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INVARIANT AND HYPERINVARIANT SUBSPACES FOR AMENABLE OPERATORS

LUO YI SHI and YU JING WU

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ABSTRACT. There has been a long-standing conjecture in Banach algebras that every amenable operator is similar to a normal operator. In this paper, we study the structure of amenable operators on Hilbert spaces. At first, we show that the conjecture is equivalent to the statement that every non-scalar amenable operator has a non-trivial hyperinvariant subspace. It is also equivalent to the statement that every amenable operator is similar to a reducible operator and has a non-trivial invariant subspace; and then, we give two decompositions for amenable operators, supporting the conjecture.

KEYWORDS: *Amenable, invariant subspaces, hyperinvariant subspaces, reduction property.*

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1. INTRODUCTION

Throughout this paper, $\mathfrak{H}$ denotes a complex separable infinite-dimensional Hilbert space and $\mathfrak{B}(\mathfrak{H})$ denotes the bounded linear operators on $\mathfrak{H}$. For an algebra $\mathfrak{A}$ in $\mathfrak{B}(\mathfrak{H})$, we write $\mathfrak{A}'$ for the commutant of $\mathfrak{A}$ (i.e., $\mathfrak{A}' = \{B \in \mathfrak{B}(\mathfrak{H}), BA = AB \text{ for all} A \in \mathfrak{A}\}$) and $\mathfrak{A}''$ for the double commutant of $\mathfrak{A}$ (i.e., $\mathfrak{A}'' = (\mathfrak{A}')'$). If $T \in \mathfrak{B}(\mathfrak{H})$, we write $\mathfrak{A}_T$ for the norm-closure of $\operatorname{span}\{T^k : k \in \{0\} \cup \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. We also write $\operatorname{Lat}\mathfrak{A}$ for the collection of those closed subspaces which are invariant for every operator in $\mathfrak{A}$. We say $\mathfrak{A}$ is *completely reducible* if for every subspace M in $\operatorname{Lat}\mathfrak{A}$ there exists N in $\operatorname{Lat}\mathfrak{A}$ such that $\mathfrak{H} = M \dot{+} N$ (i.e., $M \cap N = \{0\}$ and $\mathfrak{H}$ is the algebraic direct sum of M and N); $\mathfrak{A}$ is *reductive* if for every subspace M in $\operatorname{Lat}\mathfrak{A}$ we have $M^{\perp}$ (the orthogonal complement of M) in $\operatorname{Lat}\mathfrak{A}$; $\mathfrak{A}$ is *transitive* if $\operatorname{Lat}\mathfrak{A} = \{\{0\}, \mathfrak{H}\}$. If $T \in \mathfrak{B}(\mathfrak{H})$, we say that a subspace M of $\mathfrak{H}$ is a *hyperinvariant subspace* for T if M is invariant under each operator in $\mathfrak{A}'_T$; M is a *reducing subspace* for T if $M, M^{\perp} \in \operatorname{Lat} T$.

The concept of amenable Banach algebras was first introduced by B.E. Johnson in [12]. Suppose that $\mathfrak{A}$ is a Banach algebra. A *Banach* $\mathfrak{A}$*-bimodule* is a Banach space X that is also an algebra $\mathfrak{A}$-bimodule for which there exists a constant $K > 0$ such that $\|a \cdot x\| \leqslant K\|a\|\|x\|$ and $\|x \cdot a\| \leqslant K\|a\|\|x\|$ for all $a \in \mathfrak{A}$ and $x \in X$. We note that when X is a Banach $\mathfrak{A}$-bimodule, then X^*, the dual of X, is a Banach $\mathfrak{A}$-bimodule with respect to the dual actions $[a \cdot f](x) = f(x \cdot a), [f \cdot a](x) = f(a \cdot x), a \in \mathfrak{A}, x \in X, f \in X^*$. Such a Banach $\mathfrak{A}$-bimodule is called a *dual* $\mathfrak{A}$-*bimodule*.

A *derivation* $D : \mathfrak{A} \to X$ is a continuous linear map such that $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all $a, b \in \mathfrak{A}$. Given $x \in X$, the *inner derivation* $\delta_x : \mathfrak{A} \to X$, is defined by $\delta_x(a) = a \cdot x - x \cdot a$.

According to Johnson's original definition, a Banach algebra $\mathfrak{A}$ is *amenable* if every derivation from $\mathfrak{A}$ into the dual $\mathfrak{A}$-bimodule X^* is inner for all Banach $\mathfrak{A}$-bimodules X. If $T \in \mathfrak{B}(\mathfrak{H})$, we say that T is an *amenable operator*, if $\mathfrak{A}_T$ is an amenable Banach algebra. Ever since its introduction, the concept of amenability has played an important role in research in Banach algebras, operator algebras and harmonic analysis. There has been a long-standing conjecture in the Banach algebra community, stated as follows:

CONJECTURE 1.1. *A Banach subalgebra of* $\mathfrak{B}(\mathfrak{H})$ *is amenable if and only if it is similar to a* C^**-algebra.*

One of the first results in this direction is due to Willis [17]. Willis showed that if T is an amenable compact operator, then T is similar to a normal operator. In [8] Gifford studied the reduction property for operator algebras consisting of compact operators and showed that if such an algebra is amenable then it is similar to a C^*-algebra. In the recent papers [6], [7] Farenick, Forrest and Marcoux showed that if T is similar to a normal operator, then $\mathfrak{A}_T$ is amenable if and only if $\mathfrak{A}_T$ is similar to a C^*-algebra and the spectrum of T has connected complement and empty interior; if T is a triangular operator with respect to an orthonormal basis of $\mathfrak{H}$, then $\mathfrak{A}_T$ is amenable if and only if T is similar to a normal operator whose spectrum has connected complement and empty interior. For further details see [6] and [7].

In this paper, we give a characterization of the structure of amenable operators. First, we use the reduction theory of von Neumann to give two equivalent descriptions for Conjecture 1.1; and then, we give two decompositions for amenable operators, which support Conjecture 1.1.

2. AN EQUIVALENT FORMULATION OF CONJECTURE 1.1

In this section we use the reduction theory of von Neumann to give two equivalent descriptions for Conjecture 1.1. We obtain that every amenable operator is similar to a normal operator if and only if every non-scalar amenable operator has a non-trivial hyperinvariant subspace if and only if every amenable operator is similar to a reductive operator (i.e. $\mathfrak{A}_T$ is reductive) and has a non-trivial invariant subspace.

In order to prove the main theorem, we need to introduce von Neumann's reduction theory [16] and some lemmas.

Let $\mathfrak{H}_1 \subseteq \mathfrak{H}_2 \subseteq \cdots \subseteq \mathfrak{H}_\infty$ be a sequence of Hilbert spaces chosen once and for all, $\mathfrak{H}_n$ having dimension n. Let μ be a finite positive regular measure defined on the Borel sets of a separable metric space $\wedge$, and let $\{E_n\}_{n=1}^{\infty}$ be a collection of disjoint Borel sets of $\wedge$ with union $\wedge$. Then the symbol

$$\int_{\wedge}^{\oplus} \mathfrak{H}(\lambda) \mathrm{d}\mu(\lambda)$$

denotes the set of all functions f defined on $\wedge$ such that:

(i) $f(\lambda) \in \mathfrak{H}_n$ if $\lambda \in E_n$, $n = 1, 2, \ldots, \infty$;
(ii) $f(\lambda)$ is a μ-measurable function with values in $\mathfrak{H}_\infty$;
(iii) $\int_{\wedge}^{\oplus} |f(\lambda)|^2 \mathrm{d}\mu(\lambda) < \infty$.

We put
(iv) $(f, g) = \int_{\wedge}^{\oplus} (f(\lambda), g(\lambda)) \mathrm{d}\mu(\lambda)$.

The set of functions thus defined is called the *direct integral Hilbert space with measure* μ *and dimension sets* $\{E_n\}$ and denoted by $\mathfrak{H} = \int_{\wedge}^{\oplus} \mathfrak{H}(\lambda) \mathrm{d}\mu(\lambda)$.

An operator on $\mathfrak{H}$ is said to be *decomposable* if there exists a strongly μ-measurable operator-value function $A(\cdot)$ defined on $\wedge$ such that $A(\lambda)$ is a bounded operator on the space $\mathfrak{H}(\lambda) = \mathfrak{H}_n$ when $\lambda \in E_n$, and for all $f \in \mathfrak{H}$, $(Af)(\lambda) = A(\lambda) f(\lambda)$. We write $A = \int_{\wedge}^{\oplus} A(\lambda) \mathrm{d}\mu(\lambda)$ for the equivalence class corresponding to $A(\cdot)$. If $A(\lambda)$ is a scalar multiple of the identity on $\mathfrak{H}(\lambda)$ for almost all λ, then A is called *diagonal*. The collection of all diagonal operator is called the *diagonal algebra* of $\wedge$. In I.3 of [16], Schwartz showed that an operator A on Hilbert space $\mathfrak{H} = \int_{\wedge}^{\oplus} \mathfrak{H}(\lambda) \mathrm{d}\mu(\lambda)$ is decomposable if and only if A belongs to the commutant of the diagonal algebra of $\wedge$. Furthermore, $\|A\| = \mu\text{-esssup}_{\lambda \in \wedge} \|A(\lambda)\|$.

In [1], Azoff, Fong and Gilfeather used von Neumann's reduction theory to define the reduction theory for non-selfadjoint operator algebras: Fix a partitioned measure space $\wedge$ and let $\mathfrak{D}$ be the corresponding diagonal algebra. Given an algebra $\mathfrak{A}$ of decomposable operators, each operator $A \in \mathfrak{A}$ has a decomposition $A = \int_{\wedge}^{\oplus} A(\lambda) \mathrm{d}\mu(\lambda)$. Choose a countable generating set $\{A_n\}$ for $\mathfrak{A}$ and let $\mathfrak{A}(\lambda)$ be the strongly closed algebra generated by the $\{A_n(\lambda)\}$. Then $\mathfrak{A} \sim \int_{\wedge}^{\oplus} \mathfrak{A}(\lambda) \mathrm{d}\mu(\lambda)$ is called the *decomposition of* $\mathfrak{A}$ *respect to* $\mathfrak{D}$. A decomposition

$\mathfrak{A} \sim \int_{\wedge}^{\oplus} \mathfrak{A}(\lambda) \mathrm{d}\mu(\lambda)$ of an algebra is said to be *maximal* if the corresponding diagonal algebra is maximal among the abelian von Neumann subalgebras of $\mathfrak{A}'$. The following lemma is a basic result in [1] which will be used in this paper.

LEMMA 2.1 ([1], Theorem 4.1). *Let* $\mathfrak{A} \sim \int_{\wedge}^{\oplus} \mathfrak{A}(\lambda) \mathrm{d}\mu(\lambda)$ *be a decomposition of a reductive algebra. Then almost all of* $\{\mathfrak{A}(\lambda)\}$ *are reducible. In particular, if the decomposition is maximal, then almost all of the algebras* $\{\mathfrak{A}(\lambda)\}$ *are transitive.*

In [8] Gifford studied the reduction property for operator algebras and obtained the following result:

LEMMA 2.2 ([8], Lemma 4.4, Lemma 4.12). *If* $\mathfrak{A}$ *is a commutative amenable operator algebra, then* $\mathfrak{A}', \mathfrak{A}''$ *are completely reducible and there exists* $M \geqslant 1$ *so that for any idempotent* $p \in \mathfrak{A}''$ $\|p\| \leqslant M$.

Assume $\mathfrak{A}$ is an operator algebra, let $P(\mathfrak{A})$ denote the idempotents in $\mathfrak{A}$ and $\mathcal{P}(\mathfrak{A})$ denote the strongly closed algebra generated by $P(\mathfrak{A})$. We get the following lemma:

LEMMA 2.3. *If* $\mathfrak{A}$ *is a commutative amenable operator algebra, then* $\mathcal{P}(\mathfrak{A}'')$ *is similar to an abelian von Neumann algebra.*

Proof. By Lemma 2.2 and Corollary 17.3 of [2], it follows that there exists $X \in \mathfrak{B}(\mathfrak{H})$ such that XpX^{-1} is selfadjoint for each $p \in P(\mathfrak{A}'')$. Hence $\mathcal{P}(\mathfrak{A}'')$ is similar to an abelian von Neumann algebra. ∎

LEMMA 2.4 ([6]). *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be Banach algebras and suppose that* $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ *is a continuous homomorphism with* $\varphi(\mathfrak{A})$ *dense in* $\mathfrak{B}$. *If* $\mathfrak{A}$ *is amenable, then* $\mathfrak{B}$ *is amenable.*

NOTATION 2.5. From Lemmas 2.3, 2.4 we always assume that $\mathcal{P}(\mathfrak{A}_T'')$ is a abelian von Neumann algebra, and $\mathfrak{A}_T'$ is a reducible operator algebra in this section.

Now we will prove the main result of this section:

THEOREM 2.6. *The following are equivalent:*

(i) *every amenable operator is similar to a normal operator;*

(ii) *every non-scalar amenable operator has a non-trivial hyperinvariant subspace;*

(iii) *every amenable Banach algebra which is generated by an operator is similar to a* C^**-algebra.*

Proof. (i) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (ii) are clear by [6]. Therefore, in order to establish the theorem it suffices to show the implication (ii) $\Rightarrow$ (i).

Assume (ii), by Lemma 2.3 choose a maximal decomposition for

$$\mathfrak{A}_T' \sim \int_{\wedge}^{\oplus} \mathfrak{A}_T'(\lambda) \mathrm{d}\mu(\lambda)$$

with respect to the diagonal algebra $\mathcal{P}(\mathfrak{A}''_T)$.

Assume $T \sim \int_\wedge^\oplus T(\lambda)\mathrm{d}\mu(\lambda)$ is the decomposition for T. Let $\{p_n\}_{n=1}^\infty$ denote all the polynomials with rational coefficients. Then $p_n(T) \sim \int_\wedge^\oplus p_n(T)(\lambda)\mathrm{d}\mu(\lambda)$ is decomposable for all n and there exists a measurable $E \subseteq \wedge$ such that $\mu(\wedge - E) = 0$ and for any $\lambda \in E$ we have $p_n(T)(\lambda) = p_n(T(\lambda))$ and $\|p_n(T)(\lambda)\| \leqslant \|p_n(T)\|$ by Lemma I.3.1, I.3.2 of [16]. Define a mapping $\varphi_\lambda : \mathfrak{A}_T \to \mathfrak{A}_{T(\lambda)}$ by $\varphi_\lambda(p_n(T)) = p_n(T(\lambda))$ for each polynomial with rational coefficients p_n and $\lambda \in E$. Note that $\|p_n(T(\lambda))\| \leqslant \|p_n(T)\|$ for each polynomial with rational coefficients p_n and furthermore, $\{p_n(T)\}$ is dense in $\mathfrak{A}_T$. Hence, φ_λ is well-defined and φ_λ is a continuous homomorphism with $\varphi(\mathfrak{A}_T)$ dense in $\mathfrak{A}_{T(\lambda)}$. By Lemma 2.4, $T(\lambda)$ is amenable for almost all λ.

Now for almost all λ, $T(\lambda)$ is amenable and $\mathfrak{A}'_T(\lambda) \subseteq \mathfrak{A}'_{T(\lambda)}$ and $\mathfrak{A}'_T(\lambda)$ is transitive by Lemma 2.1. Thus almost all of $T(\lambda)$ are scalar operators, i.e. T is a normal operator. ∎

COROLLARY 2.7. *Every amenable operator is similar to a normal operator if and only if there exists a non-trivial idempotent in the double-commutant of every non-scalar amenable operator.*

REMARK 2.8. In [6], Farenick, Forrest and Marcoux showed that if $T \in \mathfrak{B}(\mathfrak{H})$ is amenable and similar to a normal operator N, then the spectrum of N has connected complement and empty interior. According to Theorem 1.23 of [13], N is a reducible operator. Hence, there exists an invertible operator $X \in \mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{A}''_{XTX^{-1}}$ is a reducible algebra. The following theorem gives a description equivalent to Conjecture 1.1 for the existence of invariant subspaces for amenable operators.

THEOREM 2.9. *The following are equivalent:*

(i) *every amenable operator is similar to a normal operator;*

(ii) *for every amenable operator $T \in \mathfrak{B}(\mathfrak{H})$, there exists an invertible operator $X \in \mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{A}''_{XTX^{-1}}$ is a reducible algebra and T has a non-trivial invariant subspace.*

Proof. (i) $\Rightarrow$ (ii) is clear by Remark 2.8.

(ii) $\Rightarrow$ (i) The proof can be adapted via trivial modifications from the proof of Theorem 2.6. ∎

REMARK 2.10. According to Theorem 2.6 and 2.9, it follows that Conjecture 1.1 for singly-generated algebras is equivalent to the following statements:

(i) every amenable operator T has a non-trivial invariant subspace and $\mathfrak{H}$ can be renormed with an equivalent Hilbert space norm so that under this norm $\operatorname{Lat}\mathfrak{A}_T$ becomes orthogonally complemented;

(ii) every non-scalar amenable operator has a non-trivial hyperinvariant subspace.

3. DECOMPOSITION OF AMENABLE OPERATORS

In this section, we get two decompositions for amenable operators and prove that the two decompositions are the same, which supports Conjecture 1.1.

At first, we summarize some of the details of multiplicity theory for abelian von Neumann algebras. For the most part, we will follow [3]. If A is an operator on a Hilbert space $\mathfrak{K}$ and n is a cardinal number, let $\mathfrak{K}^n$ denote the orthogonal direct sum of n copies of $\mathfrak{K}$, and $A^{(n)}$ be the operator on $\mathfrak{K}^n$ which is the direct sum of n copies of A. Whenever $\mathfrak{A}$ is an operator algebra on $\mathfrak{K}$, $\mathfrak{A}^{(n)}$ denotes the algebra $\{A^{(n)}, A \in \mathfrak{A}\}$. An abelian von Neumann algebra $\mathfrak{B}$ is of *uniform multiplicity* n if it is (unitary equivalent to) $\mathfrak{A}^{(n)}$ for some maximal abelian von Neumann algebra $\mathfrak{A}$. By [3], for any abelian von Neumann algebra $\mathfrak{A}$, there exists a sequence of regular Borel measures $\{\mu_n\}$ on a sequence of separable metric space $\{X_n\}$ such that $\mathfrak{A}$ is unitary equivalent to $\sum\limits_{n=1}^{\infty} \oplus \mathfrak{B}_n \oplus \mathfrak{B}_\infty$, where $\mathfrak{B}_n$ is a von Neumman algebra which has uniform multiplicity n for all $1 \leqslant n \leqslant \infty$. For further details see II.3 of [3].

PROPOSITION 3.1. *Suppose that T is an amenable operator and $\mathfrak{A}'_T$ contains a subalgebra which is similar to an abelian von Neumman algebra with no direct summand of infinite uniform multiplicity, then T is similar to a normal operator.*

Proof. For the sake of simplicity, we assume $\mathfrak{A}'_T$ contains a subalgebra $\mathfrak{B}$ which is an abelian von Neumman subalgebra with no direct summand of infinite uniform multiplicity. Trivial modifications adapt the proof to the more general case.

By II.3 of [3], there exists a sequence of regular Borel measures $\{\mu_n\}$ on a sequence of separable metric space $\{X_n\}$ such that $\mathfrak{B}$ is unitarily equivalent to $\sum\limits_{n=1}^{\infty} \oplus \mathfrak{B}_n$, where $\mathfrak{B}_n$ is a von Neumman algebra which has uniform multiplicity n for all n. Hence, $T = \sum\limits_{n=1}^{\infty} \oplus T_n$, where $T_n \in \mathfrak{B}'_n$. It suffices to show that T_n is similar to a normal operator for all n, then by Corollary 26 of [5], it follows that T is similar to a normal operator.

Since $T \in \mathfrak{B}'_n$, according to Theorem 7.20 of [13], for any $1 \leqslant n < \infty$ there exists a unitary operator $U_n \in \mathfrak{B}'_n$ such that

$$U_n T_n (U_n)^{-1} = \begin{bmatrix} N_{11} & N_{12} & \cdots & \cdots & N_{1n} \\ 0 & N_{22} & \cdots & \cdots & N_{2n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & N_{nn} \end{bmatrix}$$

where N_{ij} is a normal operator for all $1 \leqslant i, j \leqslant n$. By Proposition 3.1 of [11], it follows that T_n is similar to $\bigoplus_{i=1}^{n} N_{ii}$, i.e. T_n is similar to a normal operator for all n. ∎

COROLLARY 3.2. *Assume T is an amenable operator. Then there exists hyperinvariant subspaces M, N of T such that T has the form $T = T_1 \dot{+} T_2$ with respect to the space decomposition $\mathfrak{H} = M \dot{+} N$, where T_1, T_2 are amenable operators, T_1 is similar to a normal operator and $\mathcal{P}(\mathfrak{A}''_{T_2})$ is similar to an abelian von Neumman algebra with uniform infinite multiplicity.*

The proof of the following lemma is straightforward and we omit it.

LEMMA 3.3. *Suppose that $\mathfrak{A}$ is a completely reductive operator algebra and $p \in P(\mathfrak{A}')$. Then $p\mathfrak{A}$ is a completely reductive operator algebra on* $\operatorname{Ran} p$.

We are in need of the following propositions before we can address the main theorem of this section.

PROPOSITION 3.4. *Assume that T is an amenable operator and there exists a space decomposition $\mathfrak{H} = M \dot{+} N$ such that T has the matrix form $T = \begin{array}{c} \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix} \end{array}\begin{array}{c} M \\ N \end{array}$. Then T is similar to a normal operator if and only if T_1 and T_2 are similar to normal operators.*

Proof. Assume that T has the matrix form

$$T = \begin{bmatrix} T_1 & T_{12} \\ & \widetilde{T}_2 \end{bmatrix} \begin{array}{c} M \\ M^{\perp} \end{array}$$

with respect to the space decomposition $\mathfrak{H} = M \oplus M^{\perp}$. By Lemma 2.8 of [11], there exists an invertible operator $S = \begin{bmatrix} I & S_{12} \\ & I \end{bmatrix} \begin{array}{c} M \\ M^{\perp} \end{array}$ such that $S^{-1}TS = \begin{bmatrix} T_1 & \\ & \widetilde{T}_2 \end{bmatrix} \begin{array}{c} M \\ M^{\perp} \end{array}$. Assume that S has the matrix form $S = \begin{array}{cc} M & M^{\perp} \\ \hline \end{array}$ $\begin{bmatrix} I & \\ & S_1 \end{bmatrix} \begin{array}{c} M \\ N \end{array}$, we obtain that $T_2 = S_1 \widetilde{T}_2 S_1^{-1}$. By Propsition 6.5 of [10], we get that T is similar to a normal operator if and only if T_1 and T_2 are similar to normal operators. ∎

PROPOSITION 3.5. *Suppose that T is an amenable operator, $M_1 \in \operatorname{Lat} \mathfrak{A}'_T$ and $M_2 \in \operatorname{Lat} \mathfrak{A}''_T$. Then $M_1 + M_2$ is closed.*

Moreover, if $T|_{M_1}$ and $T|_{M_2}$ are similar to normal operators, then $T|_{M_1+M_2}$ is similar to a normal operator.

Proof. Let $N_0 = M_1 \cap M_2$, according to Lemma 3.3, there exists $N \in \operatorname{Lat} \mathfrak{A}''_T$ such that $M_2 = N_0 \dot{+} N$. Choose $q \in P(\mathfrak{A}'_T)$, such that $\operatorname{Ran} q = N$. By the assumption, $M_1 \in \operatorname{Lat} \mathfrak{A}'_T$. Hence $qM_1 \subset M_1 \cap N = \{0\}$. Therefore $M_1 \subset (I - q)\mathfrak{H}$. We see that $M_1 + M_2 = M_1 \dot{+} N$ is closed. This establishes the first statement of the proposition.

Since $T|_{M_2}$ is similar to a normal operator, by Proposition 3.4, we get that $T|_N$ is similar to a normal operator. By the assumption that $T|_{M_1}$ is similar to a normal operator, using Proposition 3.4 again, we obtain that $T|_{M_1+M_2} = T|_{M_1\dot{+}N}$ is similar to a normal operator. ∎

Now we will obtain the main theorem of this section.

THEOREM 3.6. *Assume T is an amenable operator, then there exists hyperinvariant subspaces M_1, M_2 of T such that T has the form $T = T_1\dot{+}T_2$ with respect to the space decomposition $\mathfrak{H} = M_1\dot{+}M_2$ and satisfies the following:*

(i) *T_1, T_2 are amenable operators;*

(ii) *if M is a hyperinvariant subspace of T and $T|_M$ is similar to a normal operator, then $M \subseteq M_1$, i.e. M_1 is the largest hyperinvariant subspace on which T is similar to a normal operator;*

(iii) *for any $q \in P(\mathfrak{A}''_{T_2})$, $T_2|_{\operatorname{Ran} q}$ is not similar to a normal operator;*

(iv) *$\mathcal{P}(\mathfrak{A}''_{T_2})$ is similar to an abelian von Neumman algebra with uniform infinite multiplicity;*

(v) $\mathfrak{A}'_T = \mathfrak{A}'_{T_1}\dot{+}\mathfrak{A}'_{T_2}$, $\mathfrak{A}''_T = \mathfrak{A}''_{T_1}\dot{+}\mathfrak{A}''_{T_2}$;

(vi) *there exists no nonzero compact operator in $\mathfrak{A}'_{T_2}$.*

Proof. Case 1. For any $p \in P(\mathfrak{A}''_T)$, $T|_{\operatorname{Ran} p}$ is not similar to a normal operator. According to the proof of Proposition 3.1, we obtain that $\mathcal{P}(\mathfrak{A}''_T)$ is similar to an abelian von Neumman algebra with uniform infinite multiplicity. Let $M_1 = 0$.

Case 2. There exists $p \in P(\mathfrak{A}''_T)$ such that $T|_{\operatorname{Ran} p}$ is similar to a normal operator. Then, by Zorn's Lemma and the same method in the proof of Corollary 26 in [5], we can show that there exists an element $p_0 \in P(\mathfrak{A}''_T)$ which is maximal with respect to the property that $T|_{\operatorname{Ran} p_0}$ is similar to a normal operator. Using Proposition 3.5, $\operatorname{Ran} p_0$ is the largest hyperinvariant subspace of T on which T is similar to a normal operator. Hence, T has the form $T = T_1\dot{+}T_2$ with respect to the space decomposition $\mathfrak{H} = \operatorname{Ran} p_0\dot{+}\operatorname{Ker} p_0$ where T_1 is similar to a normal operator, T_1, T_2 are amenable operators. Let $M_1 = \operatorname{Ran} p_0, M_2 = \operatorname{Ker} p_0$.

Next we will prove that for any $q \in P(\mathfrak{A}''_{T_2})$, $T_2|_{\operatorname{Ran} q}$ is not similar to a normal operator. Then according to Proposition 3.1 $\mathcal{P}(\mathfrak{A}''_{T_2})$ is similar to an abelian von Neumman algebra with uniform infinite multiplicity.

Indeed, if there exists $q \in P(\mathfrak{A}''_{T_2})$ such that $T_2|_{\operatorname{Ran} q}$ is similar to a normal operator and q has the form $q = \begin{array}{@{}c@{}l@{}} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{l} \operatorname{Ran} q \\ \operatorname{Ker} q \end{array} \end{array}$. Then for any $A \in \mathfrak{A}'_T$, A has the form

$$A = \begin{array}{@{}c@{}l@{}} \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} & \begin{array}{l} \operatorname{Ran} p_0 \\ \operatorname{Ran} q \\ \operatorname{Ker} q \end{array} \end{array}.$$

Let

$$R = \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix} \begin{matrix} \mathrm{Ran} p_0 \\ \mathrm{Ran}\, q\,. \\ \mathrm{Ker} q \end{matrix}$$

Then $R \in P(\mathfrak{A}''_T)$. By the assumption $T|_{\mathrm{Ran} R}$ is similar to a normal operator which contradicts to the maximal property of p_0.

At last we will prove that there exists no nonzero compact operator in $\mathfrak{A}'_{T_2}$.

Indeed, if there exists a nonzero compact operator $k_0 \in \mathfrak{A}'_{T_2}$, let L_1 denote the subspace spanned by the ranges of all compact operators in $\mathfrak{A}'_{T_2}$, and L_2 be the intersection of their kernel, by Lemma 3.1 of [15], both L_1, L_2 lie in $\mathrm{Lat}\,\mathfrak{A}'_{T_2}$ and $L_1 \dot{+} L_2 = \mathrm{Ker} p_0$. Considering the restriction $T_2|_{L_1}$, assume $T_{21} = T_2|_{L_1}$, then T_{21} is an amenable operator and $\mathfrak{A}'_{T_{21}}$ contain a sufficient set of compact operators. By Lemma 2.2 and Theorem 9 of [14], T_{21} is similar to a normal operator which contradicts to the above discussion. ∎

Using trivial modifications, we can adapt the proof of Theorem 3.6 to obtain the following theorem, which decomposes amenable operators according to their invariant subspaces.

THEOREM 3.7. *Assume T is an amenable operator, then there exist invariant subspaces N_1, N_2 of T such that T has the form $T = A_1 \dot{+} A_2$ with respect to the space decomposition $\mathfrak{H} = N_1 \dot{+} N_2$ and satisfies the following:*

(i) *A_1, A_2 are amenable operators;*

(ii) *if N is an invariant subspace of T such that $N_1 \subseteq N$ and $T|_N$ is similar to a normal operator, then $N = N_1$, i.e. N_1 is the maximal invariant subspace on which T is similar to a normal operator;*

(iii) *for any $q \in P(\mathfrak{A}'_{T_2})$, $T_2|_{\mathrm{Ran}\, q}$ is not similar to a normal operator;*

(iv) *if $\mathcal{P}(\mathfrak{A}'_{T_2})$ contains a subalgebra which is similar to an abelian von Neumman algebra then the von Neumman algebra has the uniform infinite multiplicity.*

REMARK 3.8. If the answer to Conjecture 1.1 is positive, by Theorem 2.6, every amenable is similar to a normal operator. Then, for the above theorem $M_1 = N_1 = \mathfrak{H}$. That is to say, the two decompositions of Theorems 3.6 and 3.7 are the same. The remainder of this section, we will prove that the two decompositions are the same which supports Conjecture 1.1.

LEMMA 3.9 ([5]). *If $T \in \mathfrak{B}(\mathfrak{H})$ is an amenable operator and there exist a one-to-one bounded linear map $W : \mathfrak{H} \to \mathfrak{H}_2$, a bounded linear map $V : \mathfrak{H}_1 \to \mathfrak{H}$ with dense range and operators $S_1 \in \mathfrak{B}(\mathfrak{H}_1)$, $S_2 \in \mathfrak{B}(\mathfrak{H}_2)$ that are similar to normal operators such that $TV = VS_1$ and $WT = S_2W$. Then T is similar to a normal operator.*

COROLLARY 3.10. *Assume $T = B_1B_2$ is an amenable operator, where B_1, B_2 are positive operators, then T is similar to a normal operator.*

Proof. Assume B_1, B_2 have the forms

$$B_2 = \begin{bmatrix} 0 & \\ & \widetilde{B}_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix},$$

with respect to the space decomposition $\mathfrak{H} = \mathrm{Ker} B_2 \oplus (\mathrm{Ker} B_2)^\perp$ where $\widetilde{B}_2$ is one-to-one and B_{11}, B_{22} are positive operators. Thus T has the form $T = \begin{bmatrix} 0 & B_{12}\widetilde{B}_2 \\ 0 & B_{22}\widetilde{B}_2 \end{bmatrix}$ with respect to the same decomposition. Since T is an amenable operator, by Lemma 2.8 of [11], T is similar to $\begin{bmatrix} 0 & 0 \\ 0 & B_{22}\widetilde{B}_2 \end{bmatrix}$. Thus, without loss of generality, we may assume that B_2 is one-to-one.

Assume B_1, B_2 have the forms

$$B_1 = \begin{bmatrix} \widetilde{B}_1 & \\ & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix},$$

with respect to the space decomposition $\mathfrak{H} = (\mathrm{Ker} B_1)^\perp \oplus \mathrm{Ker} B_1$ where $\widetilde{B}_1$ is one-to-one and has dense range and B_{11}, B_{22} are positive operators. Thus T has the form $T = \begin{bmatrix} \widetilde{B}_1 B_{11} & \widetilde{B}_1 B_{12} \\ 0 & 0 \end{bmatrix}$ with respect to this decomposition. Since T is an amenable operator, by Lemma 2.8 of [11], T is similar to $\begin{bmatrix} \widetilde{B}_1 B_{11} & 0 \\ 0 & 0 \end{bmatrix}$ and there exists an operator S such that $\widetilde{B}_1 B_{12} = \widetilde{B}_1 B_{11} S$. Note that $\widetilde{B}_1, B_2$ are one-to-one, hence $B_{12} = B_{11}S$, and B_{11} is one-to-one. Thus without loss of generality, we may assume that B_1 has dense range and B_2 is one-to-one.

Note that $B_1^{1/2} B_2 B_1^{1/2}, B_2^{1/2} B_1 B_2^{1/2}$ are positive operators and $TB_1^{1/2} = B_1^{1/2} B_1^{1/2} B_2 B_1^{1/2}$ and $B_2^{1/2} T = B_2^{1/2} B_1 B_2^{1/2} B_2^{1/2}$, by Lemma 3.9, T is similar to a normal operator. ∎

THEOREM 3.11. *The two decompositions for an amenable operator in Theorems* 3.6, 3.7 *are the same.*

Proof. According to Theorem 3.6, Theorem3.7 and Proposition 3.5, it suffices to proof that $N_1 \in \mathrm{Lat}\,\mathfrak{A}'_T$.

Suppose otherwise. Then T has the form $T = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix} \begin{matrix} N_1 \\ N_2 \end{matrix}$ and there exists $S = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix} \begin{matrix} N_1 \\ N_2 \end{matrix} \in \mathfrak{A}'_T$ where $Y \neq 0$. Note that $Y : N_1 \to N_2$ and $\mathfrak{H} = N_1 \dot{+} \mathrm{Ran} Y \oplus (N_2 \ominus \overline{\mathrm{Ran} Y})$, so S and T have the form

$$S = \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} N_1 \\ \overline{\mathrm{Ran} Y} \\ N_2 \ominus \overline{\mathrm{Ran} Y} \end{matrix}, \quad T = \begin{bmatrix} T_1 & 0 & 0 \\ & T_{21} & T_{22} \\ & T_{23} & T_{24} \end{bmatrix} \begin{matrix} N_1 \\ \overline{\mathrm{Ran} Y} \\ N_2 \ominus \overline{\mathrm{Ran} Y} \end{matrix},$$

where $\widetilde{Y}$ has dense range. Note that $TS = ST$, we get that $T_{23} = 0$. Since T is amenable, by Lemma 2.8 of [11] there exists an operator $B : N_2 \ominus \overline{\mathrm{Ran}Y} \to \overline{\mathrm{Ran}Y}$ such that

$$\begin{bmatrix} I & 0 & 0 \\ & I & B \\ & & I \end{bmatrix} \begin{bmatrix} T_1 & 0 & 0 \\ & T_{21} & T_{22} \\ & & T_{24} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & I & -B \\ & & I \end{bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ & T_{21} & 0 \\ & & T_{24} \end{bmatrix}.$$

Moreover,

$$\begin{bmatrix} I & 0 & 0 \\ & I & B \\ & & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & I & -B \\ & & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we can assume that Y has dense range. Using T is amenable again, there exists $L = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{matrix} N_1 \\ N_2 \end{matrix} \in \mathfrak{A}'_T$, where $X \neq 0$, by Lemma 4.11 of [8]. Similar to the decomposition to S and T, we get that S, L and T have the form

$$S = \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y_1} & 0 & 0 \\ \widetilde{Y_2} & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & \widetilde{X} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 & 0 \\ & T_{31} & T_{32} \\ & T_{33} & T_{34} \end{bmatrix},$$

with respect to the space decomposition $\mathfrak{H} = N_1 \dot{+} \mathrm{Ker}X \oplus (N_2 \ominus \mathrm{Ker}X)$, where $\widetilde{X}$ is one-to-one, and $\widetilde{Y_1}, \widetilde{Y_2}$ has dense range. Note that $LT = TL$, we get that $T_{33} = 0$. Using T is amenable again, there exists an operator $C : N_2 \ominus \mathrm{Ker}X \to \mathrm{Ker}X$ such that

$$\begin{bmatrix} I & 0 & 0 \\ & I & C \\ & & I \end{bmatrix} \begin{bmatrix} T_1 & 0 & 0 \\ & T_{31} & T_{32} \\ & & T_{34} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & I & -C \\ & & I \end{bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ & T_{31} & 0 \\ & & T_{34} \end{bmatrix},$$

$$\begin{bmatrix} I & 0 & 0 \\ & I & C \\ & & I \end{bmatrix} \begin{bmatrix} 0 & 0 & \widetilde{X} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & I & -C \\ & & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & \widetilde{X} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} I & 0 & 0 \\ & I & C \\ & & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y_1} & 0 & 0 \\ \widetilde{Y_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & I & -C \\ & & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{Y_1} + C\widetilde{Y_2} & 0 & 0 \\ \widetilde{Y_2} & 0 & 0 \end{bmatrix}.$$

Moreover, $\widetilde{Y_2}T_1 = T_{34}\widetilde{Y_2}$, $T_1\widetilde{X} = \widetilde{X}T_{34}$, and T_1 is similar to a normal operator. By Lemma 3.9, T_{34} is similar to a normal operator, which contradicts Theorem 3.7. ∎

COROLLARY 3.12. *Assume T is an amenable operator, then M is a maximal invariant subspace such that $T|_M$ is similar to a normal operator if and only if M is the largest invariant subspace such that $T|_M$ is similar to a normal operator.*

COROLLARY 3.13. *Assume T is an amenable operator and which is quasisimilar to a compact operator, then T is similar to a normal operator.*

Proof. Suppose, $TV = VK, WT = KW$ with V, W injective operators having dense ranges and K is a compact operator. Then $TVKW = VKWT$. Let $C = VKW$, $C \in \mathfrak{A}'_T$, and C is a compact operator. According to Theorem 3.6, C has the form $\begin{bmatrix} C_1 & \\ & 0 \end{bmatrix}$ with respect to the decomposition $\mathfrak{H} = M_1 \dot{+} M_2$ following from that theorem. If $Cx = 0$, $VWTx = Cx = 0$, thus $Tx = 0$. It follows that there is no part of T_2, i.e. T is similar to a normal operator. ∎

4. (ESSENTIAL) OPERATOR VALUED ROOTS OF ABELIAN ANALYTIC FUNCTIONS

In this section, we will study the structure of an operator which is an (essential) operator-valued root of an abelian analytic function and then we get that if such an operator is also amenable, then it is similar to a normal operator. In [9] Gilfeather introduce the concept of operator-valued roots of abelian analytic functions as follows: let $\mathfrak{A}$ be an abelian von Neumann algebra and $\psi(z)$, an $\mathfrak{A}$ valued analytic function on a domain $\mathcal{D}$ in the complex plane. We may decompose $\mathfrak{A}$ into a direct integral of factors such that for $A \in \mathfrak{A}$, there exists a unique $g \in L_\infty(\wedge, \mu)$ such that $\mathfrak{A} = \int_\wedge^\oplus g(\lambda)I(\lambda)\mathrm{d}\mu(\lambda)$. If $T \in \mathfrak{A}'$ and $\sigma(T) \subseteq \mathcal{D}$, let

$$\psi(T) = (2\pi \mathrm{i})^{-1} \int_\wedge (T - zI)^{-1} \psi(z) \mathrm{d}z.$$

An operator T is called an (essential) root of the abelian analytic function ψ, if $\psi(T) = 0$ (if $\psi(T)$ is compact, respectively). The structure of roots of a locally nonzero abelian analytic function has been given in [9]. In this section we mainly study the structure of essential roots of a locally nonzero abelian analytic function.

LEMMA 4.1. *Assume $T \in \mathfrak{B}(\mathfrak{H})$, f is a locally nonzero analytic function on the neighborhood of $\sigma(T)$ and assume $f(T)$ is a compact operator, then T is a polynomial compact operator.*

Proof. Let $\widehat{T}$ denote the image of T in the Calkin algebra, then $\widehat{f(T)} = 0$. Since f is a locally nonzero analytic function on $\sigma(T)$, there exists a polynomial p such that $\widehat{p(T)} = 0$. i.e. T is a polynomial compact operator. ∎

THEOREM 4.2. *Let ψ be a locally nonzero abelian analytic function on $\mathcal{D}$ taking values in the von Neumann algebra $\mathfrak{A}$. If T is an essential root of ψ and is amenable, then T is similar to a normal operator.*

Proof. Since $\mathfrak{A}$ is an abelian von Neumann algebra, $\mathfrak{A}$ is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus \mathfrak{B}_n \oplus \mathfrak{B}_\infty$, where $\mathfrak{B}_n$ is a von Neumann algebra which has uniform multiplicity n for all $1 \leqslant n \leqslant \infty$. Note $T \in \mathfrak{A}'$ is an amenable operator, thus

$T = T_1 \oplus T_2$, where T_1 is similar to a normal operator, and $T_1 \in \left(\sum_{n=1}^{\infty} \oplus \mathfrak{B}_n\right)'$, $T_2 \in \mathfrak{B}'_\infty$. Let $\sigma_1(\sigma_2)$ denote the continuous (atom, respectively) parts of the spectrum of $\mathfrak{B}_\infty$, then $\mathfrak{B}_\infty = \mathfrak{C}_\infty \oplus \mathfrak{D}_\infty$, where $\mathfrak{C}_\infty$ and $\mathfrak{D}_\infty$ are uniform multiplicity ∞ von Neumman algebra and $\sigma(\mathfrak{C}_\infty) = \sigma_1, \sigma(\mathfrak{D}_\infty) = \sigma_2$ and $T_2 = T_3 \oplus T_4$, where $T_3 \in \sigma(\mathfrak{C}_\infty)', T_4 \in \sigma(\mathfrak{D}_\infty)'$.

Assume ψ is a locally nonzero abelian analytic function on $\mathcal{D}$ and $\sigma(T) \subseteq \mathcal{D}$, then $\psi(T) = \psi(T_1) \oplus \psi(T_3) \oplus \psi(T_4)$, note that $\psi(T_3)$ is a compact operator and $\sigma(\mathfrak{C}_\infty) = \sigma_1$, so $\psi(T_3) = 0$. Since $\mathfrak{D}_\infty$ are uniform multiplicity ∞ and $\sigma(\mathfrak{D}_\infty) = \sigma_2$, by Lemma 4.1, it follows that T_4 is a direct sum of polynomial compact operators. According to Theorem 2.1 of [9], there exists a sequence of mutually orthogonal projections $\{P_n, Q_m\}$ in $\mathfrak{A}$ with $I = \sum P_n + \sum Q_m$ so that $T|_{P_n}$ is a finite type spectral operator and $T|_{Q_m}$ is a polynomial compact operator. By Theorems 3.5, 4.5 of [11], we get that T is similar to a normal operator. ∎

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LUO YI SHI, DEPARTMENT OF MATHEMATICS, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300160, P.R. CHINA
E-mail address: sluoyi@yahoo.cn

YU JING WU, TIANJIN VOCATIONAL INSTITUTE, TIANJIN 300410, P.R. CHINA
E-mail address: wuyujing111@yahoo.cn

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