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COMPARISON OF MATRIX NORMS ON BIPARTITE SPACES

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ABSTRACT. Two non-commutative versions of the classical $L^q(L^p)$ norm on the product matrix algebras $\mathcal{M}_n \otimes \mathcal{M}_m$ are compared. The first norm was defined recently by Carlen and Lieb, as a byproduct of their analysis of certain convex functions on matrix spaces. The second norm was defined by Pisier and others using results from the theory of operator spaces. It is shown that the second norm is upper bounded by a constant multiple of the first for all $1 \leqslant p \leqslant 2$, $q \geqslant 1$. In one case ($2 = p < q$) it is also shown that there is no such lower bound, and hence that the norms are inequivalent. It is conjectured that the norms are inequivalent in all cases.

KEYWORDS: *Schatten norm, operator space, monotonicity.*

MSC (2000): 47A30, 47L25, 15A60.

1. INTRODUCTION

Let $\mathcal{M}_n$ denote the algebra of $n \times n$ complex-valued matrices. The Schatten norm on $\mathcal{M}_n$ provides a non-commutative version of the classical L^p norm. It is defined for $p \geqslant 1$ by

$$\|A\|_p = (\mathrm{Tr}|A|^p)^{1/p}.$$

Many of the standard properties of the classical norm extend to the Schatten norm, including monotonicity, convexity, Hölder's inequality and duality.

In this paper we compare two non-commutative versions of the classical $L^q(L^p)$ norm. The first version was introduced by Carlen and Lieb in their recent paper [1], where it arose out of ideas connected with the central theme of strong subadditivity of entropy. The second version arises from the work of Pisier and others on operator spaces [7], [8]. Norms of this second type were used in the paper [2] to prove results about completely positive maps on matrix algebras. Thus both norms are connected to important ideas in quantum information theory, and this motivates our study of the similarities and differences between them.

Recall that in the classical (commutative) case a function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, may also be regarded as a map $f : \mathbb{R} \to L^p(\mathbb{R})$, $x \mapsto f(x, \cdot) \equiv f_x(\cdot)$. The L^q norm of this map is

$$\Big(\int \|f_x\|_p^q \mathrm{d}x \Big)^{1/q} = \Big(\int \Big(\int |f(x,y)|^p \mathrm{d}y \Big)^{q/p} \mathrm{d}x \Big)^{1/q} \tag{1.1}$$

and this defines the $L^q(L^p)$ norm of f.

The right side of (1.1) suggests a possible non-commutative version of this norm for the bipartite space $\mathcal{M}_{nm} = \mathcal{M}_n \otimes \mathcal{M}_m$, namely $(\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q}$, where Tr_1 is the partial trace over the first or "outside" space $\mathcal{M}_n$, and Tr_2 is the partial trace over the second or "inside" space $\mathcal{M}_m$. While this specific expression does not lead to a norm, Carlen and Lieb have recently constructed a norm based on this idea. For positive semidefinite Y they define

$$\Psi_{p,q}(Y) = (\mathrm{Tr}_1(\mathrm{Tr}_2 Y^p)^{q/p})^{1/q}. \tag{1.2}$$

In Theorem 1.1 in the paper [1] it is proved that for all $1 \leqslant p \leqslant 2$ and $q \geqslant 1$ the function $\Psi_{p,q}$ is convex on the set of positive semidefinite matrices in $\mathcal{M}_n \otimes \mathcal{M}_m$.

Building on the convexity properties of (1.2), Carlen and Lieb defined a new norm on $\mathcal{M}_n \otimes \mathcal{M}_m$ in the following way. First define for Hermitian matrices X the quantity

$$|||X|||_{p,q} = \inf_{A,B} \{\Psi_{p,q}(A) + \Psi_{p,q}(B) : X + A = B, A \geqslant 0, B \geqslant 0\}.$$

Then the Carlen–Lieb norm is defined for $1 \leqslant p \leqslant 2$, $q \geqslant 1$ as

$$\|Y\|_{\mathrm{CL}} = \frac{1}{2} \Big|\Big|\Big| \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix} \Big|\Big|\Big|_{p,q}. \tag{1.3}$$

On the right side of (1.3) the function $\Psi_{p,q}$ is applied to a matrix in $\mathcal{M}_{2nm}$. This space must be split as a bipartite space in order to apply the definition. Carlen and Lieb choose the splitting $\mathcal{M}_{2nm} = \mathcal{M}_n \otimes (\mathcal{M}_m \otimes \mathcal{M}_2)$. In other words, in the definition (1.2) the inside space $\mathcal{M}_m$ is replaced by $\mathcal{M}_m \otimes \mathcal{M}_2$, and the outside space is still $\mathcal{M}_n$.

A different approach to the question of defining norms for these bipartite matrix spaces arises from the work of Effros and Ruan [3], [4], Pisier [7], [8] and Junge [5] on operator spaces. Several alternative formulations of such norms were used in the paper [2] to analyze CB norms of completely positive maps. In this paper we will present and analyze these Pisier-type norms using matrix analytic methods, without relying on results from the general theory of operator spaces. Define the value r by

$$\frac{1}{r} = \max\Big\{\frac{1}{p}, \frac{1}{q}\Big\} - \min\Big\{\frac{1}{p}, \frac{1}{q}\Big\}. \tag{1.4}$$

The expression for the norm differs for the cases $p < q$ and $p > q$ (the subscript "NC" stands for "non-commutative"). The first expression (1.5) was used in [2], while the second expression (1.6) is a modified version of one used in [2] (the expression used in [2] had only one term inside the infimum).

Case $1 \leqslant p \leqslant q \leqslant \infty$.

$$\|Y\|_{\mathrm{NC}} = \sup_{A,B \in \mathcal{M}_n} \Big\{ \frac{\|(A \otimes I_m) Y (B \otimes I_m)\|_p}{\|A\|_{2r} \|B\|_{2r}} \Big\}. \tag{1.5}$$

Case $1 \leqslant q \leqslant p \leqslant \infty$.

$$\|Y\|_{\mathrm{NC}} = \inf_{\substack{A_i, B_i \in \mathcal{M}_n \\ Z_i \in \mathcal{M}_{nm}}} \Big\{ \sum_i \|A_i\|_{2r} \|B_i\|_{2r} \|Z_i\|_p : Y = \sum_i (A_i \otimes I_m) Z_i (B_i \otimes I_m) \Big\}. \tag{1.6}$$

REMARK 1.1. Without loss of generality the sup on the right side of (1.5) may be restricted to the set of positive semidefinite matrices $A, B \geqslant 0$. This may be seen by writing $A = U(A^*A)^{1/2}$ and $B = (BB^*)^{1/2}V$ where U, V are unitaries, and observing that $\|UC\|_t = \|CU\|_t = \|C\|_t$ for all t, C and unitary U.

REMARK 1.2. For positive semidefinite matrices $Y \geqslant 0$,

$$\begin{aligned} \|(A \otimes I_m) Y (B \otimes I_m)\|_p &= \|(A \otimes I_m) Y^{1/2} Y^{1/2} (B \otimes I_m)\|_p \\ &\leqslant \|(A \otimes I_m) Y (A^* \otimes I_m)\|_p^{1/2} \|(B^* \otimes I_m) Y (B \otimes I_m)\|_p^{1/2}. \end{aligned}$$

Thus the supremum on the right side of (1.5) may be restricted to $A = B \geqslant 0$. Furthermore, letting

$$C = A^{2r} (\|A\|_{2r})^{-2r} \tag{1.7}$$

it follows that $\mathrm{Tr} C = 1$, and therefore for $p \leqslant q$

$$\begin{aligned} \|Y\|_{\mathrm{NC}} &= \sup_{A \geqslant 0 \in \mathcal{M}_n} \Big\{ \frac{\|(A \otimes I_m) Y (A \otimes I_m)\|_p}{\|A\|_{2r}^2} \Big\} \\ &= \sup_{C \geqslant 0, \mathrm{Tr} C = 1} \{ \|(C^{1/2r} \otimes I_m) Y (C^{1/2r} \otimes I_m)\|_p \}. \end{aligned}$$

REMARK 1.3. In some cases we will need to refer to the values p, q in the norm; we do this by writing $\|\cdot\|_{\mathrm{NC}:p,q}$.

Our first result establishes basic properties of $\|Y\|_{\mathrm{NC}}$. These properties help to motivate the definitions (1.5) and (1.6).

LEMMA 1.4. *Assume* $p, q \geqslant 1$.

(i) [Triangle inequality] *For any* $Y, W \in \mathcal{M}_{nm}$

$$\|Y + W\|_{\mathrm{NC}} \leqslant \|Y\|_{\mathrm{NC}} + \|W\|_{\mathrm{NC}}.$$

(ii) [Hölder's inequality] *Define the usual conjugate values for* p, q*:*

$$\frac{1}{p'} = 1 - \frac{1}{p}, \quad \frac{1}{q'} = 1 - \frac{1}{q}.$$

Then for all $Y, W \in \mathcal{M}_{nm}$*,*

$$|\mathrm{Tr}(YW)| \leqslant \|Y\|_{\mathrm{NC}:p,q} \|W\|_{\mathrm{NC}:p',q'}.$$

(iii) [Duality] *For all* $Y \in \mathcal{M}_{nm}$

$$\|Y\|_{\mathrm{NC}:p,q} = \sup_{W \in \mathcal{M}_{nm}} \{|\mathrm{Tr}(YW)| : \|W\|_{\mathrm{NC}:p',q'} \leqslant 1\}.$$

(iv) [Product form for product matrices] *For any* $Y_1 \in \mathcal{M}_n$, $Y_2 \in \mathcal{M}_m$

$$\|Y_1 \otimes Y_2\|_{\mathrm{NC}} = \|Y_1\|_q \|Y_2\|_p.$$

(v) [Reduction to Schatten norm at $p = q$] *For any* $Y \in \mathcal{M}_{nm}$, *and for* $p = q$,

$$\|Y\|_{\mathrm{NC}} = \|Y\|_p.$$

(vi) [Reduction to classical norm on diagonal matrices] *Choose orthonormal bases* $\{e_i\}$ *in* $\mathbb{C}^n$ *and* $\{f_j\}$ *in* $\mathbb{C}^m$. *Then for any matrix* $Y \in \mathcal{M}_{nm}$ *which is diagonal in the basis* $\{e_i \otimes f_j\}$,

$$\|Y\|_{\mathrm{NC}} = (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q}.$$

In order for this paper to be self-contained, we include the proof of Lemma 1.4 in the Appendix.

The main purpose of this paper is to compare the norms $\|\cdot\|_{\mathrm{CL}}$ and $\|\cdot\|_{\mathrm{NC}}$, in particular to investigate whether the norms are equivalent. The next result provides a bound for $\|Y\|_{\mathrm{NC}}$ in terms of $\|Y\|_{\mathrm{CL}}$.

THEOREM 1.5. *Assume* $1 \leqslant p \leqslant 2$ *and* $q \geqslant 1$.

(i) *Let* $Y \geqslant 0$ *be positive semidefinite, then*

$$\|Y\|_{\mathrm{NC}} \leqslant \Psi_{p,q}(Y). \tag{1.8}$$

(ii) *For all matrices* Y *in* $\mathcal{M}_n \otimes \mathcal{M}_m$,

$$\|Y\|_{\mathrm{NC}} \leqslant 2^{3-1/p} \|Y\|_{\mathrm{CL}}. \tag{1.9}$$

REMARK 1.6. The bound (1.8) is sharp, since both sides agree on product matrices $Y = Y_1 \otimes Y_2$.

Recall that a function $f : \mathcal{M}_n \to \mathbb{R}$, $A \mapsto f(A)$ is monotone if $0 \leqslant A \leqslant B$ implies $f(A) \leqslant f(B)$. In the paper [1], Remark 1.5, the authors point out that the function $A \mapsto \Psi_{p,q}(A)$ is not monotone, but then state that the CL-norm is monotone. In fact as the following lemma shows the CL-norm is non-monotone in some cases.

LEMMA 1.7. *For all* $1 \leqslant p \leqslant 2$ *and* $p \leqslant q$ *the function* $A \mapsto \|A\|_{\mathrm{CL}}$ *is not monotone.*

REMARK 1.8. In the case $p \leqslant q$ it is clear from (1.5) that the NC-norm is monotone. Thus Lemma 1.7 implies that the CL- and NC-norms are different in this case. The following lemma addresses the question of whether the norms are equivalent.

THEOREM 1.9. *Assume* $p < q$.

(i) *Let* $1 \leqslant p \leqslant 2$. *For all* $k \geqslant 1$ *there exist integers* $\{n_k, m_k\}$ *and positive semidefinite matrices* $0 \leqslant Y^{(k)} \in \mathcal{M}_{n_k} \otimes \mathcal{M}_{m_k}$, *such that*

$$\frac{\Psi_{p,q}(Y^{(k)})}{\|Y^{(k)}\|_{\mathrm{NC}}} \to \infty \quad \text{as } k \to \infty. \tag{1.10}$$

(ii) *Let* $p = 2$. *Then for all* $k \geqslant 1$ *there exist integers* $\{n_k, m_k\}$ *and matrices* $Y^{(k)} \in \mathcal{M}_{n_k} \otimes \mathcal{M}_{m_k}$, *such that*

$$\frac{\|Y^{(k)}\|_{\mathrm{CL}}}{\|Y^{(k)}\|_{\mathrm{NC}}} \to \infty \quad \text{as } k \to \infty. \tag{1.11}$$

Theorem 1.9(ii) implies that the norms $\|Y\|_{\mathrm{NC}}$ and $\|Y\|_{\mathrm{CL}}$ are not equivalent when $p = 2$ and $q > 2$, in the sense that there do not exist non-zero finite constants c_1 and c_2 such that $\|Y\|_{\mathrm{NC}} \leqslant c_1\|Y\|_{\mathrm{CL}} \leqslant c_2\|Y\|_{\mathrm{NC}}$ for all matrices Y. We conjecture that the norms are not equivalent for all $1 < p \leqslant 2$ and all $q \geqslant 1$.

The paper is organized as follows. In Section 2 we derive an alternative expression for the Carlen–Lieb norm for the case where the matrix is Hermitian. In Section 3 we use this expression to first prove Theorem 1.5 for positive semidefinite matrices, and then extend the result to general matrices. The main technical tool in the proof is an application of the Lieb–Thirring inequality. Section 4 presents a construction of the counterexamples which prove Theorem 1.9, and the Appendix contains the proofs of Lemma 1.4 and Lemma 1.7.

2. REPRESENTATION FOR CL-NORM OF HERMITIAN MATRIX

In this section we derive a simplified representation for the Carlen–Lieb norm in the case where the matrix is Hermitian. We assume throughout this section that $1 \leqslant p \leqslant 2$ and $q \geqslant 1$.

LEMMA 2.1. *For a Hermitian matrix* $Y = Y^* \in \mathcal{M}_{mn}$,

$$\|Y\|_{\mathrm{CL}} = \inf_{A \geqslant 0, Y+A \geqslant 0} \Psi_{p,q}\begin{pmatrix} Y+A & 0 \\ 0 & A \end{pmatrix}. \tag{2.1}$$

Proof. Recall the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix} = Y \otimes \sigma_x.$$

Define

$$\mathcal{G}(Y) = \{A \in \mathcal{M}_{mn} \otimes \mathbb{C}^2 : A \geqslant 0, Y \otimes \sigma_x + A \geqslant 0\}.$$

Then

$$\|Y\|_{\mathrm{CL}} = \frac{1}{2} \inf_{A \in \mathcal{G}(Y)} (\Psi_{p,q}(Y \otimes \sigma_x + A) + \Psi_{p,q}(A)). \tag{2.2}$$

In general $A \in \mathcal{G}(Y)$ has the form $A = A_1 \otimes I_2 + A_2 \otimes \sigma_x + A_3 \otimes \sigma_y + A_4 \otimes \sigma_z$, where σ_y, σ_z are the other Pauli matrices. The function $\Psi_{p,q}$ is invariant under unitary transformations on each of the spaces $\mathcal{M}_m$ and $\mathcal{M}_n$. Hence $\Psi_{p,q}(A)$ and $\Psi_{p,q}(Y \otimes \sigma_x + A)$ are unchanged if A is replaced by $A' = (I \otimes \sigma_x)A(I \otimes \sigma_x) = A_1 \otimes I_2 + A_2 \otimes \sigma_x - A_3 \otimes \sigma_y - A_4 \otimes \sigma_z$. The function $\Psi_{p,q}$ is also convex, and hence the expression $\Psi_{p,q}(Y \otimes \sigma_x + A) + \Psi_{p,q}(A)$ on the right side of (2.2) can only be lowered by replacing A by $(A + A')/2 = A_1 \otimes I_2 + A_2 \otimes \sigma_x$. Therefore the infimum on the right side of (2.2) may be restricted to matrices of the form $A = A_1 \otimes I + A_2 \otimes \sigma_x$, where A_1, A_2 satisfy

$$A_1 \geqslant |A_2|, \quad A_1 \geqslant |Y + A_2|.$$

The matrix A is unitarily equivalent to $A_1 \otimes I_2 + A_2 \otimes \sigma_z$, and similarly $Y \otimes \sigma_x + A$ is unitarily equivalent to $A_1 \otimes I_2 + (Y + A_2) \otimes \sigma_z$. Let $C = A_1 + A_2 \geqslant 0$ and $D = A_1 - A_2 \geqslant 0$. Then (2.2) can be written as

$$\|Y\|_{\mathrm{CL}} = \frac{1}{2} \inf_{\{C,D,C+Y,D-Y\geqslant 0\}} \left(\Psi \begin{pmatrix} C+Y & 0 \\ 0 & D-Y \end{pmatrix} + \Psi \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right).$$

Alternatively, since $D = Y + E$ with $E \geqslant 0$ this can be written as

$$\|Y\|_{\mathrm{CL}} = \frac{1}{2} \inf_{\{C,E,C+Y,E+Y\geqslant 0\}} \left(\Psi \begin{pmatrix} C+Y & 0 \\ 0 & E \end{pmatrix} + \Psi \begin{pmatrix} C & 0 \\ 0 & E+Y \end{pmatrix} \right). \tag{2.3}$$

Note that

$$\Psi \begin{pmatrix} C & 0 \\ 0 & E+Y \end{pmatrix} = \Psi \begin{pmatrix} E+Y & 0 \\ 0 & C \end{pmatrix}$$

and so (2.3) may be written as

$$\|Y\|_{\mathrm{CL}} = \frac{1}{2} \inf_{\{C,E\geqslant 0,C+Y\geqslant 0,E+Y\geqslant 0\}} \left(\Psi \begin{pmatrix} C+Y & 0 \\ 0 & E \end{pmatrix} + \Psi \begin{pmatrix} E+Y & 0 \\ 0 & C \end{pmatrix} \right). \tag{2.4}$$

Convexity of $\Psi_{p,q}$ implies that

$$\frac{1}{2} \left(\Psi \begin{pmatrix} C+Y & 0 \\ 0 & E \end{pmatrix} + \Psi \begin{pmatrix} E+Y & 0 \\ 0 & C \end{pmatrix} \right) \geqslant \Psi \begin{pmatrix} F+Y & 0 \\ 0 & F \end{pmatrix}$$

where $F = (C + E)/2$. Hence the infimum on the right side of (2.4) may be restricted to $C = E$ and this gives (2.1). ∎

3. PROOF OF THEOREM 1.5

3.1. THE BOUND FOR $Y \geqslant 0$.

Case $p < q$. Note that

$$\Psi_{p,q}(Y)^p = (\mathrm{Tr}_1 (\mathrm{Tr}_2 Y^p)^{q/p})^{p/q}. \tag{3.1}$$

Define t by

$$\frac{1}{t} + \frac{p}{q} = 1$$

then by duality

$$\Psi_{p,q}(Y)^p = \sup_B \mathrm{Tr}_1[B^{1/t}\mathrm{Tr}_2(Y^p)] \tag{3.2}$$

where the supremum runs over positive matrices $B \geqslant 0$ satisfying $\mathrm{Tr}B = 1$. Let $r = pt$ then

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$$

and (3.2) can be written as

$$\Psi_{p,q}(Y)^p = \sup_B \mathrm{Tr}_1[B^{p/r}\mathrm{Tr}_2(Y^p)] = \sup_B \mathrm{Tr}[(B^{p/r} \otimes I_m)Y^p]. \tag{3.3}$$

Turning now to the NC norm, by Remark 1.2 this may be rewritten as a supremum over positive matrices $B \geqslant 0$ with $\mathrm{Tr}B = 1$:

$$\|Y\|_{\mathrm{NC}} = \sup_{B \geqslant 0, \mathrm{Tr}B=1} \|(B^{1/2r} \otimes I_m)Y(B^{1/2r} \otimes I_m)\|_p.$$

Hence

$$\|Y\|_{\mathrm{NC}}^p = \sup_{B \geqslant 0, \mathrm{Tr}B=1} \mathrm{Tr}[(B^{1/2r} \otimes I_m)Y(B^{1/2r} \otimes I_m)]^p.$$

The Lieb–Thirring inequality [6] implies that for $Y, B \geqslant 0$ and $p \geqslant 1$,

$$\mathrm{Tr}[(B^{1/2r} \otimes I_m)Y(B^{1/2r} \otimes I_m)]^p \leqslant \mathrm{Tr}[(B^{p/2r} \otimes I_m)Y^p(B^{p/2r} \otimes I_m)].$$

Comparing with (3.3) yields the inequality.

Case $p > q$. Given $0 < t < 1$ define the conjugate value t' by

$$\frac{1}{t'} = \frac{1}{t} - 1.$$

Then for any non-negative sequence $a_1, \ldots, a_n$ it is easy to check that

$$\Big(\sum_{i=1}^n a_i^t\Big)^{1/t} = \inf\Big\{\sum_{i=1}^n a_i b_i : b_i \geqslant 0, \sum_{i=1}^n b_i^{-t'} = 1\Big\}. \tag{3.4}$$

Given a positive semidefinite matrix $A \geqslant 0$, $A \in \mathcal{M}_n$, define its positive commutant

$$\mathrm{Comm}_+[A] = \{B \in \mathcal{M}_n : B \geqslant 0, AB = BA\}.$$

Then (3.4) implies that for $A \geqslant 0$

$$\begin{aligned}\|A\|_t &= \inf\{\mathrm{Tr}(AB) : \mathrm{Tr}B^{-t'} = 1, B \in \mathrm{Comm}_+[A]\} \\ &= \inf\{\mathrm{Tr}(AC^{-1/t'}) : \mathrm{Tr}C = 1, C \in \mathrm{Comm}_+[A]\}.\end{aligned}$$

Applying this to (1.2) with $t = q/p$ and $A = \mathrm{Tr}_2(Y^p)$ gives

$$\begin{aligned}\Psi_{p,q}(Y)^p &= \inf_{\mathrm{Tr}C=1, C\in\mathrm{Comm}_+[A]} \mathrm{Tr}_1[C^{-p/r}\mathrm{Tr}_2(Y^p)] \\ &= \inf_{\mathrm{Tr}C=1, C\in\mathrm{Comm}_+[A]} \mathrm{Tr}[(C^{-p/2r} \otimes I_m)Y^p(C^{-p/2r} \otimes I_m)] \\ &\geqslant \inf_{\mathrm{Tr}C=1, C\geqslant 0} \mathrm{Tr}[(C^{-p/2r} \otimes I_m)Y^p(C^{-p/2r} \otimes I_m)].\end{aligned} \tag{3.5}$$

We obtain an upper bound for $\|Y\|_{\mathrm{NC}}$ by restricting the infimum on the right side of (1.6) to a single term $Y = (A \otimes I_m) Z (B \otimes I_m)$ with $A = B > 0$, which leads to the bound

$$\|Y\|_{\mathrm{NC}}^p \leqslant \inf_{\mathrm{Tr}B=1, B \geqslant 0} \mathrm{Tr}[(B^{-1/2r} \otimes I_m) Y (B^{-1/2r} \otimes I_m)]^p. \tag{3.6}$$

Since $p > 1$ the Lieb–Thirring bound again implies that (3.5) upper bounds (3.6) and thus the result follows.

3.2. THE BOUND FOR GENERAL Y. *Case* $p < q$. First we establish the bound for Hermitian matrices. Suppose $Y = Y^*$ then

$$\|Y\|_{\mathrm{NC}} = \sup_{A,B \geqslant 0} \frac{\|(A \otimes I_m) Y (B \otimes I_m)\|_p}{\|A\|_{2r} \|B\|_{2r}}. \tag{3.7}$$

Write $Y = Z - W$ where both Z, W are positive, then

$$\begin{aligned}
\|(A \otimes I_m) Y (B \otimes I_m)\|_p &= \|(A \otimes I_m) Z (B \otimes I_m) - (A \otimes I_m) W (B \otimes I_m)\|_p \\
&\leqslant \|(A \otimes I_m) Z (B \otimes I_m)\|_p + \|(A \otimes I_m) W (B \otimes I_m)\|_p \\
&\leqslant \|(A \otimes I_m) Z^{1/2}\|_{2p} \|(B \otimes I_m) Z^{1/2}\|_{2p} \\
&\qquad + \|(A \otimes I_m) W^{1/2}\|_{2p} \|(B \otimes I_m) W^{1/2}\|_{2p} \\
&\leqslant (\|(A \otimes I_m) Z^{1/2}\|_{2p} + \|(A \otimes I_m) W^{1/2}\|_{2p}) \\
&\qquad \times (\|(B \otimes I_m) Z^{1/2}\|_{2p} + \|(B \otimes I_m) W^{1/2}\|_{2p}).
\end{aligned}$$

Taking the supremum over A, B gives the bound

$$\|Y\|_{\mathrm{NC}} \leqslant \sup_A (\|(A \otimes I_m) Z^{1/2}\|_{2p} + \|(A \otimes I_m) W^{1/2}\|_{2p})^2 (\|A\|_{2r})^{-2}.$$

As noted in Remark 1.1 we can assume that $A \geqslant 0$. Let C be defined as in (1.7), then

$$\begin{aligned}
\|Y\|_{\mathrm{NC}} &\leqslant \sup_C (\|(C^{1/2r} \otimes I_m) Z^{1/2}\|_{2p} + \|(C^{1/2r} \otimes I_m) W^{1/2}\|_{2p})^2 \\
&\leqslant 2 \sup_C (\|(C^{1/2r} \otimes I_m) Z^{1/2}\|_{2p}^2 + \|(C^{1/2r} \otimes I_m) W^{1/2}\|_{2p}^2)
\end{aligned}$$

where the supremum is taken over positive matrices with $\mathrm{Tr}C = 1$. Note that

$$\|(C^{1/2r} \otimes I_m) Z^{1/2}\|_{2p}^2 = \|(C^{1/2r} \otimes I_m) Z (C^{1/2r} \otimes I_m)\|_p$$

and further that for all $x, y \geqslant 0$ we have the inequality

$$x^{1/p} + y^{1/p} \leqslant 2^{1-1/p} (x + y)^{1/p}.$$

Hence

$$\|Y\|_{\mathrm{NC}}^p \leqslant 2^{2p-1} \sup_C (\mathrm{Tr}[(C^{1/2r} \otimes I_m) Z (C^{1/2r} \otimes I_m)]^p + \mathrm{Tr}[(C^{1/2r} \otimes I_m) W (C^{1/2r} \otimes I_m)]^p)$$

$$= 2^{2p-1} \sup_C \mathrm{Tr}\Big[(C^{1/2r} \otimes I_{2m}) \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} (C^{1/2r} \otimes I_{2m})\Big]^p$$

$$= 2^{2p-1} \Big\| \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \Big\|_{\mathrm{NC}}^p \leqslant 2^{2p-1} \Psi_{p,q} \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}^p \tag{3.8}$$

where the norm on the right side is computed for the decomposition $\mathcal{M}_{2nm} = \mathcal{M}_n \otimes \mathcal{M}_{2m}$, and where we used Theorem 1.5(i) at the last step. Since this holds for every pair of positive matrices Z, W satisfying $Y = Z - W$ we have by Lemma 2.1

$$\|Y\|_{\mathrm{NC}} \leqslant 2^{2-1/p} \inf_{W \geqslant 0, Y+W \geqslant 0} \Psi_{p,q} \begin{pmatrix} Y+W & 0 \\ 0 & W \end{pmatrix} = 2^{2-1/p} \|Y\|_{\mathrm{CL}}. \tag{3.9}$$

This establishes the bound for Hermitian matrices. Now consider a general matrix Y, and write $Y = Y_1 + \mathrm{i}Y_2$ where Y_1 and Y_2 are Hermitian. Then by the above bound,

$$\|Y\|_{\mathrm{NC}} \leqslant \|Y_1\|_{\mathrm{NC}} + \|Y_2\|_{\mathrm{NC}} \leqslant 2^{2-1/p} \|Y_1\|_{\mathrm{CL}} + 2^{2-1/p} \|Y_2\|_{\mathrm{CL}}. \tag{3.10}$$

Consider now the definition of $\|Y_1 + \mathrm{i}Y_2\|_{\mathrm{CL}}$. There are matrices Z_0, Z_1, Z_2, Z_3 such that

$$\|Y_1 + \mathrm{i}Y_2\|_{\mathrm{CL}} = \frac{1}{2}(\Psi(W) + \Psi(V))$$

where

$$W = Z_0 \otimes I_2 + Z_3 \otimes \sigma_z + (Z_1 + Y_1) \otimes \sigma_x - (Z_2 + Y_2) \otimes \sigma_y,$$
$$V = Z_0 \otimes I_2 + Z_3 \otimes \sigma_z + Z_1 \otimes \sigma_x - Z_2 \otimes \sigma_y.$$

Define

$$\widetilde{W} = Z_0 \otimes I_2 - Z_3 \otimes \sigma_z + (Z_1 + Y_1) \otimes \sigma_x + (Z_2 + Y_2) \otimes \sigma_y,$$
$$\widetilde{V} = Z_0 \otimes I_2 - Z_3 \otimes \sigma_z + Z_1 \otimes \sigma_x + Z_2 \otimes \sigma_y.$$

Since W, V are positive, and since $\widetilde{W}$ and $\widetilde{V}$ are obtained by conjugation with the unitary $I_{mn} \otimes \sigma_x$, it follows that $\widetilde{W}$ and $\widetilde{V}$ are also positive. Then invariance of Ψ under local unitaries implies that

$$\Psi(W) = \Psi(\widetilde{W}), \quad \Psi(V) = \Psi(\widetilde{V}).$$

By convexity of Ψ it follows that

$$\|Y_1 + \mathrm{i}Y_2\|_{\mathrm{CL}} \geqslant \frac{1}{2}\Big(\Psi\Big(\frac{W + \widetilde{W}}{2}\Big) + \Psi\Big(\frac{V + \widetilde{V}}{2}\Big)\Big) \geqslant \|Y_1\|_{\mathrm{CL}}.$$

A similar argument shows that $\|Y_1 + \mathrm{i}Y_2\|_{\mathrm{CL}} \geqslant \|Y_2\|_{\mathrm{CL}}$, hence from (3.10) it follows that

(3.11)
$$\|Y\|_{\mathrm{NC}} \leqslant 2^{2-1/p}(\|Y_1\|_{\mathrm{CL}} + \|Y_2\|_{\mathrm{CL}}) \leqslant 2^{3-1/p}\|Y_1 + \mathrm{i}Y_2\|_{\mathrm{CL}} = 2^{3-1/p}\|Y\|_{\mathrm{CL}}.$$

Case $p > q$. As in the previous case it is sufficient to consider a Hermitian matrix $Y = Z - W$ with $Z, W \geqslant 0$, and to show the analog of (3.9), that is

$$\|Y\|_{\mathrm{NC}} \leqslant 2^{2-1/p}\Psi_{p,q}\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}. \tag{3.12}$$

Once (3.12) is shown, the argument leading from (3.10) to (3.11) can be repeated and the result then follows. In order to show (3.12), we restrict the infimum on the right side of (1.6) to obtain

$$\|Y\|_{\mathrm{NC}} \leqslant \inf_{C>0,\mathrm{Tr}C=1} \|(C^{-1/2r} \otimes I_m)Y(C^{-1/2r} \otimes I_m)\|_p.$$

The steps leading from (3.7) to (3.8) can now be repeated, leading to the conclusion

$$\|Y\|_{\mathrm{NC}}^p \leqslant 2^{2p-1} \inf_{C>0,\mathrm{Tr}C=1} \mathrm{Tr}\Big[(C^{-1/2r} \otimes I_m)\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}(C^{-1/2r} \otimes I_m)\Big]^p. \tag{3.13}$$

Since $\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}$ is positive semidefinite, we may use the Lieb–Thirring bound, as we did for (3.6), to conclude that

$$\|Y\|_{\mathrm{NC}}^p \leqslant 2^{2p-1}\Psi_{p,q}\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}^p$$

and this completes the proof.

4. PROOF OF THEOREM 1.9

Next we demonstrate (1.10) with a family of examples. There are 2^n diagonal $n \times n$ matrices with ± 1 on the diagonal. Denote these unitary matrices as $\{U_a\}$, with $U_1 = I_n$. Then for any $n \times n$ matrix A,

$$\sum_{a=1}^{2^n} 2^{-n} U_a A U_a = A_{\mathrm{diag}} \tag{4.1}$$

where A_{diag} is the diagonal matrix obtained by replacing all off-diagonal entries of A with zero.

Let $|\psi\rangle \in \mathbb{C}^d$ be a unit vector. Define

$$Y_1 = |\psi\rangle\langle\psi|, \quad Y_a = U_a Y_1 U_a, \quad a = 1, \ldots, 2^n.$$

Note that Y_a is a pure state for all a and hence $Y_a^p = Y_a$ for all p. Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of Y_1, then define

$$D = \sum_{a=1}^{2^n} 2^{-n} Y_a = (Y_1)_{\text{diag}} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Let $m = 2^n$ and define the $mn \times mn$ block diagonal matrix

$$Y = \sum_{a=1}^{m} Y_k \otimes |a\rangle\langle a| = \begin{pmatrix} Y_1 & 0 & & \cdots \\ 0 & Y_2 & 0 & \cdots \\ \vdots & & \ddots & \\ 0 & & & Y_m \end{pmatrix}.$$

Then $Y \geqslant 0$ and $Y^p = Y$, hence for $1 \leqslant p \leqslant 2$ and $q \geqslant 1$

$$\Psi_{p,q}(Y)^p = (\mathrm{Tr}_1(\mathrm{Tr}_2 Y^p)^{q/p})^{p/q} = \Big(\mathrm{Tr}\Big(\sum_{a=1}^{m} Y_a\Big)^{q/p}\Big)^{p/q} = 2^n \|D\|_{q/p}. \tag{4.2}$$

Furthermore, for $p < q$

$$\begin{aligned} \|Y\|_{\text{NC}}^p &= \sup_{C \geqslant 0, \mathrm{Tr} C = 1} \|(C^{1/2r} \otimes I_m) Y (C^{1/2r} \otimes I_m)\|_p^p \\ &= \sup_{C \geqslant 0, \mathrm{Tr} C = 1} \sum_{a=1}^{m} \mathrm{Tr}(C^{1/2r} Y_a C^{1/2r})^p \end{aligned} \tag{4.3}$$

where $r^{-1} = p^{-1} - q^{-1}$. Note that for any $b \in \{1, \dots, 2^n\}$,

$$\{Y_a\}_{a=1}^{2^n} = \{U_b Y_a U_b\}_{a=1}^{2^n} \tag{4.4}$$

that is conjugation by U_b permutes the set of matrices Y_a. The property (4.4) implies that for all C and all $b \in \{1, \dots, 2^n\}$,

$$\begin{aligned} \sum_{a=1}^{m} \mathrm{Tr}(C^{1/2r} Y_a C^{1/2r})^p &= \sum_{a=1}^{m} \mathrm{Tr}(C^{1/2r} U_b Y_a U_b C^{1/2r})^p \\ &= \sum_{a=1}^{m} \mathrm{Tr}((U_b C U_b)^{1/2r} Y_a (U_b C U_b)^{1/2r})^p. \end{aligned} \tag{4.5}$$

Furthermore since $r^{-1} \leqslant p/r \leqslant 1$ the map

$$C \mapsto \mathrm{Tr}(C^{1/2r} Y C^{1/2r})^p = \mathrm{Tr}(Y^{1/2} C^{1/r} Y^{1/2})^p$$

is concave on the positive matrices [1]. Together with (4.5) and (4.1) this implies that for all positive C,

$$\sum_{a=1}^{m} \mathrm{Tr}(C^{1/2r} Y_a C^{1/2r})^p \leqslant \sum_{a=1}^{m} \mathrm{Tr}(C_{\text{diag}}^{1/2r} Y_a C_{\text{diag}}^{1/2r})^p.$$

Hence the supremum in (4.3) is achieved on diagonal matrices. Therefore

$$\|Y\|_{\mathrm{NC}}^p = \sup_{\{p_1,\dots,p_n \geqslant 0\}} \sum_{a=1}^m \Big(\sum_{j=1}^n p_j^{1/r} \lambda_j \Big)^p = 2^n \sup_{\{p_1,\dots,p_n \geqslant 0\}} \Big(\sum_{j=1}^d p_j^{1/r} \lambda_j \Big)^p \tag{4.6}$$

where the sup runs over positive vectors satisfying $\sum_j p_j = 1$. Using Hölder's inequality gives

$$\sum_{j=1}^n p_j^{1/r} \lambda_j \leqslant \Big(\sum_{j=1}^d \lambda_j^{r'} \Big)^{1/r'}$$

where r' is conjugate to r. Therefore

$$\|Y\|_{\mathrm{NC}}^p \leqslant 2^n \Big(\sum_{j=1}^n \lambda_j^{r'} \Big)^{p/r'} = 2^n \|D\|_{r'}^p. \tag{4.7}$$

Combining with (4.2) gives

$$\frac{\Psi_{p,q}(Y)^p}{\|Y\|_{\mathrm{NC}}^p} \geqslant \frac{\|D\|_{q/p}}{\|D\|_{r'}^p}. \tag{4.8}$$

Now consider the values

$$\lambda_j = \frac{c}{j}, \quad j = 1, \dots, n.$$

The normalization condition $\sum \lambda_j = 1$ implies that $c \leqslant 1/\ln n$. For $t > 1$ define

$$h(t) = \Big(\sum_{k=1}^\infty k^{-t} \Big)^{1/t}$$

then it follows that $c \leqslant \|D\|_t \leqslant c h(t)$. Since $p < q$ it follows that $r < \infty$ and $r' > 1$, and therefore

$$\frac{\|D\|_{q/p}}{\|D\|_{r'}^p} \geqslant c^{1-p} \frac{1}{h(r')^p} \geqslant (\ln n)^{p-1} \frac{1}{h(r')^p}.$$

Since $p > 1$, $(\ln n)^{p-1}$ diverges as $n \to \infty$. Therefore the left side of (4.8) is not uniformly bounded, and we have a sequence of positive semidefinite matrices $\{Y^{(k)}\}$ such that

$$\frac{\Psi_{p,q}(Y^{(k)})}{\|Y^{(k)}\|_{\mathrm{NC}}} \to \infty \tag{4.9}$$

as $k \to \infty$. This proves (1.10).

In order to prove (1.11) we use the result of Lemma 4.1 below, which shows that for $p = 2$ and $q \geqslant 2$ we can lower bound $\|Y\|_{\mathrm{CL}}$ by $2^{-1/2}\Psi_{p,q}(Y)$. Inserting this bound in (4.9) completes the proof of Theorem 1.9.

LEMMA 4.1. *For $2 = p \leqslant q$ and for all $Y \geqslant 0$,*

$$\|Y\|_{\mathrm{CL}} \geqslant \frac{1}{\sqrt{2}} \Psi_{p,q}(Y).$$

Proof. Recall that for a positive Hermitian matrix Y,

$$\begin{aligned}\|Y\|_{\mathrm{CL}} &= \inf_{A\geqslant 0} \Psi_{p,q}\begin{pmatrix} Y+A & 0 \\ 0 & A\end{pmatrix} \\ &= \inf_{A\geqslant 0}[\mathrm{Tr}_1(\mathrm{Tr}_2(Y+A)^p + \mathrm{Tr}_2 A^p)^{q/p}]^{1/q} \\ &= \inf_{A\geqslant 0} \|\mathrm{Tr}_2(Y+A)^p + \mathrm{Tr}_2 A^p\|_{q/p}^{1/p}.\end{aligned}$$

Note that for $p=2$,

$$(Y+A)^2 + A^2 = Y^2 + 2\Big(A+\frac{Y}{2}\Big)^2 - \frac{Y^2}{2} \geqslant \frac{Y^2}{2}.$$

The partial trace preserves positivity, and since $q \geqslant 2$ the $q/2$-norm is monotonic, hence, as claimed,

$$\|\mathrm{Tr}_2(Y+A)^2 + \mathrm{Tr}_2 A^2\|_{q/2} \geqslant \frac{1}{2}\|\mathrm{Tr}_2 Y^2\|_{q/2} = \frac{1}{2}\Psi_{2,q}(Y)^2. \quad ∎$$

APPENDIX A: PROOF OF LEMMA 1.4

Because the norm is given by the different expressions (1.5) and (1.6) depending on the relative sizes of p and q, the proofs are given separately for the two cases.

(i) *Case* 1. $p \leqslant q$. For any matrices $A, B \in \mathcal{M}_n$ and $Y, W \in \mathcal{M}_{nm}$,

$$\begin{aligned}\|(A\otimes I_m)(Y+W)(B\otimes I_m)\|_p &\leqslant \|(A\otimes I_m)Y(B\otimes I_m)\|_p + \|(A\otimes I_m)W(B\otimes I_m)\|_p \\ &\leqslant \|A\|_{2r}\|B\|_{2r}(\|Y\|_{\mathrm{NC}} + \|W\|_{\mathrm{NC}}).\end{aligned}$$

Dividing both sides by $\|A\|_{2r}\|B\|_{2r}$ and taking the sup over A, B gives the bound.

Case 2. $q \leqslant p$. Given $\varepsilon > 0$ there are matrices A_i, B_i, Z_i such that

$$Y = \sum_i (A_i \otimes I_m) Z_i (B_i \otimes I_m), \quad \sum_i \|A_i\|_{2r}\|B_i\|_{2r}\|Z_i\|_p < \|Y\|_{\mathrm{NC}} + \varepsilon.$$

Similarly there are matrices C_j, D_j, X_j such that

$$W = \sum_j (C_j \otimes I_m) X_j (D_j \otimes I_m), \quad \sum_j \|C_j\|_{2r}\|D_j\|_{2r}\|X_j\|_p < \|W\|_{\mathrm{NC}} + \varepsilon.$$

Therefore

$$\begin{aligned}\|Y+W\|_{\mathrm{NC}} &\leqslant \sum_i \|A_i\|_{2r}\|B_i\|_{2r}\|Z_i\|_p + \sum_j \|C_j\|_{2r}\|D_j\|_{2r}\|X_j\|_p \\ &< \|Y\|_{\mathrm{NC}} + \|W\|_{\mathrm{NC}} + 2\varepsilon.\end{aligned}$$

Since this holds for all $\varepsilon > 0$ the result follows.

(ii) *Case* 1. Without loss of generality assume that $p \leqslant q$, then it follows that $q' \leqslant p'$. Consider any decomposition of W:

$$W = \sum_i (A_i \otimes I_m) Z_i (B_i \otimes I_m). \tag{4.10}$$

Hence

$$\begin{aligned} |\mathrm{Tr}(YW)| &= \Big| \sum_i \mathrm{Tr}[Y(A_i \otimes I_m) Z_i (B_i \otimes I_m)] \Big| = \Big| \sum_i \mathrm{Tr}[Z_i (B_i \otimes I_m) Y (A_i \otimes I_m)] \Big| \\ &\leqslant \sum_i \|Z_i\|_{p'} \|(B_i \otimes I_m) Y (A_i \otimes I_m)\|_p \leqslant \sum_i \|Z_i\|_{p'} \|B_i\|_{2r} \|A_i\|_{2r} \|Y\|_{\mathrm{NC}:p,q}. \end{aligned}$$

Noting that $r^{-1} = p^{-1} - q^{-1} = (q')^{-1} - (p')^{-1}$ we may take the inf over A_i, B_i, Z_i satisfying (4.10) to conclude that

$$|\mathrm{Tr}(YW)| \leqslant \|Y\|_{\mathrm{NC}:p,q} \|W\|_{\mathrm{NC}:p',q'}.$$

(iii) *Case* 1. $p \leqslant q$. From the definition (1.5) we obtain

$$\begin{aligned} \|Y\|_{\mathrm{NC}:p,q} &= \sup_{A,B} \{ \|(A \otimes I_m) Y (B \otimes I_m)\|_p : \|A\|_{2r} \|B\|_{2r} \leqslant 1 \} \\ &= \sup_{A,B,Z} \{ |\mathrm{Tr}[(A \otimes I_m) Y (B \otimes I_m) Z]| : \|A\|_{2r} \|B\|_{2r} \leqslant 1, \|Z\|_{p'} \leqslant 1 \} \\ &\leqslant \sup_{A,B,Z} \{ |\mathrm{Tr}[Y (B \otimes I_m) Z (A \otimes I_m)]| : \|A\|_{2r} \|B\|_{2r} \|Z\|_{p'} \leqslant 1 \} \\ &= \sup_{W,A,B,Z} \{ |\mathrm{Tr}(YW)| : W = (B \otimes I_m) Z (A \otimes I_m), \|A\|_{2r} \|B\|_{2r} \|Z\|_{p'} \leqslant 1 \} \\ &\leqslant \sup_W \{ |\mathrm{Tr}(YW)| : \|W\|_{\mathrm{NC}:p',q'} \leqslant 1 \} \end{aligned}$$

where in the last inequality we used again $r^{-1} = (q')^{-1} - (p')^{-1}$. Using Hölder's inequality, the condition $\|W\|_{\mathrm{NC}:p',q'} \leqslant 1$ implies

$$|\mathrm{Tr}(YW)| \leqslant \|Y\|_{\mathrm{NC}:p,q} \|W\|_{\mathrm{NC}:p',q'} \leqslant \|Y\|_{\mathrm{NC}:p,q}.$$

Combining these inequalities we deduce that equality must hold, and hence the result follows.

Case 2. $q \leqslant p$. From Hölder's inequality we know that

$$\sup_W \{ |\mathrm{Tr}(YW)| : \|W\|_{\mathrm{NC}:p',q'} \leqslant 1 \} \leqslant \|Y\|_{\mathrm{NC}:p,q}$$

so it is sufficient to show that there is a matrix W such that

$$|\mathrm{Tr}(YW)| = \|Y\|_{\mathrm{NC}:p,q} \|W\|_{\mathrm{NC}:p',q'}. \tag{4.11}$$

Consider the space $X = \mathcal{M}_{nm}$ equipped with the norm (1.6). Every linear functional on X may be written as $f(\cdot) = \mathrm{Tr}(\cdot W)$ for some matrix W, therefore the norm of f is

$$\|f\| = \sup_Z \{ |\mathrm{Tr}(WZ)| : \|Z\|_{\mathrm{NC}:p,q} \leqslant 1 \} = \|W\|_{\mathrm{NC}:p',q'}$$

where we used the previous duality result. So it is sufficient to show that for every $Y \in X$ there is a linear functional f with $f(Y) = \|f\| \|Y\|_{\mathrm{NC}:p,q}$, as this will imply (4.11). But the existence of such a functional is a well-known corollary of the Hahn–Banach Theorem (see for example [9]).

(iv) *Case* 1. $p \leqslant q$. First note that for any matrices A, B

$$\|(A \otimes I_m)(Y_1 \otimes Y_2)(B \otimes I_m)\|_p = \|AY_1B\|_p \|Y_2\|_p.$$

Furthermore $p^{-1} = r^{-1} + q^{-1}$ hence by Hölder's inequality

$$\|AY_1B\|_p \|Y_2\|_p \leqslant \|A\|_{2r} \|Y_1\|_q \|B\|_{2r} \|Y_2\|_p.$$

Therefore

$$\frac{\|(A \otimes I_m)(Y_1 \otimes Y_2)(B \otimes I_m)\|_p}{\|A\|_{2r} \|B\|_{2r}} \leqslant \|Y_1\|_q \|Y_2\|_p \tag{4.12}$$

and taking the sup over A, B gives

$$\|Y_1 \otimes Y_2\|_{\mathrm{NC}} \leqslant \|Y_1\|_q \|Y_2\|_p.$$

It remains to show that equality can be achieved in (4.12) by a suitable choice of A, B. Without loss of generality we can assume that $Y_1 \geqslant 0$, in which case we define

$$A = B = Y_1^{(q-p)/2p}.$$

Then $AY_1B = Y_1^{q/p}$, and $\|A\|_{2r} = \|B\|_{2r} = \|Y_1\|_q^{q/2r}$, which yields

$$\frac{\|(A \otimes I_m)(Y_1 \otimes Y_2)(B \otimes I_m)\|_p}{\|A\|_{2r} \|B\|_{2r}} = \|Y_1^{q/p}\|_p \|Y_1\|_q^{-q/r} \|Y_2\|_p = \|Y_1\|_q \|Y_2\|_p.$$

Case 2. $q \leqslant p$. There are matrices W_1, W_2 satisfying

$$\|Y_1\|_q = \mathrm{Tr} Y_1 W_1, \quad \|W_1\|_{q'} = 1, \quad \|Y_2\|_p = \mathrm{Tr} Y_2 W_2, \quad \|W_2\|_{p'} = 1,$$

where q', p' are the conjugate values for q, p respectively. Note that $p' \leqslant q'$. Consider any set of matrices A_i, B_i, Z_i for which

$$Y_1 \otimes Y_2 = \sum_i (A_i \otimes I_m) Z_i (B_i \otimes I_m) \tag{4.13}$$

then it follows that

$$\begin{aligned} \|Y_1\|_q \|Y_2\|_p &= \mathrm{Tr}(Y_1 \otimes Y_2)(W_1 \otimes W_2) = \sum_i \mathrm{Tr}(A_i \otimes I_m) Z_i (B_i \otimes I_m)(W_1 \otimes W_2) \\ &= \sum_i \mathrm{Tr} Z_i ((B_i W_1 A_i) \otimes W_2) \leqslant \sum_i \|Z_i\|_p \|B_i W_1 A_i\|_{p'} \|W_2\|_{p'} \\ &\leqslant \sum_i \|Z_i\|_p \|B_i\|_{2r'} \|W_1\|_{q'} \|A_i\|_{2r'} \|W_2\|_{p'} \qquad (4.14) \end{aligned}$$

where in the last line the value r' is given by

$$\frac{1}{r'} = \frac{1}{p'} - \frac{1}{q'} = 1 - \frac{1}{p} - \left(1 - \frac{1}{q}\right) = \frac{1}{r}.$$

Hence $r' = r$ and therefore (4.14) gives

$$\|Y_1\|_q \|Y_2\|_p \leqslant \sum_i \|Z_i\|_p \|B_i\|_{2r} \|A_i\|_{2r}.$$

Since this holds for every decomposition (4.13) it follows that

$$\|Y_1\|_q \|Y_2\|_p \leqslant \|Y_1 \otimes Y_2\|_{\mathrm{NC}}.$$

To complete the proof, it is sufficient to find matrices A, B, Z such that $Y_1 = AZB$ and

$$\|Y_1\|_q = \|A\|_{2r} \|Z\|_p \|B\|_{2r}.$$

Without loss of generality we may again assume that $Y_1 \geqslant 0$, then taking

$$A = B = Y_1^{(p-q)/2p}, \quad Z = Y_1^{q/p}$$

it follows that $AZB = Y_1$ and $\|A\|_{2r}^2 = \|Y_1\|_q^{q/r}$, and hence

$$\|A\|_{2r} \|Z\|_p \|B\|_{2r} = \|Y_1\|_q^{q/r} \|Z\|_p = \|Y_1\|_q^{q/r} \|Y_1\|_q^{q/p} = \|Y_1\|_q.$$

(v) *Case* 1. $p \leqslant q$. We must show that the expression (1.5) reduces to $\|Y\|_p$ when $p = q$. In this case $r = \infty$, and so

$$\|Y\|_{\mathrm{NC}} = \sup_{A,B} \frac{\|(A \otimes I_m) Y (B \otimes I_m)\|_p}{\|A\|_\infty \|B\|_\infty}.$$

Note that $\|A \otimes I_m\|_\infty = \|A\|_\infty$ and similarly for B, therefore by Hölder's inequality

$$\|(A \otimes I_m) Y (B \otimes I_m)\|_p \leqslant \|A \otimes I_m\|_\infty \|B \otimes I_m\|_\infty \|Y\|_p = \|A\|_\infty \|B\|_\infty \|Y\|_p$$

which leads to

$$\|Y\|_{\mathrm{NC}} \leqslant \|Y\|_p. \tag{4.15}$$

Taking $A = B = I_n$ shows that equality is achieved in (4.15).

Case 2. $q \leqslant p$. Now we show that (1.6) also reduces to $\|Y\|_p$ when $p = q$. Consider any decomposition $Y = \sum_i (A_i \otimes I_m) Z_i (B_i \otimes I_m)$ then

$$\begin{aligned} \|Y\|_p &\leqslant \sum_i \|(A_i \otimes I_m) Z_i (B_i \otimes I_m)\|_p \leqslant \sum_i \|A_i \otimes I_m\|_\infty \|Z_i\|_p \|B_i \otimes I_m\|_\infty \\ &= \sum_i \|A_i\|_\infty \|Z_i\|_p \|B_i\|_\infty \end{aligned}$$

and hence

$$\|Y\|_p \leqslant \|Y\|_{\mathrm{NC}}.$$

Taking $A_1 = B_1 = I_n$ and $Z_1 = Y$ shows that equality is achieved, and hence the result follows.

(vi) *Case* 1. $p \leqslant q$. Let Y be diagonal in the product basis with entries $\{y_{ij}\}$. Let $A \in \mathcal{M}_n$ be the diagonal matrix with entries

$$a_i = w_i^{q/2pr}\Big(\sum_i w_i^{q/p}\Big)^{-1/2r} \tag{4.16}$$

where $w_i = \sum_j |y_{ij}|^p$. Then $\|A\|_{2r} = \Big(\sum_i |a_i|^{2r}\Big)^{1/2r} = 1$ and

$$\mathrm{Tr}|(A \otimes I_m)Y(A \otimes I_m)|^p = \sum_i |a_i|^{2p} w_i = \Big(\sum_i w_i^{q/p}\Big)^{p/q} = (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{p/q}.$$

Hence we deduce that

$$\|Y\|_{\mathrm{NC}} \geqslant (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q}. \tag{4.17}$$

Furthermore, write $Y = |Y|V$ where V is a diagonal unitary matrix, then for any A, B

$$\begin{aligned}\|(A\otimes I_m)Y(B\otimes I_m)\|_p &= \|(A \otimes I_m)|Y|^{1/2}|Y|^{1/2}V(B \otimes I_m)\|_p \\ &\leqslant \|(A\otimes I_m)|Y|(A^*\otimes I_m)\|_p^{1/2}\|(B^*\otimes I_m)V^*|Y|V(B\otimes I_m)\|_p^{1/2} \\ &= \|(A \otimes I_m)|Y|(A^* \otimes I_m)\|_p^{1/2}\|(B^* \otimes I_m)|Y|(B \otimes I_m)\|_p^{1/2}\end{aligned}$$

since V is diagonal and therefore commutes with $|Y|$. Therefore we deduce that

$$\begin{aligned}\|Y\|_{\mathrm{NC}} &\leqslant \sup_A \frac{\|(A \otimes I_m)|Y|(A^* \otimes I_m)\|_p}{\|A\|_{2r}^2} = \sup_{A\geqslant 0} \frac{\|(A \otimes I_m)|Y|(A \otimes I_m)\|_p}{\|A\|_{2r}^2} \\ &= \sup_\sigma (\mathrm{Tr}((\sigma^{1/2r} \otimes I_m)|Y|(\sigma^{1/2r} \otimes I_m))^p)^{1/p} \\ &= \sup_\sigma (\mathrm{Tr}(|Y|^{1/2}(\sigma^{1/r} \otimes I_m)|Y|^{1/2})^p)^{1/p} \end{aligned} \tag{4.18}$$

where the final sup runs over positive semidefinite matrices with trace one. Define

$$F(\sigma) = \mathrm{Tr}(|Y|^{1/2}(\sigma^{1/r} \otimes I_m)|Y|^{1/2})^p.$$

Since $r \geqslant p$ the results of Theorem 1.1 in [1] imply that F is concave. Furthermore there are 2^n diagonal matrices $\{U_j\}$ with ± 1 on the diagonals such that

$$(\sigma)_{\mathrm{diag}} = 2^{-n}\sum_j U_j \sigma U_j^*.$$

Since $|Y|$ is also diagonal this implies that

$$F((\sigma)_{\mathrm{diag}}) = F\Big(2^{-n}\sum_j U_j\sigma U_j^*\Big) \geqslant 2^{-n}\sum_j F(U_j\sigma U_j^*) = 2^{-n}\sum_j F(\sigma) = F(\sigma).$$

Hence the sup in (4.18) is achieved on the diagonal matrices. Therefore

$$\|Y\|_{\mathrm{NC}} \leqslant \sup_{c_i\geqslant 0}\Big(\sum_{i,j}|c_i y_{ij}|^p\Big)^{1/p} = \sup_{c_i\geqslant 0}\Big(\sum_i |c_i|^p w_i\Big)^{1/p}$$

where the $\{c_i\}$ satisfy $\sum_i |c_i|^r = 1$. Using Hölder's inequality with $s = r/p, s' = q/p$ gives

$$\sum_i |c_i|^p w_i \leqslant \Big(\sum_i |c_i|^r\Big)^{1/s}\Big(\sum_i w_i^{s'}\Big)^{1/s'} \leqslant \Big(\sum_i w_i^{q/p}\Big)^{p/q} = (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{p/q}$$

and therefore

$$\|Y\|_{\mathrm{NC}} \leqslant (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q}. \tag{4.19}$$

Combining (4.17) and (4.19) gives the result.

Case 2. $q \leqslant p$. By using the same diagonal matrix A defined for the $p \leqslant q$ case in (4.16), we can write $Y = (A \otimes I_m)Z(A \otimes I_m)$. Then we have $\|A\|_{2r} = 1$ and

$$\|Z\|_p^p = \sum_{i,j} |z_{ij}|^p = \sum_i \frac{w_i}{|a_i|^{2p}} = \Big(\sum_i w_i^{q/p}\Big)^{p/q} = (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{p/q}.$$

Therefore

$$\|Y\|_{\mathrm{NC}} \leqslant (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q}. \tag{4.20}$$

Furthermore, by classical duality there is a diagonal matrix W satisfying

$$(\mathrm{Tr}_1(\mathrm{Tr}_2|W|^{p'})^{q'/p'})^{1/q'} = 1, \quad (\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q} = |\mathrm{Tr}(YW)|.$$

Consider any decomposition $Y = \sum_i (A_i \otimes I_m)Z_i(B_i \otimes I_m)$, then

$$\begin{aligned}|\mathrm{Tr}(YW)| &= \Big|\sum_i \mathrm{Tr}[Z_i(B_i \otimes I_m)W(A_i \otimes I_m)]\Big| \leqslant \sum_i \|Z_i\|_p \|(B_i \otimes I_m)W(A_i \otimes I_m)\|_{p'} \\ &\leqslant \sum_i \|Z_i\|_p \|B_i\|_{2r'} \|A_i\|_{2r'} \|W\|_{\mathrm{NC}:p',q'} = \sum_i \|Z_i\|_p \|B_i\|_{2r'} \|A_i\|_{2r'}\end{aligned}$$

where in the last line we used the previous result that for diagonal matrices

$$\|W\|_{\mathrm{NC}:p',q'} = (\mathrm{Tr}_1(\mathrm{Tr}_2|W|^{p'})^{q'/p'})^{1/q'}$$

since $p' \leqslant q'$. Also $r' = r$ and so

$$|\mathrm{Tr}(YW)| \leqslant \sum_i \|Z_i\|_p \|B_i\|_{2r} \|A_i\|_{2r}.$$

Since this holds for every decomposition we deduce that

$$(\mathrm{Tr}_1(\mathrm{Tr}_2|Y|^p)^{q/p})^{1/q} \leqslant \|Y\|_{\mathrm{NC}}. \tag{4.21}$$

Together (4.20) and (4.21) imply equality.

APPENDIX B: PROOF OF LEMMA 1.7

Let $Y \geqslant 0$ be a positive semidefinite matrix. Recall Lemma 2.1. Setting $A = 0$ on the right side of (2.1) shows that

$$\|Y\|_{\mathrm{CL}} \leqslant \Psi_{p,q} \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} = \Psi_{p,q}(Y). \tag{4.22}$$

Furthermore, for all $A \geqslant 0$,

$$\Psi_{p,q} \begin{pmatrix} Y+A & 0 \\ 0 & A \end{pmatrix}^q = \mathrm{Tr}_1 (\mathrm{Tr}_2 (Y+A)^p + \mathrm{Tr}_2 A^p)^{q/p}. \tag{4.23}$$

The Schatten norm $A \mapsto \|A\|_t$ is monotone for all $t \geqslant 1$. Thus the condition $q \geqslant p$ implies from (4.23) that

$$\Psi_{p,q} \begin{pmatrix} Y+A & 0 \\ 0 & A \end{pmatrix}^q \geqslant \mathrm{Tr}_1 (\mathrm{Tr}_2 A^p)^{q/p} = \Psi_{p,q}(A)^q.$$

Hence the infimum on the right side of (2.1) may be restricted to positive matrices A satisfying $\Psi_{p,q}(A) \leqslant \Psi_{p,q}(Y)$. This is a compact set, and the function $\Psi_{p,q}$ is continuous, hence the infimum is achieved on a positive semidefinite matrix B. Furthermore if $C, D \geqslant 0$ are nonzero positive semidefinite matrices and $t \geqslant 1$ then $\|C\|_t < \|C + D\|_t$ (this can be seen by reducing to the case where D has rank one, and then using the interlacing condition for the eigenvalues of C and $C + D$). If we assume $B \neq 0$ then this implies

$$\|Y\|_{\mathrm{CL}} = \Psi_{p,q} \begin{pmatrix} Y+B & 0 \\ 0 & B \end{pmatrix} > \Psi_{p,q} \begin{pmatrix} Y+B & 0 \\ 0 & 0 \end{pmatrix} = \Psi_{p,q}(Y+B) \geqslant \|Y+B\|_{\mathrm{CL}} \tag{4.24}$$

where in the first inequality we used strict monotonicity of the map $A \mapsto \|A\|_{q/p}$, and the second inequality follows from (4.22). Since $Y \leqslant Y + B$ this shows that $\| \cdot \|_{\mathrm{CL}}$ is not monotone unless $B = 0$ for every $Y \geqslant 0$.

But if $B = 0$ for all $Y \geqslant 0$ it would follow from (4.24) that $\|Y\|_{\mathrm{CL}} = \Psi_{p,q}(Y)$ for all $Y \geqslant 0$. However it is easy to construct examples which show that the function $Y \mapsto \Psi_{p,q}(Y)$ is not monotone (see below for one such construction), and therefore also in this case we conclude that $\| \cdot \|_{\mathrm{CL}}$ is not monotone.

We now present a numerical example showing that $\Psi_{p,q}$ is not monotone. Let $w = 3 - \sqrt{10}$ and define the positive matrices

$$W = \begin{pmatrix} w^2 & w & 0 & 0 \\ w & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & /12 \end{pmatrix}.$$

Define the function

$$g(t) = \Psi_{p,q}(Y + tW).$$

Then $g'(t) < 0$ for small values of t, and a range of values of p, q.

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