-(j-i-1))!(2\nu+2j)_{m-(j-i-1)}}.$$

To sum the double series $\sum_{m=0}^{\infty} \sum_{j \geqslant i+1}$ in (4.2) we change variables $m = j-i-1+p$, with $j \geqslant i+1$, $p \geqslant 0$ and write it as $\sum_{j \geqslant i+1} \sum_{p=0}^{\infty}$. The factors depending on p are

$$\begin{aligned}\frac{m!(2\nu+2i)_m}{(m-j+i+1)!(2\nu+2j)_{m-j+i+1}} &= (j-i-1)!(2\nu+2i)_{j-i-1} \frac{(j-i)_p(2\nu+j+i-1)_p}{p!(2\nu+2j)_p} \\ &= \Gamma(j-i) \frac{\Gamma(2\nu+j+i-1)}{\Gamma(2\nu+2i)} \frac{(j-i)_p(2\nu+j+i-1)_p}{p!(2\nu+2j)_p}.\end{aligned}$$

The sum over p,

$$\sum_{p=0}^{\infty} \frac{(j-i)_p(2\nu+j+i-1)_p}{p!(2\nu+2j)_p}$$

is the hypergeometric series $F(j-i, 2\nu+j+i-1; 2\nu+2j; 1)$, which is convergent and whose value, again by the Gauss summation formula, is

$$\frac{\Gamma(2\nu+2j)}{\Gamma(2\nu+j+i)\Gamma(j-i+1)}.$$

The factors $\frac{(\nu+i)_{j-i}}{(\nu)_j}$ are bounded and the summation over j is equivalent to

$$\sum_{j\geqslant n+1}\frac{\Gamma(2\nu+2j)\Gamma(j+1)\Gamma(j-i)\Gamma(2\nu+j+i-1)}{\Gamma(2\nu+2j-1)\Gamma(2\nu-1+j)\Gamma(2\nu+j+i)\Gamma(j-i+1)}$$
$$=\sum_{j\geqslant n+1}\frac{(2\nu+2j-1)}{(j-i)(2\nu+j+i-1)}\frac{\Gamma(j+1)}{\Gamma(2\nu-1+j)}.$$

Each term can be estimated using the Stirling formula,

$$\frac{(2\nu+2j-1)}{(j-i)(2\nu+j+i-1)}\frac{\Gamma(j+1)}{\Gamma(2\nu-1+j)}\approx\frac{1}{j^{2\nu-1}},$$

and thus the series is convergent if and only if $\nu>1$. ∎

Now we prove Theorem 4.1. The operator $H_f^*H_f$ is a trace class operator for any polynomial $f(z)$. To see this we let, for any bounded holomorphic function $F(z,w)$,

$$R_F=M_F|_M, S_F=P_{M^\perp}M_F|_{M^\perp}$$

be the restriction on the submodule $M=[(z-w)^{N+1}]$ and compression on the quotient $M^\perp$ of M_F. An easy matrix computation show that

$$H_{z^2}=R_zH_z+H_zS_z.$$

Since $H_z\in\mathcal{L}^2$ from Lemma 4.2 we have $H_{z^2}\in\mathcal{L}^2$. Similarity, $H_{z^n}\in\mathcal{L}^2$ for any n. This implies that $H_f^*H_f$ is also a trace class operator for any polynomial $f(z)$. Therefore,

$$(f,g)\to\operatorname{Tr}H_g^*H_f$$

defines an invariant pre-Hilbert norm on the space of all polynomials $f(z)$. For any Möbius transformation ϕ and $g(z)=f(\phi(z))$, we have that H_f and H_g are unitarily equivalent. Thus the trace of $H_f^*H_f$ is Möbius invariant whenever it exists. Now Theorem 4.1 can be proved by using the same method as that of Theorem 3.4.

We consider now the case $\nu=1$. We need some simple facts on the computation of the Dixmier trace. We call an operator T on H sub-diagonal if there exist an orthonormal basis $\{e_i\}_{i=1}^\infty$ of H and an integer N such that $\langle Te_i,e_j\rangle=0$ for $|i-j|>N$.

LEMMA 4.3. *Suppose T is sub-diagonal with the corresponding orthonormal basis $\{e_i\}_{i=0}^\infty$. If $T\in\mathcal{L}^{1,\infty}$ and $\langle Te_i,e_i\rangle=0$ for any $i\geqslant 0$, then* $\operatorname{Tr}_\omega T=0$.

Proof. Since a sub-diagonal operator T is a sum of finitely many unilateral shift operators, it suffices to consider the case that T is a weighted unilateral shift satisfying $T\in\mathcal{L}^{1,\infty}$ with weight $\{0,a_1,0,a_3,0,\ldots\}$. We have then $\operatorname{Tr}_\omega(T+T^*)=0$. In fact, a direct computation shows that the eigenvalues of $T+T^*$ are $\{-|a_1|,|a_1|,-|a_3|,|a_3|,\ldots\}$. This implies that the positive part $(T+T^*)_+$ and negative part $(T+T^*)\_$ of $T+T^*$ have the same eigenvalues distribution. Therefore,

$\mathrm{Tr}_\omega(T+T^*)_+ = \mathrm{Tr}_\omega(T+T^*)_-$ and $\mathrm{Tr}_\omega(T+T^*) = 0$. A similar argument shows that $\mathrm{Tr}_\omega \frac{T-T^*}{i} = 0$. Thus $\mathrm{Tr}_\omega T = 0$, as desired. ∎

PROPOSITION 4.4. *Let $\nu = 1$. The operator $H_z^* H_z$ is in the weak trace class $\mathcal{L}^{1,\infty}$ but not in $\mathcal{L}^1$, and* $\mathrm{Tr}_\omega H_z^* H_z = (N+1)^2$.

Proof. By the definition of the operator ideal $\mathcal{L}^{p,\infty}$, it suffices to show that the Hankel operator $H_z = \sum_{i=0}^{N} P_M M_z P_i \in \mathcal{L}^{2,\infty}$. The operator $P_i M_z^* P_M M_z P_i = (P_M M_z P_i)^*(P_M M_z P_i)$ is diagonal under the orthonormal basis $\{e_m^i\}$ given in (2.7), and we need only to show that

$$\|P_i M_z^* P_M M_z P_i e_m^i\| = \|P_M M_z P_i e_m^i\|^2 = O\Big(\frac{1}{m}\Big).$$

As $P_k M_z P_l = 0$ for $k < l$ we have

$$\begin{aligned} P_M M_z P_i &= (I - P_0 - \cdots - P_N) M_z P_i = (I - P_i - P_{i+1} - \cdots - P_N) M_z P_i \\ &= (I - P_i) M_z P_i - S_{i+1,i} - \cdots - S_{N,i}, \end{aligned}$$

and consequently

$$\|P_M M_z e_m^i\|^2 = \|(I-P_i) M_z e_m^i\|^2 - \|S_{i+1,i} e_m^i\|^2 - \cdots - \|S_{N,i} e_m^i\|^2. \tag{4.3}$$

The first term above is

$$\begin{aligned} \|(I-P_i) M_z e_m^i\|^2 &= \|M_z e_m^i\|^2 - \|P_i M_z e_m^i\|^2 = 1 - \|S_{i,i} e_m^i\|^2 = 1 - \frac{m+1}{2+2i+m} \\ &= \frac{1+2i}{2+2i+m} = \frac{1+2i}{m} + O\Big(\frac{1}{m^2}\Big), \end{aligned}$$

where we have used Lemma 2.2 that $S_{i,i}$ is unitarily equivalent to the multiplication operator by z on H_{2+2i}, and the fact that M_z is an isometry when $\nu = 1$. The remaining terms are estimated in (3.2), viz

$$\|S_{k,i} e_m^i\|^2 \leqslant C \frac{1}{m^2}, \quad 0 \leqslant i < k \leqslant N.$$

Thus

$$\|P_i M_z^* P_M M_z P_i e_m^i\| = \|P_M M_z P_i e_m^i\|^2 = \frac{1+2i}{m} + O\Big(\frac{1}{m^2}\Big),$$

and the operator $P_M M_z P_i \in \mathcal{L}^{2,\infty}$. This completes the proof of the first claim. Furthermore, by Lemma 4.3 we have the following that complets the proof:

$$\mathrm{tr}_\omega H_z^* H_z = \mathrm{tr}_\omega \sum_{i=0}^{N} P_i M_z^* P_M M_z P_i = \sum_{i=0}^{N} (1+2i) = (N+1)^2. \quad ∎$$

By similar methods as in the proof of Theorem 4.1 we can prove that $H_f^* H_f$ is in $\mathcal{L}^{1,\infty}$. To compute its Dixmier trace we observe that

$$H_f^* H_f = P_{M^\perp} [M_f^*, M_f] P_{M^\perp} - [S_f^*, S_f],$$

and thus $\mathrm{tr}_\omega P_{M^\perp} [M_f^*, M_f] P_{M^\perp} = \mathrm{tr}_\omega H_f^* H_f$, since $[S_f^*, S_f] \in \mathcal{L}^1$. We have therefore

THEOREM 4.5. *Let $\nu = 1$. For any polynomial $f(z)$ we have*

$$\operatorname{tr}_\omega H_f^* H_f = \operatorname{tr}_\omega P_{M^\perp}[M_f^*, M_f]P_{M^\perp} = (N+1)^2 \frac{1}{\pi} \int_D |f'(z)|^2 \mathrm{d}m(z).$$

5. SUBMODULES OF HARDY SPACES GENERATED BY HOMOGENEOUS POLYNOMIALS

In this section we consider a submodule M of $H_\nu \otimes H_\nu$ generated by homogeneous polynomials. For computational convenience, we shall only consider the case $\nu = 1$, i.e., the Hardy space on the bidisk. As is shown in [25], [7], up to a finite dimensional subspace, M is of the form $M = [p]$ for a single homogeneous polynomial p with $p = p_1 p_2$, where the zero sets $Z(p_1)$ and $Z(p_2)$ have the properties that

$$Z(p_1) \cap \partial D^2 = Z(p_1) \cap T^2 \tag{5.1}$$

and respectively

$$Z(p_2) \cap \partial D^2 = Z(p_2) \cap (\partial D^2 \setminus T^2), \tag{5.2}$$

where ∂D^2 is the topological boundary of D^2, so that $\partial D^2 \setminus T^2 = (T \times D) \cup (D \times T)$. We recall the following result from [17].

THEOREM 5.1. *The quotient module $[p]^\perp$ is compact if and only if $p = p_1 p_2$, with p_2 being one of the following polynomials:*

$$1,\ (z - \alpha w), \quad (w - \beta z), \quad (z - \alpha w)(w - \beta z), \quad \text{for } |\alpha| < 1, |\beta| < 1.$$

We will thus only consider quotient modules classified in the above theorem and study further their $\mathcal{L}^{p,q}$ properties, in particular their trace class properties.

THEOREM 5.2. *Suppose $M = [p]$ with p as in Theorem* 5.1.

(i) *The quotient module is trace class if and only if p is one of the following polynomials:*

$$(z - \alpha_1 w)^{n+1}, \quad (z - \alpha w), \quad (w - \beta z), \quad (z - \alpha w)(w - \beta z),$$

with $|\alpha_1| = 1$, $|\alpha| < 1, |\beta| < 1$.

(ii) *The quotient module is in the weak trace class if and only if p is one of the following polynomials:*

$$\prod_{j=1}^{k} (z - \alpha_j w)^{n_k+1}, \quad (z - \alpha w), \quad (w - \beta z), \quad (z - \alpha w)(w - \beta z),$$

with $|\alpha_j| = 1, \forall j$, and $|\alpha| < 1, |\beta| < 1$.

We divide the proof into several steps. We note that the results in the previous sections are clearly valid for the submodule $M = [p]$ generated by $p = (z - \alpha w)^{n+1}$ for some α with $|\alpha| = 1$. We consider first the case when $p = z - \alpha w$ with $|\alpha| < 1$.

LEMMA 5.3. *Let $p = z - \alpha w$ for some $|\alpha| < 1$. On the quotient module $M^\perp = [p]^\perp$, we have that $S_z = \alpha S_w$ and S_w is unitarily equivalent to the multiplication operator M_w on the Bergman space $L^2_a(D, \mathrm{d}\mu)$, where μ is a probability measure defined by*

$$\mathrm{d}\mu(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1 - |\alpha^2|}{2\pi}\Big(\sum_{j=0}^{\infty} |\alpha^{2j}|\, \delta_{|\alpha|^j}(r)\Big) \times \mathrm{d}\theta,$$

where δ_x is the delta measure supported at x. In particular, the quotient $M^\perp \in \mathcal{L}^1$.

Proof. By direct computation, the polynomials

$$e_{n,\alpha}(z, w) := e_n(z, w) := \sqrt{\frac{1 - |\alpha^2|}{1 - |\alpha^2|^{n+1}}}\, \frac{(\overline{\alpha} z)^{n+1} - w^{n+1}}{\overline{\alpha} z - w} \tag{5.3}$$

form an orthonormal basis for the space $M^\perp$ (see [17]). The operator S_w on e_n is a weighted shift

$$S_w e_n = \sqrt{\frac{1 - |\alpha^2|^{n+1}}{1 - |\alpha^2|^{n+2}}}\, e_{n+1}.$$

On the other hand the functions $\{w^n\}$ form an orthogonal basis and its norm square in $L^2_a(D, \mathrm{d}\mu)$ is

$$\|w^n\|^2 = (1 - |\alpha^2|) \sum_{j=0}^{\infty} |\alpha^2|^j |\alpha|^{2nj} = (1 - |\alpha^2|) \frac{1}{1 - |\alpha^2|^{n+1}},$$

from which it follows that the mapping $e_n \mapsto \frac{w^n}{\|w^n\|}$ realizes the unitary equivalence of the operators S_w and M_w. ∎

REMARK 5.4. If $M = [z - \alpha w]$ as above we have

$$\mathrm{Tr}[S^*_{f(w)}, S_{f(w)}] = \int_D |f'(w)|^2 \mathrm{d}m(w)$$

by the general result in [4]. For any polynomial $F(z, w)$ we have

$$\mathrm{Tr}[S^*_F, S_F] = \int_D |f'(w)|^2 \mathrm{d}m(w), \quad f(w) = F(\alpha w, w)$$

since $S_z = \alpha S_w$. Also the above results are obviously valid in the case of $p = w - \beta z$ for some $|\beta| < 1$.

Now we consider the general cases. For the conceptual clarity we introduce the following; see [17] for the compact case, namely when $(p, q) = (\infty, \infty)$.

DEFINITION 5.5. Let N_1 and N_2 be two closed subspaces of a Hilbert space N and P_1, P_2 the corresponding orthogonal projections. They are called (p, q)-*orthogonal* if $P_1 P_2 \in \mathcal{L}^{p,q}$.

The definition is clearly independent of the Hilbert space N.

PROPOSITION 5.6. *Let N, N_1 and N_2 be three quotient modules of $H \otimes H$ such that $N_1 + N_2$ is dense in N. If N_1 and N_2 are (p,q)-orthogonal and (p,q)-essentially normal, then N is also (p,q)-essentially normal.*

Proof. Let P_1, P_2, P denote the projections from $H \otimes H$ onto N_1, N_2, N respectively. Then $P_1P_2 \in \mathcal{L}^{p,q}$. This implies [17] in particular that P_1P_2 is compact and hence $N_1 + N_2$ is closed and $N_1 \cap N_2$ is of finite dimension. Without loss of generality we may assume $N_1 \cap N_2 = 0$ and then $N = N_1 \dot{+} N_2$ is a direct sum decomposition.

Define $Q := P_1 + P_2 : N \to N$. Q is then an invertible operator on N. Moreover,

$$Q(P - Q) = Q^2 - Q = P_1P_2 + P_2P_1$$

is in $\mathcal{L}^{p,q}$, and so is also $P - Q = P - (P_1 + P_2)$.

Let $S_f^{N_1} := P_{N_1}M_f|_{N_1}, S_f^{N_2} := P_{N_2}M_f|_{N_2}, S_f^N := P_N M_f|_N$ be the compressions of the multiplication operator M_f with symbol f on N_1, N_2, N, respectively. For any polynomial f the commutators

$$[S_f^{N_1*}, S_f^{N_1}], [S_f^{N_2*}, S_f^{N_2}] \in \mathcal{L}^{(p,q)}$$

since N_1, N_2 is (p,q)-essentially normal. Moreover, since N_1, N_2 are co-invariant subspaces and $Q - P, P_1P_2 \in \mathcal{L}^{(p,q)}$, we have

$$\begin{aligned} S_f^N - S_f^{N_1} - S_f^{N_2} &= (PM_fP - QM_fQ) + (QM_fQ - P_1M_fP_1 - P_2M_fP_2) \\ &= (PM_fP - QM_fQ) + P_1M_fP_2 + P_2M_fP_1 \\ &= (PM_fP - QM_fQ) + P_1M_fP_1P_2 + P_2M_fP_2P_1 \in \mathcal{L}^{(p,q)}. \end{aligned}$$

This implies that $[S_f^{N*}, S_f^N] \in \mathcal{L}^{(p,q)}$ for any polynomial f. In particular, $[S_z^{N*}, S_z^N]$, $[S_w^{N*}, S_w^N] \in \mathcal{L}^{(p,q)}$. Furthermore,

$$[S_z^{N*}, S_w^N] = \frac{1}{4}\{[S_{z+w}^{N*}, S_{z+w}^N] - [S_{z-w}^{N*}, S_{z-w}^N] + i[S_{z+iw}^{N*}, S_{z+iw}^N] - i[S_{z-iw}^{N*}, S_{z-iw}^N]\} \in \mathcal{L}^{(p,q)}.$$

Therefore, the quotient module N is $\mathcal{L}^{(p,q)}$, as desired. ∎

LEMMA 5.7. (i) *The subspaces $[(z - \alpha w)^{n+1}]^\perp$ and $[(z - \beta w)^{N+1}]^\perp$ are $(1,\infty)$-orthogonal if $\alpha \neq \beta, |\alpha| = |\beta| = 1$.*

(ii) *The subspaces $[z - \alpha w]^\perp$ and $[(z - \beta w)^n]^\perp$ are $(2,\infty)$-orthogonal if $|\alpha| < |\beta| = 1$.*

(iii) *The subspaces $[z - \alpha w]^\perp$ and $[(w - \beta z)]^\perp$ are p-orthogonal for all $p > 0$ if $|\alpha|, |\beta| < 1$.*

Proof. (i) By the rotational invariance we may assume $\beta = 1$. Let P, P' be the orthogonal projection onto $[(z - w)^{N+1}]^\perp, [(z - \alpha w)^{n+1}]^\perp$, respectively. Then

$$P = \sum_{i=0}^{N} P_i, \quad P' = \sum_{j=0}^{n} P'_j,$$

where P_i and P'_j are the orthogonal projections onto $[(z-w)^{i+1}]^\perp \ominus [(z-w)^i]^\perp$ and $[(z-\alpha w)^{j+1}]^\perp \ominus [(z-\alpha w)^j]^\perp$, respectively. Denote $\{\widetilde{e}^i_m, m=0,1,\dots\}_{i=0}^N$ the orthonormal basis $[(z-w)^N]^\perp = \sum_{i=0}^N [(z-w)^{i+1}]^\perp \ominus [(z-w)^i]^\perp$ given in (2.7). Replacing z by $\overline{\alpha} z$ we get an orthonormal basis $\{e^j_m, m=0,1,\dots\}_{j=0}^n$ of $[(z-\alpha w)^{n+1}]^\perp$. Then

$$P_i = \bigoplus_{m\geqslant 0} \widetilde{e}^i_m \otimes \widetilde{e}^i_m, \quad P'_j = \bigoplus_{m\geqslant 0} e^j_m \otimes e^j_m. \tag{5.4}$$

Here $u\otimes v$ denotes as usual the rank one operator $x \to (x,v)u$.

We claim that

$$\|Pe_m\| \leqslant C\frac{1}{m} \tag{5.5}$$

for some C independent of m. This implies then the required result that $P'P \in \mathcal{L}^{1,\infty}$. In fact, by the rotational invariance and (5.4),

$$|P'_jP_i|^2 = P_iP'_jP_i = \bigoplus_{m\geqslant 0} P_ie^j_m \otimes P_ie^j_m = \bigoplus_{m\geqslant 0} |\langle \widetilde{e}^i_{m+j-i}, e^j_m\rangle|^2\, \widetilde{e}^i_{m+j-i} \otimes \widetilde{e}^i_{m+j-i}.$$

Therefore, $|P'_jP_i| = \bigoplus_{m\geqslant 0} |\langle \widetilde{e}^i_{m+j-i}, e^j_m\rangle|\, \widetilde{e}^i_{m+j-i} \otimes \widetilde{e}^i_{m+j-i}$. The estimate (5.5) concludes that $|\langle \widetilde{e}^i_{m+j-i}, e^j_m\rangle| = O(\frac{1}{m})$. Hence $|P'_jP_i| \in \mathcal{L}^{1,\infty}$ and $P'P \in \mathcal{L}^{1,\infty}$.

The proof of (5.5) involves some rather delicate computations. To ease the notation, we will suppress the index k in e^k_m since only m is relevant. We write e_m as

$$e_m = c_m \sum_{l=0}^m a_{m,l} (\overline{\alpha} z - w)^k (\overline{\alpha} z)^l w^{m-l}, \quad m=0,1,\dots.$$

Here

$$c_m = \frac{k!}{\sqrt{k!(1+k)_k}} \sqrt{\frac{m!}{(2+2k)_m}} = Cm^{-(2+2k-1)/2}\Big(1+O\Big(\frac{1}{m}\Big)\Big)$$

for some C independent of m, and $a_{m,l} = \frac{(1+k)_l}{l!}\frac{(1+k)_{m-l}}{(m-l)!}$. We rewrite $a_{m,l}$ as

$$\begin{aligned} a_{m,l} &= \frac{1}{k!^2}(l+1)\cdots(l+k)\,(m-l+1)\cdots(m-l+k) \\ &= \frac{1}{k!^2}(l^k + c_1 l^{k-1} + \cdots + c_k)((m-l)^k + c_1(m-l)^{k-1} + \cdots + c_k). \end{aligned}$$

Thus $a(m,l)$ is a linear combination of $l^{k_1}(m-l)^{k_2}$, $k_1,k_2 \leqslant k$ with coefficients independent of (m,l). Similarly $(\overline{\alpha} z - w)^k$ is such a linear combination of $z^{k_3}w^{k_4}$ with $k_3+k_4 = k$. Thus the function e_m above is a linear combination of the functions

$$e'_m := e'_m(k_1,k_2,k_3,k_4) := c_m \sum_{l=0}^m l^{k_1}(m-l)^{k_2}\overline{\alpha}^l z^{k_3+l} w^{k_4+m-l}, \quad m=0,1,\dots$$

with coefficients dominated by constants independent of m and l. To obtain (5.5), it suffices to show

$$\|Pe'_m\| \leqslant C\frac{1}{m}.$$

Here as well as below C denote any constant independent of (m, l).

By Lemma 2.1 we have

$$P = \sum_{j=0}^{N} T_j^* T_j.$$

We shall estimate $\|T_j e'_m\|$ and prove that

$$\|T_j e'_m\| \leqslant C\frac{1}{m},$$

which then implies the estimates for $\|Pe'_m\|$ and $\|Pe_m\|$.

By the rotational invariance of $T_j e'_m$, we see that $T_j e'_m$ is a scalar multiple of z^{m+k-j} in the Bergman space H_{2+2j}. Now T_j is a linear combination of the differential operators $f(z,w) \mapsto (\partial_z^i \partial_w^{j-i} f)(z,z)$, and each operator maps e'_m to $c_m d_m z^{m+k-j}$ with

$$d_m = \sum_{l=0}^{m} b_m(l)\, \overline{\alpha}^l,$$

here

$$\begin{aligned} b_m(l) := {}& l^{k_1}(l+k_3)\cdots(l+k_3-i+1) \\ & \times (m-l)^{k_2}(m-l+k_4)\cdots(m-l+k_4-(j-i)+1) \end{aligned}$$

is a polynomial of l. (To ease notation we have suppressed indexes $k_1, k_2, k_3, k_4 \leqslant k$ within d_m and $b_m(l)$).

The series d_m is a trigonometric series in α with coefficients $b_m(l)$. We will use now the Abel partial summation formula $2k + j$ times to reduce d_m to the geometric series $\sum\limits_{l=0}^{m} \overline{\alpha}^l$ multiplied by m^q with $q \leqslant k + j$. To bound the boundary terms in the Abel partial summation we need to keep track of the evaluations of discrete differentiation $b_m(l)$ as a function of l at the end points $l = 0$ and $l = m$. We write $\partial b(l) := b(l) - b(l+1)$ for the discrete differentiation. The key observation is that the differentiations $\partial^q b_m(l)$ of all degrees q at the point $l = 0$ and m are all dominated by m^{k+j}, namely

$$|\partial^q b_m(l)| \leqslant Cm^{k+j}, \quad l = 0, m, \ q \leqslant 2k + j \tag{5.6}$$

with C independent of m. We prove this for the end point $l = 0$ and the other end point is exactly the same by changing the variable l to $m - l$. If $q = 0$ then $\partial^q b_m(l) = b_m(l)$, and its values at the end point $l = 0$ are zero unless $k_1 = 0$ in which case

$$b_m(0) = k_3 \cdots (k_3 - i + 1) m^{k_2}(m + k_4)\cdots(m + k_4 - j + i + 1) \leqslant Cm^{k_2+j-i} \leqslant Cm^{k+j},$$

and (5.6) is indeed true. For general $q \leqslant 2k+j$ we observe that $b_m(l)$ is a polynomial in l of maximum degree $2k+j$ with coefficients being polynomials of m of maximum degree $k+j$. Each discrete differentiation in l reduces the degree of $b_m(l)$ by one whose evaluation at $l=0$ is still a polynomial of m of maximum degree $k+j$. Repeating the argument we see that (5.6) is true as $2k+j$ is fixed and independent of m.

We perform now the Abel summation. Notice the partial sums of the series $\sum_j \overline{\alpha}^j$ are $\frac{1-\overline{\alpha}^{l+1}}{1-\overline{\alpha}} = \frac{1}{1-\overline{\alpha}} - \frac{\overline{\alpha}}{1-\overline{\alpha}}\overline{\alpha}^l$, which is again a geometric series apart from the constant term. Thus for $m > 2k+j$,

$$\begin{aligned} d_m &= \sum_{l=0}^{m-1} \partial b_m(l) \frac{1-\overline{\alpha}^{l+1}}{1-\overline{\alpha}} + b_m(m)\frac{1-\overline{\alpha}^{m+1}}{1-\overline{\alpha}} \\ &= \frac{-\overline{\alpha}}{1-\overline{\alpha}} d'_m + \frac{1}{1-\overline{\alpha}} b_m(0) + b_m(m)\frac{-\overline{\alpha}^{m+1}}{1-\overline{\alpha}} \end{aligned}$$

with the leading term

$$d'_m := \sum_{l=0}^{m-1} \partial b_m(l)\overline{\alpha}^l,$$

which is again a trigonometric series of $\overline{\alpha}$ and its coefficients $\partial_l b_m(l)$ are polynomials of l of maximum degree $2k+j-1$. Using (5.6) we see that the error term

$$\Big|\frac{1}{1-\overline{\alpha}} b_m(0) + b_m(m)\frac{-\overline{\alpha}^{m+1}}{1-\overline{\alpha}}\Big| \leqslant Cm^{k+j}.$$

Thus

$$|d_m| \leqslant |d'_m| + Cm^{k+j}.$$

Applying the partial summation $2k+j$ times we see that $|d_m| \leqslant Cm^{k+j}$. Consequently

$$|c_m d_m| \leqslant Cc_m m^{k+j} \leqslant Cm^{-(2+2k-1)/2} m^{k+j} = Cm^{-(1/2)+j}.$$

The norm $T_j e_m$ is then

$$\|T_j e'_m\| = \|d_m z^{m+k-j}\| \leqslant Cm^{-(1/2)+j} m^{-(2j+1)/2} = Cm^{-1},$$

completing the proof.

(ii) Using the similar argument as in (i), we see the inner product of e_m with $e_{m+j,\alpha}(z,w)$ in (5.3) satisfies

$$|\langle e_m, e_{m+j,\alpha}\rangle| \leqslant C\frac{1}{\sqrt{m}}.$$

Let P be the orthogonal projection onto $[(z-\alpha w)]^{\perp}$. Then

$$\|Pe_m\| \leqslant C\frac{1}{\sqrt{m}},$$

which implies the desired result by the similar argument as in (i).

(iii) We may assume $1 > |\alpha| \geqslant |\beta|$. The polynomials e_n in (5.3) and

$$f_n(z,w) = \sqrt{\frac{1-|\beta^2|}{1-|\beta^2|^{n+1}}}\frac{(\overline{\beta}w)^{n+1} - z^{n+1}}{\overline{\beta}w - z}$$

form an normalized basis of the subspaces $[z-\alpha w]^\perp$ and $[(w-\beta z)]^\perp$, respectively. It is easy to see that

$$|\langle e_n, f_n\rangle| \leqslant Cn|\alpha|^n$$

for some constant C independent of n. Thus $P_2P_1 \in \mathcal{L}^p$ for any $p > 0$, where P_1, P_2 are the projections of $[z-\alpha w]^\perp$ and $[(w-\beta z)]^\perp$ respectively. ∎

REMARK 5.8. Let $p = \prod_{j=1}^{k}(z-\alpha_j w)^{n_j}$ with $|\alpha_j| \leqslant 1$ and P the orthogonal projection onto $[p]^\perp$. Note that our proof depends only on the estimates of a trigonometric series. The same proof and its iteration then yield the following estimate: For the given basis of e_m of $[(z-\alpha w)^n]$, with $|\alpha| = 1$ and α not being one of α_j,

$$\|P(M_z^*)^a M_z^b e_m\| \leqslant C\frac{1}{m}, \tag{5.7}$$

if all $|\alpha_j| = 1$;

$$\|P(M_z^*)^a M_z^b e_m\| \leqslant C\frac{1}{\sqrt{m}}, \tag{5.8}$$

if one of $|\alpha_j| < 1$. Here a, b are non-negative integers.

LEMMA 5.9. *Suppose S is a (p,q)-essentially normal operator on N, and $N = N_1 \oplus N_2$ with N_2 being an invariant subspace of S. Write $S = \begin{pmatrix} S_1 & 0 \\ S_{21} & S_2 \end{pmatrix}$ with $S_1 = P_{N_1}S|_{N_1}$, $S_2 = S|_{N_2}$. If one of the operators S_1 and S_2 is (p,q)-essentially normal then so is the other.*

Proof. Indeed, $[S^*, S]$ has diagonal entries $[S_1^*, S_1] + S_{21}^*S_{21}$ and $[S_2^*, S_2] - S_{21}S_{21}^*$, which are all in $\mathcal{L}^{p,q}$ since $[S^*, S]$ is. Thus if one of the commutators, say $[S_1^*, S_1]$ is in the class, then so is $S_{21}^*S_{21}$, and consequently $S_{21}S_{21}^*$ and $[S_2^*, S_2]$ are in the same class. ∎

We consider now the module generated by a polynomial with two simple factors $(z-w)$ and $(z-\alpha w)$.

LEMMA 5.10. *Let $M = [p]$, $p = (z-w)(z-\alpha w)$.*

(i) *If $\alpha \neq 1$, $|\alpha| = 1$ then the quotient module $M^\perp = [p]^\perp$ is $\mathcal{L}^{1,\infty}$, but not $\mathcal{L}^1$-essentially normal.*

(ii) *If $|\alpha| < 1$ then the quotient module $M^\perp = [p]^\perp$ is $\mathcal{L}^{2,\infty}$, but not $\mathcal{L}^2$-essentially normal.*

Proof. The positive part of the two claims are consequences of Lemmas 5.3 and 5.7, Proposition 5.6, and the results of Section 3. To prove the negative claim in (i) we choose the orthonormal basis $f_n(z,w) = \frac{1}{\sqrt{n+1}}\frac{z^{n+1}-w^{n+1}}{(z-w)}$ of $[(z-w)]^{\perp}$, and $e_n = \frac{(\overline{\alpha}z)^{n+1}-w^{n+1}}{(\overline{\alpha}z-w)\sqrt{n+1}}$ of $[(z-\alpha w)]^{\perp}$ as before. Let $S_z = P_{M^{\perp}}M_z|_{M^{\perp}}$. We compute the inner product $\langle([S_z^*, S_z]f_n, e_{n,\alpha}\rangle$ and find that

$$\langle [S_z^*, S_z]f_n, e_{n,\alpha}\rangle = \frac{\alpha-1}{n} + O\Big(\frac{1}{n^2}\Big);$$

we omit the elementary routine computation. Thus, by Theorem 1.4.8 of [27], the operator $[S_z^*, S_z]$ is not in $\mathcal{L}^1$.

The similar argument works also for the negative claim in (ii). Let e_n' be the orthonormal basis of $[(z-\alpha w)]^{\perp}$ given by (5.3) for $|\alpha| < 1$. A direct computation shows that

$$\langle [S_z^*, S_z]f_n, e_{n,\alpha}'\rangle = \frac{(\alpha-1)\sqrt{1-|\alpha^2|}}{\sqrt{n}} + O\Big(\frac{1}{n}\Big);$$

Thus $[S_z^*, S_z]$ is not in $\mathcal{L}^2$. ∎

We prove now Theorem 5.2.

Proof. The sufficiency is a consequence of Theorem 3.3, Lemmas 5.3, Proposition 5.6 and Lemmas 5.7. We prove now the necessity in part (i), and part (ii) is the same.

Let $p = p_1p_2$ be as in Theorem 5.1. We consider two cases.

Case 1. $p_2 = 1$, that is, $p = p_1 = (z-\alpha_1 w)^{n_1}\cdots(z-\alpha_l w)^{n_l}$ with different $\alpha_1, \dots \alpha_m$ and $|\alpha_1| = \cdots = |\alpha_l| = 1$. We will prove that if $S := S_z$ is 1-essentially normal then $l = 1$, i.e., $p = (z-\alpha_1 w)^{n_1}$ with only one factor of multiplicity n_1. Suppose the contrary, that $l > 1$. We prove that sub-quotient module $[(z-\alpha_1 w)(z-\alpha_2 w)]^{\perp}$ is 1-essentially normal, a contradiction to Lemma 5.10(i).

Denote the last factor $(z-\alpha_l w)^{n_l}$ by $(z-\alpha w)^{k+1}$, $k \geqslant 0$, and write

$$p = (z-\alpha_1 w)^{n_1}\cdots(z-\alpha w)^{k+1} = q(z-\alpha w),$$
$$q = (z-\alpha_1 w)^{n_1}\cdots(z-\alpha_{l-1} w)^{n_{l-1}}(z-\alpha_l w)^{k}.$$

We decompose N as

$$N = [p]^{\perp} = N_1 \oplus N_2, \quad N_1 = [q]^{\perp}, \quad N_2 = N \ominus N_1.$$

Then N_2 is an invariant subspace of S and S is a lower triangular matrix under the above decomposition,

$$S = \begin{pmatrix} S_1 & 0 \\ S_{21} & S_2 \end{pmatrix},$$

where $S_2 = S|_{N_2}$ and $S_z = P_{N_1}S|_{N_1}$. We will prove by using Lemma 5.9 that S_2 and thus S_1 is 1-essentially normal.

Let e_m be the orthonormal basis of $[(z-\alpha w)^{k+1}]^\perp \ominus [(z-\alpha w)^k]^\perp$ given in (2.7). We claim that $P := P_{N_1}$ satisfies

$$\|Pe_m\| \leqslant C\frac{1}{m}, \quad \|PSe_m\| \leqslant C\frac{1}{m}, \quad \|PS^* e_m\| \leqslant C\frac{1}{m}. \tag{5.9}$$

We factorize q further as

$$q = q_1(z-\alpha w)^k, \quad q_1 := (z-\alpha_1 w)^{n_1}\cdots(z-\alpha_{l-1}w)^{n_{l-1}}.$$

Thus $M_1 := [(z-\alpha w)^k]^\perp$, and $M_2 := [q_1]^\perp$ are two subspaces of N_1 and $N_1 = M_1 + M_2$; M_1 and M_2 are $\mathcal{L}^{1,\infty}$ orthogonal, by Lemma 5.7, and the sum $P_1 + P_2$ of the corresponding projections $P_1 := P_{M_1}$ and $P_2 := P_{M_1}$ is then invertible on N_1. Thus there exists an operator T such that

$$T(P_1+P_2) = P. \tag{5.10}$$

Moreover, by (3.2), we have that

$$P_1 e_m = 0, \quad P_1 S e_m = 0, \quad \|P_1 S^* e_m\| \leqslant C\frac{1}{m}. \tag{5.11}$$

Recall the formula (5.7) that,

$$\|P_2 e_m\| \leqslant C\frac{1}{m}, \quad \|P_2 S e_m\| \leqslant C\frac{1}{m}, \quad \|P_2 S^* e_m\| \leqslant C\frac{1}{m}. \tag{5.12}$$

The claim (5.9) follows immediately from the formulas (5.10), (5.11), and (5.12).

Let $e'_m = c_m(e_m - Pe_m)$ be the normalized projection of e_m on the subspace N_2, where $c_m = \frac{1}{\|e_m - Pe_m\|}$. It is easy to show that $\{e'_m\}_{m=0}^\infty$ is an orthonormal basis of N_2; the orthogonality followed by the different homogeneous degrees of e_m and the invariance of P under the circle action. The operator S is then a weighted shift on N_2, i.e,

$$S_2 e'_m = \langle S_2 e'_m, e'_{m+1}\rangle e'_{m+1} = \langle S e'_m, e'_{m+1}\rangle e'_{m+1}$$

with

$$\begin{aligned}\langle Se'_m, e'_{m+1}\rangle &= c_m c_{m+1}\langle S(e_m - Pe_m), e_{m+1} - Pe_{m+1}\rangle \\ &= c_m c_{m+1}(\langle Se_m, e_{m+1}\rangle - \langle Se_m, Pe_{m+1}\rangle - \langle SPe_m, e_{m+1}\rangle + \langle SPe_m, Pe_{m+1}\rangle).\end{aligned}$$

The first term $\langle Se_m, e_{m+1}\rangle$ above, by Lemma 2.2, is

$$\langle Se_m, e_{m+1}\rangle = \langle M_z e_m, e_{m+1}\rangle = \sqrt{\frac{m+1}{1+2k+m}} = 1 + \frac{1}{m} + O\Big(\frac{1}{m^2}\Big).$$

since the compression of S acting on $\{e_m\}$ is the Bergman shift M_z in the space H_{2+2k}. The remaining terms are all of order $O(\frac{1}{m^2})$ in view of (5.9) and the Schwartz inequality. The normalization constant $c_m = (1 - \|Pe_m\|^2)^{-1/2} = 1 + O(\frac{1}{m^2})$ by the estimate (5.5). Putting those computations together we have proved

$$S_2 e'_m = \Big(1 + \frac{1}{m} + O\Big(\frac{1}{m^2}\Big)\Big)\, e'_{m+1},$$

and that $[S_2^*, S_2]$ is of trace class. From Lemma 5.9, $S_1 = P_{N_1} S P_{N_1} = P_{[q]^\perp} M_z P_{[q]^\perp}$ is also 1-essentially normal.

By continuing this procedure of restricting the action of S on sub-quotient modules, we prove that $[(z-\alpha_1 w)(z-\alpha_2 w)]$ is 1-essentially normal, contradicting to Lemma 5.10(i).

Case 2. $p_2 = z - \alpha w$ or $(z-\alpha w)(w-\beta z)$. This can be treated by the same method and we omit the details here. (Actually one can prove that the corresponding quotient $[p]^\perp$, $p = p_1 p_2$, is in $\mathcal{L}^{2,\infty}$, but not in $\mathcal{L}^2$, in particular not $\mathcal{L}^1$.) ∎

In what follows, we will consider trace formulas in quotient modules. In the case of $p = (z-\alpha w)^{n+1}$ with $|\alpha| = 1$, the trace formula of $[S_F^*, S_F]$ is treated in Theorem 3.4. For more general cases, we have the following result. The formula in (ii) is shown in Remark 5.4. The other cases are much the same and we omit it.

THEOREM 5.11. *Let $F(z,w)$ be a polynomial.*

(i) *If $p = (z-\alpha w)^{N+1}$ for some $|\alpha| = 1$, then*

$$\mathrm{Tr}[S_F^*, S_F] = (N+1)\int_D |f'(w)|^2 \mathrm{d}m(w), \quad f(w) = F(\alpha w, w).$$

(ii) *If $p = z - \alpha w$ for some $|\alpha| < 1$, then*

$$\mathrm{Tr}[S_F^*, S_F] = \int_D |f'(w)|^2 \mathrm{d}m(w), \quad f(w) = F(\alpha w, w).$$

(iii) *If $p = (z-\alpha w)(w-\beta z)$ for some $|\alpha| < 1, |\beta| < 1$, then*

$$\mathrm{Tr}[S_F^*, S_F] = \int_D |f_1'(w)|^2 \mathrm{d}m(w) + \int_D |f_2'(z)|^2 \mathrm{d}m(z),$$

where $f_1(w) = F(\alpha w, w)$, $f_2(z) = F(z, \beta z)$.

We may also consider the Dixmier trace of the related operators. In the case of $p = \prod_{j=1}^{k} (z-\alpha_j w)^{n_j}$, $|\alpha_j| = 1$ (equivalently, when p has the property (5.1)), then $[S_z^*, S_z]$ is of weak trace class from Theorem 5.2(ii). However, using the computations in the proof of Theorem 5.2, we find that $\mathrm{Tr}_\omega[S_z^*, S_z] = 0$, giving a trivial quantity.

REMARK 5.12. We note that an algebraic variety Z with the property (5.1) is called a distinguished variety and it has been studied by Agler–McCarthy [1]. We may thus ask the following question: Is a (non-homogeneous) module $M = [p]$ with property (5.1) always in the weak trace class? We consider an example of quasi-homogeneous module where the answer is indeed positive.

EXAMPLE 5.13. Let $k, l > 1$ be two co-prime positive integers. We consider the quotient module $[z^k - w^l]^\perp$ of the Hardy space $H^2(D^2)$. The rotation group acts unitarily on H^2 and on the quotient by $f(z,w) \to f(e^{il\theta}z, e^{ik\theta}w)$. Denote $K(z,w;\xi,\eta) = (1-z\overline{\xi})^{-1}(1-w\overline{\eta})^{-1}$ the reproducing kernel of H^2. We observe first that the restrictions $K(z,w;\lambda^l,\lambda^k)$ of K on the zero set of $\xi^k - \eta^l$ are in the quotient $[z^k - w^l]^\perp$ and generate a dense subset. Indeed suppose f in the quotient is orthogonal to all $K(z,w;\lambda^l,\lambda^k)$. Write f as an orthogonal sum $\sum_n f_n$, where f_n is the quasi-homogeneous component of f defined by the circle group action. Clearly f_n are polynomials in the quotient and orthogonal to all $K(z,w;\lambda^l,\lambda^k)$. Thus $f_n(\lambda^l,\lambda^k) = 0$, namely f_n is vanishing on the zero set of $z^k - w^l$. But the ideal $(z^k - w^l)$ is prime so that $f_n(z,w)$ is in the ideal, thus is zero. The reproducing kernel $K(\cdot,\cdot;\lambda^l,\lambda^k)$ on the zero set has an expansion

$$K(z^l,z^k;\lambda^l,\lambda^k) = \sum_{s=0}^{\infty} z^s \overline{\lambda}^s N_s$$

where $N_s = \#\{(m,n); m,n \geqslant 0, s = ml + nk\}$. We thus define a Hilbert space $H_{k,l}(D)$ of holomorphic functions, a posterior, on D, such that

$$\|z^s\|^2 = \frac{1}{N_s}$$

for those s with $N_s \neq 0$. The restriction operator $f(z,w) \mapsto f(z^l,z^k)$ on H^2 induces then a unitary operator R from the quotient to $H_{k,l}(D)$, so that S_z and S_w on the quotient are unitarily equivalent to M_{z^l} and M_{z^k} on $H_{k,l}(D)$. Thus $T = M_{z^k}$ is a shift operator

$$T\Big(\frac{z^s}{\sqrt{N_s}}\Big) = \sqrt{\frac{N_{s+k}}{N_s}}\Big(\frac{z^{s+k}}{\sqrt{N_{s+k}}}\Big),$$

and $[T^*,T]$ is a diagonal operator with diagonal entries

$$\frac{N_s}{N_{s+k}} - \frac{N_{s-k}}{N_s}.$$

As N_s is approximately linear in s we have $[T^*,T]$ is of weak trace class. Choosing $s = klj$, we have $N_s = j+1$ and the above is $1 - \frac{j-1}{j+1} = \frac{2}{j+1}$, and $[T^*,T]$ is not of trace class.

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