J. OPERATOR THEORY
69:1(2013), 3–16
doi: 10.7900/jot.2010jun01.1866

STABLE AND NORM-STABLE INVARIANT SUBSPACES

ALEXANDER BORICHEV, DON HADWIN and HASSAN YOUSEFI

Communicated by Nikolai K. Nikolski

ABSTRACT. We prove that if T is an operator on an infinite-dimensional Hilbert space whose spectrum and essential spectrum are both connected and whose *Fredholm index* is only 0 or 1, then the only nontrivial *norm-stable invariant subspaces* of T are the finite-dimensional ones. We also characterize norm-stable invariant subspaces of any weighted unilateral shift operator. We show that *quasianalytic shift operators* are points of norm continuity of the lattice of the invariant subspaces. We also provide a necessary condition for strongly stable invariant subspaces for certain operators.

KEYWORDS: *Stable invariant subspaces, weighted shift operator.*

MSC (2010): Primary 47A15; Secondary 47B37.

1. INTRODUCTION

In this paper we continue work started in [11] on stable invariant subspaces of Hilbert-space operators. In Section 2, we show that if T is an operator on a separable Hilbert space whose Fredholm index at every point in its semi-Fredholm domain is either 0 or 1 and whose spectrum and essential spectrum are connected, then the proper norm stable invariant subspaces of T must be finite-dimensional (Theorem 2.5). As a consequence, in Section 3, we completely characterize the norm-stable invariant subspaces of every weighted unilateral shift operator (Theorem 3.5). In Section 4 we prove a semicontinuity result (Lemma 4.1) for an index invariant (see, for instance, [22]), and we use it to provide a necessary condition for invariant subspaces of certain operators to be (strongly) stable. Several open questions are given in Section 5.

In our work, we need a variant of Y. Domar's result answering a problem of A. Shields (Problem 17 in [25], 1974; see also the updated edition of [25] published in 1979): the nonzero invariant subspaces of a quasianalytic weighted unilateral shift operator all have finite co-dimension (Theorem 5.1 in Appendix).

Throughout this paper, $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ is the set of all (bounded, linear) operators on $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ is the space of compact operators on $\mathcal{H}$. A closed linear subspace M of $\mathcal{H}$ will be identified with the orthogonal projection P_M onto M. If $T \in \mathcal{B}(\mathcal{H})$, then $\mathrm{Lat}\,(T)$ denotes the set of all closed invariant subspaces of T; alternatively, $\mathrm{Lat}\,(T)$ is the set of all projections P in $\mathcal{B}(\mathcal{H})$ such that $(1-P)TP = 0$. We denote the spectrum of T by $\sigma(T)$ and the spectral radius of T by $r(T)$. The image of an operator $T \in \mathcal{B}(\mathcal{H})$ in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted by $\widetilde{T}$ and $\sigma_{\mathrm{e}}(T)$ denotes the essential spectrum of T, which by definition, equals $\sigma(\widetilde{T})$. Recall that $T \in \mathcal{B}(\mathcal{H})$ is a *semi-Fredholm operator* if $\mathrm{Range}\,(T)$ is closed and either $\dim(\mathrm{Ker}\,(T))$ or $\dim(\mathrm{Ker}\,(T^*))$ is finite. In this case, the *index* of T is defined by

$$\mathrm{ind}\,(T) = \dim(\mathrm{Ker}\,(T)) - \dim(\mathrm{Ker}\,(T^*)).$$

The *semi-Fredholm domain* of T is denoted by $\rho_{\text{s-F}}(T)$ and is defined by

$$\rho_{\text{s-F}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is semi-Fredholm}\}.$$

If $\mathrm{ind}\,(\lambda - T) \neq \pm\infty$ for every $\lambda \in \rho_{\text{s-F}}(T)$, then

$$\mathbb{C}\backslash\rho_{\text{s-F}}(T) = \sigma_{\mathrm{e}}(T).$$

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *biquasitriangular* if and only if $\mathrm{ind}\,(T-\lambda) = 0$ for every $\lambda \in \rho_{\text{s-F}}(T)$. The *similarity orbit* of T is defined to be

$$\mathcal{S}(T) = \{ATA^{-1} : A \in \mathcal{B}(\mathcal{H}) \text{ is invertible}\}.$$

One of the deepest results in approximation theory for operators is the *Similarity Theorem* (Theorem 9.2 in [2]). For our purposes we need only the special case taken from [4], which we state here.

PROPOSITION 1.1 ([4]). *Suppose $B, A \in \mathcal{B}(\mathcal{H})$.*

(i) *If A is a normal operator and $\sigma(A) \subset \mathbb{C}\backslash\rho_{\text{s-F}}(B)$, then $\mathcal{S}(B)$ and $\mathcal{S}(A \oplus B)$ have the same norm closures.*

(ii) *If*

(a) *$\sigma_{\mathrm{e}}(B)$ has no isolated points,*

(b) *either $\sigma_p(B) = \emptyset$ or $\sigma_p(B^*) = \emptyset$,*

(c) *each component of $\sigma_{\mathrm{e}}(A)$ meets $\sigma_{\mathrm{e}}(B)$,*

(d) *$\sigma_{\mathrm{e}}(B) \subset \sigma_{\mathrm{e}}(A)$,*

(e) *$\rho_{\text{s-F}}(A) \subset \rho_{\text{s-F}}(B)$ and* $\mathrm{ind}\,(A-\lambda) = \mathrm{ind}\,(B-\lambda) \neq \pm\infty$ *for every $\lambda \in \rho_{\text{s-F}}(A)$.*

Then A is in the norm closure of $\mathcal{S}(B)$.

An invariant subspace $P \in \mathrm{Lat}\,(T)$ is called (strongly) *stable* if, whenever there is a sequence $\{T_n\}$ in $\mathcal{B}(\mathcal{H})$ such that $\|T_n - T\| \to 0$, there is a sequence $\{P_n\}$ with $P_n \in \mathrm{Lat}\,(T_n)$ for $n \geqslant 1$ such that $P_n \to P$ in the strong operator topology (SOT). We say that P is *norm stable* if we can always choose $\{P_n\}$ so that $\|P_n - P\| \to 0$. We let $\mathrm{Lat}_{\,\mathrm{s}}(T)$ be the collection of stable invariant subspaces of T and $\mathrm{Lat}_{\,\mathrm{ns}}(T)$ be the collection of norm-stable invariant subspaces of T. It is clear that $\mathrm{Lat}_{\,\mathrm{ns}}(T) \subset \mathrm{Lat}_{\,\mathrm{s}}(T)$ and it is easy to show that $\mathrm{Lat}_{\,\mathrm{ns}}(T)$ contains $\{0\}$

and $\mathcal{H}$. It is also easy to show that $\mathrm{Lat}_{\mathrm{ns}}(T)$ is norm closed and that $\mathrm{Lat}_{\mathrm{s}}(T)$ is SOT-closed (see [11]). The following question was posed in [11]:

QUESTION: Is $\mathrm{Lat}_{\mathrm{s}}(T)$ always the SOT-closure of $\mathrm{Lat}_{\mathrm{ns}}(T)$?

J. Conway and D. Hadwin [11] gave an affirmative answer to this question when T is normal or an unweighted unilateral shift of finite multiplicity.

In the finite-dimensional setting, the stable invariant subspaces of an operator were characterized in [5], [9], and [2]. In [3] C. Apostol, C. Foiaş and N. Salinas showed that if T is a normal operator, then the projections in $\mathrm{Lat}_{\mathrm{ns}}(T)$ are precisely the spectral subspaces corresponding to clopen subsets of $\sigma(T)$. Moreover, they proved that a quasitriangular operator with connected spectrum has no nontrivial norm-stable invariant subspace (i.e., $\mathrm{Lat}_{\mathrm{ns}}(T) = \{0, 1\}$). The question for $\mathrm{Lat}_{\mathrm{s}}(T)$ is much more delicate and is related to the invariant subspace problem [11].

2. RESULTS ON $\mathrm{Lat}_{\mathrm{ns}}$

We begin with a topological lemma for compact subsets of the plane. If $K \subset \mathbb{C}$ and $\varepsilon > 0$, then we define $K_\varepsilon = \{z \in \mathbb{C} : \mathrm{dist}\,(z, K) < \varepsilon\}$.

We use the notation $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{T} = \partial\mathbb{D}$.

LEMMA 2.1. *Suppose that K is a nonempty compact connected subset of $\mathbb{C}$, $\delta > 0$, and $\mathcal{U}$ is a collection of bounded connected components of $\mathbb{C}\backslash K$ such that each $U \in \mathcal{U}$ contains a point whose distance to K is greater than δ. Let $V = \bigcup_{U \in \mathcal{U}} U$. Then there is an $r > 1$ and a univalent (analytic) function φ on $r\mathbb{D}$ such that:*

(i) *$\varphi(\mathbb{T}) \subset K_\delta$, and*

(ii) *$V \backslash K_\delta \subset \varphi(\mathbb{D}) \subset K_\delta \cup V$.*

Proof. If $\mathcal{U} = \varnothing$, then $V = \varnothing$, and we can choose an $a \in K$ and define $\varphi(z) = a + (\delta/r)z$. Now suppose that $\mathcal{U} \neq \varnothing$. Let $V_1, \ldots, V_n$ be the elements of $\mathcal{U}$ that contain a closed disc of radius δ. Since K is connected, each V_j is simply connected, since it has a connected complement. Thus there is a univalent function f_j from the unit disc $\mathbb{D}$ to V_j for $1 \leqslant j \leqslant n$. Since $\{f_j(r\mathbb{D}) : 0 < r < 1\}$ is an open cover of V_j, there is a t, $0 < t < 1$, such that

$$\Big\{z \in V_j : \mathrm{dist}\,(z, K) \geqslant \frac{\delta}{2}\Big\} \subset f_j(t\mathbb{D}) \equiv W_j$$

for $1 \leqslant j \leqslant n$. Since

$$\Omega = K_\delta \cup \Big(\bigcup_{1 \leqslant j \leqslant n} W_j \Big)$$

is an open and connected set containing the disjoint closed contractible "discs" $\overline{W}_1, \ldots, \overline{W}_n$, there is a simple closed curve γ in Ω that winds around each point in $\bigcup_{1 \leqslant j \leqslant n} \overline{W}_j$ and doesn't wind around any point outside Ω. (This is extremely easy to see if we first shrink each $\overline{W}_j$ to a point.) Thus there is a univalent function f on $\mathbb{D}$ such that $f(\mathbb{D})$ is the set of points inside γ. As above we can choose s, $0 < s < 1$, so that $\bigcup_{1\leqslant j\leqslant n} \overline{W}_j \subset f(s\mathbb{D})$. It is clear that defining $\varphi(z) = f(z/s)$ and $r = 1/s$ yields the desired function. ∎

LEMMA 2.2. *Suppose $S \in \mathcal{B}(\mathcal{H})$, $r > 1 \geqslant r(S)$, and φ is a univalent function on $r\mathbb{D}$. Then:*

(i) $\sigma(\varphi(S)) = \varphi(\sigma(S))$, $\sigma_{\mathrm{e}}(\varphi(S)) = \varphi(\sigma_{\mathrm{e}}(S))$,

(ii) *for every* $\lambda \in \overline{\mathbb{D}}$, $\mathrm{Ker}\,(S-\lambda) = \mathrm{Ker}\,(\varphi(S) - \varphi(\lambda))$, $\mathrm{Ker}\,(S-\lambda)^* = \mathrm{Ker}\,(\varphi(S) - \varphi(\lambda))^*$, $\mathrm{ind}\,(S-\lambda) = \mathrm{ind}\,(\varphi(S) - \varphi(\lambda))$,

(iii) $\mathrm{Lat}\,(S) = \mathrm{Lat}\,(\varphi(S))$.

Proof. Statement (i) is the spectral mapping theorem for the Riesz–Dunford functional calculus. Since φ is univalent, $\varphi(r\mathbb{D})$ is simply connected, and it follows from Runge's Theorem that φ and φ^{-1} are limits of polynomials that converge uniformly on compact sets. This shows that (ii) holds and then S and $\varphi(S)$ generate the same norm-closed unital algebras, from which (iii) easily follows. ∎

Let $\mathcal{Q}(\mathcal{H})$ denote the set of all operators in $\mathcal{B}(\mathcal{H})$ whose nontrivial invariant subspaces all have finite co-dimension.

LEMMA 2.3. *Suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space and $T \in \mathcal{Q}(\mathcal{H})$. Then:*

(i) $\sigma(T)$ *is connected,*

(ii) $\mathrm{ind}\,(T-\lambda) \in \{0, -1\}$ *for every* $\lambda \in \rho_{\text{s-F}}(T)$.

Proof. If $\sigma(T)$ were disconnected, then the Riesz–Dunford functional calculus would give a spectral idempotent P that commutes with T and either $\mathrm{Ker}\,P$ or $\mathrm{Ker}\,(1-P)$ would be a nonzero invariant subspace with infinte co-dimension.

Clearly, T cannot have any eigenvalues, and T^* cannot have an eigenvalue with an infinite-dimensional eigenspace. Hence, $\mathrm{ind}\,(T-\lambda) \leqslant 0$ whenever $\lambda \in \rho_{\text{s-F}}(T)$. Suppose that $\mathrm{ind}\,(T-\lambda) \leqslant -2$. Then $A = T - \lambda$ is injective, $A(\mathcal{H})$ is closed and $\dim A(\mathcal{H})^{\perp} \geqslant 2$. It follows from the injectivity of A that $\dim(A^n(\mathcal{H}) \ominus A^{n+1}(\mathcal{H})) \geqslant 2$ for every $n \geqslant 1$. Let $x_0, f \in A(\mathcal{H})^{\perp}$ be two orthogonal unit vectors, and, for $n \geqslant 1$, take a unit vector $x_n \in A^n(\mathcal{H}) \ominus A^{n+1}(\mathcal{H})$ such that $x_n \perp A^n f$. Let $M = \operatorname{clos\,span}\{A^n f : n \geqslant 0\}$. Then M is an invariant subspace for T and if $x = \sum \alpha_n x_n \in M$, then, by successively projecting onto $A^n(\mathcal{H})^{\perp}$, $n = 1, 2, \ldots$, we see that $x = 0$. Hence M has infinite co-dimension, which is a contradiction to the assumption $T \in \mathcal{Q}$. Hence statement (ii) holds. ∎

THEOREM 2.4. *The norm closure of the set $\mathcal{R}$ of operators with connected essential spectrum and whose nontrivial invariant subspaces all have finite co-dimension is the set $\mathcal{T}$ of all operators $T \in \mathcal{B}(\mathcal{H})$, such that $\sigma(T)$ and $\sigma_{\mathrm{e}}(T)$ are both connected and such* *that, for every* $\lambda \in \rho_{\text{s-F}}(T)$*,*

$$\operatorname{ind}(T-\lambda) \in \{0,-1\}.$$

Proof. The inclusion $\operatorname{clos}\mathcal{R} \subset \mathcal{T}$ follows from Lemma 2.3 and the fact that $\mathcal{T}$ is norm closed. Suppose $T \in \mathcal{T}$ and $\varepsilon > 0$. Let N be a normal operator such that $\sigma(N) = \sigma_e(N) = \sigma_e(T)$. It follows from Proposition 1.1 that T is in the closure of the similarity orbit of $T \oplus N$. We let $K = \sigma_e(T)$, and let $\mathcal{U}$ be the set of those components of $\mathbb{C}\backslash K$ where $\operatorname{ind}(T-\lambda) = -1$. Now we apply Lemma 2.1 to obtain V, r, φ, and let $\gamma = \varphi(\mathbb{T})$. Let S be a quasianalytic shift operator satisfying the conditions of Theorem 5.1 in Appendix. It follows from Lemma 2.2 and Theorem 5.1 that $\operatorname{Lat} S = \operatorname{Lat}\varphi(S)$, $\varphi(S) \in \mathcal{R}$, $\sigma(\varphi(S)) = \varphi(\sigma(S)) = \varphi(\overline{\mathbb{D}})$, $\sigma_e(\varphi(S)) = \varphi(\sigma_e(S)) = \varphi(\mathbb{T}) = \gamma$, $\operatorname{ind}(\varphi(S)-\lambda) = -1$ exactly when $\lambda \in \varphi(\mathbb{D})$.

Choose a normal operator N_ε such that $\|N - N_\varepsilon\| \leqslant \varepsilon$ and $\sigma(N_\varepsilon) = \operatorname{clos}[\sigma_e(T)_\varepsilon]$. It follows from Proposition 1.1 that $T \oplus N_\varepsilon$ is in the closure of the similarity orbit of $\varphi(S)$; hence $T \oplus N_\varepsilon \in \operatorname{clos}\mathcal{R}$. Since $\|T \oplus N - T \oplus N_\varepsilon\| \leqslant \varepsilon$ and $\varepsilon > 0$ was arbitrary, we know that $T \oplus N \in \operatorname{clos}\mathcal{R}$. Since $T \in \operatorname{clos}\mathcal{S}(T \oplus N)$, we know that $T \in \operatorname{clos}\mathcal{R}$. Thus, $\mathcal{T} \subset \operatorname{clos}\mathcal{R}$. ∎

We can now prove the main theorem of this section.

THEOREM 2.5. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and suppose that* $\sigma_e(T)$ *and* $\sigma(T)$ *are both connected. Then:*

(i) *If* $\operatorname{ind}(T-\lambda) \in \{0,-1\}$ *for every* $\lambda \in \rho_{\text{s-F}}(T)$ *and if* $0 \neq M \in \operatorname{Lat}_{\text{ns}}(T)$*, then* $\dim M^\perp < \infty$ *and* $\sigma(T^*|M^\perp) \subset \operatorname{clos}\{\overline{\lambda} \in \mathbb{C} : \operatorname{ind}(T-\lambda) = -1\}$.

(ii) *If* $\operatorname{ind}(T-\lambda) \in \{0,1\}$ *for every* $\lambda \in \rho_{\text{s-F}}(T)$ *and if* $\mathcal{H} \neq M \in \operatorname{Lat}_{\text{ns}}(T)$*, then* $\dim M < \infty$ *and* $\sigma(T|M) \subset \operatorname{clos}\{\lambda \in \mathbb{C} : \operatorname{ind}(T-\lambda) = 1\}$.

Proof. (i) In this case it follows from Theorem 2.4 that T is a norm limit of operators whose nonzero invariant subspaces all have finite co-dimension. Since the set of projections with finite co-dimension is norm closed, it follows that any nonzero norm-stable invariant subspace of T has finite co-dimension. If $U = \{\lambda \in \mathbb{C} : \operatorname{ind}(T-\lambda) = -1\} = \varnothing$, then it follows from Theorem 2.4 that T and T^* are norm limits of operators whose nontrivial invariant subspaces all have finite co-dimension; whence, $\operatorname{Lat}_{\text{ns}}(T) = \{0, 1\}$. Suppose $U \neq \varnothing$. If $\varepsilon > 0$, then U has only finitely many connected components $U_1, \ldots, U_m$ containing a disc of radius ε. Since $\sigma(T)$ and $\sigma_e(T)$ are connected, the components $U_1, \ldots, U_m$ are simply connected. For each k, $1 \leqslant k \leqslant m$, there is an $r_k > 1$ and a univalent mapping $\varphi_k : r_k\mathbb{D} \to U_k$ such that $U_k\backslash\varphi_k(\mathbb{D}) \subset \{z \in U_k : \operatorname{dist}(z, \mathbb{C}\backslash U_k) < \varepsilon\}$. Let S be a quasianalytic shift operator satisfying the conditions of Theorem 5.1, and let $A = \varphi_1(S) \oplus \cdots \oplus \varphi_m(S) \oplus N_\varepsilon$, where N_ε is a normal operator with no eigenvalues whose spectrum is the closure of $\sigma_e(T)_\varepsilon$. It follows from Proposition 1.1 that $T \oplus N_\varepsilon$ is in the closure of the similarity orbit of A. Also we can choose a normal operator N with $\sigma(N) = \sigma_e(N) = \sigma_e(T)$ such that $\|N - N_\varepsilon\| < \varepsilon$. Since, by Proposition 1.1, T is in the closure of the similarity orbit of $T \oplus N$, and $\sigma_p(A^*) \subset$

$\left\{\overline{\lambda} : \lambda \in \bigcup_{k=1}^{m} \varphi_k(\overline{\mathbb{D}})\right\} \subset \{\overline{\lambda} : \lambda \in U\}$, it follows that T is a limit of operators T_n such that $\sigma_p(T_n^*|M_n^\perp) \subset \{\overline{\lambda} : \lambda \in U\}$ for every $M_n \in \mathrm{Lat}\, T_n$ with $\mathrm{codim}\,(M_n) < \infty$. Thus statement (i) is proved.

(ii) In this case T^* satisfies the conditions of part (i), and since $\mathrm{Lat}_{\mathrm{ns}}(T^*)$ is clearly $\{M^\perp : M \in \mathrm{Lat}_{\mathrm{ns}}(T)\}$, the desired conclusion follows from part (i). ∎

3. $\mathrm{Lat}_{\mathrm{ns}}$ FOR WEIGHTED UNILATERAL SHIFTS

We can use Theorem 2.5 to completely characterize $\mathrm{Lat}_{\mathrm{ns}}(T)$ whenever T is a weighted unilateral shift operator. The first step is the following

COROLLARY 3.1. *If T is an injective unilateral weighted shift operator, then every nonzero norm-stable invariant subspace of T has finite co-dimension.*

Proof. If the weights of T are not bounded away from 0, then a compact perturbation of T, obtained by replacing a subsequence of weights with zeros, is quasidiagonal, and hence biquasitriangular. Since weighted shifts have connected spectrum, it follows from Corollary 3.9 in [3] that T has no nontrivial norm-stable invariant subspaces. If the weights of T are bounded away from 0, then T satisfies condition (i) in Theorem 2.5. ∎

The following lemma is a simple application of the Gram–Schmidt process.

LEMMA 3.2. *Suppose $\{u_1, \ldots, u_n\}$ is a linear basis for a subspace M of $\mathcal{H}$, and suppose, for $k \in \mathbb{N}$, that M_k is the span of $\{u_{k1}, \ldots, u_{kn}\}$. If $\lim_{k\to\infty} \|u_j - u_{kj}\| = 0$ for $1 \leqslant j \leqslant n$, then $\|P_{M_k} - P_M\| \to 0$.*

COROLLARY 3.3. *If $T \in \mathcal{B}(\mathcal{H})$ and $M_1, \ldots, M_k$ are one-dimensional subspaces in $\mathrm{Lat}_{\mathrm{ns}}(T)$ (respectively, $\mathrm{Lat}_{\mathrm{s}}(T)$), then $M_1 + \cdots + M_k$ belongs to $\mathrm{Lat}_{\mathrm{ns}}(T)$ (respectively, $\mathrm{Lat}_{\mathrm{s}}(T)$).*

LEMMA 3.4. *Suppose T is a bounded operator on $\mathcal{H}$, and M is a nonzero finite-dimensional cyclic invariant subspace for T^* such that $\sigma(T^*|M) \cap \sigma_{\mathrm{e}}(T^*) = \varnothing$ and $\sigma(T^*|M) \cap \{\overline{\lambda} : \lambda \in \sigma_p(T)\} = \varnothing$. Then $M \in \mathrm{Lat}_{\mathrm{ns}}(T^*)$.*

Proof. We know that if $p(z)$ is the minimal polynomial for $T^*|M$, then its set of roots is $\sigma(T^*|M)$ and $M \subset \mathrm{Ker}\,(p(T^*))$. Since $\sigma(T^*|M) \cap \{\overline{\lambda} : \lambda \in \sigma_p(T)\} = \varnothing$, we know that $\mathrm{Ker}\,(p(T^*)^*) = 0$. Moreover, $\sigma(T^*|M) \cap \sigma_{\mathrm{e}}(T^*) = \varnothing$ implies that $p(T^*)$ is Fredholm, and we know that $\mathrm{Range}\,(p(T^*)) = \mathrm{Ker}\,(p(T^*)^*)^\perp = \mathcal{H}$, thus $p(T^*)$ is surjective. If $\{A_k\}$ is a sequence converging in norm to T^*, then $\|p(A_k) - p(T^*)\| \to 0$, and, by Lemma 1.6 in [11], the projection Q_k onto $\mathrm{Ker}\,(p(A_k))$ converge in norm to the projection Q onto $\mathrm{Ker}\,(p(T^*))$. Let e be a cyclic vector for $T^*|M$. If $m = \deg(p)$, then $\{e, T^*e, \ldots, (T^*)^{m-1}e\}$ is a basis for M, and since $Q_k e \in \mathrm{Ker}\, p(A_k)$, the set $\{Q_k e, A_k Q_k e, \ldots, A_k^{m-1} Q_k e\}$ spans a subspace

$M_k \in \mathrm{Lat}\,(A_k)$ for each $k \geqslant 1$. It follows from Lemma 3.2 that $\|P_{M_k} - P_M\| \to 0$. Thus $M \in \mathrm{Lat}_{\mathrm{ns}}(T^*)$. ∎

We are now ready to completely characterize $\mathrm{Lat}_{\mathrm{ns}}(T)$ for every unilateral shift operator T.

THEOREM 3.5. *Suppose T is a weighted unilateral shift operator and let $r_0 = \inf\{|\lambda| : \lambda \in \sigma_{\mathrm{e}}(T)\}$. If $r_0 = 0$, then $\mathrm{Lat}_{\mathrm{ns}}(T) = \{0, 1\}$. If $r_0 > 0$, then a subspace $M \neq \mathcal{H}$ is in $\mathrm{Lat}_{\mathrm{ns}}(T^*)$ if and only if $\dim M < \infty$ and $|\lambda| \leqslant r_0$ for every $\lambda \in \sigma(T^*|M)$.*

Proof. The "only if" part follows from Theorem 2.5. Suppose that $\{e_0, e_1, \ldots\}$ is an orthonormal basis, and $\{\alpha_n\}_{n\geqslant 0}$ is a sequence of positive numbers such that $Te_n = \alpha_n e_{n+1}$ for all $n \geqslant 0$. Then $T^* e_0 = 0$ and $T^* e_{n+1} = \alpha_n e_n$ for all $n \geqslant 0$. Suppose that $\lambda \in \mathbb{C}$ and $\lambda \in \sigma_p(T^*)$. Then there is a vector $0 \neq f_{\lambda,1} = (\beta_0, \beta_1, \ldots)$ such that $(T^* - \lambda) f_{\lambda,1} = 0$. It is clear that $\beta_0 \neq 0$, so we can assume $\beta_0 = 1$, and, for $n \geqslant 1$,

$$\beta_n = \frac{\lambda^n}{\alpha_0 \cdots \alpha_{n-1}}.$$

It follows that $\mathrm{Ker}\,(T^* - \lambda)$ is 1-dimensional. Thus, if M is a finite-dimensional invariant subspace for T^*, then each eigenvalue for $T^*|M$ has exactly one Jordan block in its Jordan form. Thus $T^*|M$ is cyclic. It follows from Lemma 3.4 that if $\sigma(T^*|M) \cap \sigma_{\mathrm{e}}(T^*) = \varnothing$, then $M \in \mathrm{Lat}_{\mathrm{ns}}(T^*)$. Next suppose that there exists an $f_{\lambda,2}$ such that $(T^* - \lambda) f_{\lambda,2} = f_{\lambda,1}$. Then we can choose $f_{\lambda,2} = (0, \gamma_1, \gamma_2, \ldots)$, and we see the γ_k's are uniquely determined and, for $n \geqslant 1$,

$$\gamma_n = \frac{n\lambda^{n-1}}{\alpha_0 \cdots \alpha_{n-1}}.$$

More generally, if we have $f_{\lambda,1}, \ldots, f_{\lambda,m}$ such that, for $1 \leqslant k < m$,

$$(T^* - \lambda) f_{\lambda,k+1} = f_{\lambda,k}$$

and the first k coordinates of $f_{\lambda,k+1}$ are 0, then

$$f_{\lambda,k} = (0, \ldots, 0, c_{k,k}, c_{k,k+1}\lambda, c_{k,k+2}\lambda^2, \ldots),$$

where the positive numbers $c_{k,j}$ ($j \geqslant k$) depend only on the weights $\{\alpha_n\}$ and not on λ. Note that $\|f_{\lambda,k}\|$ depends on $|\lambda|$, so that if $\|f_{\lambda_0,k}\| < \infty$ for some λ_0, then $\|f_{\lambda,k}\| < \infty$ for all λ with $|\lambda| \leqslant |\lambda_0| = r$. Moreover, the map $\lambda \mapsto \|f_{\lambda,k}\|$ is continuous on $\overline{r\mathbb{D}}$ (by the dominated convergence theorem). If a sequence $\{h_n\}$ of vectors in a Hilbert space converges weakly to h and if $\|h_n\| \to \|h\|$, then $\|h_n - h\| \to 0$. It follows that the map $\lambda \mapsto f_{\lambda,k}$ is norm continuous on $\overline{r\mathbb{D}}$.

Now suppose $M \in \mathrm{Lat}\,(T^*)$ is finite-dimensional and $|\lambda| \leqslant r_0$ for every $\lambda \in \sigma(T^*|M)$. If $p(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_s)^{m_s}$ is the minimal polynomial for $T^*|M$, then $\{f_{\lambda_j,k} : 1 \leqslant j \leqslant s, 1 \leqslant k \leqslant m_j\}$ is a linear basis for M. The desired conclusion follows from Lemma 3.2. ∎

We see that quasianalytic shifts are points of norm continuity of Lat.

COROLLARY 3.6. *Suppose T is a quasianalytic unilateral shift operator satisfying the conditions of Theorem* 5.1. *Then* $\mathrm{Lat}\,(T) = \mathrm{Lat}_{\mathrm{ns}}(T)$.

4. A RESULT FOR $\mathrm{Lat}_{\mathrm{s}}$.

Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $P \in \mathrm{Lat}\,(T)$. We define the *index of T relative to P* by

$$\mathrm{ind}\,(T,P) = \dim P(\mathcal{H}) \ominus \mathrm{clos}\, TP(\mathcal{H}).$$

If T is bounded from below, i.e., $\mathrm{Ker}\, T = 0$ and $T(\mathcal{H})$ is closed, then $\mathrm{ind}\,(T,P)$ can be defined in terms of the Fredholm index, namely $\mathrm{ind}\,(T,P) = -\,\mathrm{ind}\,(T|P)$. We first prove a semicontinuity result.

LEMMA 4.1. *Suppose that $T, S_1, S_2, \ldots \in \mathcal{B}(\mathcal{H})$ and $P \in \mathrm{Lat}\, T$ and $P_n \in \mathrm{Lat}\, S_n$ for $n \geqslant 1$ are such that:*

(i) *T is bounded from below,*
(ii) $\|S_n - T\| \to 0$, *and*
(iii) *$P_n \to P$ in the strong operator topology.*

Then

$$\mathrm{ind}\,(T,P) \leqslant \liminf_{n\to\infty} \mathrm{ind}\,(S_n, P_n).$$

Proof. Let $\mathrm{ind}\,(T,P) \geqslant k$, and let E be a linear subspace of $P(\mathcal{H}) \ominus TP(\mathcal{H})$, $\dim E = k$. Since $\dim E < \infty$, $P_n|E$ converge to $1|E$ in norm.

Suppose that $P_nE \cap S_nP_n(\mathcal{H}) \neq \{0\}$ for large n. Passing to a subsequence, we can find $e_n \in E$, $\|e_n\| = 1$, such that

$$P_ne_n \in S_nP_n(\mathcal{H}), \quad e_n \to e \in E,$$

and hence $Pe = e \neq 0$. Denote $u_n = S_n^{-1}P_ne_n \in P_n(\mathcal{H})$. Then $\{u_n\}$ is a bounded sequence, and then, a Cauchy sequence (since $S_nu_n \to e$, $n \to \infty$); denote $f = \lim_{n\to\infty} u_n$. We have $Tf = e \in E$, $f \notin P(\mathcal{H})$. Furthermore, $u_n = P_nu_n \to Pf \neq f$. This contradiction shows that $P_nE \cap S_nP_n(\mathcal{H}) = \{0\}$ for large n, and, hence,

$$\dim P_n(\mathcal{H}) \ominus S_nP_n(\mathcal{H}) \geqslant \dim E = k. \quad \blacksquare$$

LEMMA 4.2. *Suppose that S is the (unweighted) unilateral shift operator, $r > 1$, and $\varphi : r\mathbb{D} \to \mathbb{C}$ is univalent with $\varphi(0) = 0$. Then for every $P \in \mathrm{Lat}\, S$ we have* $\mathrm{ind}\,(\varphi(S), P) \leqslant 1$.

Proof. It follows from Lemma 2.2 that $\mathrm{Lat}\, S = \mathrm{Lat}\, \varphi(S)$. Since φ is univalent and $\varphi(0) = 0$, we have $\varphi(z) = z\psi(z)$ where $\psi(z) \neq 0$ for every $z \in r\mathbb{D}$. Hence $\varphi(S) = SA$ with A invertible and A and $A^{-1} = (1/\psi)(S)$ are in the weakly closed algebra generated by S. Thus $\varphi(S)P(\mathcal{H}) = SP(\mathcal{H})$, and we conclude that $\mathrm{ind}\,(\varphi(S), P) = \mathrm{ind}\,(S,P) \leqslant 1$. The last inequality follows from Beurling's characterization of $\mathrm{Lat}\, S$ [6]. $\blacksquare$

THEOREM 4.3. *Suppose that* $T \in \mathcal{B}(\mathcal{H})$, $\sigma(T)$ *and* $\sigma_e(T)$ *are both connected, and* $\operatorname{ind}(T-\lambda) \in \{-1,0\}$ *for every* $\lambda \in \rho_{\text{s-F}}(T)$, *and suppose that* $\operatorname{ind}(T) = -1$ *and* $\operatorname{Ker}(T) = 0$. *If* $P \in \operatorname{Lat}_s(T)$, *then* $\operatorname{ind}(T,P) \leqslant 1$.

Proof. We imitate the proof of Theorem 2.4 replacing a quasianalytic shift with the unweighted unilateral shift S. Since $\operatorname{ind}(T) = -1$, we can assume (by composing with a disc automorphism) that $\varphi(0) = 0$. It follows that T is the norm limit of a sequence $\{S_n\}$, where each S_n is similar to $\varphi_n(S)$ for some univalent functions φ_n on a neighborhood of $\overline{\mathbb{D}}$ with $\varphi_n(0) = 0$. The desired conclusion follows from Lemmas 4.1 and 4.2. ∎

Suppose $M \in \operatorname{Lat}(T)$ and $N = \operatorname{clos} T(M)$. With respect to the decomposition $\mathcal{H} = N \oplus (M \ominus N) \oplus M^{\perp}$, T has an operator matrix

$$\begin{pmatrix} A & B & C \\ 0 & 0 & D \\ 0 & 0 & E \end{pmatrix}$$

where the size of the 0 in the $(2,2)$-entry is $\operatorname{ind}(T,M)$. In [1] C. Apostol, H. Bercovici, C. Foiaş, and C. Pearcy introduced a class $\mathbb{A}_{\aleph_0}$ of contraction operators and they proved that if $T \in \mathbb{A}_{\aleph_0}$, then T has invariant subspaces with arbitrary index. They also proved that if T is a contractive unilateral weighted shift whose weights converge to 1, then either $T \in \mathbb{A}_{\aleph_0}$ or T is similar to the unweighted unilateral shift. Thus the Bergman shift with weights $\sqrt{(n+1)/(n+2)}$ is in $\mathbb{A}_{\aleph_0}$.

COROLLARY 4.4. *Suppose T is a unilateral shift whose weights converge to* $\|T\| \neq 0$. *Then the following are equivalent:*

(i) $\operatorname{Lat}(T) = \operatorname{Lat}_s(T)$;

(ii) *T is similar to $\|T\|$ times the unweighted unilateral shift;*

(iii) $\operatorname{ind}(T,P) \leqslant 1$ *for every* $P \in \operatorname{Lat}(T)$.

REMARK 4.5. If, in the proof of Theorem 4.3, we replace the role of the unweighted unilateral shift S with a direct sum $S^{(n)}$ of $n \geqslant 1$ copies of S, then we can show that if $\operatorname{ind}(T) = -n$, $\operatorname{Ker}(T) = 0$, $\operatorname{ind}(T-\lambda) \in \{-n,0\}$ for every $\lambda \in \rho_{\text{s-F}}(T)$, and $\sigma(T)$ and $\sigma_e(T)$ are both connected, then $\operatorname{ind}(T,P) \leqslant n$ for every $P \in \operatorname{Lat}_s(T)$.

5. QUESTIONS

We conclude with some open questions.

QUESTION 1. Suppose that $T \in \mathcal{Q}(\mathcal{H})$, i.e., the nontrivial invariant subspaces of T all have finite co-dimension, must $\sigma_e(T)$ be connected? In particular, is there an operator $T \in \mathcal{Q}(\mathcal{H})$ whose spectrum is an annulus and essential spectrum is its boundary with Fredholm index -1 inside the annulus?

QUESTION 2. Suppose that T is a weighted unilateral shift with closed range. If $P \in \mathrm{Lat}\,(T)$ and $\mathrm{ind}\,(T,P) = 1$, must $P \in \mathrm{Lat}_{\mathrm{s}}(T)$?

QUESTION 3. What is $\mathrm{Lat}_{\mathrm{s}}(T)$ when T is a unilateral weighted shift? What if T is the Bergman shift?

APPENDIX. QUASIANALYTIC SHIFTS

A key ingredient of the proof of our main results on stable invariant subspaces involves properties of *quasianalytic shift operators*, which are weighted unilateral shifts with weights converging to 1, whose essential spectrum is the unit circle, whose spectrum is the closed unit disc, whose Fredholm index is -1 on the open unit disc. An example of a quasianalytic shift has weights $\exp(\sqrt{n+1} - \sqrt{n})$. These shifts were used in [17] and [16] to show that results of Lomonosov [20], [21] did not lead to an immediate solution of the invariant subspace problem. The most important property of quasianalytic shifts from our point of view concerns their invariant subspaces. Question 17 in the seminal 1974 paper [25] on weighted shift operators by Allen Shields is whether the nonzero invariant subspaces of quasianalytic shifts all have finite co-dimension. In this section we establish a version of Domar's result answering Shields' question in the affirmative.

Let us introduce some definitions. Given a function $\omega : \mathbb{Z}_+ \to [1, +\infty)$ such that

$$0 < \inf_{n\geqslant 0} \frac{\omega(n+1)}{\omega(n)} \leqslant \sup_{n\geqslant 0} \frac{\omega(n+1)}{\omega(n)} < \infty, \quad \lim_{n\to\infty} \omega(n)^{1/n} = 1,$$

we consider the unilateral shift operator $S_\omega : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, $S_\omega e_n = (\omega(n+1)/\omega(n))e_{n+1}$, where $e_n = \{\delta_{mn}\}_{m\geqslant 0} \in \ell^2(\mathbb{Z}_+)$.

Let A^2_ω be the Beurling space of the functions f analytic in the unit disc $\mathbb{D}$, with $f(z) = \sum_{n\geqslant 0} \widehat{f}(n) z^n$, $z \in \mathbb{D}$, such that

$$\|f\|^2_{A^2_\omega} = \sum_{n\geqslant 0} |\widehat{f}(n)|^2 \omega(n)^2 < \infty.$$

The operator $M_z : f \mapsto zf$ of multiplication by the independent variable on A^2_ω is isomorphic to S_ω. Denote $\omega_s(n) = \omega(n)(1+n)^{-s}$, and suppose that the sequence $\log \omega_1(n)$ is concave for large n. Then $\omega_1(n+m) \leqslant c\, \omega_1(n)\omega_1(m)$, $n \geqslant 0$, $m \geqslant 0$, and A^2_ω is a Banach algebra with respect to the convolution multiplication:

$$\begin{aligned}\|fg\|^2_{A^2_\omega} &= \sum_{n\geqslant 0} |\widehat{fg}(n)|^2 \omega(n)^2 = \sum_{n\geqslant 0} \Big| \sum_{0\leqslant k\leqslant n} \widehat{f}(k)\widehat{g}(n-k) \Big|^2 \omega(n)^2 \\ &\leqslant \sum_{n\geqslant 0} \Big[\sum_{0\leqslant k\leqslant n} \Big(\frac{\omega(n)}{\omega(k)\omega(n-k)} \Big)^2 \Big] \times \Big[\sum_{0\leqslant k\leqslant n} |\widehat{f}(k)|^2 \omega(k)^2 |\widehat{g}(n-k)|^2 \omega(n-k)^2 \Big]\end{aligned}$$

$$\leqslant c \sum_{m\geqslant 0} |\widehat{f}(m)|^2\omega(m)^2 \cdot \sum_{v\geqslant 0} |\widehat{g}(v)|^2\omega(v)^2 \times \max_{n\geqslant 0} \sum_{0\leqslant k\leqslant n} \frac{(n+1)^2}{(k+1)^2(n-k+1)^2}$$
$$\leqslant c\|f\|^2_{A^2_\omega}\|g\|^2_{A^2_\omega}.$$

By Corollary 1 on p. 94 in [25], the space of maximal ideals of A^2_ω is the closed unit disc $\overline{\mathbb{D}}$.

Next, we suppose that ω is log-convex, that is it extends to $\mathbb{R}_+$ in such a way that $t \mapsto \log\omega(\exp t)$ is convex for large t, and that

$$\sum_{n\geqslant 0} \frac{\log\omega(n)}{n^{3/2}+1} = \infty.$$

In this case, the space A^2_ω is quasi-analytic, in the sense that $A^2_\omega \subset C^\infty(\overline{\mathbb{D}})$, and if $f \in A^2_\omega$, $z \in \overline{\mathbb{D}}$, $f^{(n)}(z) = 0$ for all $n \geqslant 0$, then $f = 0$, see [10], [23], [19].

Let us assume that ω satisfies an additional property: for every polynomial p, there exists $c(p) < \infty$ such that

$$\|pf_1f_2\|_{A^2_\omega} \leqslant c(p)\|pf_1\|_{A^2_\omega}\|pf_2\|_{A^2_\omega}, \quad f_1, f_2 \in A^2_\omega. \tag{5.1}$$

Then by Theorem 1 of Domar in [13], every non-trivial closed M_z-invariant subspace of A^2_ω has finite co-dimension.

By Remarks 1 and 2 in [13], we need only to verify (5.1) for $p(z) = (z-\lambda)^n$ with $\lambda \in \overline{\mathbb{D}}$. In fact, Domar gives (Theorem 3, Remarks 3 and 4 in [13], see also [12]) a sufficient condition on ω in order that the space

$$A^1_\omega = \Big\{ \sum_{n\geqslant 0} \widehat{f}(n)z^n : \sum_{n\geqslant 0} |\widehat{f}(n)|\omega(n) < \infty \Big\}$$

would satisfy the property that every non-trivial closed M_z-invariant subspace of A^1_ω has finite co-dimension.

Here we present two arguments giving (5.1). For simplicity, let us assume that $p(z) = z - 1$. We need to prove that

$$\|(z-1)f_1f_2\|_{A^2_\omega} \leqslant c\|(z-1)f_1\|_{A^2_\omega} \cdot \|(z-1)f_2\|_{A^2_\omega}, \quad f_1, f_2 \in A^2_\omega. \tag{5.2}$$

The first argument. It is known (see, for example, Lemma 5.2 in [15] adapting a general construction in Proposition B.1 in [8]) that there exists a positive continuous function $\varphi : (1,2] \to (0,\infty)$ such that

$$\int_1^2 \varphi(r)r^{-2n}\mathrm{d}r \asymp \frac{1}{\omega(n)}, \quad n \geqslant 0.$$

Let $\Omega = \{z : 1 < |z| < 2\}$. Given a function g in

$$L^2(\varphi,\Omega) = \Big\{h : \int_\Omega |h|^2\varphi^2 < \infty\Big\},$$

we define

$$\mathcal{C}g(z)=\frac{1}{\pi}\int\limits_{1<|w|<2}\frac{g(w)}{z-w}\,\mathrm{d}m_2(w).$$

Then the space A^2_ω coincides ([14], [8]) with the image space $\{\mathcal{C}g : g\in L^2(\varphi,\Omega)\}$,

$$\|f\|_{A^2_\omega}\asymp \inf\{\|g\|_{L^2(\varphi,\Omega)} : \mathcal{C}g|\mathbb{D}=f\}.$$

Now, let $g_1(z)=(z-1)f_1(z)$, $g_2(z)=(z-1)f_2(z)$, where $g_1,g_2\in L^2(\varphi,\Omega)$ are canonical densities (Section 2 in [8]). Then

$$\frac{\mathcal{C}g_1\cdot\mathcal{C}g_2}{\cdot-1}=\mathcal{C}\Bigl(\bar\partial\Bigl(\frac{\mathcal{C}g_1\cdot\mathcal{C}g_2}{\cdot-1}\Bigr)\Bigr),$$

and to prove (5.2) we need only to verify that

$$\Bigl\|g_1\cdot\frac{\mathcal{C}g_2}{\cdot-1}\Bigr\|_{L^2(\varphi,\Omega)}\leqslant c\|g_1\|_{L^2(\varphi,\Omega)}\cdot\|g_2\|_{L^2(\varphi,\Omega)}.$$

Finally, the equality

$$\frac{(\mathcal{C}g_2)(z)}{z-1}=\frac{\mathcal{C}g_2(z)-\mathcal{C}g_2(1)}{z-1}=\Bigl(\mathcal{C}\Bigl(\frac{g_2}{\cdot-1}\Bigr)\Bigr)(z),$$

Lemma 8.2, Proposition 8.4, and Lemma B.6 in [8] give us the estimate

$$\Bigl\|\frac{\mathcal{C}g_2}{\cdot-1}\Bigr\|_{L^\infty(\Omega)}\leqslant c\Bigl\|\frac{g_2}{\cdot-1}\Bigr\|_{L^3(\Omega)}\leqslant c_1\|g_2\|_{L^2(\varphi,\Omega)},$$

if $\log\omega_3(n)$ is concave for large n.

The second argument was proposed to us by Omar El-Fallah.

First of all, if $\omega_2(n)$ increases for large n, then

$$\|(z-1)f\|_{A^2_\omega}\geqslant c\|f\|_{A^2_{\omega_1}},\quad f\in A^2_\omega. \tag{5.3}$$

Indeed, if $g(z)=(z-1)f(z)$, $g(z)=\sum\limits_{n\geqslant0}\widehat{g}(n)z^n$, then

$$f(z)=\frac{g(z)-g(1)}{z-1}=\sum_{0\leqslant k<n}\widehat{g}(n)z^k,$$

$$\|f\|^2_{A^2_{\omega_1}}=\sum_{n\geqslant0}|\widehat{f}(n)|^2\omega_1(n)^2=\sum_{n\geqslant0}\Bigl|\sum_{k>n}\widehat{g}(k)\Bigr|^2\omega_1(n)^2\leqslant c\sum_{n\geqslant0}\frac{\omega_1(n)^2}{n+1}\sum_{k>n}|\widehat{g}(k)|^2k^2$$

$$=c\sum_{k>0}|\widehat{g}(k)|^2k^2\sum_{0\leqslant n<k}\frac{\omega_1(n)^2}{n+1}\leqslant c\sum_{k>0}|\widehat{g}(k)|^2\omega(k)^2=c\|g\|^2_{A^2_\omega}.$$

Next, we note that

$$\|f\|_{A^2_\omega}\asymp|f(0)|+\|f'\|^2_{A^2_{\omega_1}},\quad f\in A^2_\omega.$$

Therefore, by (5.3), to prove (5.2) we need only to verify that

$$\|[(z-1)f]'g\|^2_{A^2_{\omega_1}}\leqslant c\|[(z-1)f]'\|^2_{A^2_{\omega_1}}\cdot\|g\|^2_{A^2_{\omega_1}};$$

this estimate follows immediately if $\log\omega_2(n)$ is concave for large n.

Summing up, we obtain the following result.

THEOREM 5.1. *Let $\omega : \mathbb{Z}_+ \to [1, +\infty)$ be such that $\lim_{n\to\infty} \omega(n)^{1/n} = 1$, for every $k \geqslant 0$, the sequence $\omega(n)(n+1)^{-k}$ is concave for large n, ω is log-convex, and*

$$\sum_{n\geqslant 0} \frac{\log \omega(n)}{n^{3/2}+1} = \infty.$$

Then every nontrivial closed invariant subspace of S_ω has finite co-dimension.

Acknowledgements. The first author was partially supported by the ANR projects DYNOP and FRAB (ANR-09-BLAN-0058-01).

REFERENCES

[1] C. APOSTOL, H. BERCOVICI, C. FOIAŞ, C. PEARCY, Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I, *J. Funct. Anal.* **63**(1985), 369–404.

[2] C. APOSTOL, L. FIALKOW, D. HERRERO, D. VOICULESCU, *Approximation of Hilbert Space Operators*, Vol. II, Pitman Res. Notes Math., vol. 102, Pitman, Boston 1984.

[3] C. APOSTOL, C. FOIAŞ, N. SALINAS, On stable invariant subspaces, *Integral Equations Operator Theory* **8**(1985), 721–750.

[4] C. APOSTOL, D. HERRERO, D. VOICULESCU, The closure of the similarity orbit of a Hilbert space operator, *Bull. Amer. Math. Soc. (NS)* **6**(1982), 421–426.

[5] H. BART, I. GOHBERG, M. KAASHOEK, Stable factorizations of monic matrix polynomials and stable invariant subspaces, *Integral Equations Operator Theory* **1**(1978), 496–517.

[6] A. BEURLING, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81**(1949), 239–255.

[7] A. BORICHEV, On convolution equations with restrictions on supports, *Algebra i Analiz* **14**(2002), 1–10; *St. Petersburg Math. J.* **14**(2003), 179–187.

[8] A. BORICHEV, H. HEDENMALM, Completeness of translates in weighted spaces on the half-line, *Acta Math.* **174**(1995), 1–84.

[9] S. CAMPBELL, J. DAUGHTRY, The stable solutions of quadratic matrix equations, *Proc. Amer. Math. Soc.* **74**(1979), 19–23.

[10] L. CARLESON, Sets of uniqueness for functions regular in the unit circle, *Acta Math.* **87**(1952), 325–345.

[11] J. CONWAY, D. HADWIN, Stable invariant subspaces for operators on Hilbert space, *Ann. Polon. Math.* **66**(1997), 49–61.

[12] Y. DOMAR, Primary ideals in Beurling algebras, *Math. Z.* **126**(1972), 361–367.

[13] Y. DOMAR, On spectral synthesis in commutative Banach algebras using closed ideals of finite co-dimension, in *L'analyse harmonique dans le domaine complexe (Actes Table*

Ronde Internat., CNRS, Montpellier, 1972), Lecture Notes in Math., vol. 336, Springer, Berlin 1973, pp. 63–78.

[14] E.M. DYNKIN, Extensions and integral representations of smooth functions of one complex variable [Russian], Ph.D. Dissertation, Steklov Mathematical Institiute, Leningrad Branch, Leningrad 1972.

[15] J. ESTERLE, Singular inner functions and biinvariant subspaces for dissymmetric weighted shifts, *J. Funct. Anal.* **144**(1997), 64–104.

[16] D. HADWIN, An operator still not satisfying Lomonosov's hypothesis, *Proc. Amer. Math. Soc.* **123**(1995), 3039–3041.

[17] D. HADWIN, E. NORDGREN, H. RADJAVI, P. ROSENTHAL, An operator not satisfying Lomonosov's hypothesis, *J. Funct. Anal.* **38**(1980), 410–415.

[18] D. HERRERO, *Approximation of Hilbert Space Operators*, Vol. I, Pitman Res. Notes Math., vol. 72, Pitman, Boston 1982.

[19] B. KORENBLUM, Quasianalytic classes of functions in a circle, *Dokl. Akad. Nauk SSSR* **164**(1965), 36–39; English *Soviet Math. Dokl.* **6**(1965), 1155–1158.

[20] V. LOMONOSOV, Invariant subspaces of the family of operators that commute with a completely continuous operator [Russian], *Funkcional. Anal. i Prilozen.* **7**(1973), 55–56.

[21] V. LOMONOSOV, On real invariant subspaces of bounded operators with compact imaginary part, *Proc. Amer. Math. Soc.* **115**(1992), 775–777.

[22] S. RICHTER, Invariant subspaces in Banach spaces of analytic functions, *Trans. Amer. Math. Soc.* **304**(1987), 585–616.

[23] B. SALINAS, Functions with null moments [Spanish], *Rev. Acad. Cienc. Madrid* **49**(1955), 331–368.

[24] D. SARASON, Invariant subspaces and unstarred operator algebras, *Pacific J. Math.* **17**(1966) 511–517.

[25] A. SHIELDS, Weighted shift operators and analytic function theory, in *Topics in Operator Theory*, Math. Surveys, vol. 13, Amer. Math. Soc., Providence, R.I. 1974, pp. 49–128.

ALEXANDER BORICHEV, CENTRE DE MATHÉMATIQUES ET INFORMATIQUE, AIX-MARSEILLE UNIVERSITÉ, 39 RUE FRÉDÉRIC JOLIOT-CURIE, 13453 MARSEILLE, FRANCE

E-mail address: borichev@cmi.univ-mrs.fr

DON HADWIN, MATHEMATICS DEPARTMENT, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NH 03824, U.S.A.

E-mail address: don@unh.edu

HASSAN YOUSEFI, MATHEMATICS DEPARTMENT, CALIFORNIA STATE UNIVERSITY, FULLERTON, CA 92831, U.S.A.

E-mail address: hyousefi@fullerton.edu

Received June 1, 2010; posted on February 8, 2013.