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THE RELATIVE TENSOR PRODUCT AND A MINIMAL FIBER PRODUCT IN THE SETTING OF C^*-ALGEBRAS

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ABSTRACT. We introduce a relative tensor product of C^*-bimodules and a spatial fiber product of C^*-algebras that are analogues of Connes' fusion of correspondences and the fiber product of von Neumann algebras introduced by Sauvageot, respectively. These new constructions form the basis for our approach to quantum groupoids in the setting of C^*-algebras that is published separately.

KEYWORDS: *Hilbert module, bimodule, relative tensor product, fiber product, quantum groupoid.*

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1. INTRODUCTION

The relative tensor product of Hilbert modules over von Neumann algebras was introduced by Connes in an unpublished manuscript [4], [10], [20] and later used by Sauvageot to define a fiber product of von Neumann algebras relative to a common (commutative) von Neumann subalgebra [21]. These constructions and Haagerup's theory of operator-valued weights on von Neumann algebras [12], [13] form the basis for the theory of measured quantum groupoids developed by Enock, Lesieur and Vallin [8], [9], [18], [30], [31].

In this article, we introduce a new notion of a bimodule in the setting of C^*-algebras, construct relative tensor products of such bimodules, and define a fiber product of C^*-algebras represented on such bimodules. These constructions form the basis for a series of articles on quantum groupoids in the setting of C^*-algebras, individually addressing fundamental unitaries [29], axiomatics of the compact case [27], and coactions of quantum groupoids on C^*-algebras [28]. Moreover, our previous approach to quantum groupoids in the setting of C^*-algebras [25] embeds functorially into this new framework [26], and the latter overcomes the serious restrictions of the former one.

Already in the definition of a quantum groupoid, the relative tensor product and a fiber product appear as follows. Roughly, such an object consists of the following ingredients: an algebra B, thought of as the functions on the unit space, an algebra A, thought of as functions on the total space, a homomorphism $r\colon B \to A$ and an antihomomorphism $s\colon B \to A$ corresponding to the range and the source map, and a comultiplication $\Delta\colon B \to A \underset{B}{*} A$ corresponding to the multiplication of the quantum groupoid. Here, $A \underset{B}{*} A$ is a fiber product whose precise definition depends on the class of the algebras involved. In the setting of operator algebras, A acts naturally on some bimodule H and product $A \underset{B}{*} A$ is a certain subalgebra of operators acting on a relative tensor product $H \underset{B}{\otimes} H$. This relative tensor product is important also because it forms the domain or range of the fundamental unitary of the quantum groupoid.

Let us now sketch the problems and constructions studied in this article.

The first problem is the construction of a tensor product $H \underset{B}{\otimes} K$ of modules H, K over some algebra B. In the algebraic setting, $H \underset{B}{\otimes} K$ is simply a quotient of the full tensor product $H \otimes K$. In the setting of von Neumann algebras, H and K are Hilbert spaces, and Connes explained that the right tensor product is not a completion of the algebraic one but something more complicated. If B is commutative and of the form $B = L^\infty(X, \mu)$, then the modules H, K can be disintegrated into two measurable fields of Hilbert spaces in the form $H = \int_X^\oplus H_x \mathrm{d}\mu(x)$ and $K = \int_X^\oplus K_x \mathrm{d}\mu(x)$, and $H \underset{B}{\otimes} K$ is obtained by taking tensor products of the fibers and integrating again: $H \underset{B}{\otimes} K = \int_X^\oplus H_x \otimes K_x \mathrm{d}\mu(x)$. For the situation where B is a C^*-algebra, we propose an approach that is based on the internal tensor product of Hilbert C^*-modules and essentially consists of an algebraic reformulation of Connes' fusion. Central to this approach is a new notion of a bimodule in the setting of C^*-algebras.

The second problem is the construction of a fiber product $A \underset{B}{*} C$ of two algebras A, C relative to a subalgebra B. If B is central in A and the opposite B^{op} is central in C, this fiber product is just a relative tensor product. In the algebraic setting, it coincides with the tensor product of modules; in the setting of operator algebras, it can be obtained via disintegration and a fiberwise tensor product again. This approach was studied by Sauvageot for Neumann algebras [21], and by Blanchard [1] for C^*-algebras.

The case where the subalgebra B^{op} is no longer central in A or C is more difficult. In the algebraic setting, the fiber product was introduced by Takeuchi [24] and is, roughly, the largest subalgebra of the relative tensor product $A \underset{B}{\otimes} C$ where componentwise multiplication is still well defined. In the setting of von Neumann algebras, Sauvageot's definition of the fiber product carries over to the general case and takes the form $A \underset{B}{*} C = (A' \underset{B}{\otimes} C')'$, where A and C are represented on Hilbert spaces H and K, respectively, and $A' \underset{B}{\otimes} C'$ acts on Connes' relative tensor product $H \underset{B}{\otimes} K$. Here, it is important to note that $A' \underset{B}{\otimes} C'$ is a completion of an algebraic tensor product spanned by elementary tensors, but in general, $A \underset{B}{*} C$ is not. Similarly, in the setting of C^*-algebras, one cannot start from some algebraic tensor product and define the fiber product to be some completion; rather, a new idea is needed. We propose such a new fiber product for C^*-algebras represented on the new class of modules mentioned above. Unfortunately, several important questions concerning this construction remain open, but the applications in [27], [28], [29] already prove its usefulness.

This article is organized as follows.

The introduction ends with a short summary on terminology and some background on Hilbert C^*-modules.

Section 2 is devoted to the relative tensor product in the setting of C^*-algebras. It starts with some motivation, then presents a new notion of modules and bimodules in the setting of C^*-algebras, and finally gives the construction and its formal properties like functoriality, associativity and unitality.

Section 3 introduces a minimal fiber product of C^*-algebras. It begins with an overview and then proceeds to C^*-algebras represented on the class of modules and bimodules introduced in Section 2. The fiber product is first defined and studied for such represented C^*-algebras, including a discussion of functoriality, slice maps, lack of associativity, and unitality. A natural extension to non-represented C^*-algebras is indicated at the end.

Section 4 relates our constructions for the setting of C^*-algebras to the corresponding constructions for the setting of von Neumann algebras. Adapting our constructions to von Neumann algebras, one recovers Connes' fusion and Sauvageot's fiber product; moreover, the constructions are related by functors going from the C^*-level to the W^*-level. The section ends with a categorical interpretation of Sauvageot's fiber product.

Section 5 shows that for a commutative base $B = C_0(X)$, the relative tensor product of the new class of modules corresponds to the fiberwise tensor product of continuous Hilbert bundles over X, and the fiber product of represented C^*-algebras is related to the relative tensor product of continuous $C_0(X)$-algebras studied by Blanchard.

We use the following conventions and notation.

Given a category $\mathbf{C}$, we write $A, B \in \mathbf{C}$ to indicate that A, B are objects of $\mathbf{C}$, and denote by $\mathbf{C}(A, B)$ the associated set of morphisms.

Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y.

All sesquilinear maps like inner products on Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one.

Given a Hilbert space H and an element $\xi \in H$, we define ket-bra operators $|\xi\rangle\colon \mathbb{C} \to H$, $\lambda \mapsto \lambda\xi$, and $\langle\xi| = |\xi\rangle^*\colon H \to \mathbb{C}$, $\xi' \mapsto \langle\xi|\xi'\rangle$.

We shall make extensive use of (right) Hilbert C^*-modules; a standard reference is [16].

Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote by $\mathcal{L}(E,F)$ the space of all adjointable operators from E to F. Let E and F be Hilbert C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}(F)$ be a $*$-homomorphism. Then the internal tensor product $E \otimes_\pi F$ is a Hilbert C^*-module over B ([16], Section 4) and the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi'\rangle = \langle \xi | \pi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. We denote the internal tensor product by "$\ogreaterthan$" and drop the index π if the representation is understood; thus, $E \ogreaterthan F = E \ogreaterthan_\pi F = E \otimes_\pi F$.

We define a *flipped internal tensor product* $F {}_\pi\!\olessthan E$ as follows. We equip the algebraic tensor product $F \odot E$ with a product $\langle \xi \odot \eta | \xi' \odot \eta'\rangle := \langle \xi | \pi(\langle \eta|\eta'\rangle)\xi'\rangle$ and a module structure via $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-B-module $F {}_\pi\!\olessthan E$ which is the closed linear span of elements $\xi {}_\pi\!\olessthan \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi {}_\pi\!\olessthan \eta | \xi' {}_\pi\!\olessthan \eta'\rangle = \langle \xi | \pi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\xi {}_\pi\!\olessthan \eta)b = \xi b {}_\pi\!\olessthan \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we usually drop the index π and simply write "$\olessthan$" instead of "${}_\pi\!\olessthan$". Evidently, there exists a unitary $\Sigma\colon F \ogreaterthan E \xrightarrow{\cong} E \olessthan F$, $\eta \ogreaterthan \xi \mapsto \xi \olessthan \eta$.

Let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with $*$-homomorphisms $\pi_i\colon A \to \mathcal{L}(F_i)$ for $i = 1, 2$, and let $S \in \mathcal{L}(E_1, E_2)$, $T \in \mathcal{L}(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \ogreaterthan T \in \mathcal{L}(E_1 \ogreaterthan F_1, E_2 \ogreaterthan F_2)$ such that $(S \ogreaterthan T)(\eta \ogreaterthan \xi) = S\eta \ogreaterthan T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \ogreaterthan T)^* = S^* \ogreaterthan T^*$ ([7], Proposition 1.34).

2. THE RELATIVE TENSOR PRODUCT IN THE SETTING OF C^*-ALGEBRAS

2.1. MOTIVATION. The aim of this section is to construct a relative tensor product of suitably defined left and right modules over a general C^*-algebra B such that (i) the construction shares the main properties of the ordinary tensor product of bimodules over rings like functoriality and associativity and (ii) the modules admit representations of C^*-algebras that do not commute with the module structures. The latter condition will be needed to construct fiber products of C^*-algebras; see Section 3.

The internal tensor product of Hilbert C^*-modules meets condition (i) but not (ii) because C^*-algebras represented on such modules necessarily commute with the right module structure. An approach to quantum groupoids based on the internal tensor product was developed in [25] but remained restricted to very special cases.

What we are looking for is an analogue of Connes' fusion of correspondences. Here, B is a von Neumann algebra, and left and right modules are Hilbert spaces equipped with suitable representation or antirepresentation of B, respectively. The relative tensor product of a right module H and a left module K is then constructed as follows. Choose a normal, semi-finite, faithful (n.s.f.) weight μ on B, construct a B-valued inner product $\langle \cdot | \cdot \rangle_\mu$ on the dense subspace $H_0 \subseteq H$ of all bounded vectors, and define $H \underset{\mu}{\otimes} K$ to be the separated completion of the algebraic tensor product $H_0 \odot K$ with respect to the sesquilinear form given by $\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \langle \eta | \langle \xi | \xi' \rangle_\mu \eta' \rangle$. The definition of bounded vectors involves the GNS-space $\mathfrak{H} := H_\mu$ for μ which, by Tomita–Takesaki theory, is bimodule over B, and each bounded vector $\xi \in H_0$ gives rise to a map $L(\xi) \in \mathcal{L}(\mathfrak{H}_B, H_B)$ of right B-modules such that $\langle \xi | \xi' \rangle_\mu = L(\xi)^* L(\xi') \in B \subseteq \mathcal{L}(\mathfrak{H})$.

EXAMPLE 2.1. Assume that $B = L^\infty(X, \mu)$ for some nice measure space (X, μ), and denote the weight on B given by integration by μ as well. Then $\mathfrak{H} = L^2(X, \mu)$, and we can disintegrate H and K into measurable fields $(H_x)_x$ and $(K_x)_x$ of Hilbert spaces over X such that $H \cong \int_X^\oplus H_x \mathrm{d}\mu(x)$ and $K \cong \int_X^\oplus K_x \mathrm{d}\mu(x)$. Each vector ξ of H or K corresponds to a measurable section $x \mapsto \xi(x)$ with square-integrable norm function $|\xi| \colon x \mapsto \|\xi_x\|$, and is bounded with respect to μ if and only if this norm function is essentially bounded. Then for all $\xi, \xi' \in H_0$, $x \in X$, $\eta, \eta' \in K$,

$$\langle \xi | \xi' \rangle_\mu(x) = \langle \xi(x) | \xi'(x) \rangle_{H_x},$$
$$\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \int_X \langle \xi(x) | \xi'(x) \rangle \langle \eta(x) | \eta'(x) \rangle \mathrm{d}\mu(x),$$

and $H \underset{\mu}{\otimes} K \cong \int_X^\oplus H_x \otimes K_x \mathrm{d}\mu(x)$. Note that the sesquilinear form above need not extend to $H \odot K$ because the integrand need not be in $L^1(X, \mu)$ for arbitrary $\xi, \xi' \in H$ and $\eta, \eta' \in K$.

For our purpose, the following algebraic description of $H \underset{\mu}{\otimes} K$ is useful. This relative tensor product can be identified with the separated completion of algebraic tensor product

$$\mathcal{L}(\mathfrak{H}_B, H_B) \odot \mathfrak{H} \odot \mathcal{L}({}_B\mathfrak{H}, {}_BK) \tag{2.1}$$

with respect to the sesquilinear form

$$\langle S \odot \zeta \odot T | S' \odot \zeta' \odot T' \rangle = \langle \zeta | S^* S' T^* T' \zeta' \rangle = \langle \zeta | T^* T' S^* S' \zeta' \rangle,$$

where $\mathcal{L}(\mathfrak{H}_B, H_B)$ and $\mathcal{L}({}_B\mathfrak{H}, {}_BK)$ are all bounded maps of right or left B-modules, respectively. We adapt this definition to the setting of C^*-algebras, making the following modifications:

(A) The construction above depends on the choice of some n.s.f. weight μ or, more precisely, the triple $(H_\mu, \pi_\mu(B), \pi_\mu(B)')$, but any other μ yields a triple which is unitarily equivalent. In the setting of C^*-algebras, such a canonical triple does not exist but has to be chosen.

(B) The module structure of H and K can equivalently be described in terms of (anti)representations of B or in terms of the spaces $\mathcal{L}(\mathfrak{H}_B, H_B)$ and $\mathcal{L}({}_B\mathfrak{H}, {}_BK)$. In the setting of C^*-algebras, this equivalence breaks down, and we shall make suitable closed subspaces of intertwiners the primary object. In the commutative case, a representation corresponds to a measurable field of Hilbert spaces, and the subspaces fix a continuous structure.

(C) If H and K are bimodules, then so is $H \underset{\mu}{\otimes} K$. Here, a bimodule structure on H is given by the additional choice of a representation of some von Neumann algebra A that commutes with the antirepresentation of B or, equivalently, satisfies $A\mathcal{L}(\mathfrak{H}_B, H_B) = \mathcal{L}(\mathfrak{H}_B, H_B)$. If we pass to C^*-algebras, then commutation is too weak, and we shall adopt the second condition, where $\mathcal{L}(\mathfrak{H}_B, H_B)$ is replaced by the subspace of intertwiners mentioned above.

2.2. MODULES AND BIMODULES OVER C^*-BASES. Observation (A) leads us to adopt the following terminology.

DEFINITION 2.2. A *C^*-base* $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ consists of a Hilbert space $\mathfrak{H}$ and commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$, respectively. The *opposite* of $\mathfrak{b}$ is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$. A C^*-base $(\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ is *equivalent* to $\mathfrak{b}$ if $\mathrm{Ad}_V(\mathfrak{A}) = \mathfrak{B}$ and $\mathrm{Ad}_V(\mathfrak{A}^\dagger) = \mathfrak{B}^\dagger$ for some unitary $V \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$.

Clearly, the Hilbert space $\mathbb{C}$ and twice the algebra $\mathbb{C} \equiv \mathcal{L}(\mathbb{C})$ form a trivial C^*-base $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$.

EXAMPLE 2.3. Let μ be a proper, faithful KMS-weight on a C^*-algebra A [15] with GNS-space H_μ, GNS-representation $\pi_\mu \colon A \to \mathcal{L}(H_\mu)$, modular conjugation $J_\mu \colon H_\mu \to H_\mu$, and opposite GNS-representation $\pi_{\mu^{\mathrm{op}}} \colon A^{\mathrm{op}} \to \mathcal{L}(H_\mu)$, $a \mapsto J_\mu \pi_\mu(a^*) J_\mu$. Then the triple $(H_\mu, \pi_\mu(A), \pi_{\mu^{\mathrm{op}}}(A^{\mathrm{op}}))$ is a C^*-base. Its opposite is equivalent to the C^*-base associated to the opposite weight μ^{op} on A^{op}. Indeed, H_μ can be considered as the GNS-space for μ^{op} via the opposite GNS-map $\Lambda_{\mu^{\mathrm{op}}} \colon \mathfrak{N}_{\mu^{\mathrm{op}}} \to H_\mu$, $a^{\mathrm{op}} \mapsto J_\mu \Lambda_\mu(a^*)$, and then $J_{\mu^{\mathrm{op}}} \pi_{\mu^{\mathrm{op}}}(A^{\mathrm{op}}) J_{\mu^{\mathrm{op}}} = \pi_\mu(A)$.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. We define C^*-modules over $\mathfrak{b}$ as indicated in comment (B).

DEFINITION 2.4. A *C^*-$\mathfrak{b}$-module* $H_\alpha = (H, \alpha)$ is a Hilbert space H with a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, $[\alpha^*\alpha] = \mathfrak{B}$. A *semi-morphism* between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$. If additionally $T^*\beta \subseteq \alpha$, we call T a *morphism*. We denote the set of all (semi-)morphisms by $\mathcal{L}_{(\mathrm{s})}(H_\alpha, K_\beta)$.

Evidently, the class of all C^*-$\mathfrak{a}$-modules forms a category with respect to all semi-morphisms, and a C^*-category in the sense of [11] with respect to all morphisms.

LEMMA 2.5. (i) *Let H, K be Hilbert spaces and $I \subseteq \mathcal{L}(H, K)$ such that $[IH] = K$. Then there exists a unique normal, unital $*$-homomorphism $\rho_I\colon (I^*I)' \to (II^*)'$ such that $\rho_I(x)S = Sx$ for all $x \in (I^*I)'$, $S \in I$.*

(ii) *Let H, K, L be Hilbert spaces and $I \subseteq \mathcal{L}(H, K)$, $J \subseteq \mathcal{L}(K, L)$ such that $[IH] = K$, $[JK] = L$, and $J^*JI \subseteq I$. Then $\rho_I((I^*I)') \subseteq (J^*J)'$ and $\rho_J \circ \rho_I = \rho_{JI}$.*

Proof. (i) Uniqueness is evident. Let $x \in (I^*I)'$ and $S_1, \ldots, S_n \in I$, $\xi_1, \ldots, \xi_n \in H$. Since x^*x commutes with each $S_i^*S_j$, the matrix $(S_i^*S_jx^*x)_{i,j} \in M_n(\mathcal{L}(H))$ is dominated by $\|x^*x\|(S_i^*S_j)_{i,j}$, and

$$\Big\|\sum_i S_i x\xi_i\Big\|^2 = \sum_{i,j}\langle\xi_i|S_i^*S_jx^*x\xi_j\rangle \leqslant \|x\|^2\sum_{i,j}\langle\xi_i|S_i^*S_j\xi_j\rangle = \|x\|^2\Big\|\sum_i S_i\xi_i\Big\|^2.$$

Hence, there exists an operator $\rho_I(x) \in \mathcal{L}(K)$ as claimed. One easily verifies that the assignment $x \mapsto \rho_I(x)$ is a $*$-homomorphism. It is normal because $[IH] = K$ and for all $S, T \in I$, $\xi, \eta \in K$, the functional $x \mapsto \langle S\xi|\rho_I(x)T\eta\rangle = \langle\xi|xS^*T\eta\rangle$ is normal.

(ii) Let $x \in (I^*I)'$. Then $\rho_I(x) \in J^*J$ since $S^*T\rho_I(x)R = S^*TRx = \rho_I(x)S^*TR$ for all $S, T \in J$, $R \in I$, and $\rho_{JI}(x) = \rho_J(\rho_I(x))$ because $\rho_{JI}(x)TR = TRx = \rho_J(\rho_I(x))TR$ for all $T \in J$, $R \in I$. ∎

LEMMA 2.6. *Let H_α be a C^*-$\mathfrak{b}$-module.*

(i) *α is a Hilbert C^*-$\mathfrak{B}$-module with inner product $(\xi, \xi') \mapsto \xi^*\xi'$.*

(ii) *There exist isomorphisms $\alpha \otimes \mathfrak{K} \to H$, $\xi \otimes \zeta \mapsto \xi\zeta$, and $\mathfrak{K} \otimes \alpha \to H$, $\zeta \otimes \xi \mapsto \xi\zeta$.*

(iii) *There exists a unique normal, unital and faithful representation $\rho_\alpha\colon \mathfrak{B}' \to \mathcal{L}(H)$ such that $\rho_\alpha(x)(\xi\zeta) = \xi x\zeta$ for all $x \in \mathfrak{B}'$, $\xi \in \alpha$, $\zeta \in \mathfrak{K}$.*

(iv) *Let K_β be a C^*-$\mathfrak{b}$-module and $T \in \mathcal{L}_{\mathrm{s}}(H_\alpha, K_\beta)$. Then $T\rho_\alpha(x) = \rho_\beta(x)T$ for all $x \in \mathfrak{B}'$. If additionally $T \in \mathcal{L}(H_\alpha, K_\beta)$, then left multiplication by T defines an operator in $\mathcal{L}_{\mathfrak{B}}(\alpha, \beta)$, again denoted by T.*

Proof. Assertions (i) and (ii) are obvious, and (iii) follows from the preceding lemma. To prove (iv), let $x \in \mathfrak{B}', \xi \in \alpha, \zeta \in \mathfrak{K}$. Then $T\xi \in \beta$ and $T\rho_\alpha(x)\xi\zeta = T\xi x\zeta = \rho_\beta(x)T\xi\zeta$. ∎

EXAMPLE 2.7. Let Z be a locally compact Hausdorff space, μ a Radon measure on Z of full support, and $\mathcal{H} = (H_z)_z$ a continuous bundle of Hilbert spaces on Z with full support. Then the Hilbert space $\mathfrak{K} = L^2(Z, \mu)$ together with the C^*-algebras $\mathfrak{B} = \mathfrak{B}^\dagger = C_0(Z) \subseteq \mathcal{L}(\mathfrak{K})$ forms a C^*-base. Let $H = \int_Z^\oplus H_z \mathrm{d}\mu(z)$ and $\alpha = m(\Gamma_0(\mathcal{H}))$, where for each section $\xi \in \Gamma_0(\mathcal{H})$, the operator $m(\xi) \in \mathcal{L}(\mathfrak{K}, H)$ is given by pointwise multiplication, $m(\xi)f = (\xi(z)f(z))_{z\in Z}$. Then H_α is a C^*-$\mathfrak{b}$-module and $\rho_\alpha\colon \mathfrak{B}' = L^\infty(Z, \mu) \to \mathcal{L}(H)$ is given by pointwise multiplication of sections by functions. Every C^*-$\mathfrak{b}$-module arises in this way from a continuous bundle; see Section 5.

Let also $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ be a C^*-base. We define C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-bimodules as indicated in (C).

DEFINITION 2.8. A C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$*-module* is a triple ${}_\alpha H_\beta = (H, \alpha, \beta)$, where H is a Hilbert space, (H, α) a C^*-$\mathfrak{a}^\dagger$-module, (H, β) a C^*-$\mathfrak{b}$-module, and $[\rho_\alpha(\mathfrak{A})\beta] = \beta$ and $[\rho_\beta(\mathfrak{B}^\dagger)\alpha] = \alpha$. The set of *(semi-)morphisms* between C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules ${}_\alpha H_\beta$ and ${}_\gamma K_\delta$ is $\mathcal{L}_{(\mathrm{s})}({}_\alpha H_\beta, {}_\gamma K_\delta) := \mathcal{L}_{(\mathrm{s})}(H_\alpha, K_\gamma) \cap \mathcal{L}_{(\mathrm{s})}(H_\beta, K_\delta)$.

REMARK 2.9. By Lemma 2.6, we have $[\rho_\alpha(\mathfrak{A}), \rho_\beta(\mathfrak{B}^\dagger)] = 0$ for every C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module ${}_\alpha H_\beta$.

Again, the class of all C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules forms a category with respect to all semi-morphisms, and a C^*-category with respect to all morphisms.

EXAMPLE 2.10. (i) $\mathfrak{H}_\mathfrak{A}$ is a C^*-$\mathfrak{a}$-module, $\rho_\mathfrak{A}(x) = x$ for all $x \in \mathfrak{A}'$, and ${}_{\mathfrak{A}^\dagger}\mathfrak{H}_\mathfrak{A}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{a})$-module because $[\rho_{\mathfrak{A}^\dagger}(\mathfrak{A})\mathfrak{A}] = [\mathfrak{A}\mathfrak{A}] = \mathfrak{A}$ and $[\rho_\mathfrak{A}(\mathfrak{A}^\dagger)\mathfrak{A}^\dagger] = \mathfrak{A}^\dagger$.

(ii) Let H_β be a C^*-$\mathfrak{b}$-module, let $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$ be the trivial C^*-base, and let $\alpha = \mathcal{L}(\mathbb{C}, H)$. Then ${}_\alpha H_\beta$ is a C^*-$(\mathfrak{t}, \mathfrak{b})$-module.

(iii) Let $(\mathcal{H}_i)_i$ be a family of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules, where $\mathcal{H}_i = (H_i, \alpha_i, \beta_i)$ for each i. Denote by $\boxplus_i \alpha_i \subseteq \mathcal{L}(\mathfrak{H}, \oplus_i H_i)$ the norm-closed linear span of all operators of the form $\zeta \mapsto (\xi_i \zeta)_i$, where $(\xi_i)_i$ is in the algebraic direct sum $\bigoplus_i^{\mathrm{alg}} \alpha_i$, and similarly define $\boxplus_i \beta_i \subseteq \mathcal{L}(\mathfrak{K}, \oplus_i H_i)$. Then the triple $\boxplus_i \mathcal{H}_i := (\oplus_i H_i, \boxplus_i \alpha_i, \boxplus_i \beta_i)$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, for each j, the canonical inclusions $\iota_j \colon H_j \to \oplus_i H_i$ and projection $\pi_j \colon \oplus_i H_i \to H_j$ are morphisms $\mathcal{H}_j \to \boxplus_i \mathcal{H}_i$ and $\boxplus_i \mathcal{H}_i \to \mathcal{H}_j$, and with respect to these maps, $\boxplus_i \mathcal{H}_i$ is the direct sum of the family $(\mathcal{H}_i)_i$.

The following example shows how bimodules arise from conditional expectations.

EXAMPLE 2.11. Let B be a C^*-algebra with a KMS-state μ and associated C^*-base $\mathfrak{b}$ (Example 2.3), let A be a unital C^*-algebra containing B such that $1_A \in B$, and let $\phi \colon A \to B$ be a faithful conditional expectation such that $\nu := \mu \circ \phi$ is a KMS-state and $\phi \circ \sigma_t^\nu = \sigma_t^\mu \circ \phi$ for all $t \in \mathbb{R}$. Fix a GNS-construction $\pi_\nu \colon A \to \mathcal{L}(H_\nu)$ for ν with modular conjugation $J_\nu \colon H_\nu \to H_\nu$, and define $\pi_\nu^{\mathrm{op}} \colon A^{\mathrm{op}} \to \mathcal{L}(H_\nu)$ by $a \mapsto J_\nu \pi_\nu(a^*) J_\nu$. Then the inclusion $B \hookrightarrow A$ extends to an isometry $\zeta \colon \mathfrak{K} = H_\mu \hookrightarrow H_\nu = H$, and we obtain a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-module ${}_\alpha H_\beta$, where $H = H_\nu$, $\alpha = [J_\nu \pi_\nu(A)\zeta]$, $\beta = [\pi_\nu(A)\zeta]$, and $\rho_\alpha \circ \pi_{\mu^{\mathrm{op}}} = \pi_\nu^{\mathrm{op}}$, $\rho_\beta \circ \pi_\mu = \pi_\nu$. Moreover,

$$\pi_\nu(A) + \pi_\nu^{\mathrm{op}}((A \cap B')^{\mathrm{op}}) \subseteq \mathcal{L}(H_\alpha), \quad \pi_{\nu^{\mathrm{op}}}(A^{\mathrm{op}}) + \pi_\nu(A \cap B') \subseteq \mathcal{L}(H_\beta).$$

For details, see Section 2–3 of [27].

2.3. THE RELATIVE TENSOR PRODUCT. The concepts introduced above allow us to adapt the algebraic formulation of Connes' fusion to the setting of C^*-algebras as follows. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β a C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. Then the *relative tensor product* of H_β and K_γ is the Hilbert space

$$H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K := \beta \varodot \mathfrak{K} \varodot \gamma,$$

which is spanned by elements $\xi \varodot \zeta \varodot \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, the inner product being given by $\langle \xi \varodot \zeta \varodot \eta | \xi' \varodot \zeta' \varodot \eta' \rangle = \langle \zeta | \xi^* \xi' \eta^* \eta' \zeta' \rangle = \langle \zeta | \eta^* \eta' \xi^* \xi' \zeta' \rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$.

EXAMPLE 2.12. (i) If $\mathfrak{b}$ is the trivial C^*-base $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$, then $\beta = \mathcal{L}(\mathbb{C}, H)$, $\gamma = \mathcal{L}(\mathbb{C}, K)$, and $H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K \cong H \otimes K$ via $\xi \varodot \zeta \varodot \eta \mapsto \xi\zeta \otimes \eta 1 = \xi 1 \otimes \eta\zeta$.

(ii) Let Z be a locally compact Hausdorff space, μ a Radon measure on Z of full support, $\mathcal{H} = (H_z)_z$ and $\mathcal{K} = (K_z)_z$ continuous bundles of Hilbert spaces on Z with full support, and H_α, K_β the associated C^*-$\mathfrak{b}$-modules as defined in Example 2.7. One easily checks that then we have an isomorphism

$$H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K \to \int_Z^\oplus H_z \otimes K_z \, \mathrm{d}\mu(z), \quad m(\xi) \varodot \zeta \varodot m(\eta) \mapsto (\xi(z)\zeta(z) \otimes \eta(z))_{z \in Z}.$$

Let us list some easy observations and a few definitions.

(i) The isomorphisms in Lemma 2.6(ii), applied to H_β and K_γ, respectively, yield the following identifications which we shall use without further notice:

$$\beta \varodot_{\rho_\gamma} K \cong H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K \cong H_{\rho_\beta} \varodot \gamma, \quad \xi \varodot \eta\zeta \equiv \xi \varodot \zeta \varodot \eta \equiv \xi\zeta \varodot \eta.$$

(ii) For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to \beta \varodot_{\rho_\gamma} K = H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, \quad \omega \mapsto \xi \varodot \omega,$$
$$|\eta\rangle_2 \colon H \to H_{\rho_\beta} \varodot \gamma = H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, \quad \omega \mapsto \omega \varodot \eta,$$

whose adjoints $\langle \xi|_1 := |\xi\rangle_1^*$ and $\langle \eta|_2 := |\eta\rangle_2^*$ are given by

$$\langle \xi|_1 \colon \xi' \varodot \omega \mapsto \rho_\gamma(\xi^*\xi')\omega, \quad \langle \eta|_2 \colon \omega \varodot \eta' \mapsto \rho_\beta(\eta^*\eta')\omega.$$

We put $|\beta\rangle_1 := \{|\xi\rangle_1 \mid \xi \in \beta\} \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

(iii) For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \varodot \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \varodot \gamma) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K), \quad \mathrm{id} \varodot T \in \mathcal{L}(\beta \varodot_{\rho_\gamma} K) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K).$$

If these operators commute, we let $S \underset{\mathfrak{b}}{\otimes} T := (S \varodot \mathrm{id})(\mathrm{id} \varodot T) = (\mathrm{id} \varodot T)(S \varodot \mathrm{id})$. The commutativity condition holds in each of the following cases:

(a) $S \in \mathcal{L}_{\mathrm{s}}(H_\beta)$; then $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \varodot \omega) = S\xi \varodot T\omega$ for each $\xi \in \beta, \omega \in K$;

(b) $T \in \mathcal{L}_{\mathrm{s}}(K_\gamma)$; then $(S \underset{\mathfrak{b}}{\otimes} T)(\omega \varodot \eta) = S\omega \varodot T\eta$ for each $\omega \in H, \eta \in \gamma$;

(c) $(\mathfrak{B}^\dagger)' = \mathfrak{B}''$; then for all $\xi, \xi' \in \beta$ and $\eta, \eta' \in \gamma$, the elements $\eta^* T \eta' \in \mathfrak{B}'$ and $\xi^* S \xi' \in (\mathfrak{B}^\dagger)'$ commute, and if $\zeta, \zeta' \in \mathfrak{K}$ and $\omega = \xi \olessthan \zeta \ogreaterthan \eta$, $\omega' = \xi' \olessthan \zeta' \ogreaterthan \eta'$, then

$$\begin{aligned} \langle \omega | (\mathrm{id} \ogreaterthan T)(S \olessthan \mathrm{id}) \omega' \rangle &= \langle \zeta | (\eta^* T \eta')(\xi^* S \xi') \zeta' \rangle \\ &= \langle \zeta | (\xi^* S \xi')(\eta^* T \eta') \zeta' \rangle = \langle \omega | (S \olessthan \mathrm{id})(\mathrm{id} \ogreaterthan T) \omega' \rangle. \end{aligned}$$

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be further C^*-bases. Then the relative tensor product of bimodules over $(\mathfrak{a}^\dagger, \mathfrak{b})$ and $(\mathfrak{b}^\dagger, \mathfrak{c})$ is a bimodule over $(\mathfrak{a}^\dagger, \mathfrak{c})$:

PROPOSITION 2.13. *Let $\mathcal{H} = {}_\alpha H_\beta$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, $\mathcal{K} = {}_\gamma K_\delta$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module, and*

$$\alpha \triangleleft \gamma := [|\gamma\rangle_2 \alpha] \subseteq \mathcal{L}(\mathfrak{H}, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K), \quad \beta \triangleright \delta := [|\beta\rangle_1 \delta] \subseteq \mathcal{L}(\mathfrak{L}, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K). \tag{2.2}$$

Then $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K} := {}_{(\alpha \triangleleft \gamma)}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{(\beta \triangleright \delta)}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-module and

$$\rho_{(\alpha\triangleleft\gamma)}(x) = \rho_\alpha(x) \olessthan \mathrm{id} \ \text{ for all } x \in (\mathfrak{A}^\dagger)', \quad \rho_{(\beta\triangleright\delta)}(y) = \mathrm{id} \ogreaterthan \rho_\delta(y) \ \text{ for all } y \in \mathfrak{C}'. \tag{2.3}$$

Proof. The pair $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, \alpha \triangleleft \gamma)$ is a C^*-$\mathfrak{a}^\dagger$-module since $[\alpha^* \langle \gamma|_2 |\gamma\rangle_2 \alpha] = [\alpha^* \rho_\beta(\mathfrak{B}^\dagger) \alpha] = \mathfrak{A}^\dagger$, $[|\gamma\rangle_2 \alpha \mathfrak{A}^\dagger] = [|\gamma\rangle_2 \alpha]$, and $[|\gamma\rangle_2 \alpha \mathfrak{H}] = [|\gamma\rangle_2 H] = H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K$. Likewise, $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, \beta \triangleright \delta)$ is a C^*-$\mathfrak{c}$-module. For all $x \in (\mathfrak{A}^\dagger)'$, $\zeta \in \mathfrak{H}$, $\theta \in \alpha$, $\eta \in \gamma$, we have $|\eta\rangle_2 \theta \in \alpha \triangleleft \gamma$ and hence

$$\rho_{(\alpha\triangleleft\gamma)}(x)(\theta\zeta \olessthan \eta) = \rho_{(\alpha\triangleleft\gamma)}(x) |\eta\rangle_2 \theta \zeta = |\eta\rangle_2 \theta x \zeta = \rho_\alpha(x) \theta \zeta \olessthan \eta = (\rho_\alpha(x) \olessthan \mathrm{id})(\theta \zeta \olessthan \eta).$$

The first equation in (2.3) follows, and a similar agument proves the second one. Finally, ${}_{(\alpha\triangleleft\gamma)}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{(\beta\triangleright\delta)}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-module because $[\rho_{(\alpha\triangleleft\gamma)}(\mathfrak{A}) |\beta\rangle_1 \delta] = [|\rho_\alpha(\mathfrak{A})\beta\rangle_1 \delta] = [|\beta\rangle_1 \delta]$ and $[\rho_{(\beta\triangleright\delta)}(\mathfrak{C}^\dagger) |\gamma\rangle_2 \alpha] = [|\gamma\rangle_2 \alpha]$. ∎

In the situation above, we call $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$ the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$. Note the following commutative diagram of Hilbert spaces and closed spaces of operators between them:

$$\begin{tikzcd}
\mathfrak{H} \arrow[dr, "\alpha"] \arrow[ddrr, bend right, "\alpha\triangleleft\gamma"'] & & \mathfrak{K} \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & & \mathfrak{L} \arrow[dl, "\delta"'] \arrow[ddll, bend left, "\beta\triangleright\delta"] \\
& H \arrow[dr, "|\gamma\rangle_2"] & & K \arrow[dl, "|\beta\rangle_1"'] & \\
& & H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K & &
\end{tikzcd}$$

Given a C^*-$\mathfrak{b}$-module $\mathcal{H} = H_\beta$ and a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module $\mathcal{K} = {}_\gamma K_\delta$, we abbreviate $H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta := (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{\beta\triangleright\delta}$. Likewise, we write ${}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K$ for $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{\alpha\triangleleft\gamma}$ and ${}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta$ for ${}_{\alpha\triangleleft\gamma}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{\beta\triangleright\delta}$.

The relative tensor product is functorial, associative, unital, and compatible with direct sums in the following sense:

PROPOSITION 2.14. *Let $\mathcal{H} = {}_\alpha H_\beta$ and $\mathcal{H}^1 = {}_{\alpha_1}H^1_{\beta_1}$, $\mathcal{H}^2 = {}_{\alpha_2}H^2_{\beta_2}$ be C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules, $\mathcal{K} = {}_\gamma K_\delta$, $\mathcal{K}^1 = {}_{\gamma_1}K^1_{\delta_1}$, $\mathcal{K}^2 = {}_{\gamma_2}K^2_{\delta_2}$ C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-modules, and $\mathcal{L} = {}_\varepsilon L_\phi$ a C^*-$(\mathfrak{c}^\dagger, \mathfrak{d})$-module.*

(i) *$S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}(\mathcal{H}^1 \underset{\mathfrak{b}}{\otimes} \mathcal{K}^1, \mathcal{H}^2 \underset{\mathfrak{b}}{\otimes} \mathcal{K}^2)$ for all $S \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^2)$, $T \in \mathcal{L}(\mathcal{K}^1, \mathcal{K}^2)$.*

(ii) *The composition of the isomorphisms $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta) \underset{\mathfrak{c}}{\otimes} {}_\varepsilon L \cong (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{\rho_{(\beta \triangleright \delta)}} \otimes \varepsilon \cong \beta \otimes_{\rho_\gamma} K_{\rho_\delta} \otimes \varepsilon$ and $\beta \otimes_{\rho_\gamma} K_{\rho_\delta} \otimes \varepsilon \cong \beta \otimes_{\rho_{(\gamma \triangleleft \varepsilon)}} (K_\delta \underset{\mathfrak{c}}{\otimes} {}_\varepsilon L) \cong H_\beta \underset{\mathfrak{b}}{\otimes} ({}_\gamma K_\delta \underset{\mathfrak{c}}{\otimes} {}_\varepsilon L)$ is an isomorphism of C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-modules $a_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}(\mathcal{L}, \mathcal{K}, \mathcal{H}) \colon (\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}) \underset{\mathfrak{c}}{\otimes} \mathcal{L} \to \mathcal{H} \underset{\mathfrak{b}}{\otimes} (\mathcal{K} \underset{\mathfrak{c}}{\otimes} \mathcal{L})$.*

(iii) *Put $\mathcal{U} := {}_{\mathfrak{B}^\dagger}\mathfrak{K}_{\mathfrak{B}}$. Then there exist isomorphisms*

$$r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \colon \mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{U} \to \mathcal{H}, \quad \xi \otimes \zeta \otimes b^\dagger \mapsto \xi b^\dagger \zeta = \rho_\beta(b^\dagger)\xi\zeta,$$
$$l_{\mathfrak{b},\mathfrak{c}}(\mathcal{K}) \colon \mathcal{U} \underset{\mathfrak{b}}{\otimes} \mathcal{K} \to \mathcal{K}, \quad b \otimes \zeta \otimes \eta \mapsto \eta b \zeta = \rho_\gamma(b)\eta\zeta.$$

(iv) *Let $(\mathcal{H}^i)_i$ be a family of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules and $(\mathcal{K}^j)_j$ a family of C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-modules. For each i, j, denote by $\iota^i_{\mathcal{H}} \colon \mathcal{H}^i \to \boxplus_{i'} \mathcal{H}^{i'}$, $\iota^j_{\mathcal{K}} \colon \mathcal{K}^j \to \boxplus_{j'} \mathcal{K}^{j'}$ and $\pi^i_{\mathcal{H}} \colon \boxplus_{i'} \mathcal{H}^{i'} \to \mathcal{H}^i$, $\pi^j_{\mathcal{K}} \colon \boxplus_{j'} \mathcal{K}^{j'} \to \mathcal{K}^j$ the canonical inclusions and projections, respectively. Then there exist inverse isomorphisms $\boxplus_{i,j}(\mathcal{H}^i \underset{\mathfrak{b}}{\otimes} \mathcal{K}^j) \leftrightarrows (\boxplus_i \mathcal{H}^i) \underset{\mathfrak{b}}{\otimes} (\boxplus_j \mathcal{K}^j)$, given by $(\omega_{i,j})_{i,j} \mapsto \sum_{i,j} (\iota^i_{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \iota^j_{\mathcal{K}})(\omega_{i,j})$ and $((\pi^i_{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \pi^j_{\mathcal{K}})(\omega))_{i,j} \leftarrow \omega$, respectively.*

Proof. (i) If S, T are as above and $\mathcal{H}^i = {}_{\alpha_i}H^i_{\beta_i}$, $\mathcal{K}^j = {}_{\gamma_j}K^j_{\delta_j}$ for $i, j = 1, 2$, then $(S \underset{\mathfrak{b}}{\otimes} T)|\gamma_1\rangle_2\alpha_1 = |T\gamma_1\rangle_2 S\alpha_1 \subseteq |\gamma_2\rangle_2\alpha_2$ and similarly $(S \underset{\mathfrak{b}}{\otimes} T)|\beta_1\rangle_1\delta_1 \subseteq |\beta_2\rangle_1\delta_2$, $(S \underset{\mathfrak{b}}{\otimes} T)^*|\gamma_2\rangle_2\alpha_2 \subseteq |\gamma_1\rangle_2\alpha_1$, $(S \underset{\mathfrak{b}}{\otimes} T)^*|\beta_2\rangle_1\delta_2 \subseteq |\beta_1\rangle_1\delta_1$.

(ii) Straightforward.

(iii) $r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \cdot (\alpha \triangleleft \mathfrak{B}^\dagger) = [\rho_\beta(\mathfrak{B}^\dagger)\alpha] = \alpha$ and $r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \cdot (\beta \triangleright \mathfrak{B}) = [\beta\mathfrak{B}] = \beta$. For $l_{\mathfrak{b},\mathfrak{c}}(\mathcal{K})$, the arguments are similar.

(iv) Straightforward. ∎

REMARK 2.15. The relative tensor product of modules and morphisms can be considered as composition in a bicategory as follows. Recall that a bicategory $\mathbf{B}$ consists of a class of objects $\operatorname{ob}\mathbf{B}$, a category $\mathbf{B}(A, B)$ for each $A, B \in \operatorname{ob}\mathbf{B}$ whose objects and morphisms are called *1-cells* and *2-cells*, respectively, a functor $c_{A,B,C} \colon \mathbf{B}(B, C) \times \mathbf{B}(A, B) \to \mathbf{B}(A, C)$ ("composition") for each $A, B, C \in \operatorname{ob}\mathbf{B}$, an object $1_A \in \mathbf{B}(A, A)$ ("identity") for each $A \in \operatorname{ob}\mathbf{B}$, an isomorphism $a_{A,B,C,D}(f, g, h) \colon c_{A,B,D}(c_{B,C,D}(h, g), f) \to c_{A,C,D}(h, c_{A,B,C}(g, f))$ in $\mathbf{B}(A, D)$ ("associativity") for each triple of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathbf{B}$, and isomorphisms $l_A(f) \colon c_{A,A,B}(f, 1_A) \to f$ and $r_B(f) \colon c_{A,B,B}(1_B, f) \to f$ in $\mathbf{B}(A, B)$ for each 1-cell $A \xrightarrow{f} B$ in $\mathbf{B}$, subject to several axioms [17]. Tedious but straightforward calculations show that there exists a bicategory $\mathbf{C^*}$-$\mathbf{bimod}$ such that

(i) the objects are all C^*-bases and $\mathbf{C^*\text{-}bimod}(\mathfrak{a},\mathfrak{b})$ is the category of all C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-modules with morphisms (not semi-morphisms) for all C^*-bases $\mathfrak{a},\mathfrak{b}$;

(ii) the functor $c_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ is given by $({}_\gamma K_\delta, {}_\alpha H_\beta) \mapsto {}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta$ and $(T,S) \mapsto S \underset{\mathfrak{b}}{\otimes} T$, respectively, and the identity $1_\mathfrak{a}$ is ${}_{\mathfrak{A}^\dagger}\mathfrak{H}_{\mathfrak{A}}$ for all C^*-bases $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$;

(iii) a,r,l are as in Proposition 2.14.

3. THE SPATIAL FIBER PRODUCT OF C^*-ALGEBRAS

3.1. BACKGROUND. We now use the relative tensor product to construct a fiber product of C^*-algebras that are represented on C^*-modules over C^*-bases. To motivate our approach, let us first review several related constructions. In each case, the task is to construct a relative tensor product or "fiber product" of two algebras A and C with respect to a common subalgebra B.

First, assume that we are working in the category of unital commutative rings. Then the fiber product is just the push-out of the diagram formed by A,B,C. Explicitly, it is the algebraic tensor product $A \underset{B}{\odot} C$, where A and C are considered as modules over B, and the multiplication is defined componentwise. In the category of commutative C^*-algebras, the push-out is the maximal completion of the algebraic tensor product $A \underset{B}{\odot} C$ and, as usual in the setting of C^*-algebras, also other interesting completions exist [1]. For example, if $B = C_0(X)$ for some locally compact Hausdorff space and if A and C are represented on Hilbert spaces H and K, respectively, then H and K can be disintegrated over X with respect to some measure μ (see Subsection 2.1), and the algebra $A \underset{B}{\odot} C$ has a natural representation π on the relative tensor product $H \underset{\mu}{\otimes} K = \int_X^\oplus H_x \otimes K_x \mathrm{d}\mu(x)$, leading to a minimal completion $\overline{\pi(A \underset{B}{\odot} C)}$. In the setting of von Neumann algebras, H and K are intrinsic, and the desired fiber product is $\pi(A \underset{B}{\odot} C)'' \subseteq \mathcal{L}(H \underset{\mu}{\otimes} K)$. Note that all of these constructions do not depend on commutativity of A and C and make sense as long as B is central in A and in C.

Next, consider the case where A,B,C are non-commutative, B is a subalgebra of A, and the opposite B^{op} is a subalgebra of C. Then one can consider A and C as modules over B via right multiplication, and form the algebraic tensor product $A \underset{B}{\odot} C$, but componentwise multiplication is well defined only on the subspace $A \underset{B}{\times} C \subseteq A \underset{B}{\odot} C$ which consists of all elements $\sum_i a_i \odot c_i$ satisfying $\sum_i ba_i \odot c_i = \sum_i a_i \odot b^{\mathrm{op}} c_i$ for all $b \in B$. This subspace was first considered by Takeuchi and provides the right notion of a fiber product for the algebraic theory of quantum groupoids [2], [32]. In the setting of C^*-algebras, the Takeuchi product $A \underset{B}{\times} C$ may be 0 even when we expect a nontrivial fiber product on the level of C^*-algebras; therefore, the latter cannot be obtained as the completion of the former. In the setting of von Neumann algebras, a fiber product can be constructed as follows [21]. If A and C act on Hilbert spaces H and K, respectively, one can form the Connes fusion $H \underset{\mu}{\otimes} K$ with respect to some weight μ on B and the actions of B on H and B^{op} on K which, by functoriality, carries a representation $\pi\colon A' \odot C' \to \mathcal{L}(H \underset{\mu}{\otimes} K)$, and the desired fiber product is $A \underset{\mu}{*} C = \pi(A' \odot C')'$. A categorical interpretation of this construction is given in Theorem 4.3.

We modify the last construction to define a fiber product for C^*-algebras A and C as follows:

(A) We assume that A and C are represented on a C^*-$\mathfrak{b}$-module H_β and a C^*-$\mathfrak{b}^\dagger$-module K_γ, respectively, where $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ is a C^*-base, such that $\rho_\beta(\mathfrak{B})$ and $\rho_\gamma(\mathfrak{B}^\dagger)$ take the places of B and B^{op}, respectively.

(B) On $H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K$, we define two C^*-algebras $\mathrm{Ind}_{|\gamma\rangle_2}(A)$ and $\mathrm{Ind}_{|\beta\rangle_1}(C)$ which, roughly, take the places of $\pi(A' \odot \mathrm{id}_K)'$ and $\pi(\mathrm{id}_H \odot C')'$.

(C) The fiber product is then $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B = \mathrm{Ind}_{|\gamma\rangle_2}(A) \cap \mathrm{Ind}_{|\beta\rangle_1}(C) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$.

3.2. C^*-ALGEBRAS REPRESENTED ON C^*-MODULES. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. As indicated in step (A), we adopt the following terminology.

DEFINITION 3.1. A *C^*-$\mathfrak{B}^\dagger$-algebra* (A, ρ), briefly written A_ρ, is a C^*-algebra A with a $*$-homomorphism $\rho\colon \mathfrak{B}^\dagger \to M(A)$. A *morphism* of C^*-$\mathfrak{B}^\dagger$-algebras A_ρ and B_σ is a $*$-homomorphism $\pi\colon A \to B$ satisfying $\sigma(x)\pi(a) = \pi(\rho(x)a)$ for all $x \in \mathfrak{B}^\dagger, a \in A$. We denote the category of all C^*-$\mathfrak{B}^\dagger$-algebras by $\mathbf{C}^*_{\mathfrak{B}^\dagger}$.

A *(nondegenerate) C^*-$\mathfrak{b}$-algebra* is a pair $A^\alpha_H = (H_\alpha, A)$, where H_α is a C^*-$\mathfrak{b}$-module, $A \subseteq \mathcal{L}(H)$ a (nondegenerate) C^*-algebra, and $\rho_\alpha(\mathfrak{B}^\dagger)A \subseteq A$. A (semi-)morphism between C^*-$\mathfrak{b}$-algebras A^α_H, B^β_K is a $*$-homomorphism $\pi\colon A \to B$ satisfying the condition $\beta = [\mathcal{L}^\pi_{(\mathrm{s})}(H_\alpha, K_\beta)\alpha]$, where $\mathcal{L}^\pi_{(\mathrm{s})}(H_\alpha, K_\beta) := \{T \in \mathcal{L}_{(\mathrm{s})}(H_\alpha, K_\beta) : \forall a \in A : Ta = \pi(a)T\}$. We denote the category of all C^*-$\mathfrak{b}$-algebras together with all (semi-)morphisms by $\mathbf{C}^{*\,(\mathrm{s})}_{\mathfrak{b}}$.

We first give some examples of C^*-$\mathfrak{b}$-algebras and then study the relation between $\mathbf{C}^*_{\mathfrak{B}^\dagger}$ and $\mathbf{C}^*_{\mathfrak{b}}$.

EXAMPLE 3.2. (i) If H is a Hilbert space and $A \subseteq \mathcal{L}(H)$ a C^*-algebra, then A^α_H is a C^*-$\mathfrak{t}$-algebra, where $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$ denotes the trivial C^*-base and $\alpha = \mathcal{L}(\mathbb{C}, H)$.

(ii) Let A^α_H be a nondegenerate C^*-$\mathfrak{b}$-algebra. If we identify $M(A)$ with a C^*-subalgebra of $\mathcal{L}(H)$ in the canonical way, $M(A)^\alpha_H$ becomes a C^*-$\mathfrak{b}$-algebra.

(iii) Let $(\mathcal{A}_i)_i$ be a family of C^*-$\mathfrak{b}$-algebras, where $\mathcal{A}_i = (\mathcal{H}_i, A_i)$ for each i. Then the c_0-sum $\bigoplus_i A_i$ and the l^∞-product $\prod_i A_i$ are naturally represented on the underlying Hilbert space of $\boxplus_i \mathcal{H}_i$, and we obtain C^*-$\mathfrak{b}$-algebras $\boxplus_i \mathcal{A}_i := (\boxplus_i \mathcal{H}_i, \bigoplus_i A_i)$

and $\prod_i \mathcal{A}_i := (\boxplus_i \mathcal{H}_i, \prod_i A_i)$. For each j, the canonical maps $A_j \to \bigoplus_i A_i \to \prod_i A_i \to A_j$ are evidently morphisms of C^*-$\mathfrak{b}$-algebras $\mathcal{A}_j \to \boxplus_i \mathcal{A}_i \to \prod_i \mathcal{A}_i \to \mathcal{A}_j$.

The following example is a continuation of Example 2.11.

EXAMPLE 3.3. Let B be a C^*-algebra with a KMS-state μ and associated C^*-base $\mathfrak{b}$, and let A be a C^*-algebra containing B with a conditional expectation $\phi\colon A \to B$ as in Example 2.11. With the notation introduced before, $\pi_\nu(A)^\beta_H$ is a nondegenerate C^*-$\mathfrak{b}$-algebra because $\rho_\beta(\mathfrak{B})\pi_\nu(A) = \pi_\nu(B)\pi_\nu(A) \subseteq \pi_\nu(A)$, and similarly, $(\pi^{\mathrm{op}}_\nu(A^{\mathrm{op}}))^\alpha_H$ is a nondegenerate C^*-$\mathfrak{b}^\dagger$-algebra ([27], Section 2–3).

The categories $\mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$ and $\mathbf{C}^*_{\mathfrak{B}^\dagger}$ are related by a pair of adjoint functors, as we shall see now.

LEMMA 3.4. *Let π be a semi-morphism of C^*-$\mathfrak{b}$-algebras A^α_H and B^β_K. Then π is normal and $\pi(a\rho_\alpha(x)) = \pi(a)\rho_\beta(x)$ for all $x \in \mathfrak{B}^\dagger$, $a \in A$.*

Proof. Let $T, T' \in \mathcal{L}^\pi_{\mathrm{s}}(H_\alpha, K_\beta)$, $\xi, \xi' \in \alpha$, $\zeta, \zeta' \in \mathfrak{K}$, $a \in A$, and $x \in \mathfrak{B}^\dagger$. Then $\langle T\xi\zeta | \pi(a)T'\xi'\zeta'\rangle = \langle \xi\zeta | aT^*T'\xi'\zeta'\rangle$ and

$$\pi(a\rho_\alpha(x))T\xi\zeta = Ta\rho_\alpha(x)\xi\zeta = \pi(a)T\xi x\zeta = \pi(a)\rho_\beta(x)T\xi\zeta$$

because $T\xi \in \beta$. Now, the assertions follow since $K = [\mathcal{L}^\pi_{\mathrm{s}}(H_\alpha, K_\beta)\alpha\mathfrak{K}]$. ∎

The preceding lemma shows that there exists a forgetful functor

$$\mathbf{U}_{\mathfrak{b}} \colon \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}} \to \mathbf{C}^*_{\mathfrak{B}^\dagger}, \quad \begin{cases} A^\alpha_H \mapsto A_{\rho_\alpha} & \text{for each object } A^\alpha_H, \\ \pi \mapsto \pi & \text{for each morphism } \pi. \end{cases}$$

We shall see that this functor has a partial adjoint that associates to a C^*-$\mathfrak{B}^\dagger$-algebra a universal representation on a C^*-$\mathfrak{b}$-module. For the discussion, we fix a C^*-$\mathfrak{B}^\dagger$-algebra C_σ.

DEFINITION 3.5. A *representation of C_σ in $\mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$* is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A} = A^\alpha_H \in \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$ and $\phi \in \mathbf{C}^*_{\mathfrak{B}^\dagger}(C_\sigma, \mathbf{U}\mathcal{A})$. Denote by $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ the category of all such representations, where the *morphisms* between objects $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$ are all $\pi \in \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}(\mathcal{A}, \mathcal{B})$ satisfying $\psi = \mathbf{U}\pi \circ \phi$.

Note that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is just the comma category $(C_\sigma \downarrow \mathbf{U}_{\mathfrak{b}})$ [19]. Unfortunately, we have no general method like the GNS-construction to produce representations of C_σ in in $\mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$. In particular, we have no good criteria to decide whether there are any and, if so, whether there exists a faithful one. However, we now show that if there are any representations, then there also is a universal one. The proof involves the following direct product construction.

EXAMPLE 3.6. Let $(\mathcal{A}_i, \phi_i) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ for all i, where $\mathcal{A}_i = (\mathcal{H}_i, A_i)$, and define $\phi\colon C \to \prod_i A_i$ by $c \mapsto (\phi_i(c))_i$. Then $\prod_i(\mathcal{A}_i, \phi_i) := (\prod_i \mathcal{A}_i, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$, and the canonical maps $\mathcal{A}_j \to \prod_i \mathcal{A}_i \to \mathcal{A}_j$ are morphisms between $(\mathcal{A}_j, \phi_j)$ and $(\prod_i \mathcal{A}_i, \phi)$ for each j.

PROPOSITION 3.7. *If the category $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is non-empty, then it has an initial object.*

Proof. Assume that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is non-empty. We first use a cardinality argument to show that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ has an initial set of objects, and then apply the direct product construction to this set to obtain an initial object.

Given a topological vector space X and a cardinal number c, let us call X *c-separable* if X has a linearly dense subset of cardinality c. Choose a cardinal number d such that $\mathfrak{B}$ and $C \times \mathfrak{K}$ are d-separable, and let $e := |\mathbb{N}| \sum_n d^n$. Then the isomorphism classes of e-separable Hilbert C^*-$\mathfrak{B}$-modules form a set, and hence there exists a set $\mathcal{R}$ of objects in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ such that each $(A^\alpha_H, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ with e-separable α is isomorphic to some element of $\mathcal{R}$. Let $(A^\alpha_H, \phi) = \boxplus_{R \in \mathcal{R}} R$. We show that $(\phi(C)^\alpha_H, \phi)$ is initial in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$.

Let $(B^\beta_K, \psi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$. We shall show that there exists a morphism $\pi \in \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}(\phi(C)^\alpha_H, B^\beta_K)$ such that $\psi = \pi \circ \phi$. Uniqueness of such a π is evident. Let $\xi \in \beta$ be given. Since $\mathfrak{B}$ and $C \times \mathfrak{K}$ are d-separable, we can inductively choose subspaces $\beta_0 \subseteq \beta_1 \subseteq \cdots \subseteq \beta$ and cardinal numbers $d_0, d_1, \ldots$ such that $\xi \in \beta_0$, $[\beta_0^* \beta_0] = \mathfrak{B}$, $d_0 \leqslant 2d + 1$, β_0 is d_0-separable and for all $n \geqslant 0$,

$$\beta_n \mathfrak{B} \subseteq \beta_{n+1}, \quad \psi(C)\beta_n \mathfrak{K} \subseteq [\beta_{n+1}\mathfrak{K}], \quad d_{n+1} \leqslant |\mathbb{N}| d d_n, \quad \beta_{n+1} \text{ is } d_{n+1}\text{-separable.}$$

Let $\widetilde{\beta} := [\bigcup_n \beta_n] \subseteq \beta$ and $\widetilde{K} := [\widetilde{\beta}\mathfrak{K}] \subseteq K$. By construction, $[\widetilde{\beta}^* \widetilde{\beta}] = \mathfrak{B}$, $\widetilde{\beta}\mathfrak{B} \subseteq \widetilde{\beta}$, $\psi(C)\widetilde{K} \subseteq \widetilde{K}$, so that $(\psi(C)|_{\widetilde{K}})^{\widetilde{\beta}}_{\widetilde{K}}$ is in $\mathbf{C}^*_{\mathfrak{b}}$. Define $\widetilde{\psi} \colon C \to \psi(C)|_{\widetilde{K}}$ by $c \mapsto \psi(c)|_{\widetilde{K}}$. Then $(\widetilde{\psi}(C)^{\widetilde{\beta}}_{\widetilde{K}}, \widetilde{\psi})$ is in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$. Since $\widetilde{\beta}$ is e-separable, $(\widetilde{\psi}(C)^{\widetilde{\beta}}_{\widetilde{K}}, \widetilde{\psi})$ is isomorphic to some element of $\mathcal{R}$. Hence, there exists a surjection $\widetilde{T} \colon H \to \widetilde{K}$ such that $\widetilde{T}\alpha = \widetilde{\beta}$, and the composition with the inclusion $\widetilde{K} \to K$ gives an operator $T \in \mathcal{L}_{\mathrm{s}}(H_\alpha, K_\beta)$ such that $\psi(c)T = T\phi(c)$ for all $c \in C$. Since $\xi \in \widetilde{\beta} = T\alpha$ and $\xi \in \beta$ was arbitrary, we can conclude the existence of π as desired. ∎

Evidently, every morphism Φ between C^*-$\mathfrak{B}^\dagger$-algebras C_σ and D_τ yields a functor

$$\Phi^* \colon \mathbf{Rep}_{\mathfrak{b}}(D_\tau) \to \mathbf{Rep}_{\mathfrak{b}}(C_\sigma), \quad \begin{cases} (A^\alpha_H, \phi) \mapsto (A^\alpha_H, \phi \circ \Phi) & \text{for objects } (A^\alpha_H, \phi), \\ \pi \mapsto \pi & \text{for morphisms } \pi. \end{cases}$$

Denote by $\mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger}$ the full subcategory of $\mathbf{C}^*_{\mathfrak{B}^\dagger}$ formed by all objects C_σ for which $\mathbf{Rep}(C_\sigma)$ is non-empty.

THEOREM 3.8. *There exist a functor $\mathbf{R}_{\mathfrak{b}} \colon \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger} \to \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$ and natural transformations $\eta \colon \mathrm{id}_{\mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger}} \to \mathbf{U}_{\mathfrak{b}}\mathbf{R}_{\mathfrak{b}}$ and $\varepsilon \colon \mathbf{R}_{\mathfrak{b}}\mathbf{U}_{\mathfrak{b}} \to \mathrm{id}_{\mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}}$ such that for every $C_\sigma, D_\tau \in \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger}$, $\Phi \in \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger}(C_\sigma, D_\tau)$, $A^\alpha_H \in \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$,*

(i) $\mathbf{R}_{\mathfrak{b}}(C_\sigma) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ *is an initial object and* $\mathbf{R}_{\mathfrak{b}}(\Phi)$ *is the unique morphism from* $\mathbf{R}_{\mathfrak{b}}(C_\sigma)$ *to* $\Phi^*(\mathbf{R}_{\mathfrak{b}}(D_\tau))$*;*

(ii) $\eta_{C_\sigma} = \phi$ *if* $\mathbf{R}_{\mathfrak{b}}(C_\sigma) = (B_K^\beta, \phi)$*, and* $\varepsilon_{A_H^\alpha}$ *is the unique morphism from* $\mathbf{R}_{\mathfrak{b}}\mathbf{U}_{\mathfrak{b}}(A_H^\alpha)$ *to* $(A_H^\alpha, \mathrm{id}_A)$*.*

Moreover, $\mathbf{R}_{\mathfrak{b}}$ *is left adjoint to* $\mathbf{U}_{\mathfrak{b}}$ *and* η, ε *are the unit and counit of the adjunction, respectively.*

The proof follows from Proposition 3.7 and Section IV Theorem 2 of [19].

We next consider C^*-algebras acting on C^*-bimodules. Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ be a C^*-base.

DEFINITION 3.9. A C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-*algebra* is a triple (A, ρ, σ), briefly written $A_{\rho,\sigma}$, where A_ρ is a C^*-$\mathfrak{A}$-algebra, A_σ is a C^*-$\mathfrak{B}^\dagger$-algebra, and $[\rho(\mathfrak{A}), \sigma(\mathfrak{B}^\dagger)] = 0$. A *morphism* of C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebras is a morphism of the underlying C^*-$\mathfrak{A}$-algebras and C^*-$\mathfrak{B}^\dagger$-algebras. We denote the category of all C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebras by $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$.

A *(nondegenerate)* C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-*algebra* is a pair $A_H^{\alpha,\beta} = ({}_\alpha H_\beta, A)$, where ${}_\alpha H_\beta$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, A_H^α is a (nondegenerate) C^*-$\mathfrak{a}^\dagger$-algebra, and A_H^β is a C^*-$\mathfrak{b}$-algebra. A (semi-)morphism of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$ is a $*$-homomorphism $\pi\colon A \to B$ satisfying $\gamma = [\mathcal{L}^\pi_{(\mathrm{s})}({}_\alpha H_\beta, {}_\gamma K_\delta)\alpha]$ and $\delta = [\mathcal{L}^\pi_{(\mathrm{s})}({}_\alpha H_\beta, {}_\gamma K_\delta)\beta]$, where $\mathcal{L}^\pi_{(\mathrm{s})}({}_\alpha H_\beta, {}_\gamma K_\delta) := \{T \in \mathcal{L}_{(\mathrm{s})}({}_\alpha H_\beta, {}_\gamma K_\delta) : \forall a \in A : Ta = \pi(a)T\}$. We denote the category of all C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras together with all (semi-)morphisms by $\mathbf{C}^{*(\mathrm{s})}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$.

REMARK 3.10. Note that the condition on a (semi-)morphism between C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras above is stronger than just being a (semi-)morphism of the underlying C^*-$\mathfrak{a}^\dagger$-algebras and C^*-$\mathfrak{b}$-algebras.

Examples 3.2(ii) and (iii) naturally extend to C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras, and the categories $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ and $\mathbf{C}^{*\mathrm{s}}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$ are again related by a pair of adjoint functors.

THEOREM 3.11. *There exists a functor* $\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})}\colon \mathbf{C}^{*\mathrm{s}}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \to \mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$*, given by* $A_H^{\alpha,\beta} \mapsto A_{\rho_\alpha,\rho_\beta}$ *on objects and* $\pi \mapsto \pi$ *on morphisms. Denote by* $\mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ *the full subcategory of* $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ *formed by all objects* $C_{\sigma,\rho}$ *for which the comma category* $(C_{\sigma,\rho} \downarrow \mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})})$ *is non-empty. Then the corestriction of* $\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$ *to* $\mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ *has a left adjoint* $\mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})}\colon \mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \to \mathbf{C}^{*\mathrm{s}}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$*.*

Proof. The proof proceeds as in the case of C^*-$\mathfrak{b}$-algebras with straightforward modifications, so we only indicate the necessary changes for the second half of the proof of Proposition 3.7. Given a C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebra $C_{\sigma,\tau}$ and a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra $B_K^{\gamma,\delta}$ with a morphism $\psi\colon C_{\sigma,\tau} \to B_{\rho_\gamma,\rho_\delta}$, one constructs $\widetilde{\gamma} \subseteq \gamma$ and $\widetilde{\delta} \subseteq \delta$ for given $\xi \in \gamma$, $\eta \in \delta$ as follows. One first fixes a cardinal number d such that

$\mathfrak{A}, \mathfrak{A}^\dagger, \mathfrak{H}, \mathfrak{B}, \mathfrak{B}^\dagger, \mathfrak{H}$ are d-separable, and then inductively chooses cardinal numbers $d_0, d_1, \dots$ and closed subspaces $\gamma_0 \subseteq \gamma_1 \subseteq \cdots \subseteq \gamma$ and $\delta_0 \subseteq \delta_1 \subseteq \cdots \subseteq \delta$ such that, for all $n \geqslant 0$,

$$\xi \in \gamma_0, \quad \eta \in \delta_0, \quad [\gamma_0^* \gamma_0] = \mathfrak{A}^\dagger, \quad [\delta_0^* \delta_0] = \mathfrak{B}, \quad d_0 \leqslant 2d+1, \quad \gamma_0, \delta_0 \text{ are } d_0\text{-separable},$$
$$\rho_\delta(\mathfrak{B}^\dagger)\gamma_n + \gamma_n \mathfrak{A}^\dagger \subseteq \gamma_{n+1}, \quad \rho_\gamma(\mathfrak{A})\gamma_n + \delta_n \mathfrak{B} \subseteq \delta_{n+1},$$
$$\psi(C)\gamma_n \mathfrak{H} + \psi(C)\delta_n \mathfrak{K} \subseteq [\gamma_{n+1}\mathfrak{H}] \cap [\delta_{n+1}\mathfrak{K}],$$
$$d_{n+1} \leqslant |\mathbb{N}| d^2 d_n, \quad \gamma_{n+1}, \delta_{n+1} \text{ are } d_{n+1}\text{-separable},$$

and finally lets $\widetilde{\gamma} := [\bigcup_n \gamma_n], \widetilde{\delta} := [\bigcup_n \delta_n], \widetilde{K} := [\widetilde{\gamma}\mathfrak{H}] = [\widetilde{\delta}\mathfrak{K}]$. ∎

REMARK 3.12. Let $C_{\rho,\sigma}$ be a C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebra, $A_H^{\alpha,\beta} = \mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})}(C_{\rho,\sigma})$, and $\phi = \eta_{C_{\rho,\sigma}} : C_{\rho,\sigma} \to A_{\rho_\alpha,\rho_\beta}$ the morphism given by the unit of the adjunction above. Then $(A^\alpha, \phi) \in \mathbf{Rep}_{\mathfrak{a}^\dagger}(C_\rho)$ and $(A^\beta, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ and therefore, we have semi-morphisms $\mathbf{R}_{\mathfrak{a}^\dagger}(C_\sigma) \to A_H^\alpha$ and $\mathbf{R}_{\mathfrak{b}}(C_\rho) \to A_H^\beta$.

3.3. THE SPATIAL FIBER PRODUCT FOR C^*-ALGEBRAS ON C^*-MODULES. Our definition of the fiber product of C^*-algebras represented on C^*-modules, more precisely, step (B) in the introduction, involves the following construction.

Let H and K be Hilbert spaces, $I \subseteq \mathcal{L}(H, K)$ a subspace and $A \subseteq \mathcal{L}(H)$ a C^*-algebra such that $[IH] = K$, $[I^*K] = H$, $[II^*I] = I$, $I^*IA \subseteq A$. We define a new C^*-algebra

$$\mathrm{Ind}_I(A) := \{T \in \mathcal{L}(K) : TI + T^*I \subseteq [IA]\} \subseteq \mathcal{L}(K).$$

DEFINITION 3.13. The *I-strong-$*$, I-strong, and I-weak topology* on $\mathcal{L}(K)$ are the topologies induced by the families of semi-norms $T \mapsto \|T\xi\| + \|T^*\xi\|$ $(\xi \in I)$, $T \mapsto \|T\xi\|$ $(\xi \in I)$, and $T \mapsto \|\xi^* T \xi'\|$ $(\xi, \xi' \in I)$, respectively. Given a subset $X \subseteq \mathcal{L}(K)$, denote by $[X]_I$ the closure of $\operatorname{span} X$ with respect to the I-strong-$*$ topology.

Evidently, the multiplication in $\mathcal{L}(K)$ is separately continuous with respect to the topologies introduced above, and the involution $T \mapsto T^*$ is continuous with respect to the I-strong-$*$ and the I-weak topology. Define $\rho_I : (I^*I)' \to \mathcal{L}(K)$ as in Lemma 2.5.

LEMMA 3.14. (i) $[I^* \mathrm{Ind}_I(A) I] \subseteq A$ *and* $\mathrm{Ind}_I(A) = [IAI^*]_I$.
(ii) $\mathrm{Ind}_I(M(A)) \subseteq M(\mathrm{Ind}_I(A))$.
(iii) $\mathrm{Ind}_I(A) \subseteq \mathcal{L}(K)$ *is nondegenerate if and only if* $A \subseteq \mathcal{L}(H)$ *is nondegenerate.*
(iv) *If* $A \subseteq \mathcal{L}(H)$ *is nondegenerate, then* $A' \subseteq (I^*I)'$ *and* $\mathrm{Ind}_I(A) \subseteq \rho_I(A')'$.

Proof. (i) First, $[I^* \mathrm{Ind}_I(A) I] \subseteq [I^*IA] \subseteq A$ by definition and $[IAI^*]_I \subseteq \mathrm{Ind}_I(A)$ because $[IAI^*]_I I \subseteq [IAI^*I] \subseteq [IA]$. To see that $[IAI^*]_I \supseteq \mathrm{Ind}_I(A)$, choose a bounded approximate unit $(u_\nu)_\nu$ for the C^*-algebra $[II^*]$ and observe that for each $T \in \mathrm{Ind}_I(A)$, the net $(u_\nu T u_\nu)_\nu$ lies in the space $[II^* \mathrm{Ind}_I(A) II^*] \subseteq$

$[IAI^*]$ and converges to T in the I-strong-$*$ topology because $\lim_\nu T^{(*)}u_\nu\xi = T^{(*)}\xi \in [IA]$ for all $\xi \in I$ and $\lim_\nu u_\nu\omega = \omega$ for all $\omega \in [IA]$.

(ii) If $S \in \mathrm{Ind}_I(M(A))$ and $T \in \mathrm{Ind}_I(A)$, then $ST \in \mathrm{Ind}_I(A)$ because $STI \subseteq [SIA] \subseteq [IM(A)A] = [IA]$ and $T^*S^*I \subseteq [TIM(A)] \subseteq [IAM(A)] = [IA]$.

(iii) If $\mathrm{Ind}_I(A) \subseteq \mathcal{L}(K)$ is nondegenerate, then $[AH] \supseteq [I^*\,\mathrm{Ind}_I(A)IH] = [I^*\,\mathrm{Ind}_I(A)K] = [I^*K] = H$. Conversely, if A is nondegenerate, then $[IAI^*]$ and hence also $\mathrm{Ind}_I(A)$ is nondegenerate.

(iv) Assume that A is nondegenerate. Then $I^*I \subseteq M(A) \subseteq \mathcal{L}(H)$ and hence $A' \subseteq (I^*I)'$. For all $x \in \mathrm{Ind}_I(A), y \in A', S, T \in I$, we have $S^*x\rho_I(y)T = S^*xTy = yS^*xT = S^*\rho_I(y)xT$ because $S^*xT \in A$, and since $[IH] = K$, we can conclude that $x\rho_I(y) = \rho_I(y)x$. ∎

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, A^β_H a C^*-$\mathfrak{b}$-algebra, and B^γ_K a C^*-$\mathfrak{b}^\dagger$-algebra. We apply the construction above to A, B and $|\gamma\rangle_2 \subseteq \mathcal{L}(H, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$, $|\beta\rangle_1 \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$, respectively, and define the *fiber product* of A^β_H and B^γ_K to be the C^*-algebra

$$\begin{aligned} A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B &:= \mathrm{Ind}_{|\gamma\rangle_2}(A) \cap \mathrm{Ind}_{|\beta\rangle_1}(B) \\ &= \{T \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K) : T^{(*)}|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A] \text{ and } T^{(*)}|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]\}. \end{aligned}$$

The spaces of operators involved are visualized as arrows in the following diagram:

$$\begin{tikzcd} H \arrow[r, "|\gamma\rangle_2"] \arrow[d, "A"'] & H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K \arrow[d, "A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B"] & K \arrow[l, "|\beta\rangle_1"'] \arrow[d, "B"] \\ H \arrow[r, "|\gamma\rangle_2"] & H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K & K \arrow[l, "|\beta\rangle_1"'] \end{tikzcd}$$

Even in very special situations, it seems to be difficult to give a more explicit description of the fiber product. The main drawback of the definition above is that apart from special situations, we do not know how to produce elements of the fiber product.

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be further C^*-bases.

PROPOSITION 3.15. *Let $\mathcal{A} = A^{\alpha,\beta}_H$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra and $\mathcal{B} = B^{\gamma,\delta}_K$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra. Then $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} := ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-algebra.*

Proof. The product $X := \rho_{(\alpha\triangleleft\gamma)}(\mathfrak{A}^\dagger)(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)$ is contained in $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$ because

$$X|\beta\rangle_1 \subseteq [|\rho_\alpha(\mathfrak{A})\beta\rangle_1 B] = [|\beta\rangle_1 B], \quad X^*|\beta\rangle_1 = (A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\rho_\alpha(\mathfrak{A})\beta\rangle_1 \subseteq [|\beta\rangle_1 B],$$

$$X|\gamma\rangle_2 \subseteq [|\gamma\rangle_2\rho_\alpha(\mathfrak{A})A] \subseteq [|\gamma\rangle_2 A], \quad X^*|\gamma\rangle_2 = (A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\gamma\rangle_2\rho_\alpha(\mathfrak{A}) \subseteq [|\gamma\rangle_2 A],$$

by equation (2.3). A similar argument proves $\rho_{(\beta\triangleright\delta)}(\mathfrak{C}^\dagger)(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B) \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$. ∎

In the situation above, we call $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}$ the *fiber product* of $\mathcal{A}$ and $\mathcal{B}$. Forgetting α or δ, we obtain a C^*-$\mathfrak{c}$-algebra $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B_\delta := A^\beta_H \underset{\mathfrak{b}}{*} B^{\gamma,\delta}_H := (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)$ and a C^*-$\mathfrak{a}^\dagger$-algebra ${}_\alpha A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B = A^{\alpha,\beta}_H \underset{\mathfrak{b}}{*} B^\gamma_K$.

Denote by $A' \subseteq \mathcal{L}(H)$ and $B' \subseteq \mathcal{L}(K)$ the commutants of A and B, respectively, and let

$$A^{(\beta)} := A \cap \mathcal{L}(H_\beta), \quad B^{(\gamma)} := B \cap \mathcal{L}(K_\gamma), \quad X := (A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}) + (\mathrm{id} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}),$$
$$M_{\mathrm{s}}(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}) := \{T \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K) : TX, XT \subseteq A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}\}.$$

LEMMA 3.16. *The following relations hold:*

(i) $\langle\beta|_1(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\beta\rangle_1 \subseteq B$ *and* $\langle\gamma|_2(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\gamma\rangle_2 \subseteq A$ *and* $M(A)_\beta \underset{\mathfrak{b}}{*} {}_\gamma M(B) \subseteq M(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)$.

(ii) $A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$.

(iii) *If* $[A^{(\beta)}\beta] = \beta$ *and* $[B^{(\gamma)}\gamma] = \gamma$, *then* $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$ *is nondegenerate and* $M_{\mathrm{s}}(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}) \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$.

(iv) *If* $\rho_\beta(\mathfrak{B}^\dagger) \subseteq A$, *then* $\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$. *If* $\rho_\gamma(\mathfrak{B}) \subseteq B$, *then* $A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$.

(v) $\mathrm{id}_{(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)} \in A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$ *if and only if* $\rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ *and* $\rho_\gamma(\mathfrak{B}) \subseteq B$.

(vi) *If* $A^{\alpha,\beta}_H$ *is a* C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-*algebra and* $B^{\gamma,\delta}_K$ *a* C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-*algebra such that* $\rho_\alpha(\mathfrak{A}) + \rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ *and* $\rho_\gamma(\mathfrak{B}) + \rho_\delta(\mathfrak{C}^\dagger) \subseteq B$, *then* $\rho_{(\alpha\triangleleft\gamma)}(\mathfrak{A}) + \rho_{(\beta\triangleright\delta)}(\mathfrak{C}^\dagger) \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$.

(vii) *If* $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$ *is nondegenerate, then the* C^*-*algebra* $[\beta^* A \beta] \cap [\gamma^* B \gamma] \subseteq \mathcal{L}(\mathfrak{K})$ *is nondegenerate.*

(viii) *If* A *and* B *are nondegenerate, then* $A' \subseteq \rho_\beta(\mathfrak{B}^\dagger)'$, $B' \subseteq \rho_\gamma(\mathfrak{B})'$, *and* $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \subseteq \rho_{|\gamma\rangle_2}(A') \cap \rho_{|\beta\rangle_1}(B') = (A' \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)' \cap (\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')'$.

Proof. (i) Immediate from Lemma 3.14.

(ii) Follows from $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\beta\rangle_1 \subseteq [|A^{(\beta)}\beta\rangle_1 B^{(\gamma)}] \subseteq [|\beta\rangle_1 B]$ and $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\gamma\rangle_2 \subseteq [|B^{(\gamma)}\gamma\rangle_1 A^{(\beta)}] \subseteq [|\gamma\rangle_2 A]$.

(iii) Assume $[A^{(\beta)}\beta] = \beta$ and $[B^{(\gamma)}\gamma] = \gamma$. Then $A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B$ is nondegenerate and for each $T \in M_{\mathrm{s}}(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})$, we have $T|\beta\rangle_1 \subseteq [T(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\beta\rangle_1] \subseteq [(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\beta\rangle_1] \subseteq [|\beta\rangle_1 B]$ and similarly $T^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]$, $T|\gamma\rangle_2 + T^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$.

(iv) If $\rho_\gamma(\mathfrak{B}) \subseteq B$, then $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)|\gamma\rangle_2 = |\gamma\rangle_2 A^{(\beta)}$ and $[(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)|\beta\rangle_1] \subseteq |\beta\rangle_1 = [|\beta\mathfrak{B}\rangle_1] = [|\beta\rangle_1 \rho_\gamma(\mathfrak{B})] \subseteq [|\beta\rangle_1 B]$. The second assertion follows similarly.

(v) If $\mathrm{id}_{(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)} \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, then $\rho_\beta(\mathfrak{B}^\dagger) = [\langle\gamma|_2|\gamma\rangle_2] \subseteq A$, $\rho_\gamma(\mathfrak{B}) = [\langle\beta|_1|\beta\rangle_1] \subseteq B$ by (i). Conversely, if the last two inclusions hold, then $|\gamma\rangle_2 = [|\gamma\mathfrak{B}^\dagger\rangle_2] = [|\gamma\rangle_2 \rho_\beta(\mathfrak{B}^\dagger)] \subseteq [|\gamma\rangle_2 A]$ and similarly $|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]$, whence $\mathrm{id}_{(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)} \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$.

(vi) Immediate from (iv).

(vii) The C^*-algebra $C := [\beta^* A \beta] \cap [\gamma^* B \gamma]$ contains $\beta^*\langle\gamma|_2(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\gamma\rangle_2\beta = \gamma^*\langle\beta|_1(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\beta\rangle_1\gamma$. Hence, $[C\mathfrak{K}] \supseteq [\beta^*\langle\gamma|_2(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)] = \mathfrak{K}$ if $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ is nondegenerate.

(viii) Immediate from Lemma 3.14. ∎

Even in the case of a trivial C^*-base, we have no explicit description of the fiber product.

EXAMPLE 3.17. Let H and K be Hilbert spaces, $\beta = \mathcal{L}(\mathbb{C}, H)$, $\gamma = \mathcal{L}(\mathbb{C}, K)$, $\mathfrak{b} = \mathfrak{t}$ the trivial C^*-base $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, and identify $H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K$ with $H \otimes K$ as in Example 2.12.

(i) Let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be nondegenerate C^*-algebras. Then $A^{(\beta)} = A$, $B^{(\gamma)} = B$, and by Lemma 3.16, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ contains the minimal tensor product $A \otimes B \subseteq \mathcal{L}(H \otimes K)$ and $M_{\mathrm{s}}(A \otimes B) = \{T \in \mathcal{L}(H \otimes K) : T^{(*)}(1 \otimes B), T^{(*)}(A \otimes 1) \subseteq A \otimes B\}$. If A or B is non-unital, then $\mathrm{id}_{H \otimes K} \notin A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ by Lemma 3.16 and so $M(A \otimes B) \not\subseteq A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$. In Example 5.3(iii), we shall see that also $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \not\subseteq M(A \otimes B)$ is possible.

(ii) Assume that $H = K = l^2(\mathbb{N})$ and identify $\beta = \gamma = \mathcal{L}(\mathbb{C}, H)$ with H. Then the flip $\Sigma\colon H \otimes H \to H \otimes H$, $\zeta \otimes \eta \mapsto \eta \otimes \zeta$, is not contained in $\mathcal{L}(H)_\beta \underset{\mathfrak{b}}{*}{}_\gamma \mathcal{L}(H)$. Indeed, let $(\zeta_\nu)_\nu$ be an orthonormal basis for H and let $\eta \in H$ be non-zero. Then $\langle\zeta_\nu|_1 \Sigma|\eta\rangle_1 = |\eta\rangle\langle\zeta_\nu|$ for each ν and hence $\sum_\nu \langle\zeta_\nu|_1\Sigma|\eta\rangle_1$ does not converge in norm. On the other hand, one easily verifies that $\sum_\nu \langle\zeta_\nu|_1 S$ converges in norm for each $S \in [|H\rangle_1 \mathcal{L}(H)]$. Hence, $\Sigma|\eta\rangle_1 \notin [|H\rangle_1 \mathcal{L}(H)]$.

3.4. FUNCTORIALITY AND SLICE MAPS. We show that the fiber product is functorial, and consider various slice maps. The results concerning functoriality were stated in slightly different form in [27], [28], [29] with proofs referring to unpublished material. We use the opportunity to rectify this situation. As before, let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$, $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$, $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be C^*-bases.

LEMMA 3.18. *Let π be a (semi-)morphism of C^*-$\mathfrak{b}$-algebras A^β_H and C^λ_L, let ${}_\gamma K_\delta$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module, and let* $I := \mathcal{L}^\pi_{(\mathrm{s})}(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id} \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$.

(i) *The pairs* $\mathcal{X} := (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, (I^*I)')$ *and* $\mathcal{Y} := (L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, (II^*)')$ *are nondegenerate* C^*-$\mathfrak{c}$*-algebras.*

(ii) *There is a unique* $\rho_I \in \mathrm{Mor}_{(\mathrm{s})}(\mathcal{X}, \mathcal{Y})$ *satisfying* $\rho_I(x)S = Sx$ *for all* $x \in (I^*I)', S \in I$.

(iii) *There is a unique linear contraction* $j_\pi \colon [|\gamma\rangle_2 A] \to [|\gamma\rangle_2 C]$ *given by* $|\eta\rangle_2 a \mapsto |\eta\rangle_2 \pi(a)$.

(iv) $\mathrm{Ind}_{|\gamma\rangle_2}(A) \subseteq (I^*I)'$ *and* $\rho_I(x)|\eta\rangle_2 = j_\pi(x|\eta\rangle_2)$ *for all* $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$, $\eta \in \gamma$.

(v) *Let* B_K^γ *be a* C^*-$\mathfrak{b}^\dagger$*-algebra. Then* $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \subseteq (I^*I)'$ *and* $\rho_I(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*} {}_\gamma B$.

Proof. (i) First, $(I^*I)'$ and $(II^*)'$ are nondegenerate C^*-algebras, and second, $\rho_{(\beta \triangleright \delta)}(\mathfrak{C}^\dagger) = \mathrm{id}_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma \rho_\delta(\mathfrak{C}^\dagger) \subseteq (I^*I)'$ and $\rho_{(\lambda \triangleright \delta)}(\mathfrak{C}^\dagger) = \mathrm{id}_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma \rho_\delta(\mathfrak{C}^\dagger) \subseteq (II^*)'$.

(ii) There exists a unique $*$-homomorphism $\rho_I \colon (I^*I)' \to (II^*)'$ satisfying the formula above by Lemma 2.5, and this is a (semi-)morphism because $[I(\beta \triangleright \delta)] = [\lambda \triangleright \delta]$ by assumption on π.

(iii) For all $\eta_1, \dots, \eta_n \in \gamma$ and $a_1, \dots, a_n \in A$, we have

$$\Big\| \sum_j |\eta_j\rangle_2 \pi(a_j) \Big\|^2 = \Big\| \sum_{i,j} \pi(a_i^*) \rho_\lambda(\eta_i^* \eta_j) \pi(a_j) \Big\| \leqslant \Big\| \sum_{i,j} a_i^* \rho_\beta(\eta_i^* \eta_j) a_j \Big\| = \Big\| \sum_j |\eta_j\rangle_2 a_j \Big\|^2$$

by Lemma 3.4. The claim follows.

(iv) The first assertion follows from Lemma 3.14 and the relation $I^*I \subseteq A' \underset{\mathfrak{b}}{\otimes} \mathrm{id} = \rho_{|\gamma\rangle_2}(A')$, and the second one from the fact that for all $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A), \eta \in \gamma, S \in \mathcal{L}^\pi_{(\mathrm{s})}(H_\beta, L_\lambda)$, we have $\rho_I(x)|\eta\rangle_2 S = \rho_I(x)(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\eta\rangle_2 = (S \underset{\mathfrak{b}}{\otimes} \mathrm{id}) x |\eta\rangle_2 = j_\pi(x|\eta\rangle_2)S$.

(v) First, $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \subseteq (I^*I)'$ by Lemma 3.16. The second assertion follows from the relations

$$\begin{aligned}
&\rho_I(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\gamma\rangle_2 \subseteq \rho_I(\mathrm{Ind}_{|\gamma\rangle_2}(A))|\gamma\rangle_2 \subseteq j_\pi([|\gamma\rangle_2 A]) = [|\gamma\rangle_2 C], \\
&\rho_I(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\lambda\rangle_1 = \rho_I(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)[I|\beta\rangle_1] \subseteq [I(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)|\beta\rangle_1] \subseteq [I|\beta\rangle_1 B] = [|\lambda\rangle_1 B]. \quad \blacksquare
\end{aligned}$$

THEOREM 3.19. *Let* ϕ *be a (semi-)morphism of* C^*-$(\mathfrak{a}, \mathfrak{b})$*-algebras* $\mathcal{A} = A_H^{\alpha,\beta}$ *and* $\mathcal{C} = C_L^{\kappa,\lambda}$, *and* ψ *a (semi-)morphism of* C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$*-algebras* $\mathcal{B} = B_K^{\gamma,\delta}$ *and* $\mathcal{D} = D_M^{\mu,\nu}$. *Then there exists a unique (semi-)morphism of* C^*-$(\mathfrak{a}, \mathfrak{c})$*-algebras* $\phi * \psi$ *from* $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}$ *to* $\mathcal{C} \underset{\mathfrak{b}}{*} \mathcal{D}$ *such that*

$$(\phi * \psi)(x)R = Rx \quad \textit{for all } x \in A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \textit{ and } R \in I_M J_H + J_L I_K,$$

where $I_X = \mathcal{L}^\phi_{(\mathrm{s})}(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$ *and* $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^\psi_{(\mathrm{s})}(K_\gamma, M_\mu)$ *for* $X \in \{K, M\}, Y \in \{H, L\}$.

Proof. By Lemma 3.18, we can define $\phi * \psi$ to be the restriction of $\rho_{I_M} \circ \rho_{J_H}$ or of $\rho_{J_L} \circ \rho_{I_K}$ to $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$. Uniqueness follows from the fact that $[I_M J_H(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)] = [J_L I_K(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)] = L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\mu M$. ∎

REMARK 3.20. Let A^β_H, C^λ_L be C^*-$\mathfrak{b}$-algebras, B^γ_K, D^μ_M C^*-$\mathfrak{b}^\dagger$-algebras, and $\phi \in \mathrm{Mor}(A^\beta_H, M(C)^\lambda_L)$, $\psi \in \mathrm{Mor}(B^\gamma_K, M(D)^\mu_M)$ such that $[\phi(A)C] = C$, $[\psi(B)D] = D$. Then there exists a $*$-homomorphism $\phi \underset{\mathfrak{b}}{*} \psi\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to M(C)_\lambda \underset{\mathfrak{b}}{*}{}_\mu M(D) \hookrightarrow M(C_\lambda \underset{\mathfrak{b}}{*} {}_\mu D)$, but in general, we do not know whether this is nondegenerate.

Next, we briefly discuss two kinds of slice maps on fiber products. For applications and further details, see [29]. The first class of slice maps arises from a completely positive map on one factor and takes values in operators on a certain KSGNS-construction, that is, an internal tensor product with respect to a completely positive linear map ([16], Sections 4–5).

PROPOSITION 3.21. *Let A^β_H be a C^*-$\mathfrak{b}$-algebra, K_γ a C^*-$\mathfrak{b}^\dagger$-module, L a Hilbert space, $\phi\colon [A + \rho_\beta(\mathfrak{B}^\dagger)] \to \mathcal{L}(L)$ a c.p. map, and $\theta = \phi \circ \rho_\beta\colon \mathfrak{B}^\dagger \to \mathcal{L}(L)$. Then there exists a unique c.p. map $\phi * \mathrm{id}\colon \mathrm{Ind}_{|\gamma\rangle_2}(A) \to \mathcal{L}(L_\theta \otimes \gamma)$ such that for all $\zeta, \zeta' \in L, \eta, \eta' \in \gamma, x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$,*

$$\langle \zeta \otimes \eta | (\phi * \mathrm{id})(x)(\zeta' \otimes \eta')\rangle = \langle \zeta | \phi(\langle \eta|_2 x |\eta'\rangle_2)\zeta'\rangle. \tag{3.1}$$

If B^γ_K is a C^*-$\mathfrak{b}^\dagger$-algebra, then

$$(\phi * \mathrm{id})(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B) \subseteq (\phi(A)'{}_\theta \otimes (B' \cap \mathcal{L}(K_\gamma))' \subseteq \mathcal{L}(L_\theta \otimes \gamma).$$

Proof. Let $x = (x_{ij})_{i,j} \in M_n(\mathrm{Ind}_{|\gamma\rangle_2}(A))$ be positive, let $\zeta_1, \ldots, \zeta_n \in L$, $\eta_1, \ldots, \eta_n \in \gamma$, where $n \in \mathbb{N}$, and let $d = \mathrm{diag}(|\eta_1\rangle_2, \ldots, |\eta_n\rangle_2)$. Then $0 \leqslant (\langle \eta_i|_2 x_{ij}|\eta_j\rangle_2)_{i,j} = d^* x d \leqslant \|x\| d^* d$ and hence $0 \leqslant (\phi(\langle \eta_i|_2 x_{ij} |\eta_j\rangle_2))_{i,j} \leqslant \|x\|\phi(d^* d)$ and

$$0 \leqslant \sum_{i,j} \langle \zeta_i | \phi(\langle \eta_i|_2 x_{ij}|\eta_j\rangle_2)\zeta_j\rangle \leqslant \|x\| \sum_{i,j} \langle \zeta_i \otimes \eta_i | \zeta_j \otimes \eta_j\rangle.$$

Hence, there exists a map $\phi * \mathrm{id}$ as claimed. The verification of the assertion concerning B^γ_K is straightforward. ∎

REMARK 3.22. If C^λ_L is a C^*-$\mathfrak{b}^\dagger$-algebra and $\phi|_A$ is a semi-morphism of C^*-$\mathfrak{b}^\dagger$-algebras, then the map $\phi * \mathrm{id}$ extends the fiber product $\phi * \mathrm{id}$ defined in Theorem 3.19.

Second, we show that the fiber product is functorial with respect to the following class of maps. A *spatially implemented* map of C^*-$\mathfrak{b}$-algebras A^β_H and C^λ_L is a map $\phi\colon A \to C$ admitting sequences $(S_n)_n$ and $(T_n)_n$ in $\mathcal{L}(L_\lambda, H_\beta)$ such that:

(3.2) (i) $\sum_n S^*_n S_n, \sum_n T^*_n T_n$ converge in norm, (ii) $\phi(a) = \sum_n S^*_n a T_n$ for all $a \in A$.

Note that condition (i) implies norm-convergence of the sum in (ii). Evidently, such a map is linear, it extends to a normal map $\bar{\phi}\colon A'' \to C''$, its norm is bounded by $\left\| \sum_n S_n^* S_n \right\|^{1/2} \cdot \left\| \sum_n T_n^* T_n \right\|^{1/2}$, and the composition of spatially implemented maps is spatially implemented again.

PROPOSITION 3.23. *Let ϕ be a spatially implemented map of C^*-$\mathfrak{b}$-algebras A_H^β and C_L^λ, and let $B_K^{\gamma,\delta}$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra. Then there exists a spatially implemented map from $A_H^\beta \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ to $C_H^\lambda \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ such that $\langle \eta|_2(\phi * \mathrm{id})(x)|\eta'\rangle_2 = \phi(\langle \eta|_2 x|\eta'\rangle_2)$ for all $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, $\eta, \eta' \in \gamma$.*

Proof. Uniqueness is clear. Fix sequences $(S_n)_n$, $(T_n)_n$ as in (3.2) and let $\widetilde{S}_n := S_n \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K$, $\widetilde{T}_n := T_n \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K$ for all n. Then $\widetilde{S}_n, \widetilde{T}_n \in \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta)$ for all n, we have $\left\| \sum_n \widetilde{S}_n^* \widetilde{S}_n \right\| = \left\| \sum_n S_n^* S_n \right\|$, $\left\| \sum_n \widetilde{T}_n^* \widetilde{T}_n \right\| = \left\| \sum_n T_n^* T_n \right\|$, and the map $\phi * \mathrm{id}\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$ given by $x \mapsto \sum_n \widetilde{T}_n^* x \widetilde{S}_n$ has the desired properties. Indeed, let $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, $\eta, \eta' \in \gamma$. Then $\widetilde{S}_n|\eta\rangle_2 = |\eta\rangle_2 S_n$ and $\widetilde{T}_n|\eta'\rangle_2 = |\eta'\rangle_2 T_n$ for all n, and hence $\langle \eta|_2(\phi * \mathrm{id})(x)|\eta'\rangle_2 = \phi(\langle \eta|_2 x|\eta'\rangle_2)$. It remains to show that $(\phi * \mathrm{id})(x) \in C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$. Consider the expression $(\phi * \mathrm{id})(x)|\eta'\rangle_2 = \sum_n \widetilde{S}_n^* x|\eta'\rangle_2 T_n$. This sum converges in norm and each summand lies in $[|\gamma\rangle_2 \mathcal{L}(H)]$ because $x|\eta'\rangle_2 \in [|\gamma\rangle_2 A]$ and $[\widetilde{S}_n^*|\gamma\rangle_2] = [|\gamma\rangle_2 S_n^*]$. Since $\langle \eta''|_2(\phi * \mathrm{id})(x)|\eta'\rangle_2 \in C$ for each $\eta'' \in \gamma$, we can conclude that the sum lies in $[|\gamma\rangle_2 C]$. Finally, consider the expression $(\phi * \mathrm{id})(x)|\xi\rangle_1 = \sum_n \widetilde{S}_n x \widetilde{T}_n|\xi\rangle_1$, where $\xi \in \lambda$. Again, the sum converges in norm and each summand lies in $[|\lambda\rangle_1 B]$ because $\widetilde{S}_n^* x \widetilde{T}_n|\xi\rangle_1 = \widetilde{S}_n^* x|T_n\xi\rangle_1 \in \widetilde{S}_n^*(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\beta\rangle_1 \subseteq [\widetilde{S}_n^*|\beta\rangle_1 B] \subseteq [|\lambda\rangle_1 B]$. ∎

REMARK 3.24. (i) The map $\phi * \mathrm{id}$ constructed above is a "slice map" in the case where $C_L^\lambda = \mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}}$ and $S_n, T_n \in \beta \subseteq \mathcal{L}(\mathfrak{K}_{\mathfrak{B}}, H_\beta)$ for all n. Then, we can identify $C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$ with a C^*-subalgebra of $\mathcal{L}(K)$, and $\phi * \mathrm{id}$ is just the map $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B$ given by $x \mapsto \sum_n \langle S_n|_1 X|T_n\rangle_1$.

(ii) Assume that the extension $\widetilde{\phi}\colon [A + \rho_\beta(\mathfrak{B}^\dagger)] \to C$ given by $x \mapsto \sum_n S_n^* x T_n$ is completely positive. Here, we use the notation of the proof above. Then the map $\widetilde{\phi} * \mathrm{id}$ constructed in Proposition 3.21 extends the map $\phi * \mathrm{id}$ of Proposition 3.23 because then $\theta = \rho_\lambda$ and hence $\langle \eta|_2(\widetilde{\phi} * \mathrm{id})(x)|\eta'\rangle_2 = \widetilde{\phi}(\langle \eta|_2 x|\eta'\rangle_2)$ for all $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ and $\eta, \eta' \in \gamma$.

Of course, slice maps of the form $\mathrm{id} * \phi$ can be constructed in a similar way.

3.5. FURTHER CATEGORICAL PROPERTIES. The fiber product of C^*-algebras is neither associative, unital, nor compatible with infinite sums.

We first discuss non-associativity. Let $\mathcal{A} = A_H^{\alpha,\beta}$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra, $\mathcal{B} = B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra, and $\mathcal{C} = C_L^{\varepsilon,\phi}$ a C^*-$(\mathfrak{c}^\dagger, \mathfrak{d})$-algebra. Then we can form the fiber products $(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}) \underset{\mathfrak{c}}{*} \mathcal{C}$ and $\mathcal{A} \underset{\mathfrak{b}}{*} (\mathcal{B} \underset{\mathfrak{c}}{*} \mathcal{C})$. The following example shows that these C^*-algebras need *not* be identified by the canonical isomorphism $a_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}({}_\varepsilon L_\phi, {}_\gamma K_\delta, {}_\alpha H_\beta)$ of Proposition 2.14. A similar phenomenon occurs in the purely algebraic setting with the Takeuchi $\times_R$-product [24].

EXAMPLE 3.25. Let $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \mathfrak{d}$ be the trivial C^*-base, $H = l^2(\mathbb{N})$, $\alpha = \mathcal{L}(\mathbb{C}, H)$, $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{L}(H)_H^{\alpha,\alpha}$. Identify $H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\alpha K_\alpha \underset{\mathfrak{c}}{\otimes} {}_\alpha L \cong \alpha \otimes H \otimes \alpha$ with $H \otimes H \otimes H$ via $|\xi\rangle \otimes \zeta \otimes |\eta\rangle \equiv \xi \otimes \zeta \otimes \eta$, fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, and define $T \in \mathcal{L}(H^{\otimes 3})$ by

$$T(e_k \otimes e_l \otimes e_m) = \begin{cases} e_k \otimes e_l \otimes e_m & \text{for all } k,l,m \in \mathbb{N} \text{ such that } m \leqslant k+l, \\ e_l \otimes e_k \otimes e_m & \text{for all } k,l,m \in \mathbb{N} \text{ such that } m > k+l. \end{cases}$$

We show that T belongs to the underlying C^*-algebra of $(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}) \underset{\mathfrak{c}}{*} \mathcal{C}$, but not of $\mathcal{A} \underset{\mathfrak{b}}{*} (\mathcal{B} \underset{\mathfrak{c}}{*} \mathcal{C})$.

For each $\xi \in H$ and $\omega \in H^{\otimes 2}$, we define $|\xi\rangle_1, |\xi\rangle_3 \in \mathcal{L}(H^{\otimes 2}, H^{\otimes 3})$ and $|\omega\rangle_{12} \in \mathcal{L}(H, H^{\otimes 3})$ by $v \mapsto \xi \otimes v$, $v \mapsto v \otimes \xi$, and $\zeta \mapsto \omega \otimes \zeta$, respectively. Then for all $k,l,m \in \mathbb{N}$,

$$T|e_k \otimes e_l\rangle_{12} = |e_k \otimes e_l\rangle_{12} P_{l+k} + |e_l \otimes e_k\rangle_{12}(\mathrm{id} - P_{l+k}), \quad \text{where } P_{l+k} := \sum_{m \leqslant k+l} |e_m\rangle\langle e_m|,$$

$$T|e_m\rangle_3 = |e_m\rangle_3(\mathrm{id} + \Sigma_m), \quad \text{where } \Sigma_m := \sum_{k,l;\, k+l<m} |e_l \otimes e_k - e_k \otimes e_l\rangle\langle e_k \otimes e_l|,$$

and therefore,

$$\begin{aligned} &T|H^{\otimes 2}\rangle_{12} \in [|H^{\otimes 2}\rangle_{12}\mathcal{L}(H)], \\ &\quad T|\alpha\rangle_3 \in [|\alpha_3\rangle(\mathrm{id} + \mathcal{K}(H) \otimes \mathcal{K}(H))] \subseteq [|\alpha\rangle_3(\mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*} {}_\alpha\mathcal{L}(H))]. \end{aligned}$$

Since $T = T^*$, we can conclude that T belongs to $(\mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*} {}_\alpha\mathcal{L}(H)_\alpha) \underset{\mathfrak{b}}{*} {}_\alpha\mathcal{L}(H)$. However,

$$\begin{aligned} T|e_0\rangle_1 = |e_0\rangle_1 Q + \sum_l |e_l\rangle_1 Q_l, \ \text{ where } Q &= \sum_{m \leqslant l} |e_l \otimes e_m\rangle\langle e_l \otimes e_m| \\ \text{and } Q_l &= \sum_{m>l} |e_0 \otimes e_m\rangle\langle e_l \otimes e_m|, \end{aligned}$$

and $|e_0\rangle_1 Q \in [|\alpha\rangle_1 \mathcal{L}(H \otimes H)]$, but $\sum_l |e_l\rangle_1 Q_l \notin [|\alpha\rangle_1 \mathcal{L}(H \otimes H)]$ because the sum

$$\sum_l Q_l^* Q_l = \sum_l \sum_{m>l} |e_l \otimes e_m\rangle\langle e_l \otimes e_m|$$

does not converge in norm. Therefore, we have $T|e_0\rangle_1 \notin [|\alpha\rangle_1\mathcal{L}(H\otimes H)]$ and $T \notin \mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*} ({}_\alpha\mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*} {}_\alpha\mathcal{L}(H))$.

We next discuss unitality. A unit for the fiber product relative to $\mathfrak{b}$ would be a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $\mathcal{U} = \mathfrak{U}_{\mathfrak{K}}^{\mathfrak{B}^\dagger,\mathfrak{B}}$ such that for all C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and all C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$, we have $\mathcal{A} = \mathrm{Ad}_r(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{U})$ and $\mathcal{B} = \mathrm{Ad}_l(\mathcal{U} \underset{\mathfrak{b}}{*} \mathcal{B})$, where $r = r_{\mathfrak{a},\mathfrak{b}}({}_\alpha H_\beta)$ and $l = l_{\mathfrak{b},\mathfrak{c}}({}_\gamma K_\delta)$ (see Proposition 2.14). The relations $r|\beta\rangle_1 = \beta$, $r|\mathfrak{B}^\dagger\rangle_2 = \rho_\beta(\mathfrak{B}^\dagger)$, $l|\gamma\rangle_2 = \gamma$, $l|\mathfrak{B}\rangle_1 = \rho_\gamma(\mathfrak{B})$ imply

$$\mathrm{Ad}_r(A_\beta \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{U}) = \mathrm{Ind}_\beta(\mathfrak{U}) \cap \mathrm{Ind}_{\rho_\beta(\mathfrak{B}^\dagger)}(A),$$
$$\mathrm{Ad}_l(\mathfrak{U}_{\mathfrak{B}} \underset{\mathfrak{b}}{*} {}_\gamma B) = \mathrm{Ind}_{\rho_\gamma(\mathfrak{B})}(B) \cap \mathrm{Ind}_\gamma(\mathfrak{U}). \tag{3.3}$$

If $\mathfrak{B}^\dagger$ and $\mathfrak{B}$ are unital, then $\mathrm{Ind}_{\rho_\beta(\mathfrak{B}^\dagger)}(A) = A$ and $\mathrm{Ind}_{\rho_\gamma(\mathfrak{B})}(B) = B$, and then the C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $\mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}^\dagger,\mathfrak{B}}$ is a unit for the fiber product on the full subcategories of all $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$ satisfying $A \subseteq \mathrm{Ind}_\beta(\mathcal{L}(\mathfrak{K}))$ and $B \subseteq \mathrm{Ind}_\gamma(\mathcal{L}(\mathfrak{K}))$.

REMARK 3.26. (i) If $A \subseteq \mathrm{Ind}_\alpha(\mathcal{L}(\mathfrak{H}))$ and $B \subseteq \mathrm{Ind}_\gamma(\mathcal{L}(\mathfrak{L}))$, then we have $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \subseteq \mathrm{Ind}_{(\alpha\triangleleft\gamma)}(\mathcal{L}(\mathfrak{H})) \cap \mathrm{Ind}_{(\beta\triangleright\delta)}(\mathcal{L}(\mathfrak{K}))$.

(ii) $\mathrm{Ind}_\beta(\mathfrak{B}^\dagger) = \mathcal{L}(H_\beta)$, and if $\mathfrak{B}^\dagger$ is unital, then $\mathrm{Ad}_r(A_\beta \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{B}^\dagger) = A \cap \mathcal{L}(H_\beta) = A^{(\beta)}$.

(iii) $\mathrm{Ad}_r(\mathfrak{B}_{\mathfrak{B}} \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{B}^\dagger) = \mathcal{L}(\mathfrak{K}_{\mathfrak{B}}) \cap \mathcal{L}(\mathfrak{K}_{\mathfrak{B}^\dagger}) = M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$.

We finally discuss compatibility with sums and products. First, the fiber product is compatible with finite sums in the following sense. Let $(\mathcal{A}^i)_i$ be a finite family of C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-algebras and $(\mathcal{B}^j)_j$ a finite family of C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras. For each i,j, denote by

$$\iota^i_{\mathcal{A}}\colon \mathcal{A}^i \to \boxplus_{i'}\mathcal{A}^{i'}, \quad \iota^j_{\mathcal{B}}\colon \mathcal{B}^j \to \boxplus_{j'}\mathcal{B}^{j'}, \quad \pi^i_{\mathcal{A}}\colon \boxplus_{i'}\mathcal{A}^{i'} \to \mathcal{A}^i, \quad \pi^j_{\mathcal{B}}\colon \boxplus_{j'}\mathcal{B}^{j'} \to \mathcal{B}^j$$

the canonical inclusions and projections, respectively. One easily verifies that there exist inverse isomorphisms $\boxplus_{i,j}\mathcal{A}^i \underset{\mathfrak{b}}{*} \mathcal{B}^j \leftrightarrows (\boxplus_i\mathcal{A}^i) \underset{\mathfrak{b}}{*} (\boxplus_j\mathcal{B}^j)$, given by

$$(x_{i,j})_{i,j} \mapsto \sum_{i,j}(\iota^i_{\mathcal{A}} \underset{\mathfrak{b}}{*} \iota^j_{\mathcal{B}})(x_{i,j}) \quad \text{and} \quad ((\pi^i_{\mathcal{A}} \underset{\mathfrak{b}}{*} \pi^j_{\mathcal{B}})(y))_{i,j} \hookleftarrow y,$$

respectively. However, the fiber product is neither compatible with infinite sums nor infinite products:

EXAMPLE 3.27. Let $\mathfrak{t} = (\mathbb{C},\mathbb{C},\mathbb{C})$ be the trivial C^*-base.

(i) For each $i,j \in \mathbb{N}$, let $\mathcal{A}^i$ and $\mathcal{B}^j$ be the C^*-$\mathfrak{t}$-algebra $\mathbb{C}^{\mathbb{C}}_{\mathbb{C}}$. Identify $\bigoplus_{i,j} \mathbb{C}_{\mathbb{C}} \underset{\mathfrak{t}}{\otimes} {}_{\mathbb{C}}\mathbb{C}$ with $l^2(\mathbb{N}\times\mathbb{N})$ in the canonical way. Then $\bigoplus_{i,j}(\mathcal{A}^i \underset{\mathfrak{t}}{*} \mathcal{B}^j)$ corresponds to $C_0(\mathbb{N}\times\mathbb{N})$, represented on $l^2(\mathbb{N}\times\mathbb{N})$ by multiplication operators, but $(\bigoplus_i \mathcal{A}^i) \underset{\mathfrak{t}}{*} (\bigoplus_j \mathcal{B}^j) \cong$

$C_0(\mathbb{N}) \underset{\mathfrak{t}}{*} C_0(\mathbb{N})$ is strictly larger and contains, for example, the characteristic function of the diagonal $\{(x,x) : x \in \mathbb{N}\}$ (see Example 5.3).

(ii) Let $H = l^2(\mathbb{N})$, $\alpha = \mathcal{L}(\mathbb{C}, H)$, and let $\mathcal{A}$ and $\mathcal{B}^j$ be the C^*-t-algebra $\mathcal{K}(H)^{\alpha}_{H}$ for all j. Identify $H_{\alpha} \underset{\mathfrak{t}}{\otimes} {}_{\alpha}H$ with $H \otimes H$ as in Example 2.12(i), choose an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H, and put $y_j := |e_j \otimes e_0\rangle\langle e_0 \otimes e_0| \in \mathcal{K}(H \otimes H)$ for each $j \in \mathbb{N}$. Then $y := (y_j)_j \in \prod_j \mathcal{A} \underset{\mathfrak{t}}{*} \mathcal{B}^j$ because $y_j \in \mathcal{K}(H) \otimes \mathcal{K}(H) \subset \mathcal{A} \underset{\mathfrak{t}}{*} \mathcal{B}^j$ for all $j \in \mathbb{N}$, but with respect to the canonical identification $\bigoplus_j H \otimes H \cong H \otimes (\bigoplus_j \otimes H)$, we have $y \notin \mathcal{A} \underset{\mathfrak{t}}{*} (\prod_j \mathcal{B}^j)$ because $y|e_0\rangle_1$ corresponds to the family

$$(|e_j\rangle_1 |e_0\rangle\langle e_0|)_j \in \prod_j \mathcal{L}(H, H \otimes H) \subseteq \mathcal{L}(\textstyle\bigoplus_j H, \bigoplus_j H \otimes H)$$

which is not contained in the space $[|\alpha\rangle_1 \mathcal{L}(\bigoplus_j H)]$.

3.6. A FIBER PRODUCT OF NON-REPRESENTED C^*-ALGEBRAS. The spatial fiber product of C^*-algebras represented on C^*-modules yields a fiber product of non-represented C^*-algebras as follows.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. In Subsection 3.2, we constructed a functor $\mathbf{R}_{\mathfrak{b}} \colon \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger} \to \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$ that associates to each C^*-$\mathfrak{B}^\dagger$-algebra a universal representation in form of a C^*-$\mathfrak{b}$-algebra. Replacing $\mathfrak{b}$ by $\mathfrak{b}^\dagger$, we obtain a functor $\mathbf{R}_{\mathfrak{b}^\dagger} \colon \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}} \to \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}}$, and composition of these with the spatial fiber product gives a fiber product of non-represented C^*-algebras in form of a functor

$$\mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}} \xrightarrow{\mathbf{R}_{\mathfrak{b}} \times \mathbf{R}_{\mathfrak{b}^\dagger}} \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}} \times \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}^\dagger} \to \mathbf{C}^*, \quad (C_\sigma, D_\tau) \mapsto \mathbf{R}_{\mathfrak{b}}(C_\sigma) \underset{\mathfrak{b}}{*} \mathbf{R}_{\mathfrak{b}^\dagger}(D_\tau),$$

where $\mathbf{C}^*$ denotes the category of C^*-algebras and $*$-homomorphisms. In categorical terms, this is the right Kan extension of the spatial fiber product on $\mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}} \times \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}^\dagger}$ along the product of the forgetful functors $\mathbf{U}_{\mathfrak{b}} \times \mathbf{U}_{\mathfrak{b}^\dagger} \colon \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}} \times \mathbf{C}^{*\mathrm{s}}_{\mathfrak{b}^\dagger} \to \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}}$ ([19], Section X).

Given further C^*-bases $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$, we similarly obtain a functor

$$\mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \times \mathbf{C}^{*\mathrm{r}}_{(\mathfrak{B},\mathfrak{C}^\dagger)} \xrightarrow{\mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \times \mathbf{R}_{(\mathfrak{b}^\dagger,\mathfrak{c})}} \mathbf{C}^{*\mathrm{s}}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \times \mathbf{C}^{*\mathrm{s}}_{(\mathfrak{b}^\dagger,\mathfrak{c})} \to \mathbf{C}^{*\mathrm{s}}_{(\mathfrak{a}^\dagger,\mathfrak{c})} \xrightarrow{\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{c})}} \mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{C}^\dagger)},$$

and, using Remark 3.12, a natural transformation between the compositions in the square

$$\begin{tikzcd}
\mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \times \mathbf{C}^{*\mathrm{r}}_{(\mathfrak{B},\mathfrak{C}^\dagger)} \arrow[r] \arrow[d] & \mathbf{C}^{*\mathrm{r}}_{(\mathfrak{A},\mathfrak{C}^\dagger)}, \arrow[d] \arrow[dl, Rightarrow] \\
\mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*\mathrm{r}}_{\mathfrak{B}} \arrow[r] & \mathbf{C}^*,
\end{tikzcd}$$

where the vertical maps are the forgetful functors.

4. RELATION TO THE SETTING OF VON NEUMANN ALGEBRAS

In this section, let N be a von Neumann algebra with a n.s.f. weight μ, denote by $\mathfrak{N}_\mu, H_\mu, \pi_\mu, J_\mu$ the usual objects of Tomita–Takesaki theory [23], and define the antirepresentation $\pi_\mu^{\mathrm{op}}\colon N \to \mathcal{L}(H_\mu)$ by $x \mapsto J_\mu \pi_\mu(x^*) J_\mu$.

4.1. ADAPTATION TO VON NEUMANN ALGEBRAS. The definitions and constructions presented in Sections 2 and 3 can be adapted to a variety of other settings. We now briefly explain what happens when we pass to the setting of von Neumann algebras. Instead of a C^*-base, we start with the triple $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$, where $\mathfrak{K} = H_\mu$, $\mathfrak{B} = \pi_\mu(N)$, and $\mathfrak{B}^\dagger = J_\mu \pi_\mu(N) J_\mu$. Next, we define W^*-$\mathfrak{b}$-modules, W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules, their relative tensor product, W^*-$\mathfrak{b}$-algebras, and the fiber product by just replacing the norm closure $[\,\cdot\,]$ by the closure with respect to the weak operator topology $[\,\cdot\,]_w$ everywhere in Sections 2 and 3. We then recover Connes' fusion of Hilbert bimodules over N and Sauvageot's fiber product as follows.

MODULES. Let H be some Hilbert space. If (H, ρ) is a right N-module, then

$$\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (H, \rho)) := \{T \in \mathcal{L}(\mathfrak{K}, H) : \forall x \in N : T\pi_\mu^{\mathrm{op}}(x) = \rho(x) T\}$$

satisfies $[\alpha\mathfrak{K}] = H, [\alpha^*\alpha]_w = \mathfrak{B}, \alpha\mathfrak{B} \subseteq \alpha$, and $\rho_\alpha \circ \pi_\mu^{\mathrm{op}}$ (see Lemma 2.5) coincides with ρ. Conversely, if $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a weakly closed subspace satisfying the three preceding equations, then $(H, \rho_\alpha \circ \pi_\mu^{\mathrm{op}})$ is a right N-module and $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (H, \rho_\alpha \circ \pi_\mu^{\mathrm{op}}))$ [22]. We thus obtain a bijective correspondence between right N-modules and W^*-$\mathfrak{b}$-modules. This correspondence is an isomorphism of categories since for every other right N-module (K, σ), an operator $T \in \mathcal{L}(H, K)$ intertwines ρ and σ if and only if $T\alpha$ is contained in $\beta := \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (K, \sigma))$. For W^*-$\mathfrak{b}$-modules, the notions of morphisms and semi-morphisms coincide.

ALGEBRAS. Let H, ρ, α be as above and let $A \subseteq \mathcal{L}(H)$ be a von Neumann algebra. Then $\rho(N) \subseteq A$ if and only if $\rho_\alpha(\mathfrak{B})A \subseteq A$. Thus, W^*-$\mathfrak{b}$-algebras correspond with von Neumann algebras equipped with a normal unital embedding of N. Moreover, let K, σ, β be as above, let $B \subseteq \mathcal{L}(K)$ be a von Neumann algebra, assume $\rho(N) \subseteq A$ and $\sigma(N) \subseteq B$, and let $\pi\colon A \to B$ be a $*$-homomorphism satisfying $\pi \circ \rho = \sigma$. Then π is normal if and only if $[\mathcal{L}^\pi(H_\alpha, K_\beta)\alpha]_w = \beta$. Indeed, the "if" part is straightforward (see Lemma 3.4), and the "only if" part follows easily from the fact that every normal $*$-homomorphism is the composition of an amplification, reduction, and unitary transformation ([5], Section 4.4).

BIMODULES. Let (H, ρ) be a left N-module, let (H, σ) be a right N-module, and let $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu), (H, \rho))$ and $\beta = \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (H, \sigma))$. Then (H, ρ, σ) is an N-bimodule if and only if $\rho(N)\beta = \beta$ and $\sigma(N)\alpha = \alpha$, and thus we obtain an isomorphism between the category of N-bimodules and the category of W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules.

FUSION. The preceding considerations and (2.1) show that the relative tensor product of W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules corresponds to Connes' fusion of N-bimodules.

FIBER PRODUCT. Let (H, ρ) be a right N-module, (K, σ) a left N-module, $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (H, \rho))$ and $\beta = \mathcal{L}((\mathfrak{K}, \pi_\mu), (K, \sigma))$, and let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be von Neumann algebras satisfying $\rho(N) \subseteq H$ and $\sigma(N) \subseteq K$. One easily verifies the equivalence of the following conditions for each $x \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$:

$$\text{(i) } x|\alpha\rangle_1 \subseteq [|\alpha\rangle_1 B]_w, \qquad \text{(ii) } \langle\alpha|_1 x|\alpha\rangle_1 \subseteq B, \qquad \text{(iii) } x \in (\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')'.$$

Consequently, the fiber product of A and B, considered as a W^*-$\mathfrak{b}$-algebra and a W^*-$\mathfrak{b}^\dagger$-algebra, coincides with the fiber product $(\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')' \cap (A' \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)' = (A' \underset{\mathfrak{b}}{\otimes} B')'$ of Sauvageot.

4.2. RELATION TO CONNES' FUSION AND SAUVAGEOT'S FIBER PRODUCT. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base such that $\mathfrak{K} = H_\mu$, $\mathfrak{B}'' = \pi_\mu(N)$, $(\mathfrak{B}^\dagger)'' = \pi_\mu^{\mathrm{op}}(N) = \mathfrak{B}'$. Denote by $\mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})}$ the category of all C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules with all semi-morphisms, and by $\mathbf{W^*\text{-}bimod}_{(N, N^{\mathrm{op}})}$ the category of all N-bimodules, respectively. Lemmas 2.5 and 2.6 imply:

LEMMA 4.1. *There is a faithful functor* $\mathbf{F}\colon \mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})} \to \mathbf{W^*\text{-}bimod}_{(N, N^{\mathrm{op}})}$, *given by* ${}_\alpha H_\beta \mapsto (H, \rho_\alpha \circ \pi_\mu, \rho_\beta \circ \pi_\mu^{\mathrm{op}})$ *on objects and* $T \mapsto T$ *on morphisms.*

The categories $\mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})}$ and $\mathbf{W^*\text{-}bimod}_{(N, N^{\mathrm{op}})}$ carry the structure of monoidal categories [19], and we now show that the functor $\mathbf{F}$ above is monoidal. Let H_β be a C^*-$\mathfrak{b}$-module, K_γ a C^*-$\mathfrak{b}^\dagger$-module, and let

$$\rho = \rho_\beta \circ \pi_\mu^{\mathrm{op}}, \quad X = \mathcal{L}((\mathfrak{K}, \pi_\mu^{\mathrm{op}}), (H, \rho)), \quad \sigma = \rho_\gamma \circ \pi_\mu, \quad Y = \mathcal{L}((\mathfrak{K}, \pi_\mu), (K, \sigma)).$$

Given subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$, we define a sesquilinear form $\langle \cdot | \cdot \rangle$ on the algebraic tensor product $X_0 \odot \mathfrak{K} \odot Y_0$ such that for all $\xi, \xi' \in X_0, \zeta, \zeta' \in \mathfrak{K}, \eta, \eta' \in Y_0$,

$$\langle \xi \odot \zeta \odot \eta | \xi' \odot \zeta' \odot \eta' \rangle = \langle \zeta | (\xi^* \xi')(\eta^* \eta') \eta' \rangle = \langle \zeta | (\eta^* \eta')(\xi^* \xi') \eta' \rangle.$$

Denote by $X_0 \otimes \mathfrak{K} \otimes Y_0$ the Hilbert space obtained by forming the separated completion.

LEMMA 4.2. *Let* $X_0 \subseteq X$, $Y_0 \subseteq Y$ *be subspaces satisfying* $[X_0 \mathfrak{K}] = H$ *and* $[Y_0 \mathfrak{K}] = K$. *Then the natural map* $X_0 \otimes \mathfrak{K} \otimes Y_0 \to X \otimes \mathfrak{K} \otimes Y$ *is an isomorphism.*

Proof. Injectivity is clear. The natural map $X_0 \otimes \mathfrak{K} \otimes Y_0 \to X \otimes \mathfrak{K} \otimes Y_0$ is surjective because both spaces coincide with the separated completion of the algebraic tensor product $H \odot Y_0$ with respect to the sesquilinear inner form given by $\langle \omega \odot \eta | \omega' \odot \eta' \rangle = \langle \omega | \rho_\beta(\eta^* \eta') \omega' \rangle$, and a similar argument shows that the natural map $X \otimes \mathfrak{K} \otimes Y_0 \to X \otimes \mathfrak{K} \otimes Y$ is surjective. ∎

We conclude that Connes' original definition of the relative tensor product $H_\rho \underset{\mu}{\otimes} {}_\sigma K$ via bounded vectors coincides with the algebraic one given in (2.1) and with the relative tensor product $H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K$.

THEOREM 4.3. *There exists a natural isomorphism between the compositions in the square*

$$\begin{tikzcd}
\mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[r, "-\underset{\mathfrak{b}}{\otimes}-"] \arrow[d, "\mathbf{F}\times\mathbf{F}"'] & \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[d, "\mathbf{F}"] \arrow[dl, Rightarrow] \\
\mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{\mathrm{op}})} \times \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{\mathrm{op}})} \arrow[r, "-\underset{\mu}{\otimes}-"'] & \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{\mathrm{op}})},
\end{tikzcd}$$

given for each object $({}_\alpha H_\beta, {}_\gamma K_\delta) \in \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ *by the natural map*

$$H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K = \beta \otimes \mathfrak{K} \otimes \gamma \to X \otimes \mathfrak{K} \otimes Y = H_\rho \underset{\mu}{\otimes} {}_\sigma K. \tag{4.1}$$

With respect to this isomorphism, the functor $\mathbf{F}\colon \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \to \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{\mathrm{op}})}$ *is monoidal.*

Proof. Lemma 4.2 implies that the map (4.1) is an isomorphism. Evidently, this map is natural with respect to ${}_\alpha H_\beta$ and ${}_\gamma K_\delta$. The verification of the assertion concerning $\mathbf{F}$ is now tedious but straightforward. ∎

Denote by $\mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ the category formed by all C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebras $A^{\alpha,\beta}_H$ satisfying $\rho_\alpha(\mathfrak{B}) + \rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ and all semi-morphisms, and by $\mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$ the category of all von Neumann algebras A equipped with a normal, unital embedding and anti-embedding $\iota_A^{\mathrm{op}}\colon N \to A$ such that $[\iota_A(N), \iota_A^{\mathrm{op}}(N)] = 0$, together with all morphisms preserving these (anti-)embeddings. Lemma 3.4 implies:

PROPOSITION 4.4. *There exists a faithful functor* $\mathbf{G}\colon \mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \to \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$, *given by* $({}_\alpha H_\beta, A) \mapsto (A'', \rho_\alpha \circ \pi_\mu, \rho_\beta \circ \pi_\mu^{\mathrm{op}})$ *on objects and* $\phi \mapsto \phi''$ *on morphisms, where* ϕ'' *denotes the normal extension of* ϕ.

By Lemma 3.16, $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} \in \mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ for all $\mathcal{A}, \mathcal{B} \in \mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$, but $\mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ is not a monoidal category with respect to the fiber product because the latter is not associative (see Subsection 3.5).

PROPOSITION 4.5. *There exists a natural transformation*

$$\begin{tikzcd}
\mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[r, "-\underset{\mathfrak{b}}{*}-"] \arrow[d, "\mathbf{G}\times\mathbf{G}"'] & \mathbf{C}^{*\mathrm{s},nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[d, "\mathbf{G}"] \arrow[dl, Rightarrow] \\
\mathbf{W}^*{}_{(N,N^{\mathrm{op}})} \times \mathbf{W}^*{}_{(N,N^{\mathrm{op}})} \arrow[r, "-\underset{\mu}{*}-"'] & \mathbf{W}^*{}_{(N,N^{\mathrm{op}})},
\end{tikzcd}$$

given for each object $A_H^{\alpha,\beta}$ *and* $B_K^{\gamma,\delta}$ *by conjugation with the isomorphism* (4.1).

The proof is immediate from Theorem 4.3 and Lemma 3.16.

4.3. A CATEGORICAL INTERPRETATION OF THE FIBER PRODUCT OF VON NEUMANN ALGEBRAS. We keep the notation introduced above, denote by **Hilb** the category of Hilbert spaces and bounded linear operators, and call a subcategory of $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ a *$*$-subcategory* if it is closed with respect to the involution $T \mapsto T^*$ of morphisms.

DEFINITION 4.6. A *category over* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ is a category $\mathbf{C}$ equipped with a functor $\mathbf{U_C}\colon \mathbf{C} \to \mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ such that $\mathbf{U_C C}$ is a $*$-subcategory of $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$. Let $(\mathbf{C}, \mathbf{U_C})$ be such a category. We loosely refer to $\mathbf{C}$ as a category over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ without mentioning $\mathbf{U_C}$ explicitly, and denote by $\mathbf{H_C}$ the composition of $\mathbf{U_C}$ with the forgetful functor $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})} \to \mathbf{Hilb}$. We call an object $G \in \mathbf{C}$ *separating* if it satisfies $[\mathbf{H_C C}(G,X)(\mathbf{H_C}G)] = \mathbf{H_C}X$ for each $X \in \mathbf{C}$.

We denote by $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$ the category of all categories over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ having a separating object, where the morphisms between objects $(\mathbf{C}, \mathbf{U_C})$ and $(\mathbf{D}, \mathbf{U_D})$ are all functors $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ satisfying $\mathbf{U_D F} = \mathbf{U_C}$.

EXAMPLE 4.7. For each $A \in \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$, denote by $\mathbf{W}^*\text{-}\mathbf{mod}_A$ the category of all normal, unital representations $\pi\colon A \to \mathcal{L}(H)$ for which $\pi \circ \iota_A$ and $\pi \circ \iota_A^{\mathrm{op}}$ are faithful, and all intertwiners. This is a category over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$, where $\mathbf{U}_A\colon \mathbf{W}^*\text{-}\mathbf{mod}_A \to \mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ is given by $(L,\pi) \mapsto (L, \pi \circ \iota_A, \pi \circ \iota_A^{\mathrm{op}})$ on objects and $T \mapsto T$ on morphisms. The only non-trivial thing to check is that $\mathbf{W}^*\text{-}\mathbf{mod}_A$ has a separating object; by Lemma 2.10 of [3] or IX Theorem 1.2(iv) of [23], one can take the GNS-representation for a n.s.f. weight on A.

Each morphism $\phi\colon A \to B$ in $\mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$ yields a functor $\phi^*\colon \mathbf{W}^*\text{-}\mathbf{mod}_B \to \mathbf{W}^*\text{-}\mathbf{mod}_A$, given by $(L,\pi) \mapsto (L, \pi \circ \phi)$ on objects and $T \mapsto T$ on morphisms.

REMARK 4.8. In the definition above, $\mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C},\mathbf{D})$ need not be a set, and this may cause problems. There are several possible solutions: we can fix a "universe" to work in, or replace the category $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ by a small subcategory and require categories over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ to be small, too. It is clear how to modify the preceding example in that case.

PROPOSITION 4.9. *There exists a contravariant functor* $\mathbf{Mod}\colon \mathbf{W}^*{}_{(N,N^{\mathrm{op}})} \to \mathbf{Cat}_{(N,N^{\mathrm{op}})}$ *given by* $A \mapsto \mathbf{Mod}(A) := (\mathbf{W}^*\text{-}\mathbf{mod}_A, \mathbf{U}_A)$ *on objects and by* $\phi \mapsto \mathbf{Mod}(\phi) := \phi^*$ *on morphisms.*

For each category $\mathbf{C} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$, choose a separating object $G_\mathbf{C}$. Fix some $\mathbf{C} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$, let $\mathbf{U} = \mathbf{U_C}$, $\mathbf{H} = \mathbf{H_C}$, $G = G_\mathbf{C}$, $(H,\rho,\sigma) = \mathbf{U}G$, and define $\mathbf{End}(\mathbf{C}) := \mathbf{H}(\mathbf{C}(G,G))' \subseteq \mathcal{L}(H)$. Then $\rho(N) + \sigma(N) \subseteq \mathbf{End}(\mathbf{C})$ because

$\mathbf{H}(\mathbf{C}(G,G)) \subseteq (\rho(N)+\sigma(N))'$, and we can consider $\mathbf{End}(\mathbf{C})$ as an element of ${\mathbf{W}^*}_{(N,N^{\mathrm{op}})}$ with respect to ρ and σ.

LEMMA 4.10. *There exists a morphism $\eta_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{Mod}(\mathbf{End}(\mathbf{C}))$ in $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$, given by $X \mapsto (\mathbf{U}X, \rho^X)$ on objects and $T \mapsto \mathbf{H}T$ on morphisms, where $\rho^X = \rho_{\mathbf{HC}(G,X)}$ for each $X \in \mathbf{C}$. In particular, $\rho^X(\mathbf{End}(\mathbf{C})) \subseteq \mathbf{H}(\mathbf{C}(X,X))'$ for each $X \in \mathbf{C}$.*

Proof. Let $X \in \mathbf{C}$ and $(K,\phi,\psi) = \mathbf{U}X$. Then Lemma 2.5, applied to $I := \mathbf{HC}(G,X) \subseteq \mathcal{L}(\mathbf{H}G,\mathbf{H}X)$, gives a normal representation $\rho_I \colon (I^*I)' \to \mathcal{L}(K)$. Since $I^*I \subseteq \mathbf{HC}(G,X)$ by assumption on $\mathbf{C}$, we have $\mathbf{End}(\mathbf{C}) \subseteq (I^*I)'$ and can define $\rho^X = \rho_I|_{\mathbf{End}(\mathbf{C})}$. Each element of I intertwines ρ with ϕ and σ with ψ, whence $\mathbf{U}X = (K, \rho_I \circ \rho, \rho_I \circ \sigma) = \mathbf{U}_{\mathbf{End}(\mathbf{C})}(\eta_{\mathbf{C}}X)$.

Let $Y \in \mathbf{C}$, $T \in \mathbf{C}(X,Y)$, $J := \mathbf{HC}(G,Y)$. Then $\mathbf{H}(T)\rho_I(S) = \rho_J(S)\mathbf{H}(T)$ for all $S \in \mathbf{End}(G)$ because $\mathbf{H}(T)I \in J$, and therefore $\mathbf{H}(T)$ is a morphism from $(\mathbf{H}X,\rho^X)$ to $(\mathbf{H}Y,\rho^Y)$. By definition, $\mathbf{H}_{\mathbf{End}(\mathbf{C})}(\eta_{\mathbf{C}}(T)) = \mathbf{H}T$. ∎

REMARK 4.11. If $G' \in \mathbf{C}$ is another separating object, $\rho^{G'} \colon \mathbf{H}(\mathbf{C}(G,G))' \to \mathbf{H}(\mathbf{C}(G',G'))'$ is an isomorphism with inverse $\rho_{\mathbf{HC}(G',G)}$.

We shall eventually show that the assignment $\mathbf{C} \mapsto \mathbf{End}(\mathbf{C})$ extends to a functor $\mathbf{End} \colon \mathbf{Cat}_{(N,N^{\mathrm{op}})} \to {\mathbf{W}^*}_{(N,N^{\mathrm{op}})}$ that is adjoint to $\mathbf{Mod}$. The key is a more careful analysis of functors from a category $\mathbf{C} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$ to categories of the form $\mathbf{Mod}(A)$, where $A \in {\mathbf{W}^*}_{(N,N^{\mathrm{op}})}$. Such functors themselves can be considered as objects of a category as follows.

For all $\mathbf{C},\mathbf{D} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$, the elements of $\mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C},\mathbf{D})$ are the objects of a category, where the morphisms are all natural transformations with the usual composition.

Similarly, for all $A,B \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$, the morphisms in ${\mathbf{W}^*}_{(N,N^{\mathrm{op}})}(A,B)$ can be considered as objects of a category, where the morphisms between ϕ,ψ are all $b \in B$ satisfying $b\phi(a) = \psi(a)b$ for all $a \in A$, and where composition is given by multiplication.

PROPOSITION 4.12. *Let $A \in {\mathbf{W}^*}_{(N,N^{\mathrm{op}})}$ and $\mathbf{C} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$. Then there exists an isomorphism $\Phi_{\mathbf{C},A} \colon \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C},\mathbf{Mod}(A)) \to {\mathbf{W}^*}_{(N,N^{\mathrm{op}})}(A,\mathbf{End}(\mathbf{C}))$ with inverse $\Psi_{\mathbf{C},A} := \Phi_{\mathbf{C},A}^{-1}$ such that:*

(i) *$\Phi_{\mathbf{C},A}(\mathbf{F})$ is defined by $\mathbf{F}G_{\mathbf{C}} = (\mathbf{H}_{\mathbf{C}}G_{\mathbf{C}}, \Phi_{\mathbf{C},A}(\mathbf{F}))$ for each functor $\mathbf{F} \colon \mathbf{C} \to \mathbf{Mod}(A)$ and $\Phi_{\mathbf{C},A}(\alpha) = \alpha_{G_{\mathbf{C}}}$ for each transformation α in $\mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C},\mathbf{Mod}(A))$;*

(ii) *$\Psi_{\mathbf{C},A}(\pi) = \mathbf{Mod}(\pi) \circ \eta_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{Mod}(\mathbf{End}(\mathbf{C})) \to \mathbf{Mod}(A)$ for each object π and $\Psi_{\mathbf{C},A}(S) = (\rho^X(S))_{X \in \mathbf{C}}$ for each morphism S in ${\mathbf{W}^*}_{(N,N^{\mathrm{op}})}(A,\mathbf{End}(\mathbf{C}))$.*

Explicitly, $\Psi_{\mathbf{C},A}(\pi)$ is given by $X \mapsto (\mathbf{H}_{\mathbf{C}}X, \rho^X \circ \pi)$ on objects and $T \mapsto \mathbf{H}_{\mathbf{C}}T$ on morphisms.

The proof of Proposition 4.12 involves the following result.

LEMMA 4.13. *Write* $\mathbf{U_C}G_\mathbf{C} = (\mathbf{H_C}G_\mathbf{C}, \rho, \sigma)$*. Then the assignments* $\alpha \mapsto \alpha_{G_\mathbf{C}}$ *and* $(\rho^X(S))_{X\in\mathbf{C}} \leftarrow\!\shortmid S$ *are inverse bijections between all natural transformations* α *of* $\mathbf{H_C}$ *(or* $\eta_\mathbf{C}$*) and all elements* $S \in \mathbf{End}(G_\mathbf{C})$ *(or* $S \in \mathbf{End}(G_\mathbf{C}) \cap (\rho(N)+\sigma(N))'$*, respectively).*

Proof. A family of morphisms $(\alpha_X\colon \mathbf{H_C}X \to \mathbf{H_C}X)_{X\in\mathbf{C}}$ is a natural transformation of $\mathbf{H_C}$ if and only if $\alpha_X T = T\alpha_X$ for all $X \in \mathbf{C}$ and $T \in \mathbf{H_C}(G_\mathbf{C}, X)$, that is, if $\alpha_X = \rho^X(\alpha_{G_\mathbf{C}})$ and $\alpha_{G_\mathbf{C}} \in \mathbf{End}(\mathbf{C})$. Such a family is a natural transformation of $\eta_\mathbf{C}$ if and only if additionally, $\alpha_X = \rho^X(\alpha_{G_\mathbf{C}})$ is a morphism of $\mathbf{U_C}X$ for each $X \in \mathbf{C}$ or, equivalently, if $\alpha_{G_\mathbf{C}} \in (\rho(N)+\sigma(N))'$. ∎

Proof of Proposition 4.12. Lemma 4.13 implies that $\Psi := \Psi_{\mathbf{C},A}$ is well defined by (ii). Let us show that $\Phi := \Phi_{\mathbf{C},A}$ is well defined by (i). For each $\mathbf{F}$ as above, the image $\mathbf{H}_{\mathbf{Mod}(A)}(\mathbf{F}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))) = \mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))$ consists of intertwiners for $\Phi(\mathbf{F})$ and hence $(\Phi(\mathbf{F}))(A) \subseteq \mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))' = \mathbf{End}(\mathbf{C})$. Likewise, for each α as above, $\alpha_{G_\mathbf{C}}$ intertwines $\mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))$ and hence $\alpha_{G_\mathbf{C}} \in \mathbf{End}(\mathbf{C})$. Finally, $\Phi(\alpha \circ \beta) = \alpha_{G_\mathbf{C}} \circ \beta_{G_\mathbf{C}} = \Phi(\alpha)\Phi(\beta)$ for all composable α, β.

Next, $\Phi \circ \Psi = \mathrm{id}$ because for each π as above, $\Psi(\pi)(G_\mathbf{C}) = (\mathbf{H_C}G_\mathbf{C}, \rho^{G_\mathbf{C}} \circ \pi)$ so that $\Phi(\Psi(\pi)) = \rho^{G_\mathbf{C}} \circ \pi = \pi$, and for each S as above, the component of $(\rho^X(S))_{X\in\mathbf{C}}$ at $X = G_\mathbf{C}$ is $\rho^{G_\mathbf{C}}(S) = S$.

Finally, we prove $\Psi \circ \Phi = \mathrm{id}$. Let $\mathbf{F}$ be as above and define ϕ^X by $\mathbf{F}X = (\mathbf{H}_C X, \phi^X)$ for each $X \in \mathbf{C}$. Then $\Phi(\mathbf{F}) = \phi^{G_\mathbf{C}}$, and for each $a \in A$, the family $(\phi^X(a))_{X\in\mathbf{C}}$ is a natural transformation of $\mathbf{H}_{\mathbf{Mod}(A)} \circ \mathbf{F} = \mathbf{H_C}$ which coincides by Lemma 4.13 with $(\rho^X(\phi^{G_\mathbf{C}}(a)))_{X\in\mathbf{C}}$. Therefore, $\mathbf{F}X = (\mathbf{H_C}X, \phi^X) = (\mathbf{H_C}X, \rho^X \circ \Phi(\mathbf{F})) = \Psi(\Phi(\mathbf{F}))(X)$ for each $X \in \mathbf{C}$. On morphisms, $\Psi(\Phi(\mathbf{F}))$ and $\mathbf{F}$ coincide anyway. For each α as above, $\Psi(\Phi(\alpha)) = (\rho^X(\alpha_{G_\mathbf{C}}))_{X\in\mathbf{C}} = \alpha$ by Lemma 4.13. ∎

COROLLARY 4.14. (i) *Let* $A \in \mathbf{W^*}_{(N,N^{\mathrm{op}})}$ *and consider* id_A *as an object of* $\mathbf{C} := \mathbf{Mod}(A)$*. Then* $\Phi_{\mathbf{C},A}(\mathrm{id}_\mathbf{C})\colon A \to \mathbf{End}(\mathbf{Mod}(A))$ *is an isomorphism in* $\mathbf{W^*}_{(N,N^{\mathrm{op}})}$ *with inverse* $\varepsilon_A := \rho^{\mathrm{id}_A}$.

(ii) *Let* $A, B \in \mathbf{W^*}_{(N,N^{\mathrm{op}})}$*. Then the isomorphism* $\mathbf{Mod}_{(A,B)}$ *obtained by composing*

$$(\varepsilon_B^{-1})_*\colon \mathbf{W^*}_{(N,N^{\mathrm{op}})}(A,B) \to \mathbf{W^*}_{(N,N^{\mathrm{op}})}(A, \mathbf{End}(\mathbf{Mod}(B)))$$

with

$$\Psi_{\mathbf{Mod}(B),A}\colon \mathbf{W^*}_{(N,N^{\mathrm{op}})}(A, \mathbf{End}(\mathbf{Mod}(B))) \to \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{Mod}(B), \mathbf{Mod}(A)),$$

is given by $\phi \mapsto \mathbf{Mod}(\phi)$ *on objects and* $b \mapsto (\pi(b))_{(L,\pi)}$ *on morphisms.*

(iii) *Let* $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$*. Then the functor* $\mathbf{End}_{(\mathbf{C},\mathbf{D})}$ *obtained by composing*

$$(\eta_\mathbf{D})_*\colon \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C}, \mathbf{D}) \to \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C}, \mathbf{Mod}(\mathbf{End}(\mathbf{D})))$$

with

$$\Phi_{\mathbf{C},\mathbf{End}(\mathbf{D})}\colon \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C}, \mathbf{Mod}(\mathbf{End}(\mathbf{D}))) \to \mathbf{W^*}_{(N,N^{\mathrm{op}})}(\mathbf{End}(\mathbf{D}), \mathbf{End}(\mathbf{C}))$$

is given by $\mathbf{F} \mapsto \rho^{\mathbf{F}G_\mathbf{C}}$ *on objects and* $\alpha \mapsto \mathbf{H_D}(\alpha_{G_\mathbf{C}})$ *on morphisms.*

Proof. Assertions (i) and (iii) follow immediately from the definitions and Proposition 4.12.

Let us prove (ii). For each object ϕ, we have $G_{\mathbf{Mod}(B)} = (\mathbf{H}_{\mathbf{Mod}(B)}, \varepsilon_B^{-1})$ and $\Phi_{\mathbf{Mod}(B),A}(\mathbf{Mod}(\phi)) = \varepsilon_B^{-1} \circ \phi$, whence $\Psi_{\mathbf{Mod}(B),A}(\varepsilon_B^{-1} \circ \phi) = \mathbf{Mod}(\phi)$, and for each morphism b, the family $\alpha := (\pi(b))_{(L,\pi)}$ is a natural transformation and $\Phi_{\mathbf{Mod}(B),A}(\alpha) = \alpha_{G_{\mathbf{Mod}(B)}} = \varepsilon_B^{-1}(b)$. ∎

The relative tensor product on $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ yields a product on the category $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$ as follows. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}$. Then $\mathbf{C} \times \mathbf{D}$ and the functor

$$\mathbf{U}_{\mathbf{C}\times\mathbf{D}} = (- \underset{\mu}{\otimes} -) \circ (\mathbf{U}_{\mathbf{C}} \times \mathbf{U}_{\mathbf{D}}) \colon \mathbf{C} \times \mathbf{D} \to \mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})},$$

form a category over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{\mathrm{op}})}$ with separating object $(G_{\mathbf{C}}, G_{\mathbf{D}})$. Thus, we obtain a monoidal structure on $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$, given by $(\mathbf{C}, \mathbf{D}) \mapsto \mathbf{C} \times \mathbf{D}$ on objects and $(\mathbf{F}, \mathbf{G}) \mapsto \mathbf{F} \times \mathbf{G}$ on morphisms.

COROLLARY 4.15. *For all $A, B, C \in \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$, there exists an isomorphism*

$$\Xi \colon \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}(A, B \underset{\mu}{*} C) \to \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{Mod}(B) \times \mathbf{Mod}(C), \mathbf{Mod}(A))$$

such that

(i) *for each object π, the functor $\Xi(\pi) \colon \mathbf{Mod}(B) \times \mathbf{Mod}(C) \to \mathbf{Mod}(A)$ is given by $((L,\tau),(M,v)) \mapsto (L \underset{\mu}{\otimes} M, (\tau \underset{\mu}{*} v) \circ \pi)$ and $(S,T) \mapsto S \underset{\mu}{\otimes} T$;*

(ii) *for each morphism $x \colon \pi_1 \to \pi_2$, the transformation $\Xi(b) \colon \Xi(\pi_1) \to \Xi(\pi_2)$ is given by $\Xi(b)_{((L,\tau),(M,v))} = (\tau \underset{\mu}{*} v)(x)$.*

Proof. Let $\mathbf{B} := \mathbf{Mod}(B)$, $\mathbf{C} := \mathbf{Mod}(C)$, $\mathbf{D} := \mathbf{B} \times \mathbf{C}$. Then $G := (G_{\mathbf{B}}, G_{\mathbf{C}})$ is separating and

$$\rho^G \colon \mathbf{End}(\mathbf{D}) \to \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G,G))' = (\mathbf{End}(\mathbf{B})' \underset{\mu}{\otimes} \mathbf{End}(\mathbf{C})')' = \mathbf{End}(\mathbf{B}) \underset{\mu}{*} \mathbf{End}(\mathbf{C}) \cong B \underset{\mu}{*} C$$

is an isomorphism by Remark 4.11.

Let $X = (L,\tau) \in \mathbf{B}$ and $Y = (M,v) \in \mathbf{C}$. Then $\rho^{(X,Y)} = (\tau \underset{\mu}{*} v) \circ \rho^G$ by Lemma 2.5 because $\tau \underset{\mu}{*} v = \rho_J$, where $J = \mathbf{H}_{\mathbf{B}}(\mathbf{B}(G_{\mathbf{B}}, X)) \underset{\mu}{\otimes} \mathbf{H}_{\mathbf{C}}(\mathbf{C}(G_{\mathbf{C}}, Y))$, and $J \cdot \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G_{\mathbf{D}}, G)) \subseteq \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G_{\mathbf{D}}, (X,Y)))$. Now, the assertion follows from Proposition 4.12. ∎

The categories $\mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$ and $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$ are enriched over the monoidal category $\mathbf{Cat}$ of small categories [14], or, equivalently, are 2-categories, meaning that the morphisms between fixed objects are themselves objects of a small category, as explained before Proposition 4.12, and that the composition of morphisms between fixed objects extends to a functor, where

$$B \underset{\psi_2}{\overset{\psi_1}{\underset{\Downarrow c}{\rightrightarrows}}} C \circ A \underset{\phi_2}{\overset{\phi_1}{\underset{\Downarrow b}{\rightrightarrows}}} B = A \underset{\psi_2 \circ \phi_2}{\overset{\psi_1 \circ \phi_1}{\underset{\Downarrow \psi_2(b)c}{\rightrightarrows}}} C \text{ in } \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}, \tag{4.2}$$

$$\mathbf{C} \underset{\mathbf{G}_2}{\overset{\mathbf{G}_1}{\underset{\Downarrow \beta}{\rightrightarrows}}} \mathbf{D} \circ \mathbf{B} \underset{\mathbf{F}_2}{\overset{\mathbf{F}_1}{\underset{\Downarrow \alpha}{\rightrightarrows}}} \mathbf{C} = \mathbf{B} \underset{\mathbf{G}_2 \circ \mathbf{F}_2}{\overset{\mathbf{G}_1 \circ \mathbf{F}_1}{\underset{\Downarrow \beta_{\mathbf{F}_2} \circ \mathbf{G}_1 \alpha}{\rightrightarrows}}} \mathbf{D} \text{ in } \mathbf{Cat}_{(N,N^{\mathrm{op}})}. \tag{4.3}$$

Recall that a contravariant functor between enriched categories $\mathbf{C}, \mathbf{D}$ consists of an assignment $\mathbf{F} \colon \operatorname{ob} \mathbf{C} \to \operatorname{ob} \mathbf{D}$ and, for each pair of objects $X, Y \in \mathbf{C}$, a functor $\mathbf{F}_{(X,Y)} \colon \mathbf{C}(X,Y) \to \mathbf{D}(\mathbf{F}Y, \mathbf{F}X)$ that is compatible with composition in a natural sense. We now show that the assignments **Mod**, **End** defined above are functors in this sense and that the isomorphisms in Proposition 4.12 form part of an adjunction between **Mod** and **End**. For background on enriched categories, see [14].

THEOREM 4.16. *The assignments* **Mod** *and* **End** *define contravariant functors* $\mathbf{Mod} \colon \mathbf{W}^*{}_{(N,N^{\mathrm{op}})} \to \mathbf{Cat}_{(N,N^{\mathrm{op}})}$ *and* $\mathbf{End} \colon \mathbf{Cat}_{(N,N^{\mathrm{op}})} \to \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$ *of enriched categories. The isomorphisms* $(\Phi_{\mathbf{C},A})_{\mathbf{C},A}$ *define an adjunction whose unit is* $(\eta_{\mathbf{C}})_{\mathbf{C} \in \mathbf{Cat}_{(N,N^{\mathrm{op}})}}$ *and counit is* $(\varepsilon_A)_{A \in \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}}$.

Proof. We first show that **Mod** and **End** are functors of enriched categories. By Corollary 4.14, it suffices to prove this for **End**. Consider a diagram as in (4.3) and let $a = \mathbf{End}_{(\mathbf{B},\mathbf{C})}(\alpha)$, $b = \mathbf{End}_{(\mathbf{C},\mathbf{D})}(\beta)$, $c = \mathbf{End}_{(\mathbf{B},\mathbf{D})}(\beta_{\mathbf{F}_2} \circ \mathbf{G}_1 \alpha)$. We have to show that then the cells

$$\mathbf{End}(\mathbf{C}) \underset{\mathbf{End}_{(\mathbf{B},\mathbf{C})}(\mathbf{F}_2)}{\overset{\mathbf{End}_{(\mathbf{B},\mathbf{C})}(\mathbf{F}_1)}{\underset{\Downarrow a}{\rightrightarrows}}} \mathbf{End}(\mathbf{B}) \circ \mathbf{End}(\mathbf{D}) \underset{\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{G}_2)}{\overset{\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{G}_1)}{\underset{\Downarrow b}{\rightrightarrows}}} \mathbf{End}(\mathbf{C})$$

and

$$\mathbf{End}(D) \underset{\mathbf{End}_{(\mathbf{B},\mathbf{D})}(\mathbf{G}_2\mathbf{F}_2)}{\overset{\mathbf{End}_{(\mathbf{B},\mathbf{D})}(\mathbf{G}_1\mathbf{F}_1)}{\underset{\Downarrow c}{\rightrightarrows}}} \mathbf{End}(B)$$

are equal. By definition, $a = \mathbf{H}_{\mathbf{C}}(\alpha_{G_{\mathbf{B}}})$, $b = \mathbf{H}_{\mathbf{D}}(\beta_{G_{\mathbf{C}}})$, and by Lemma 4.13,

$$c = \mathbf{H}_{\mathbf{D}}(\beta_{\mathbf{F}_2 G_{\mathbf{B}}} \cdot \mathbf{G}_1(\alpha_{G_{\mathbf{B}}})) = \rho^{\mathbf{F}_2 G_{\mathbf{B}}}(\mathbf{H}_{\mathbf{D}}(\beta_{G_{\mathbf{C}}})) \cdot \mathbf{H}_{\mathbf{C}}(\alpha_{G_{\mathbf{B}}}) = \mathbf{End}(\mathbf{F}_2)(b) \cdot a.$$

It remains to show that for all morphisms $\phi\colon A \to B$ in $\mathbf{W}^*{}_{(N,N^{\mathrm{op}})}$ and $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{(N,N^{\mathrm{op}})}$, the diagram

$$\begin{tikzcd} \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{D},\mathbf{Mod}(B)) \arrow[r, "\Phi_{\mathbf{D},B}"] \arrow[d] & \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}(B,\mathbf{End}(\mathbf{D})) \arrow[d] \\ \mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{C},\mathbf{Mod}(A)) \arrow[r, "\Phi_{\mathbf{C},A}"] & \mathbf{W}^*{}_{(N,N^{\mathrm{op}})}(A,\mathbf{End}(\mathbf{C})) \end{tikzcd}$$

commutes, where the vertical maps are induced by $\mathbf{F}$ and $\mathbf{Mod}_{(A,B)}(\phi)$ on the left and ϕ and $\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})$ on the right, respectively, or, more precisely, that for each object $\mathbf{G}$ and each morphism α in $\mathbf{Cat}_{(N,N^{\mathrm{op}})}(\mathbf{D},\mathbf{Mod}(B))$,

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F}) \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi = \Phi_{\mathbf{C},A}(\mathbf{Mod}_{(A,B)}(\phi) \circ \mathbf{G} \circ \mathbf{F}),$$
$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})(\alpha) = \mathbf{Mod}_{(A,B)}(\phi)(\alpha_{\mathbf{F}}).$$

The second equation holds because of Lemma 4.13 and the relation

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})(\alpha_{G_{\mathbf{C}}}) = \rho^{\mathbf{F}G_{\mathbf{C}}}(\alpha_{G_{\mathbf{D}}}) = \alpha_{\mathbf{F}G_{\mathbf{C}}} = \mathbf{Mod}_{(A,B)}(\phi)(\alpha_{\mathbf{F}G_{\mathbf{C}}})$$

first one holds because by Corollary 4.14,

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F}) \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi = \rho^{\mathbf{F}G_{\mathbf{C}}} \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi,$$
$$(\mathbf{Mod}_{(A,B)}(\phi) \circ \mathbf{G} \circ \mathbf{F})(G_{\mathbf{C}}) = (\mathbf{H}_{\mathbf{C}}G_{\mathbf{C}}, \rho^{\mathbf{F}G_{\mathbf{C}}} \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi). \quad \blacksquare$$

5. THE SPECIAL CASE OF A COMMUTATIVE BASE

Let Z be a locally compact Hausdorff space with a Radon measure μ of full support, and identify $C_0(Z)$ with multiplication operators on $\mathcal{L}(L^2(Z,\mu))$. Then the relative tensor product and the fiber product over the C^*-base $\mathfrak{b} = (L^2(Z,\mu), C_0(Z), C_0(Z))$ can be related to the fiberwise product of bundles as follows.

Denote by $\mathbf{Mod}_{\mathfrak{b}}$, $\mathbf{Mod}_{C_0(Z)}$, and $\mathbf{Bdl}_Z$ the categories of all C^*-$\mathfrak{b}$-modules with all morphisms, of all Hilbert C^*-modules over $C_0(Z)$, and of all continuous Hilbert bundles over Z; for the precise definition of the latter, see [6]. Each of these categories carries a monoidal structure, where the product

(i) of $E, F \in \mathbf{Mod}_{C_0(Z)}$ is the separated completion of $E \odot F$ with respect to the inner product $\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \langle \xi | \xi' \rangle \langle \eta | \eta' \rangle$, denoted by $E \underset{C_0(Z)}{\otimes} F$;

(ii) of $\mathcal{E}, \mathcal{F} \in \mathbf{Bdl}_Z$ is the fibrewise tensor product of $\mathcal{E}$ and $\mathcal{F}$;

(iii) of $H_\beta, K_\gamma \in \mathbf{Mod}_{\mathfrak{b}}$ is $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, \beta \bowtie \gamma)$, where $\beta \bowtie \gamma := [|\gamma\rangle_2 \beta] = [|\beta\rangle_1 \gamma]$;

here, note that ${}_\beta H_\beta, {}_\gamma K_\gamma$ are C^*-$(\mathfrak{b},\mathfrak{b})$-modules.

There exist equivalences of monoidal categories

$$\mathbf{Mod}_{\mathfrak{b}} \underset{\mathbf{F}}{\overset{\mathbf{U}}{\rightleftarrows}} \mathbf{Mod}_{C_0(Z)} \underset{\Gamma_0}{\overset{\mathbf{B}}{\rightleftarrows}} \mathbf{Bdl}_Z$$

such that for each $E \in \mathbf{Mod}_{C_0(Z)}$, $\mathcal{F} \in \mathbf{Bdl}_Z$, $H_\beta \in \mathbf{Mod}_{\mathfrak{b}}$,

(i) $\mathbf{U}H_\beta = \beta \in \mathbf{Mod}_{C_0(Z)}$;

(ii) $\mathbf{F}E = (E \otimes_{C_0(Z)} L^2(Z,\mu), l(E))$, where $l(\xi)\eta = \xi \otimes_{C_0(Z)} \eta$ for each $\xi \in E, \eta \in L^2(Z,\mu)$;

(iii) $\mathbf{B}E = \bigsqcup_{z\in Z} E_z$ and $\Gamma_0(\mathbf{B}E) = \{(\xi_z)_z : \xi \in E\}$, where E_z is the completion of E with respect to the inner product $(\xi,\eta) \mapsto \langle\xi|\eta\rangle(z)$, and $\xi \mapsto \xi_z$ denotes the quotient map $E \to E_z$;

(iv) the operations on the space of sections $\Gamma_0(\mathcal{F}) \in \mathbf{Mod}_{C_0(Z)}$ are defined fiberwise.

The equivalence on the left is easily verified, and the equivalence on the right is explained in [6]. Compare also Examples 2.7 and 2.12(ii).

Denote by $\mathbf{C}^*_{C_0(Z)}$ the category of all continuous $C_0(Z)$-algebras with full support [6], where the morphisms between $A, B \in \mathbf{C}^*_{C_0(Z)}$ are all $C_0(Z)$-linear nondegenerate $*$-homomorphisms $\pi\colon A \to M(B)$, and by $\widetilde{\mathbf{C}}^*_{\mathfrak{b}}$ the category of all C^*-$\mathfrak{b}$-algebras A^β_H satisfying $[\rho_\beta(C_0(Z))A] = A$ and $[A\beta] = \beta$, where the morphisms between $A^\beta_H, B^\gamma_K \in \widetilde{\mathbf{C}}^*_{\mathfrak{b}}$ are all $\pi \in \mathbf{C}^*_{\mathfrak{b}}(A^\beta_H, M(B)^\gamma_K)$ satisfying $[\pi(A)B] = B$. Then there exists a functor $\widetilde{\mathbf{C}}^*_{\mathfrak{b}} \to \mathbf{C}^*_{C_0(Z)}$, given by $A^\beta_H \mapsto (A, \rho_\alpha)$ and $\pi \mapsto \pi$, and this functor has a full and faithful left adjoint which embeds $\mathbf{C}^*_{C_0(Z)}$ into $\widetilde{\mathbf{C}}^*_{\mathfrak{b}}$ ([28], Theorem 6.6).

We finally consider the fiber product of commutative C^*-$\mathfrak{b}$-algebras and start with preliminaries. Let Z be a locally compact space, E a Hilbert C^*-module over $C_0(Z)$, and $\mathbf{B}E = \bigsqcup_{z\in Z} E_z$ the corresponding Hilbert bundle. The topology on $\mathbf{B}E$ is generated by all open sets of the form $U_{V,\eta,\varepsilon} = \{\zeta | z \in V, \zeta \in E_z, \|\eta_z - \zeta\|_{E_z} < \varepsilon\}$, where $V \subseteq Z$ is open, $\eta \in E$, $\varepsilon > 0$. Denote by $q\colon \bigsqcup_{z\in Z} \mathcal{L}(E_z) \to Z$ the natural projection and define for each $\eta, \eta' \in E$ maps

$$\omega_{\eta,\eta'}\colon \bigsqcup_{z\in Z} \mathcal{L}(E_z) \to \mathbb{C}, \qquad T \mapsto \langle \eta_{q(T)} | T\eta'_{q(T)}\rangle,$$

$$v^{(*)}_\eta\colon \bigsqcup_{z\in Z} \mathcal{L}(E_z) \to \bigsqcup_{z\in Z} E_z, \quad T \mapsto T^{(*)}\eta_{q(T)}.$$

The *weak topology (strong-$*$-topology)* on $\bigsqcup_{z\in Z} \mathcal{L}(E_z)$ is the weakest one that makes q and all maps of the form $\omega_{\eta,\eta'}$ (of the form $v^{(*)}_\eta$) continuous.

Let A be a commutative C^*-algebra, let $\pi\colon C_0(Z) \to M(A)$ be a $*$-homomorphism, and let $\chi \in \widehat{A}$. Then we identify $E \otimes_{\phi^*} A \otimes_\chi \mathbb{C}$ with E_z, where $z \in Z$ corresponds to $\chi \circ \pi \in \widehat{C_0(Z)}$, via $\eta \otimes_\pi a \otimes_\chi \lambda \mapsto \lambda\chi(a)\eta_z$. A map $T\colon \widehat{A} \to \bigsqcup_{z\in Z} \mathcal{L}(E_z)$ is *weakly vanishing (strong-$*$-vanishing) at infinity* if for all $\eta, \eta' \in E$, the map $\omega_{\eta,\eta'} \circ T$ (the maps $\chi \mapsto \|v^{(*)}_\eta(T(\chi))\|$) vanish at infinity.

LEMMA 5.1. *Let A^{β}_{H} be a C^*-$\mathfrak{b}$-algebra, K_γ a C^*-$\mathfrak{b}^\dagger$-module, $x \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$. Assume that A is commutative, $[\rho_\beta(C_0(Z))A] = A$, and $\langle\gamma|_2 x|\gamma\rangle_2 \subseteq A$. Define $F_x\colon \widehat{A} \to \bigsqcup_{z\in Z} \mathcal{L}(\gamma_z)$ by $\chi \mapsto (\chi * \mathrm{id})(x)$. Then:*

(i) *F_x is weakly continuous, weakly vanishing at infinity.*

(ii) *$x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$ if and only if F_x is strong-$*$ continuous, strong-$*$-vanishing at infinity.*

Proof. First, note that for all $\eta, \eta' \in \gamma$ and $\chi \in \widehat{A}$,

$$\chi(\langle\eta|_2 x|\eta'\rangle_2) = \langle 1_{(\chi\circ\rho_\beta)} \otimes \eta|(\chi * \mathrm{id})(x)(1_{(\chi\circ\rho_\beta)} \otimes \eta')\rangle = \langle \eta_{(\chi\circ\rho_\beta)}|F_x(\chi)\eta'_{(\chi\circ\rho_\beta)}\rangle.$$

(i) For each $\eta', \eta \in \gamma$, the map $\chi \mapsto \langle\eta_{(\chi\circ\rho_\beta)}|F_x(\chi)\eta'_{(\chi\circ\rho_\beta)}\rangle$ equals $\langle\eta|_2 x|\eta'\rangle_2 \in A$.

(ii) Assume that F_x is strong-$*$ continuous vanishing at infinity and let $\eta \in \gamma$. Then the map $\chi \mapsto F_x(\chi)\eta_{(\chi\circ\rho_\beta)}$ lies in $\Gamma_0(\gamma \otimes_{\rho_\beta} A)$. Hence, there exists an $\omega \in \gamma \otimes_{\rho_\beta} A$ such that $F_x(\chi)\eta_{(\chi\circ\rho_\beta)} = \omega_\chi$ for all $\chi \in \widehat{A}$. We identify $\gamma \otimes_{\rho_\beta} A$ with $[|\gamma\rangle_2 A] \subseteq \mathcal{L}(H, H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$ in the canonical manner and find that $x|\eta\rangle_2 = \omega$ because $\chi(\langle\eta'|_2 x|\eta\rangle_2) = \langle\eta'_{(\chi\circ\rho_\beta)}|\omega_{(\chi\circ\rho_\beta)}\rangle = \chi(\langle\eta'|_2\omega)$ for all $\chi \in \widehat{A}$, $\eta' \in \gamma$. Since $\eta \in \gamma$ was arbitrary, we can conclude $x|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$. A similar argument, applied to x^* instead of x, shows that $x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$, and therefore $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$. Reversing the arguments, we obtain the reverse implication. ∎

Let X be a locally compact Hausdorff space with a continuous surjection $p\colon X \to Z$ and a family of Radon measures $\phi = (\phi_z)_{z\in Z}$ such that:

(i) $\operatorname{supp} \phi_z = X_z := p^{-1}(z)$ for each $z \in Z$ and

(ii) the map $\phi_*(f)\colon z \mapsto \int_{X_z} f \,\mathrm{d}\phi_z$ is continuous for each $f \in C_c(X)$.

Define a Radon measure ν_X on X such that

$$\int_X f \,\mathrm{d}\nu_X = \int_Z \phi_*(f)\mathrm{d}\mu \quad \text{for all } f \in C_c(X).$$

Then there exists a unique map

$$j_X\colon C_c(X) \to \mathcal{L}(L^2(Z,\mu), L^2(X,\nu_X))$$

such that $j_X(f)h = f p^*(h)$ and $j_X(f)^* g = \phi_*(\overline{f} g)$ for all $f, g \in C_c(X), h \in C_c(Z)$. Similarly, let Y be a locally compact Hausdorff space with a continuous map $q\colon Y \to Z$ and a family of measures $\psi = (\psi_z)_{z\in Z}$ satisfying the same conditions as X, p, ϕ, and define a Radon measure ν_Y on Y and an embedding $j_Y\colon C_c(Y) \to \mathcal{L}(L^2(Z,\mu), L^2(Y,\nu_Y))$ as above. Let

$$H := L^2(X,\nu_X), \quad \beta := [j_X(C_c(X))], \quad A := C_0(X) \subseteq \mathcal{L}(L^2(X,\nu_X) = \mathcal{L}(H),$$
$$K := L^2(Y,\nu_Y), \quad \gamma := [j_Y(C_c(Y))], \quad B := C_0(Y) \subseteq \mathcal{L}(L^2(Y,\nu_Y)) = \mathcal{L}(K).$$

Then H_β, K_γ are C^*-$\mathfrak{b}$-modules and A_H^β, B_K^γ are C^*-$\mathfrak{b}$-algebras, as one can easily check. Considering β and γ as Hilbert C^*-modules over $C_0(Z)$, we can canonically identify $\beta_z \cong L^2(X_z, \phi_z)$ and $\gamma_z \cong L^2(Y_z, \psi_z)$. Finally, define a Radon measure ν on $X_p \underset{Z}{\times} {}_qY$ such that for all $h \in C_c(X_p \underset{Z}{\times} {}_qY)$,

$$\int\limits_{X_p \underset{Z}{\times} {}_qY} h \, d\nu = \int\limits_Z \int\limits_{X_z} \int\limits_{Y_z} h(x,y) \, d\psi_z(y) \, d\phi_z(x) \, d\mu(z).$$

PROPOSITION 5.2. (i) *There exists a unitary* $U\colon H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K \to L^2(X_p \underset{Z}{\times} {}_qY, \nu)$ *such that* $(U(j_X(f) \oslash h \otimes j_Y(g)))(x,y) = f(x)h(p(x))g(y)$ *for all* $f \in C_c(X)$, $g \in C_c(Y)$, $h \in C_c(Z)$, $(x,y) \in X_p \underset{Z}{\times} {}_qY$.

(ii) $\mathrm{Ad}_U(A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B)$ *is the* C^**-algebra of all* $f \in L^\infty(X_p \underset{Z}{\times} {}_qY, \nu)$ *that have representatives* f_X, f_Y *such that the maps* $X \to \operatorname{Tot}\mathcal{L}(\gamma)$ *and* $Y \to \operatorname{Tot}\mathcal{L}(\beta)$ *given by* $x \mapsto f_X(x, \cdot) \in L^\infty(Y_{p(x)}, \psi_{p(x)})$ *and* $y \mapsto f_Y(\cdot, y) \in L^\infty(X_{q(y)}, \phi_{q(y)})$ *respectively, are strong-$*$ continuous vanishing at infinity.*

Proof. The proof of assertion (i) is straightforward, and assertion (ii) follows immediately from Proposition Lemma 3.16(viii) and Lemma 5.1(ii). ∎

EXAMPLE 5.3. (i) Let X, Y be discrete, $Z = \{0\}$, and let ϕ_0, ψ_0 be the counting measures on X, Y, respectively. Then

$$C_0(X)_\beta \underset{\mathfrak{b}}{*} {}_\gamma C_0(Y) \cong \{f \in C_b(X \times Y) : f(x, \cdot\,) \in C_0(Y) \text{ for all } x \in X, \\ f(\,\cdot\,, y) \in C_0(X) \text{ for all } y \in Y\}.$$

This follows from Proposition 5.2 and the fact that for each $f \in C_b(X \times Y)$, the maps $X \to \mathcal{L}(l^2(Y))$, $x \mapsto f(x, \cdot\,)$, and $Y \to \mathcal{L}(l^2(X))$, $y \mapsto f(\,\cdot\,, y)$, are strong-$*$ continuous vanishing at infinity if and only if $f(\,\cdot\,, y) \in C_0(X)$ and $f(x, \cdot\,) \in C_0(Y)$ for each $y \in Y$ and $x \in X$.

(ii) Let $X = \mathbb{N}$, $Z = \{0\}$, and let ϕ_0 be the counting measure. Then

$$C_0(\mathbb{N})_\beta \underset{\mathfrak{b}}{*} {}_\gamma C_0(Y) \cong \{f \in C_b(\mathbb{N} \times Y) : (f(x, \cdot))_x \text{ is a sequence in } C_0(Y) \\ \text{that converges strongly to } 0 \in \mathcal{L}(L^2(Y, \psi_0))\}$$

because for each $f \in L^\infty(\mathbb{N} \times Y)$, the map $Y \to \mathcal{L}(l^2(\mathbb{N}))$, $y \mapsto f(\,\cdot\,, y)$, is strong-$*$ continuous vanishing at infinity if and only if $f(x, \cdot\,) \in C_0(Y)$ for all $x \in \mathbb{N}$.

(iii) Let $X = Y = [0,1]$, $Z = \{0\}$, and let $\phi_0 = \psi_0$ be the Lebesgue measure. For each subset $I \subseteq [0,1]$, denote by χ_I its characteristic function. Then the function $f \in L^\infty([0,1] \times [0,1])$ given by $f(x,y) = 1$ if $y \leqslant x$ and $f(x,y) = 0$ otherwise belongs to $C([0,1])_\beta \underset{\mathfrak{b}}{*} {}_\gamma C([0,1])$ because the functions $[0,1] \to L^\infty([0,1]) \subseteq \mathcal{L}(L^2([0,1]))$ given by $x \mapsto f(x, \cdot\,) = \chi_{[0,x]}$ and $y \mapsto f(\,\cdot\,, y) = \chi_{[y,1]}$ are strong-$*$

continuous. In particular, we see that $C([0,1])_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}C([0,1]) \nsubseteq C([0,1] \times [0,1]) = C([0,1]) \otimes C([0,1])$.

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