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CROSSED PRODUCTS BY α-SIMPLE AUTOMORPHISMS ON C^*-ALGEBRAS $C(X, A)$

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ABSTRACT. Let X be a Cantor set, and let A be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem. We use $C(X, A)$ to denote the set of all continuous functions from X to A; let α be an automorphism on $C(X, A)$. Suppose that $C(X, A)$ is α-simple and $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$. We show that $C(X, A) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

KEYWORDS: *Crossed products, α-simple, tracial rank zero.*

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1. INTRODUCTION

That the transformation group C^*-algebras of minimal homeomorphisms of the Cantor set are AT algebras (direct limits of circle algebras) with real rank zero is implicit in Section 8 of [10], with the main step having been done in [22]. Elliott and Evans proved in [4] that the irrational rotation algebras are AT algebras with real rank zero. Furthermore, Lin proved in [14] that irrational higher dimensional noncommutative tori of the form $C(T^k) \rtimes_\theta \mathbb{Z}$ are in fact AT algebras. Recently Phillips constructs an inductive proof that every simple higher noncommutative torus is an AT algebra ([20]). More generally, Lin and Phillips proved the following result ([19]): Let X be an infinite compact metric space with finite covering dimensional and let α: $X \to X$ be a minimal homeomorphism, the associated crossed product C^*-algebra $A = C(X) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero whenever the image of $K_0(A)$ in $\mathrm{Aff}(T(A))$ is dense. In particular, these algebras all belong to the class known currently to be classifiable by K-theoretic invariants in the sense of the Elliott classification program ([3]).

Let A be a unital separable simple C^*-algebra with tracial rank zero, and let α be an automorphism on A. Lin proved in [18] that if α has certain Rokhlin property and there is an integer $J \geqslant 1$ such that $[\alpha^J] = [\mathrm{id}_A]$ in $KL(A, A)$. Then $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

In present, we do not know if $A \rtimes_\alpha \mathbb{Z}$ belongs to the class known currently to be classifiable by K-theoretic invariants when C^*-algebra A is neither commutative nor simple. In this paper we consider the C^*-algebra $C(X, A)$, all continuous functions from X to A, and the associated crossed product C^*-algebra $C(X, A) \rtimes_\alpha \mathbb{Z}$, where X is a Cantor set and A is a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem. When A is isomorphic to $\mathbb{C}$, it is just the case in [22]. When A is not isomorphic to $\mathbb{C}$, $C(X, A)$ is neither commutative nor simple.

In this paper, we prove the following result: let X be a Cantor set, let A be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem, and let α be an automorphism on $C(X, A)$. Suppose that $C(X, A)$ is α-simple and $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$, then $C(X, A) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero, therefore it belongs to the class known currently to be classifiable by K-theoretic invariants in the sense of the Elliott classification program [3]. Let us analysis two conditions we need in our result. The first condition is a necessary condition such that $C(X, A) \rtimes_\alpha \mathbb{Z}$ is simple. The second condition is $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$ which implys that the action on each fiber of X is a trivial element of $KL(A, A)$. An early version of the second condition is weaker and can not ensure our result which was pointed to us by Hiroki Matui.

This paper is organized as follows. In Section 2 we introduce notation and give some elementary properties of α-simple automorphisms on C^*-algebras $C(X, A)$. In Section 3, we prove that, under our hypotheses, the C^*-subalgebra $B_{\{y\}} = C^*(C(X, A), uC_0(X\backslash\{y\}, A)) \subset C(X, A) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero. Section 4 contains the proof that $C(X, A) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

2. PREMIMINARIES AND α-SIMPLE AUTOMORPHISMS ON $C(X, A)$

We will use the following convention:

(i) Let A be a C^*-algebra, let $a \in A$ be a positive element and let $p \in A$ be a projection. We write $[p] \leqslant [a]$ if there is a projection $q \in \overline{aAa}$ and a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(ii) Let A be a C^*-algebra. We denote by $\mathrm{Aut}(A)$ the automorphism group of A. If A is unital and $u \in A$ is a unitary, we denote by $\mathrm{ad}u$ the inner automorphism defined by $\mathrm{ad}u(a) = u^*au$ for all $a \in A$.

(iii) Let $x \in A$, $\varepsilon > 0$ and $\mathcal{F} \subset A$. We write $x \in_\varepsilon \mathcal{F}$, if $\mathrm{dist}(x, \mathcal{F}) < \varepsilon$, or there is $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$.

(iv) Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say A is α-simple if A does not have any non-trivial α-invariant closed two-sided ideals.

(v) Let A be a unital C^*-algebra and $T(A)$ the compact convex set of tracial states of A. If α is an automorphism of A, we use $T^\alpha(A)$ to denote the α-invariant tracial states, which is again a compact convex set. We define an affine mapping r of $T(A \rtimes_\alpha \mathbb{Z})$ into $T^\alpha(A)$ by the restriction $r(\tau) = \tau|A$. Denote by $\mathrm{Aff}(T(A))$ the normed space of all real affine continuous functions on $T(A)$. Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the homomorphism induced by $\rho_A([p])(\tau) = \tau(p)$ for $\tau \in T(A)$.

We recall the definition of tracial topological rank of C^*-algebras.

DEFINITION 2.1 ([13]). Let A be a unital simple C^*-algebra. Then A is said to have *tracial (topological) rank zero* if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$ and any nonzero positive element $a \in A$, there exists a finite dimensional C^*-subalgebra $B \subset A$ with $\mathrm{id}_B = p$ such that:
(i) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$.
(ii) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$.
(iii) $[1 - p] \leqslant [a]$.

If A has tracial rank zero, we write $\mathrm{TR}(A) = 0$.

DEFINITION 2.2. Let X be a compact metric space and let A be a C^*-algebra, we say a map $\beta : X \to \mathrm{Aut}(A)$, denoted by x to β_x, is *strongly continuous* if for any $\{x_n\}$ with $\mathrm{d}(x_n, x) \to 0$ when $n \to \infty$, we have $\|\beta_{x_n}(a) - \beta_x(a)\| \to 0$ for all $a \in A$.

LEMMA 2.3. *Let X be a compact metric space, let A be a unital simple C^*-algebra and $\alpha \in \mathrm{Aut}(C(X, A))$. Then $C(X, A)$ is α-simple if and only if there is a minimal homeomorphism σ from X to X and a strongly continuous map β from X to $\mathrm{Aut}(A)$, denote by x to β_x, such that $\alpha(f)(x) = \beta_{\sigma^{-1}(x)}(f(\sigma^{-1}(x)))$.*

The proof is a straightforward exercise, we omit it.

LEMMA 2.4. *Let X be an infinite compact metric space, let A be a unital simple C^*-algebra and $\alpha \in \mathrm{Aut}(C(X, A))$. Then $C(X, A)$ is α-simple if and only if the crossed product $C(X, A) \rtimes_\alpha \mathbb{Z}$ is simple.*

Proof. Let I be an α-invariant norm closed two-sided ideal of $C(X, A)$. Then $I \rtimes_\alpha \mathbb{Z}$ is a norm closed two-sided ideal of $C(X, A) \rtimes_\alpha \mathbb{Z}$ by Lemma 1 of [6].

Conversely, for any positive element f of the C^*-algebra $C(X, A)$, any finite set $\mathcal{F} = \{f_i; i = 1, 2, \ldots, n\} \subset C(X, A)$, any $s_i \subset \mathbb{N}, i = 1, 2, \ldots, n$, and any $\varepsilon > 0$, we claim that there exists a positive element $g \in C(X, A)$ with $\|g\| = 1$ such that

$$\|gfg\| \geqslant \|f\| - \varepsilon, \quad \|gf_i\alpha^{s_i}(g)\| \leqslant \varepsilon, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Because X is a compact set, we can get a point $x \in X$ such that $\|f(x)\| = \|f\|$. By Lemma 2.3 we can get a minimal homeomorphism σ of X. Since σ is the minimal homeomorphism of X, there exists a neighborhood $O(x)$ of x such that $\sigma^i(O(x))$ are disjoint for $i = 1, \ldots, s$, where $s = \max\{s_i, i = 1, 2, \ldots, n\}$ and $\|f(y) - f(x)\| < \varepsilon$ for all $y \in O(x)$. It is easy to find a continuous function g from

X to $[0,1]$ such that $g(x) = 1$ and $g(z) = 0$ for all $z \notin O(x)$. Then g satisfies the conditions (2.1). The claim follows.

Use the condition (2.1) and the fact that $C(X, A)$ is α-simple. We can complete the proof as the same as Theorem 3.1 in [9] and we omit it. ∎

Let u be the unitary implementing the action of α in the transformation group C^*-algebra $C(X, A) \rtimes_\alpha \mathbb{Z}$, then $ufu^* = \alpha(f)$. For a nonempty closed subset $Y \subset X$, we define the C^*-subalgebra B_Y to be

$$B_Y = C^*(C(X, A), uC_0(X \backslash Y, A)) \subset C(X, A) \rtimes_\alpha \mathbb{Z}.$$

We will often let B denote the transformation group C^*-algebra $C(X, A) \rtimes_\alpha \mathbb{Z}$. If $Y_1 \supset Y_2 \supset \cdots$ is a decreasing sequence of closed subsets of X with $\bigcap_{n=1}^{\infty} Y_n = \{y\}$, then $B_{\{y\}} = \lim B_{Y_n}$.

Let $Y \subset X$, and let $x \in Y$. If $C(X, A)$ is α-simple, by Lemma 2.3 we have a minimal homeomorphism σ of X. The first return time $\lambda_Y(x)$ (or $\lambda(x)$ if Y is understood) of x to Y is the smallest integer $n \geqslant 1$ such that $\sigma^n(x) \in Y$.

The following result is well known in the area, and is easily proved:

LEMMA 2.5. *If Y is a nonempty clopen subset and σ is a minimal homeomorphism of X. Then* $\sup_{x \in Y}\{\lambda_Y(x)\} < \infty$.

Let $Y \subset X$ is a nonempty clopen subset. Let $n(0) < n(1) < \cdots < n(l)$ be the distinct values of $\lambda(x)$ for $x \in Y$. The Rokhlin tower based on a subset $Y \subset X$ with $Y \neq \emptyset$ consist of the partition

$$Y = \coprod_{k=0}^{l} \{x \in Y : \lambda(x) = n(k)\}$$

of Y (the sets here are the base sets), and the corresponding partition of X:

$$X = \coprod_{k=0}^{l} \coprod_{j=0}^{n(k)-1} \sigma^j(\{x \in Y : \lambda(x) = n(k)\}).$$

Actually, for our purposes it is more convenient to use the partition

$$X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} \sigma^j(\{x \in Y : \lambda(x) = n(k)\}).$$

Note that

$$Y = \coprod_{k=0}^{l} \sigma^{n(k)}(\{x \in Y : \lambda(x) = n(k)\}).$$

Since Y is both closed and open, the following sets are all closed and open:

$$Y_k = \{x \in Y : \lambda(x) = n(k)\}.$$

The proof of the following theorem is the same as the one of Theorem 2.4 in [19] after we apply Lemma 2.3, so we omit it.

THEOREM 2.6. *Let X be a Cantor set, let A be a unital simple C^*-algebra and $\alpha \in \mathrm{Aut}(C(X,A))$. Suppose $C(X,A)$ is α-simple, let $Y \subset X$ be a nonempty clopen subset. Then there exists a unique isomorphism*

$$\gamma_Y : B_Y \to \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)}(A))$$

such that if $f \in C(X,A)$, then

$$\gamma_Y(f)_k = \mathrm{diag}(\alpha^{-1}(f)|_{Y_k}, \alpha^{-2}(f)|_{Y_k}, \dots, \alpha^{-n(k)}(f)|_{Y_k})$$

and if $f \in C_0(X \backslash Y, A)$, then

$$(\gamma_Y(uf))_k = s_k \gamma_Y(f)_k$$

where $s_k \in M_{n(k)} \subset C(Y_k, M_{n(k)}(A))$ is defined by

$$s_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

3. THE TRACIAL RANK OF $B_{\{y\}}$

The following lemma is similar to Proposition 2.5 in [19]. We adapt their proof to establish the simplicity of $B_{\{y\}}$.

LEMMA 3.1. *Let X be an infinite compact metric space, let A be a unital simple C^*-algebra and $\alpha \in \mathrm{Aut}(C(X,A))$. If $C(X,A)$ is α-simple, then for any $y \in X$, $B_{\{y\}}$ is simple.*

Proof. Let $I \subset B_{\{y\}}$ be a nonzero ideal. Since X is an infinite compact metric space, A is a unital simple C^*-algebra and $C(X,A)$ is α-simple, we have a minimal homeomorphism σ of X by Lemma 2.3. Then $I \cap C(X,A)$ is an ideal in $C(X,A)$. So we can write $I \cap C(X,A) = C_0(U,A)$ for some open set $U \subset X$ by the proof of Lemma 2.3, which is necessarily given by,

$$U = \{x \in X : \text{there is } f \in I \cap C(X,A) \text{ such that } f(x) \neq 0\}.$$

We first claim that $U \neq \emptyset$,

Write

$$B_{\{y\}} = \varinjlim B_{Y_m}$$

for some decreasing sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets of X with $\bigcap_{n=1}^{\infty} Y_n = \{y\}$, and $\mathrm{int}(Y_m) \neq \emptyset$,

Then there exists m such that $B_{Y_m} \cap I \neq \{0\}$ by Lemma 3.5.10 of [11]. Let a be a nonzero element of this intersection. Using Theorem 2.6, one can fairly easily prove that there is N such that every element of B_{Y_m} can be written in the form $\sum_{n=-N}^{N} f_n u^n$, with $f_n \in C(X, A)$ for $-N \leqslant n \leqslant N$. Moreover, if $a \neq 0$, and one writes $a^*a = \sum_{n=-N}^{N} f_n u^n$, then $f_0 \neq 0$. Choose $x \in X$ such that $f_0(x) \neq 0$, choose a neighborhood V of x such that the sets $h^n(V)$, for $-N \leqslant n \leqslant N$, are disjoint, and choose $g \in C(X)$ such that $\mathrm{supp}(g) \in V$ and $g(x) \neq 0$. Then $g \in B_{\{y\}}$, and one checks that

$$ga^*ag = \sum_{n=-N}^{N} g f_n u^n g = \sum_{n=-N}^{N} g(g \circ \sigma^n) f_n u^n = g^2 f_0.$$

So $g^2 f_0$ is a nonzero element of $I \cap C(X, A)$, proving the claim.

We next claim that $\sigma^{-1}(U \backslash \{\sigma(y)\}) \subset U$. So let $x \in U \backslash \{\sigma(y)\}$. Choose $f \in I \cap C(X, A)$ such that $f(x) \neq 0$, and choose $g \in C_0(X \backslash \{y\})$ such that $g(\sigma^{-1}(x)) \neq 0$. Then $ug \in B_{\{y\}}$, and

$$(ug)^* f (ug) = \bar{g} u^* f u g = |g|^2 (u^* f u).$$

Thus $|g|^2(u^* f u) \in I \cap C(X, A)$ and is nonzero at $\sigma^{-1}(x)$. This proves the claim.

We further claim that $\sigma(U \backslash \{y\}) \subset U$. The proof is similar: let $x \in U \backslash \{y\}$, let $f \in I \cap C(X, A)$ and $g \in C_0(X \backslash \{y\})$ be nonzero at x, and consider $ug \in B_{\{y\}}$,

$$(ug) f (ug)^* = ug f \bar{g} u^* = |g \circ \sigma^{-1}|^2 (u f u^*).$$

Thus $|g \circ \sigma^{-1}|^2 (u f u^*) \in I \cap C(X, A)$ and is nonzero at $\sigma(x)$. So $\sigma(U \backslash \{y\}) \subset U$.

Now set $Z = X \backslash U$. The last two claims above imply that if $x \in X$ and x is not in the orbit of y, then $\sigma^k(x) \in Z$ for all $k \in \mathbb{Z}$. Since σ is minimal, Z is closed, and $Z \neq X$, this is impossible. If $\sigma^n(y) \in Z$ for some $n > 0$, then $\sigma^{-1}(U \backslash \{\sigma(y)\}) \subset U \backslash \{\sigma(y)\}$ implies $h^k(y) \in Z$ for all $k \geqslant n$. Since σ is dense by minimality, this is also a contradiction. Similarly, if $\sigma^n(y) \in Z$ for some $n \leqslant 0$, then Z would contain the dense set $\{\sigma^k(y) : k \leqslant n\}$, again a contradiction.

So $U = X$ and $1 \in I$. Then $B_{\{y\}}$ is simple. ∎

To prove Lemma 3.6, we recall some notions and results.

Decomposition rank is a topological property, originally defined by Kirchberg and Winter, that in its lowest instance captures, like real rank, the covering dimension of the underlying space.

DEFINITION 3.2 ([8]). (i) A completely positive map $\varphi : F \to A$ has order zero if it maps orthogonal elements to orthogonal elements.

(ii) If F is a finite dimensional algebra, a c.p. map $\varphi : F \to A$ is n-decomposable if there is a decomposition $F = F_1 \oplus \cdots \oplus F_n$ such that $\varphi|F_i$ has order zero for each i.

(iii) We say that A has decomposition rank n, in symbols, dr(A)= n, if n is the smallest integer such that: for any finite subset $\mathcal{G} \subset A$ and $\varepsilon > 0$, there are a finite dimensional algebra F and c.p. contractive maps $\varphi : F \to A$ and $\psi : A \to F$ such that φ is n-decomposable and $\|\varphi \circ \psi(a) - a\| < \varepsilon$ for all $a \in \mathcal{G}$.

A C^*-algebra has finite decomposition rank if dr(A)$< +\infty$.

The following proposition shows that decomposition rank has nice permanence properties.

PROPOSITION 3.3 (3.3 of [8]). *For any two C^*-algebras A and B, we have:*

(i) $\mathrm{dr}(A \oplus B) = \max\{\mathrm{dr}(A), \mathrm{dr}(B)\}$.

(ii) $\mathrm{dr}(A) \leqslant \liminf_{n\to\infty} \mathrm{dr}(A_n)$ *if* $A = \lim_{n\to\infty} A_n$.

(iii) $\mathrm{dr}(A/J) \leqslant \mathrm{dr}(A)$ *if* $J \lhd A$ *is an ideal.*

(iv) $\mathrm{dr}(B) \leqslant \mathrm{dr}(A)$ *if* $B \subset_{\mathrm{her}} A$ *is a hereditary subalgebra.*

(v) $\mathrm{dr}(A \otimes B) \leqslant (\mathrm{dr}(A) + 1) \cdot (\mathrm{dr}(B) + 1) - 1$, *if* B *is* AF *algebra, we have* $\mathrm{dr}(A \otimes B) \leqslant \mathrm{dr}(A)$.

The Jiang–Su algebra, denoted by $\mathcal{Z}$ ([7]), occupies a central position in the structure theory of separable amenable C^*-algebras. The property of absorbing the Jiang–Su algebra tensorially ($\mathcal{Z}$-stable) is a necessary, and, in considerable generality, sufficient condition for the confirmation of Elliott's K-theoretic rigidity conjecture for simple separable amenable C^*-algebras ([23], [26]).

Winter has proved the following two remarkable results:

THEOREM 3.4 (Theorem 4.1 of [24]). *Let A be a separable simple unital C^*-algebra with finite decomposition rank n. Suppose A is $\mathcal{Z}$-stable and has real rank zero. Then A has tracial rank zero.*

THEOREM 3.5 (Theorem 5.1 of [25]). *Let A be a finite, non elementary, simple, unital C^*-algebra. If* dr(A)$< +\infty$, *then A is $\mathcal{Z}$-stable.*

LEMMA 3.6. *Let X be a Cantor set, let A be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the* UCT *(Universal Coefficient Theorem), and let $\alpha \in$* $\mathrm{Aut}(C(X, A))$. *Suppose $C(X, A)$ is α-simple. Then, for any $y \in X$, the C^*-algebra $B_{\{y\}}$ has tracial rank zero.*

Proof. Let $Y \subset X$ be a nonempty clopen subset, applying Theorem 2.6, we have

$$B_Y \cong \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)}(A)) = \bigoplus_{k=0}^{l} C(Y_k) \otimes M_{n(k)}(A).$$

Since A is a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem, by [15] A is an AH-algebra of slow dimension growth with real rank zero, furthermore, A is an AH-algebra of bounded dimension growth by [2], then by 6.1(2) of [8], A has finite decomposition rank and dr(A) is bounded by the dimensions of the base spaces. Since $C(Y_k)$ is an AF-algebra, by Proposition 3.3(5), $\mathrm{dr}(C(Y_k)\otimes M_{n(k)}(A)) \leqslant \mathrm{dr}(A)$ for $k=0,\dots,l$, then by Proposition 3.3(i), $\mathrm{dr}(B_Y)\leqslant \mathrm{dr}(A)$.

Let $y\in X$, then there exists a decreasing sequence $Z_1\supset Z_2\supset\cdots$ of clopen subsets of X with $\bigcap\limits_{n=1}^{\infty} Z_n=\{y\}$, we have $B_{\{y\}}=\lim\limits_{n\to\infty} B_{Z_n}$. By Proposition 3.3(ii), $\mathrm{dr}(B_{\{y\}})\leqslant \liminf\limits_{n\to\infty}\mathrm{dr}(B_{Z_n})\leqslant \mathrm{dr}(A)<+\infty$. By Theorem 3.5, $B_{\{y\}}$ absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially.

Since A is a unital separable simple C^*-algebra with tracial rank zero, by Theorem 3.6.11 of [11] A has real rank zero. Notice that $C(Y_k)$ is an AF-algebra, it is easy to see that B_Y has real rank zero. So for any $n\in\mathbb{N}$, B_{Z_n} has real rank zero, by Proposition 3.2.2 of [11] $B_{\{y\}}=\lim\limits_{n\to\infty} B_{Z_n}$ has also real rank zero. So the C^*-algebra $B_{\{y\}}$ has tracial rank zero by Theorem 3.4. ∎

LEMMA 3.7. *Any trace on $B_{\{y\}}$ is restricted to $C(X)$ is a σ-invariant measure on X.*

Proof. Let τ be a normalized trace on $B_{\{y\}}$, and let $f\in C(X)$. Set

$$a=u|f-f(y)|^{1/2}\quad\text{and}\quad b=u(f-f(y)\cdot 1)|f-f(y)|^{-1/2}.$$

Then a and b are both in $B_{\{y\}}$. Moreover,

$$\tau(f\circ\sigma^{-1}-f(y)\cdot 1)=\tau(u(f-f(y)\cdot 1)u^*)=\tau(ab^*)=\tau(b^*a)=\tau(f-f(y)\cdot 1).$$

Cancelling $\tau(f(y)\cdot 1)$, we get $\tau(f\circ\sigma^{-1})=\tau(f)$. ∎

4. MAIN RESULT

To prove Lemma 4.3, we firstly recall two results:

THEOREM 4.1 (Theorem 3.6 of [17]). *Let C be a unital* AH*-algebra and let A be a unital simple C^*-algebra with tracial rank zero. Suppose that $h_1,h_2: C\to A$ are two monomorphisms such that*

$$[h_1]=[h_2]\quad\text{in } KL(C,A)\quad\text{and}\quad \tau\circ h_1(a)=\tau\circ h_2(a)\quad\text{for all } a\in C \text{ and } \tau\in T(A).$$

Then h_1 and h_2 are approximately unitarily equivalent, i.e., there exists a sequence of unitaries $\{u_n\}\subset A$ such that

$$\lim_{n\to\infty}\mathrm{ad}u_n\circ h_1(a)=h_2(a)\quad\text{for all } a\in C.$$

LEMMA 4.2 (Lemma 4.1 of [18]). *Let A be a unital separable simple C^*-algebra with real rank zero and stable rank one and let $\beta\in\mathrm{Aut}(A)$. Suppose that there is a subgroup $G\subset K_0(A)$ such that $\rho_A(G)$ is dense in $\rho_A(K_0(A))$ and $(\beta)_{*0}|_G=\mathrm{id}|_G$. Then $\tau(a)=\tau(\beta(a))$ for all $a\in A$.*

LEMMA 4.3. *Let X be a Cantor set, let A be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the* UCT, *and let $\alpha \in \mathrm{Aut}(C(X,A))$. Suppose $C(X,A)$ is α-simple and $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1\otimes A, C(X,A))$. Let $B = C(X,A) \rtimes_\alpha \mathbb{Z}$, and let $y \in X$. Then for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset B$, there is a projection $p \in B_{\{y\}}$ such that:*

(i) $\|pa - ap\| < \varepsilon$ *for all* $a \in \mathcal{F}$.

(ii) $pap \in pB_{\{y\}}p$ *for all* $a \in \mathcal{F}$.

(iii) $\tau(1-p) < \varepsilon$ *for all* $\tau \in T(B_{\{y\}})$.

Proof. We may assume that $\mathcal{F} = \mathcal{G} \cup \{u\}$ for some finite subset $\mathcal{G} \subset C(X,A)$.

By Lemma 2.3, there is a minimal homomorphism σ from X to X and a strongly continuous map from X to $\mathrm{Aut}(A)$, denote by x to β_x, such that $\alpha(f)(x) = \beta_{\sigma^{-1}(x)}(f(\sigma^{-1}(x)))$.

Choose $N_0 \in \mathbb{N}$ so large that $4\pi/N_0 < \varepsilon$. Choose $\delta_0 > 0$ with $\delta_0 < (1/2)\varepsilon$ and so small that $d(x_1,x_2) < 4\delta_0$ implies $\|f(x_1) - f(x_2)\| < (1/4)\varepsilon$ for all $f \in \bigcup_{i=0}^{N_0} \alpha^{-i}(\mathcal{G})$. Choose $\delta > 0$ with $\delta \leqslant \delta_0$ and such that whenever $\mathrm{d}(x_1,x_2) < \delta$ and $0 \leqslant n \leqslant N_0$, then $\mathrm{d}(\sigma^{-n}(x_1), \sigma^{-n}(x_2)) < \delta_0$.

Since σ is minimal, there is $N > N_0 + 1$ such that $\mathrm{d}(\sigma^N(y), y) < \delta$. Since σ is free, there is a clopen neighborhood Y of y in X such that

$$\sigma^{-N_0}(Y), \sigma^{-N_0+1}(Y), \ldots, Y, \sigma(Y), \ldots, \sigma^N(Y)$$

are disjoint and all have diameter less than δ, and furthermore $\mu(Y) < \varepsilon/(N + N_0 + 1)$ for every σ-invariant Borel probability measure μ.

Define continuous functions $q_0(x) = \begin{cases} 1 & x \in Y, \\ 0 & x \in X \backslash Y. \end{cases}$

For $-N_0 \leqslant n \leqslant N$, set $q_n = \alpha^n(q_0) = \begin{cases} 1 & x \in \sigma^n(Y), \\ 0 & x \in X \backslash \sigma^n(Y), \end{cases}$ so the q_n are mutually orthogonal projections in $B_{\{y\}}$.

We now have a sequence of projections:

$$q_{-N_0}, \ldots, q_{-1}, q_0, \ldots, q_{N-N_0}, \ldots, q_{N-1}, q_N.$$

The projections q_0 and q_N live over clopen sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}. We are now going to use Berg's technique [1] to splice this sequence along the pairs of indices $(-N_0, N-N_0)$ through $(0, N)$, obtaining a loop of length N on which conjugation by u is approximately the cyclic shift.

We claim that there is a partial isometry $w \in B_{\{y\}}$ such that $w^*w = q_0$, $ww^* = q_N$ and $\|wf|_Y - f|_{\sigma^N(Y)}w\| < \varepsilon/4$ for all $f \in \bigcup_{i=0}^{N_0} \alpha^{-i}(\mathcal{G})$.

Let $x \in Y$. The first return time $\lambda_Y(x)$ (or $\lambda(x)$ if Y is understood) of x to Y is the smallest integer $n \geqslant 1$ such that $\sigma^n(x) \in Y$. By Lemma 2.5, we let $n(0) < n(1) < \cdots < n(l)$ be the distinct values of $\lambda(x)$ for $x \in Y$.

We denote $Y(k,j) = \sigma^j(\lambda^{-1}(n(k)))$, So

$$X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} \sigma^j(\{x \in Y : \lambda(x) = n(k)\}) = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} Y(k,j).$$

Define continuous functions $\chi_{Y(k,j)} = \begin{cases} 1 & x \in Y(k,j), \\ 0 & x \in X \backslash Y(k,j). \end{cases}$

Set $w' = \sum\limits_{k=0}^{l} \chi_{Y(k,N)} u^{N-n(k)}$, then $w' \in B_{\{y\}}$.

$$w'^* w' = \Big(\sum_{k=0}^{l} u^{-N+n(k)} \chi_{Y(k,N)} \Big) \Big(\sum_{k=0}^{l} \chi_{Y(k,N)} u^{N-n(k)} \Big) = \sum_{k=0}^{l} \chi_{Y(k,n(k))} = q_0,$$

$$w' w'^* = \Big(\sum_{k=0}^{l} \chi_{Y(k,N)} u^{N-n(k)} \Big) \Big(\sum_{k=0}^{l} u^{-N+n(k)} \chi_{Y(k,N)} \Big) = \sum_{k=0}^{l} \chi_{Y(k,N)} = q_N.$$

Let $a|_Y = \begin{cases} a & x \in Y, \\ 0 & x \in X \backslash Y, \end{cases}$ for all $a \in A$.

Since A is a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the UCT, by the classification theorem of [16], [15] and [5], A is a unital separable simple AH-algebra. For any $x \in X$, if ev_x is the evaluation map at X, then the composition

$$A \overset{1 \otimes \mathrm{id}}{\longrightarrow} C(X,A) \overset{\alpha}{\longrightarrow} C(X,A) \overset{\mathrm{ev}_{\sigma(x)}}{\longrightarrow} A$$

map a to $\beta_x(a)$ for all $a \in A$, the composition

$$A \overset{1 \otimes \mathrm{id}}{\longrightarrow} C(X,A) \overset{\mathrm{id}}{\longrightarrow} C(X,A) \overset{\mathrm{ev}_{\sigma(x)}}{\longrightarrow} A$$

map a to a for all $a \in A$. By the assumption $[\alpha|_{1 \otimes A}] = [\mathrm{id}|_{1 \otimes A}]$ in $KL(1 \otimes A, C(X,A))$, so $[\mathrm{ev}_{\sigma(x)} \circ \alpha|_{1 \otimes A}] = [\mathrm{ev}_{\sigma(x)} \circ \mathrm{id}|_{1 \otimes A}]$ in $KL(1 \otimes A, A)$, i.e., $[\beta_x]$ gives rise to a trivial element of $KL(A,A)$. Hence $(\beta_x)_{*0}|_{K_0(A)} = \mathrm{id}|_{K_0(A)}$ because A is a unital separable simple amenable C^*-algebra which satisfies the UCT. Moreover, Theorem 3.6.11 of [11] implies A have real rank zero and stable rank one. Applying Lemma 4.2, we get $\tau(\beta_x(a)) = \tau(a)$ for all $a \in A$ and for all $\tau \in T(A)$. Now we can use Theorem 4.1 to get that β_x is approximately unitary equivalent to the identity map for any $x \in X$, so there are a clopen neighborhood Y_x of x and an unitary $u'_j \in A$ such that $u'_j|_{\sigma^j(Y_x)} u^j a|_{Y_x} u^{*j} u'^*_j|_{\sigma^j(Y_x)} \approx_{\varepsilon/8} a|_{\sigma^j(Y_x)}$ for all $a \in \Big\{ f(y) : f \in \bigcup\limits_{i=0}^{N_0} \alpha^{-i}(\mathcal{G}) \Big\}$. Because Y is compact subset, we can get $u_j \in C(X,A)$ such that $u_j|_{\sigma^j(Y)} u^j a|_Y u^{*j} u_j^*|_{\sigma^j(Y)} \approx_{\varepsilon/8} a|_{\sigma^j(Y)}$ for all $a \in \Big\{ f(y) : f \in \bigcup\limits_{i=0}^{N_0} \alpha^{-i}(\mathcal{G}) \Big\}$.

Define $w = \Big(\sum_{k=0}^{l} u_{N-n(k)}\chi_{Y(k,N)}\Big)w'$, then $w \in B_{\{y\}}, w^*w = q_0, ww^* = q_N$ and

$$wa|_Y w^* = \sum_{k=0}^{l} u_{N-n(k)}\chi_{Y(k,N)} u^{N-n(k)} a|_Y u^{-N+n(k)} \chi_{Y(k,N)} u^*_{N-n(k)} \approx_{\varepsilon/8} a|_{\sigma^N(Y)},$$

$wa|_Y \approx_{\varepsilon/8} a|_{\sigma^N(Y)} w$, so $\|wf|_Y - f|_{\sigma^N(Y)}w\| < \varepsilon/4$ for all $f \in \bigcup_{i=0}^{N_0} \alpha^{-i}(\mathcal{G})$. The claim follows.

For $t \in \mathbb{R}$ define $v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*)$. Then $v(t)$ is a unitary in the corner $(q_0 + q_N)B_{\{y\}}(q_0 + q_N)$ whose matrix with respect to the obvious block decomposition is

$$v(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

So

$$\begin{aligned}
&\|v(t)(f|_Y + f|_{\sigma^N(Y)}) - (f|_Y + f|_{\sigma^N(Y)})v(t)\| \\
&\quad = \|(\cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*))(f|_Y + f|_{\sigma^N(Y)}) \\
&\qquad - (f|_Y + f|_{\sigma^N(Y)})(\cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*))\| \\
&\quad = \|\sin(\pi t/2)(wf|_Y - w^* f|_{\sigma^N(Y)}) - \sin(\pi t/2)(f|_{\sigma^N(Y)} w - f|_Y w^*)\| \qquad (4.1)\\
&\quad < \frac{\varepsilon}{2} \quad \text{for all } f \in \bigcup_{i=0}^{N_0} \alpha^{-i}(\mathcal{G}).
\end{aligned}$$

For $0 \leqslant k \leqslant N_0$ define $w_k = u^{-k}v(k/N_0)u^k$, so $w_k \in (q_{-k} + q_{N-k})B_{\{y\}}(q_{-k} + q_{N-k})\|uw_{k+1}u^* - w_k\| = \|v(k/N_0) - v(k-1)/N_0\| \leqslant 2\pi/N_0 < (1/2)\varepsilon$.

Now define $e_n = q_n$ for $0 \leqslant n \leqslant N - N_0$, and for $N - N_0 \leqslant n \leqslant N$ write $k = N - n$ and set $e_n = w_k q_{-k} w_k^*$. The two definitions for $n = N - N_0$ agree because $w_{N_0} q_{-N_0} w_{N_0}^* = q_{N-N_0}$, and moreover $e_N = e_0$. Therefore $\|ue_{n-1}u^* - e_n\| = 0$ for $1 \leqslant n \leqslant N - N_0$, and also $ue_N u^* = e_1$, while for $N - N_0 < n \leqslant N$ we have

$$\|ue_{n-1}u^* - e_n\| \leqslant 2\|uw_{N-n+1}u^* - w_{N-n}\| < \varepsilon.$$

Also, clearly $e_n \in B_{\{y\}}$ for all n.

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (i) through (iii).

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$.

Furthermore, $pup \in B_{\{y\}}$.

Next, let $g \in \mathcal{G}$. The sets $U_0, U_1, \ldots, U_N$ all have diameter less than δ. We have $\mathrm{d}(\sigma^N(y), y) < \delta$, so the choice of δ implies that $\mathrm{d}(\sigma^n(y), \sigma^{n-N}(y)) < \delta_0$ for

$N - N_0 \leqslant n \leqslant N$. Also, $U_{n-N} = \sigma^{n-N}(U_0)$ has diameter less than δ. Therefore $U_{n-N} \cup U_n$ has diameter less than $2\delta + \delta_0 \leqslant 3\delta_0$. Since g varies by at most $(1/4)\varepsilon$ on any set with diameter less than $4\delta_0$, and since the sets $\sigma(Y), \sigma^2(Y), \sigma^{N-N_0-1}(Y)$, $\sigma^{N-N_0}(Y) \cup \sigma^{-N_0}(Y), \sigma^{N-N_0+1}(Y) \cup \sigma^{-N_0+1}(Y), \ldots, \sigma^N(Y) \cup Y$ are disjoint.

For $0 \leqslant n \leqslant N - N_0$, we have $ge_n = gq_n = q_n g = e_n g$.

For $N - N_0 < n \leqslant N$ and any $g \in \mathcal{G}$, we use $e_n \in (q_{n-N} + q_n)B_{\{y\}}(q_{n-N} + q_n)$ and (4.1) to get $\|ge_n - e_n g\| = \|gw_k q_{-k} w_k^* - w_k q_{-k} w_k^* g\| = \|w_k^* g w_k q_{-k} - q_{-k} w_k^* g w_k\| < \varepsilon$. It follows that $\|pg - gp\| = \|ge - eg\| < \varepsilon$. That $pgp \in B_{\{y\}}$ follows from the fact that g and p are in this subalgebra. So we also have (ii) for g.

It remains only to verify (iii). Let $\tau \in T(B_{\{y\}})$, and let μ be the corresponding σ-invariant probability measure on X by Lemma 3.7. We have

$$1 - p = e \leqslant \sum_{n=-N_0}^{N} q_n,$$

so we have the following that completes the proof:

$$\tau(1-p) \leqslant \sum_{n=-N_0}^{N} \mu(\sigma^n(Y)) = (N + N_0 + 1)\mu(Y) < \varepsilon. \quad \blacksquare$$

We recall two results from Lemma 4.3 and Lemma 4.4 of [19].

LEMMA 4.4. *Let A be a unital simple C^*-algebra. Suppose that for every $\varepsilon > 0$ and every finite subset $\mathcal{F} \subset A$, there exists a unital C^*-subalgebra $B \subset A$ which has tracial rank zero and a projection $p \in B$ such that*

$$\|pa - ap\| < \varepsilon \quad \text{and} \quad \operatorname{dist}(pap, pBp) < \varepsilon$$

for all $a \in \mathcal{F}$. Then A has the local approximation property of Popa [21]*, that is for every $\varepsilon > 0$ and every finite subset $\mathcal{F} \subset A$, there exists a nonzero projection $q \in A$ and a finite dimensional unital C^*-subalgebra $D \subset qAq$ such that, for all $a \in \mathcal{F}$,*

$$\|qa - aq\| < \varepsilon \quad \text{and} \quad \operatorname{dist}(qaq, D) < \varepsilon.$$

LEMMA 4.5. *Let A be a unital simple C^*-algebra. Suppose that for every finite subset $\mathcal{F} \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital simple subalgebra $B \subset pAp$ with tracial rank zero such that:*

(i) $\|[a, p]\| < \varepsilon$ *for all* $a \in \mathcal{F}$;

(ii) $\operatorname{dist}(pap, B) < \varepsilon$ *for all* $a \in \mathcal{F}$;

(iii) $1 - p$ *is Murray–von Neumann equivalent to a projection in* $\overline{cAc}$.

Then A has tracial rank zero.

THEOREM 4.6. *Let X be a Cantor set, and let A be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfies the* UCT. *Let $C(X, A)$ denote all continuous functions from X to A and α be an automorphism of $C(X, A)$. Suppose that $C(X, A)$ is α-simple and $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$. Then $C(X, A) \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra with tracial rank zero.*

Proof. Let $B = C(X, A) \rtimes_\alpha \mathbb{Z}$, then B is a unital simple C^*-algebra by Lemma 2.4.

We verify the conditions of Lemma 4.5.

For any given point $y \in X$, $B_{\{y\}} = C^*(C(X, A), uC_0(X\backslash\{y\}, A))$ has tracial rank zero by Lemma 3.6. Thus, let $\mathcal{F} \subset B$ be a finite subset, let $\varepsilon > 0$, and let $c \in B$ be a nonzero positive element. Beyond Lemma 4.3, the main step of the proof is to find a nonzero projection in $B_{\{y\}}$ which is Murray–von Neumann equivalent to a projection in $\overline{cBc}$.

The algebra B is simple, so Lemma 4.3, Lemma 4.4 and Lemma 2.12 of [12] imply that B has property (SP). Therefore there is a nonzero projection $e \in \overline{cBc}$. Set

$$\delta_0 = \frac{1}{18} \inf_{\tau \in T(B)} \tau(e) \leqslant \frac{1}{18}.$$

By Lemma 4.3, there is a projection $q \in B_{\{y\}}$ and an element $b_0 \in qB_{\{y\}}q$ such that

$$\|qe - eq\| < \delta_0, \quad \|qeq - b_0\| < \delta_0, \quad \text{and} \quad \sup_{\tau \in T(B_{\{y\}})} \tau(1-q) \leqslant \delta_0.$$

Then $\sup_{\tau \in T(B)} \tau(1-q) \leqslant \sup_{\tau \in T(B_{\{y\}})} \tau(1-q) \leqslant \delta_0$. Replacing b_0 by $(1/2)(b_0 + b_0^*)$, we may assume that b_0 is self-adjoint. We have $-\delta_0 \leqslant b_0 \leqslant 1 + \delta_0$, so applying continuous functional calculus we may find $b \in qB_{\{y\}}q$ such that $0 \leqslant b \leqslant 1$ and $\|qeq - b\| < 2\delta_0$. Using $\|qe - eq\| < \delta_0$ on the last term in the second expression, we get

$$\|b^2 - b\| \leqslant 3\|b - qeq\| + \|(qeq)^2 - qeq\| < 3 \cdot 2\delta_0 + \delta_0 = 7\delta_0 < \frac{1}{4}.$$

Therefore there is a projection $e_1 \in qB_{\{y\}}q$ such that $\|e_1 - b\| < 14\delta_0$, giving $\|e_1 - qeq\| < 16\delta_0$. Similarly (actually, one gets a better estimate) there is a projection $e_2 \in (1-q)B(1-q)$ such that $\|e_2 - (1-q)e(1-q)\| < 16\delta_0$. Therefore $\|e_1 + e_2 - [qeq + (1-q)e(1-q)]\| < 16\delta_0$ and, using $\|qe - eq\| < \delta_0$ again, we have $\|e_1 + e_2 - e\| < 18\delta_0 \leqslant 1$. It follows that $e_1 \precsim e$. Also, for $\tau \in T(B)$, we have

$$\tau(e_1) > \tau(qeq) - 16\delta_0 = \tau(e) - \tau((1-q)e(1-q)) - 16\delta_0 \geqslant \tau(e) - \tau(1-q) - 16\delta_0 > 0,$$

so $e_1 \neq 0$.

Now set $\varepsilon_0 = \inf\{\tau(e_1) : \tau \in T(B_{\{y\}})\}$. By Lemma 4.3, there is a projection $p \in B_{\{y\}}$ such that:

(i) $\|pa - ap\| < \varepsilon$ for all $a \in \mathcal{F}$;
(ii) $pap \in pB_{\{y\}}p$ for all $a \in \mathcal{F}$;
(iii) $\tau(1-p) < \varepsilon_0$ for all $\tau \in T(B_{\{y\}})$.

Since $B_{\{y\}}$ has tracial rank zero which implies that the order on projections is determined by traces (Theorem 3.7.2 of [11]), it follows that $1 - p \precsim e_1 \precsim e$. Since $pB_{\{y\}}p$ also has tracial rank zero, we have verified the hypotheses of Lemma 4.5. Thus B has tracial rank zero. ∎

COROLLARY 4.7. *For $j = 1, 2$ let X be a Cantor set, and let A_j be a unital separable simple amenable C^*-algebra with tracial rank zero which satisfy the* UCT. *Let $C(X, A_j)$ denote all continuous functions from X to A_j, and let α_j be an automorphism of $C(X, A_j)$. Suppose that $C(X, A_j)$ is α_j-simple and $[\alpha_j|_{1\otimes A_j}] = [\mathrm{id}|_{1\otimes A_j}]$ in $KL(1 \otimes A_j, C(X, A_j))$. Let $B_j = C(X, A_j) \rtimes_{\alpha_j} \mathbb{Z}$ for $j = 1, 2$. Then $B_1 \cong B_2$ if and only if*

$$(K_0(B_1), K_0(B_1)_+, [1_{B_1}], K_1(B_1)) \cong (K_0(B_2), K_0(B_2)_+, [1_{B_2}], K_1(B_2)).$$

The proof is immediate from Theorem 5.2 of [16] and Theorem 4.6.

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