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MULTIPLICATION OPERATORS ON THE ENERGY SPACE

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ABSTRACT. We consider the multiplication operators on $\mathcal{H}_\mathcal{E}$ (the space of functions of finite energy supported on an infinite network), characterize them in terms of positive semidefinite functions. We show why they are typically not self-adjoint, and compute their adjoints in terms of a reproducing kernel. We also consider the bounded elements of $\mathcal{H}_\mathcal{E}$ and use the (possibly unbounded) multiplication operators corresponding to them to construct a boundary theory for the network. In the case when the only harmonic functions of finite energy are constant, we show that the corresponding Gel'fand space is the 1-point compactification of the underlying network.

KEYWORDS: *Multiplication operator, Dirichlet form, graph energy, discrete potential theory, graph Laplacian, weighted graph, spectral graph theory, resistance network, Gel'fand space, reproducing kernel Hilbert space.*

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1. INTRODUCTION

In this paper, we study the functions on a network, and the corresponding algebra of multiplication operators. More precisely, we consider the Hilbert space of finite-energy functions on a network, how the multiplication operators act on them, and under what conditions these operators are bounded, Hermitian, or have other properties of interest. In Theorem 3.11, we show that the multiplication operator corresponding to a function f is bounded on $\mathcal{H}_\mathcal{E}$ with $\|M\| \leqslant b$ if and only if

$$s_f(x,y) := (b^2 - f(x)\overline{f(y)})\langle v_x, v_y \rangle_\mathcal{E} \tag{1.1}$$

is a positive semidefinite function on $X \times X$ (see (3.3) for the definition of a positive semidefinite function), where $\{v_x\}_{x \in X}$ is the reproducing kernel for the Hilbert space of finite-energy functions discussed in [17]. In Theorem 4.4, we show that multiplication by a point mass gives a bounded operator, and that the bound is given in terms of the conductance of the network at x and the resistance distance to x. (While one would expect such boundedness, it is a bit surprising that the proof is not trivial.) In Theorem 4.13 we give an equivalent condition to (1.1) which is expressed in terms of an explicit matrix computation, and in Theorem 4.19 we give a sufficient condition for (1.1) to hold which is even easier to check.

Next, we study the bounded functions of finite energy, and the corresponding multiplication operators. In Theorem 5.11, we show that the boundary $\operatorname{bd} G$ developed in [22] (see also the expository paper [20]) embeds into the Gel'fand space (that is, the spectrum of a Banach algebra realized as a topological space) of the algebra of bounded harmonic functions of finite energy. In Theorem 5.12, we then see that the Gel'fand space is a 1-point compactification of G (and the unitalization of the corresponding C^*-algebra) if the only harmonic functions of finite energy are constants.

Our main results in this paper concern infinite weighted graphs, such as arise in the study of Markov processes [28], [29], [37], [38], geometric group theory [37], percolation [29], discrete harmonic analysis [35], [37], [38], and electrical networks [14], [28], [29], [35]. (This is by no means a complete catalogue of the literature, but each of the references listed in this sentence provides an excellent and extensive list of further reading.) We will need to develop some results on matrix-order and its use in the study of operators on (infinite-dimensional) separable Hilbert spaces. Aside from their applications, we hope that our separate matrix/operator results may be of independent interest. See [18] for relations to Markov processes and [21] for relations to matrix representations of operators. To make our paper accessible to separate audiences, we have included details from one area which perhaps may not be familiar to readers from the other. The literature dealing with analysis on infinite graphs is vast, and we do not attempt to cite all the subareas. The monograph [19] includes a more systematic treatment, but still slanted towards spectral theory and operators in Hilbert space. It also contains a more complete bibliography. Stressing the operator theory/algebra, and reproducing kernels, there are the papers [4], [10], [24], [34]; random walk models [9], [13], [33], [38], and references cited there; and quantum theory [12], [30], [31], [32].

2. BASIC TERMS AND PREVIOUS RESULTS

We now proceed to introduce the key notions used throughout this paper: resistance networks, the energy form $\mathcal{E}$, the Laplace operator Δ, and their elementary properties.

DEFINITION 2.1. A *(resistance) network* is a connected graph (G, c), where G is a graph with vertex set G^0, and c is the *conductance function* which defines adjacency by $x \sim y$ if and only if $c_{xy} > 0$, for $x, y \in G^0$. We assume $c_{xy} = c_{yx} \in$

$[0,\infty)$, and write $c(x) := \sum_{y\sim x} c_{xy}$. We require $c(x) < \infty$, but $c(x)$ need not be a bounded function on G. The notation c may be used to indicate the multiplication operator $(cv)(x) := c(x)v(x)$.

In this definition, connected means simply that for any $x, y \in G^0$, there is a finite sequence $\{x_i\}_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \ldots, n$. We may assume there is at most one edge from x to y, as two conductors c^1_{xy} and c^2_{xy} connected in parallel can be replaced by a single conductor with conductance $c_{xy} = c^1_{xy} + c^2_{xy}$. Also, we assume $c_{xx} = 0$ so that no vertex has a loop.

Since the edge data of (G, c) is carried by the conductance function, we will henceforth simplify notation and write $x \in G$ to indicate that x is a vertex. For any network, one can fix a reference vertex, which we shall denote by o (for "origin"). It will be apparent that our calculations depend in no way on the choice of o.

DEFINITION 2.2. The *Laplacian* on G is the linear difference operator which acts on a function $v : G \to \mathbb{R}$ by

$$(\Delta v)(x) := \sum_{y\sim x} c_{xy}(v(x) - v(y)). \tag{2.1}$$

A function $v : G \to \mathbb{R}$ is *harmonic* if and only if $\Delta v(x) = 0$ for each $x \in G$.

We have adopted the physics convention (so that the spectrum is nonnegative) and thus our Laplacian is the negative of the one commonly found in the PDE literature. The network Laplacian (2.1) should not be confused with the stochastically renormalized Laplace operator $c^{-1}\Delta$ which appears in the probability literature, or with the spectrally renormalized Laplace operator $c^{-1/2}\Delta c^{-1/2}$ which appears in the literature on spectral graph theory (e.g., [11]).

DEFINITION 2.3. The *energy* of functions $u, v : G \to \mathbb{C}$ is given by the (closed, bilinear) Dirichlet form

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x,y\in G} c_{xy}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y)), \tag{2.2}$$

with the energy of u given by $\mathcal{E}(u) := \mathcal{E}(u, u)$. The *domain* of the energy form is

$$\operatorname{dom} \mathcal{E} = \{u : G \to \mathbb{C} : \mathcal{E}(u) < \infty\}. \tag{2.3}$$

Since $c_{xy} = c_{yx}$ and $c_{xy} = 0$ for nonadjacent vertices, the initial factor of $1/2$ in (2.2) implies there is exactly one term in the sum for each edge in the network.

REMARK 2.4. To remove any ambiguity about the precise sense in which (2.2) converges, note that $\mathcal{E}(u)$ is a sum of nonnegative terms and hence converges if and only if it converges absolutely. Since the Schwarz inequality gives $\mathcal{E}(u, v)^2 \leqslant \mathcal{E}(u)\mathcal{E}(v)$, it is clear that the sum in (2.2) is well-defined whenever $u, v \in \operatorname{dom} \mathcal{E}$.

2.1. THE ENERGY SPACE $\mathcal{H}_\mathcal{E}$. The energy form $\mathcal{E}$ is sesquilinear and conjugate symmetric on $\operatorname{dom}\mathcal{E}$ and would be an inner product if it were positive definite.

DEFINITION 2.5. Let $\mathbf{1}$ denote the constant function with value 1 and recall that $\ker\mathcal{E} = \mathbb{C}\mathbf{1}$. Then $\mathcal{H}_\mathcal{E} := \operatorname{dom}\mathcal{E}/\mathbb{C}\mathbf{1}$ is a Hilbert space with inner product and corresponding norm given by

$$\langle u, v\rangle_\mathcal{E} := \mathcal{E}(u,v) \quad \text{and} \quad \|u\|_\mathcal{E} := \mathcal{E}(u,u)^{1/2}. \tag{2.4}$$

We call $\mathcal{H}_\mathcal{E}$ the *energy (Hilbert) space*.

REMARK 2.6. Since G is connected, it is possible to show (with the use of Fatou's lemma) that $\operatorname{dom}\mathcal{E}/\mathbb{C}\mathbf{1}$ is complete; see [17], [19] for further details regarding this point.

DEFINITION 2.7. Let v_x be defined to be the unique element of $\mathcal{H}_\mathcal{E}$ for which

$$\langle v_x, u\rangle_\mathcal{E} = u(x) - u(o), \quad \text{for every } u \in \mathcal{H}_\mathcal{E}. \tag{2.5}$$

The collection $\{v_x\}_{x\in G}$ forms a reproducing kernel for $\mathcal{H}_\mathcal{E}$ ([17], Corollary 2.7); we call it the *energy kernel* and (2.5) shows its span is dense in $\mathcal{H}_\mathcal{E}$.

Note that v_o corresponds to a constant function, since $\langle v_o, u\rangle_\mathcal{E} = 0$ for every $u \in \mathcal{H}_\mathcal{E}$. Therefore, v_o may often be safely ignored or omitted during calculations.

DEFINITION 2.8. A *dipole* is any $v \in \mathcal{H}_\mathcal{E}$ satisfying the pointwise identity $\Delta v = \delta_x - \delta_y$ for some vertices $x, y \in G$. One can check that $\Delta v_x = \delta_x - \delta_o$; cf. Lemma 2.13 of [17].

DEFINITION 2.9. For $v \in \mathcal{H}_\mathcal{E}$, one says that v has *finite support* if and only if there is a finite set $F \subseteq G$ for which $v(x) = k \in \mathbb{C}$ for all $x \notin F$. The set of functions of finite support in $\mathcal{H}_\mathcal{E}$ is denoted $\operatorname{span}\{\delta_x\}$, where δ_x is the Dirac mass at x, i.e., the element of $\mathcal{H}_\mathcal{E}$ containing the characteristic function of the singleton $\{x\}$. It is immediate from (2.2) that $\mathcal{E}(\delta_x) = c(x)$, whence $\delta_x \in \mathcal{H}_\mathcal{E}$. Define $\mathcal{F}in$ to be the closure of $\operatorname{span}\{\delta_x\}$ with respect to $\mathcal{E}$.

DEFINITION 2.10. The set of harmonic functions of finite energy is denoted

$$\mathcal{H}arm := \{v \in \mathcal{H}_\mathcal{E} : \Delta v(x) = 0, \text{ for all } x \in G\}. \tag{2.6}$$

LEMMA 2.11 ([17], 2.11). *For any $x \in G$, one has $\langle \delta_x, u\rangle_\mathcal{E} = \Delta u(x)$.*

The following result follows from Lemma 2.11; cf. Theorem 2.15 of [17].

THEOREM 2.12 (Royden decomposition). *$\mathcal{H}_\mathcal{E} = \mathcal{F}in \oplus \mathcal{H}arm$.*

REMARK 2.13. By combining (2.5) and the conclusion of Lemma 2.11, one can reconstruct the network (G, c) (or equivalently, the corresponding Laplacian) from the dual systems (i) $(\delta_x)_{x\in X}$ and (ii) $(v_x)_{x\in X}$. Indeed, from (ii), we obtain the (relative) reproducing kernel Hilbert space $\mathcal{H}_\mathcal{E}$ and from (ii), we get an associated operator $(\Delta u)(x) = \langle \delta_x, u\rangle_\mathcal{E}$ for $u \in \mathcal{H}_\mathcal{E}$. In other words, (i) reproduces Δ.

DEFINITION 2.14. Denote the *(free) effective resistance* between x and o by

$$R(x) := R^{\mathrm{F}}(x,o) = \mathcal{E}(v_x) = v_x(x) - v_x(o). \tag{2.7}$$

This quantity represents the voltage drop measured when one unit of current is passed into the network at x and removed at o, and the equalities in (2.7) are proved in [16] and elsewhere in the literature; see [25], [29] for different formulations.

The following results will be useful in the sequel, especially in Section 5. For further details, please see [16], [17], [19], [20], and [22].

LEMMA 2.15 ([17], Lemma 2.23). *Every v_x is $\mathbb{R}$-valued, with $v_x(y) - v_x(o) > 0$ for all $y \neq o$.*

LEMMA 2.16 ([22], Lemma 6.9). *Every v_x is bounded. In particular, $\|v_x\|_\infty \leqslant R(x)$ (see (2.7)).*

LEMMA 2.17 ([22], Lemma 6.8). *If $v \in \mathcal{H}_\mathcal{E}$ is bounded, then $P_{\mathcal{F}in} v$ is also bounded.*

DEFINITION 2.18. Let $p(x,y) := c_{xy}/c(x)$ so that $p(x,y)$ defines a random walk on the network, with transition probabilities weighted by the conductances. Then let

$$\mathbb{P}[x \to y] := \mathbb{P}_x \quad (\tau_y < \tau_x^+) \tag{2.8}$$

be the probability that the random walk started at x reaches y before returning to x. In (2.8), τ_z is the hitting time of the vertex z.

COROLLARY 2.19 ([16], Corollary 3.13 and Corollary 3.15). *For any $x \neq o$, one has*

$$\mathbb{P}[x \to o] = \frac{1}{c(x)R(x)}. \tag{2.9}$$

3. BOUNDED MULTIPLICATION OPERATORS

Henceforth, we will write $X = G \setminus \{o\}$ for brevity. Throughout the following, we use ξ to denote coefficients indexed by the vertices and write $\xi_x := \xi(x)$. Thus, ξ may or may not be an element of $\mathcal{H}_\mathcal{E}$. In order to facilitate computations which include both ξ and $u \in \mathcal{H}_\mathcal{E}$, we make the standing convention to choose the representative of u (which we also denote by u) for which

$$u(o) = 0. \tag{3.1}$$

It should be noted that under this convention, $\mathcal{F}in$ is the $\mathcal{E}$-closure of the class of functions on G which are constant (but not necessarily 0) outside of a finite set. Also, this convention allows (2.5) to be written as

$$\langle v_x, u \rangle_\mathcal{E} = u(x), \quad \text{for every } u \in \mathcal{H}_\mathcal{E}. \tag{3.2}$$

DEFINITION 3.1. A function $s : X \times X \to \mathbb{C}$ is called *positive semidefinite* (psd) if and only if for every finite subset $F \subseteq X$, one has

$$\sum_{x,y \in F} \overline{\xi}_x \xi_y s(x,y) \geqslant 0, \tag{3.3}$$

for every function $\xi : X \to \mathbb{C}$.

We shall have occasion to use basic tools from the theory of matrix-order, that is, the usual ordering of finite Hermitian matrices:

$$A \geqslant 0 \iff \langle \xi, A\xi \rangle_{\ell^2} = \sum_{x,y \in F} \overline{\xi}_x \xi_y A_{xy} \geqslant 0, \quad \forall F \text{ finite, and } \forall \xi. \tag{3.4}$$

REMARK 3.2 (The role of finite subsets of X). Definition 3.1 is a statement about all possible finite Hermitian submatrices of the matrix A with entries $A_{xy} = s_f(x,y)$. Thus, we will have frequent occasion to use the notation F to indicate a finite subset of V, and we write

$$\ell(F) = \{\xi : F \to \mathbb{C} : F \text{ is a finite subset of } X\}. \tag{3.5}$$

The order of Hermitian matrices, or of Hermitian (or self-adjoint) operators in Hilbert space is central in both harmonic analysis and in the theory of C^*-algebras. The reader may find the following references helpful: [6], [8], [15], [23], [36]. The following two lemmas are standard and proofs may be found in the references just listed.

LEMMA 3.3 (The square root lemma). *For a (finite) matrix A, one has $A \geqslant 0$ if and only if there is some $B \geqslant 0$ such that $A = B^2$.*

LEMMA 3.4. *Let A and B be finite matrices. Then with respect to the ordering* (3.4),

$$B^* AB \leqslant \|B\|^2 A \tag{3.6}$$

where the norm is the operator norm. In particular, if B is the matrix of an orthogonal projection ($B = B^* = B^2$), then $A - BAB \geqslant 0$, that is,

$$\langle u, Au \rangle \geqslant \langle u, BABu \rangle, \quad \forall u \in \ell(F). \tag{3.7}$$

DEFINITION 3.5. For a function $f : X \to \mathbb{C}$, we denote by M_f the corresponding multiplication operator:

$$(M_f u)(x) := f(x)u(x), \quad \forall x \in X. \tag{3.8}$$

(Without convention (3.1), one would replace (3.8) by $(M_f u)(x) := f(x)(u(x) - u(o))$.) When context precludes confusion, we suppress the dependence on f and just write M. The norm of M is the usual operator norm

$$\|M\| := \|M\|_{\mathcal{H}_\mathcal{E} \to \mathcal{H}_\mathcal{E}} = \sup\{\|fu\|_\mathcal{E} : \|u\|_\mathcal{E} \leqslant 1\}. \tag{3.9}$$

It is important to notice that multiplication operators are a little unusual in $\mathcal{H}_\mathcal{E}$. The following feature of $\mathcal{H}_\mathcal{E}$ operator theory contrasts sharply with the more familiar Hilbert spaces of L^2 functions, where all $\mathbb{R}$-valued functions define Hermitian multiplication operators.

REMARK 3.6. One might guess that the operator norm of M_f is computed from the sup-norm of f, but this is not the case. In Section 6, we give an example of a bounded function $f : X \to \mathbb{C}$ for which the M_f, as an operator in $\mathcal{H}_\mathcal{E}$, is unbounded.

LEMMA 3.7. *For $f : X \to \mathbb{C}$, the multiplication operator $M = M_f$ is Hermitian if and only if f is constant and $\mathbb{R}$-valued (in which case $f = 0$ in $\mathcal{H}_\mathcal{E}$).*

Proof. Choose any representatives for $u, v \in \mathcal{H}_\mathcal{E}$. From the formula (2.2),

$$\langle Mu, v\rangle_\mathcal{E} = \frac{1}{2}\sum_{x,y\in G} c_{xy}(\overline{f(x)u(x)}v(x) - \overline{f(x)u(x)}v(y) - \overline{f(y)u(y)}v(x) + \overline{f(y)u(y)}v(y)).$$

By comparison with the corresponding expression, this is equal to $\langle u, Mv\rangle_\mathcal{E}$ if and only if $(\overline{f(y)} - f(x))\overline{u(y)}v(x) = (f(y) - \overline{f(x)})\overline{u(x)}v(y)$ holds for all $x, y \in G$. However, since we are free to vary u and v, it must be the case that f is constant and $f = \overline{f}$. ∎

Since Lemma 3.7 shows that the adjoint of a multiplication operator is not what one would expect, one immediately wonders what the adjoint is, and this is the subject of our next result.

LEMMA 3.8. *Let M^* be the adjoint of the multiplication operator $M = M_f$ with respect to the energy inner product* (2.4). *Then the adjoint of M is defined by its action on the energy kernel:*

$$M^* v_x = \overline{f(x)} v_x, \quad \forall x \in X. \tag{3.10}$$

Proof. Since the energy kernel is dense in $\mathcal{H}_\mathcal{E}$, it suffices to show $\langle v_y, M^* v_x - \overline{f(x)} v_x\rangle_\mathcal{E} = 0$ for every $y \in X$. Using (3.2) for the final step, we have

$$\langle v_y, M^* v_x\rangle_\mathcal{E} = \langle M v_y, v_x\rangle_\mathcal{E} = \langle f \cdot v_y, v_x\rangle_\mathcal{E} = \overline{f(x) v_y(x)},$$

which proves (3.10) because the v_y are $\mathbb{R}$-valued by Lemma 2.15. ∎

REMARK 3.9. Note that M^* multiplies v_x by the scalar $\overline{f(x)}$, not the function $\overline{f}$.

LEMMA 3.10. *If L is an operator on a Hilbert space $\mathcal{H}$, then the following are equivalent:*

(i) *$L : \mathcal{H} \to \mathcal{H}$ is bounded with $\|L\| \leqslant b$.*
(ii) $b^2 - L^*L \geqslant 0$.
(iii) $b^2 - LL^* \geqslant 0$.

In Lemma 3.10, $L \geqslant 0$ means $\langle u, Lu\rangle \geqslant 0$ for all u in some dense subset of $\mathcal{H}$, and of course b^2 means $b^2\mathbb{I}$. The only nontrivial part of the proof of Lemma 3.10 is (ii)⇔(iii), which uses polar decomposition; see [23], for example.

THEOREM 3.11. *$M = M_f$ is bounded on $\mathcal{H}_\mathcal{E}$ with $\|M\|_{\mathcal{H}_\mathcal{E}\to\mathcal{H}_\mathcal{E}} \leqslant b$ if and only if*

$$s_f(x,y) := (b^2 - f(x)\overline{f(y)})\langle v_x, v_y\rangle_\mathcal{E} \tag{3.11}$$

is a positive semidefinite function on $X \times X$.

Proof. We will work with the dense linear subspace $\mathcal{V} := \text{span}\{v_x\}_{x\in X}$ of the energy space (density of $\mathcal{V}$ in $\mathcal{H}$ is shown in [17]). By Lemma 3.10, the first hypothesis in the statement of Theorem 3.11 is equivalent to

$$\langle u, (b^2 - MM^*)u\rangle_\mathcal{E} \geqslant 0, \quad \forall u \in \mathcal{V}. \tag{3.12}$$

Since $u \in \mathcal{V}$ means $u = \sum_{x\in F} \xi_x v_x$ for some finite set $F \subseteq X$, we can evaluate (3.12):

$$\begin{aligned}
\langle u, (b^2 - MM^*)u\rangle_\mathcal{E} &= \sum_{x,y\in F} \overline{\xi}_x \xi_y \langle v_x, (b^2 - MM^*)v_y\rangle_\mathcal{E} \\
&= \sum_{x,y\in F} \overline{\xi}_x \xi_y (\langle v_x, b^2 v_y\rangle_\mathcal{E} - \langle M^* v_x, M^* v_y\rangle_\mathcal{E}) \\
&= \sum_{x,y\in F} \overline{\xi}_x \xi_y (b^2\langle v_x, v_y\rangle_\mathcal{E} - \langle \overline{f(x)} v_x, \overline{f(y)} v_y\rangle_\mathcal{E}) \\
&= \sum_{x,y\in F} \overline{\xi}_x \xi_y (b^2 - f(x)\overline{f(y)})\langle v_x, v_y\rangle_\mathcal{E},
\end{aligned}$$

where Lemma 3.8 was used to obtain the third equality. In view of (3.3), it is now clear that (3.12) holds for every choice of coefficients ξ if and only if $s_f(x,y)$ as defined in (3.11) is a positive semidefinite function on $X \times X$. ∎

COROLLARY 3.12. *If f_1 and f_2 are functions on X and $\|M_{f_i}\| \leqslant b_i < \infty$ for $i = 1, 2$, then*

$$s_{12}(x,y) := (b_1 b_2 - (f_1 f_2)(x)\overline{(f_1 f_2)(y)})\langle v_x, v_y\rangle_\mathcal{E} \tag{3.13}$$

is a positive semidefinite function on $X \times X$, where $(f_1 f_2)(x) := f_1(x) f_2(x)$.

Proof. Since $M_{f_1} M_{f_2} = M_{(f_1 f_2)}$, we get $\|M_{(f_1 f_2)}\| \leqslant b_1 b_2$. Now Lemma 3.10 gives (3.13). ∎

REMARK 3.13. It is rather difficult to prove (3.13) from first principles.

4. ALGEBRAS OF MULTIPLICATION OPERATORS

We continue to use $X := G \setminus \{o\}$ in conjunction with convention (3.1), as discussed at the beginning of Section 3. We begin by considering the multiplication operators $M_x := M_{\delta_x}$, that is, the special case of multiplication operators corresponding to the function

$$f := \delta_x = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases} \tag{4.1}$$

REMARK 4.1. In this section, it will be helpful to think of a function f on X as a vector, corresponding to some fixed enumeration of the vertices. Then the operator A can be thought of as a matrix with entries A_{xy} (in the same fixed enumeration of the vertices) given by matrix multiplication:

$$(Af)(x) := \sum_{y \in X} A_{xy} f(y). \tag{4.2}$$

The following operator will be very useful throughout the sequel. (This operator appeared for the first time in [21], as far as we know. However, given the breadth and depth of the literature in this area, it is quite possible that it has appeared previously (in some guise) in the literature on random walks, percolation, or resistance forms.)

DEFINITION 4.2 (V and $\mathbb{V}$). Let

$$V_{xy} := \langle v_x, v_y \rangle_{\mathcal{E}}, \tag{4.3}$$

and define the inner product

$$\langle \xi, \eta \rangle_{\mathbb{V}} = \sum_{x,y \in X} \overline{\xi}_x \eta_y V_{xy} \tag{4.4}$$

and corresponding norm $\|\xi\|_{\mathbb{V}} = \sqrt{\langle \xi, \xi \rangle_{\mathbb{V}}}$. Then we have a Hilbert space

$$\mathbb{V} = \{(\xi_x)_{x \in X} : \|\xi\|_{\mathbb{V}} < \infty\}. \tag{4.5}$$

REMARK 4.3. Since V is psd, one has that $\sum_{x,y \in F} \overline{\xi}_x \xi_y V_{xy} \geqslant 0$ for every finite subset F of X, and so $\langle \xi, \xi \rangle_{\mathbb{V}}$ can be defined by (4.4) as the supremum of the finite sums over F. Then $\langle \xi, \eta \rangle_{\mathbb{V}}$ is obtained by polarization. For an alternative justification/definition, see Remark 2.4.

In view of (4.2), note also that (4.3) defines a self-adjoint operator V with $\operatorname{dom} V^{1/2} = \mathbb{V}$. We will see in Lemma 4.17 that $\mathbb{V}$ is unitarily equivalent to $\mathcal{H}_{\mathcal{E}}$.

Recall from Definition 2.14 that $R(x)$ denotes the (free) effective resistance between x and o, and note that $R(x) = v_x(x)$ under the convention (3.2). Recall also from Definition 2.18 that $\mathbb{P}[x \to o]$ denotes the probability that the random walk started at x reaches o before returning to x.

THEOREM 4.4. *For any $x \in X$, the multiplication operator M_x is bounded on $\mathcal{H}_{\mathcal{E}}$ with*

$$\|M_x\| = \sqrt{c(x)R(x)} = \mathbb{P}[x \to o]^{-1/2}. \tag{4.6}$$

Proof. Define an operator on $\mathcal{H}_{\mathcal{E}}$ via (4.2) with

$$(D_f V D_{\overline{f}})_{xy} := f(x) V_{xy} \overline{f(y)}, \quad \forall x, y \in X \times X, \tag{4.7}$$

where D_f is the diagonal operator whose x^{th} diagonal entry is $f(x)$. Consequently,

$$s_f(x,y) = (1 - f(x)\overline{f(y)})\langle v_x, v_y\rangle_{\mathcal{E}} = V_{xy} - (D_f V D_{\overline{f}})_{xy} = (V - D_f V D_{\overline{f}})_{xy}. \tag{4.8}$$

By Theorem 3.11, we need to show that $s_{\delta_{x_o}}$ is psd, but $f = \delta_{x_o}$ changes (4.8) into

$$s_{\delta_{x_o}}(x,y) = V_{xy} - V_{x_o,x_o}\delta_{(x,y),(x_o,x_o)}, \tag{4.9}$$

where $\delta_{(x,y),(x_o,x_o)}$ is a Kronecker delta for matrix position (x_o, x_o). To check that (4.9) is psd, suppose that $F \subseteq X$ is any finite subset containing x_o so that positive semidefiniteness is equivalent to

$$V - P_o V P_o \geqslant 0, \tag{4.10}$$

where P_o is the projection in $\mathbb{V}$ onto the 1-dimensional subspace of $\{f : F \to \mathbb{C}\}$ spanned by δ_{x_o}. So the boundedness of M_x follows from (3.7).

It remains to compute the norm. First, note that for any $u \in \mathcal{H}_{\mathcal{E}}$, one immediately has $\|M_x u\|_{\mathcal{E}}^2 = c(x)|u(x)|^2$ from (2.2). Then (3.1), the Schwarz inequality and (2.5) give

$$|u(x)| = |u(x) - u(o)| = |\langle v_x, u\rangle_{\mathcal{E}}| \leqslant \|v_x\|_{\mathcal{E}}\|u\|_{\mathcal{E}} = \sqrt{R(x)}\|u\|_{\mathcal{E}},$$

and (4.6) follows upon multiplying across by $\sqrt{c(x)}$ and applying Corollary 2.19. ∎

4.1. THE MULTIPLIER C^*-ALGEBRA OF $\mathcal{H}_{\mathcal{E}}$. Our work in this section is inspired in part by work on quantum graphs as systems of coherent state configurations on countable graphs. See [1], [2], [5], [26], [27], for example.

DEFINITION 4.5. Define the multiplier C^*-algebra of $\mathcal{H}_{\mathcal{E}}$ to be the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{E}})$ generated by the bounded multiplication operators M_f. We denote this algebra by

$$C^*(\mathcal{H}_{\mathcal{E}}) := \bigvee\{M_f, M_f^* : f : X \to \mathbb{C} \text{ and } M_f \text{ is bounded}\}, \tag{4.11}$$

and the relations defining this algebra are given in Corollary 4.10. In (4.11), the symbol $\bigvee$ indicates that the linear span is closed in the operator topology; i.e., the uniform norm of bounded operators.

REMARK 4.6. There is an important distinction between the abelian algebra generated by M_f (with f such that s_f is psd, as in Theorem 3.11), and the C^*-algebra generated by M_f. The first is abelian and the second very non-abelian.

REMARK 4.7. Theorem 4.4 shows that $M_x \in C^*(\mathcal{H}_{\mathcal{E}})$, and hence that $M_f \in C^*(\mathcal{H}_{\mathcal{E}})$ for every finitely supported function $f : X \to \mathbb{C}$.

Recall that $|u\rangle\langle v|$ is Dirac's notation for the rank-1 operator that sends v to u, and it is a projection if and only if both u and v are unit vectors.

LEMMA 4.8. *For any $x \in X$, M_x and M_x^* are the rank-1 operators expressed in Dirac notation by*

$$M_x = |\delta_x\rangle\langle v_x| \quad \text{and} \quad M_x^* = |v_x\rangle\langle\delta_x|. \tag{4.12}$$

Proof. It suffices to verify the second identity in (4.12) on the dense set $\operatorname{span}\{v_x\}$:

$$M_x^* v_y = \delta_x(y) v_y = \begin{cases} v_x & y = x, \\ 0 & \text{else,} \end{cases} = (\delta_y(x) - \delta_y(o)) v_x = v_x \langle \delta_x, v_y \rangle_{\mathcal{E}} = |v_x\rangle\langle\delta_x| v_y,$$

where we have used (3.2). Now the first identity in (4.12) follows from the second.

For an alternative proof, note that $M_f^* v_x = \overline{f(x)} v_x$, by Lemma 3.8, which implies that $M_x^* = |v_x\rangle\langle\delta_x|$. Then $M_x = (M_x^*)^* = |v_x\rangle\langle\delta_x|^* = |\delta_x\rangle\langle v_x|$. ∎

Note that (2.2) immediately gives

$$\langle \delta_x, \delta_y \rangle_{\mathcal{E}} = \begin{cases} -c_{xy} & x \neq y, \\ c(x) & x = y. \end{cases} \tag{4.13}$$

REMARK 4.9. One can prove Theorem 4.4 from Lemma 4.8:

$$\|M_x\|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} = \| |\delta_x\rangle\langle v_x| \|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} = \|\delta_x\|_{\mathcal{E}} \|v_x\|_{\mathcal{E}} = \sqrt{c(x)}\sqrt{R(x)}. \tag{4.14}$$

COROLLARY 4.10. *$C^*(\mathcal{H}_{\mathcal{E}})$ is the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{E}})$ with generators $\{M_x, M_x^*\}_{x \in X}$ and relations*

$$M_x^* M_y = \langle \delta_x, \delta_y \rangle_{\mathcal{E}} |v_x\rangle\langle v_y|, \tag{4.15}$$

$$M_x M_y^* = \langle v_x, v_y \rangle_{\mathcal{E}} |\delta_x\rangle\langle \delta_y|, \tag{4.16}$$

where $\langle \delta_x, \delta_y \rangle_{\mathcal{E}}$ is as in (4.13).

Proof. The computations are direct applications of (4.12) and Dirac's notation:

$$M_x^* M_y = (|v_x\rangle\langle\delta_x|)(|\delta_y\rangle\langle v_y|) = |v_x\rangle\langle\delta_x|\delta_y\rangle\langle v_y| = \langle \delta_x, \delta_y \rangle_{\mathcal{E}} |v_x\rangle\langle v_y|,$$

and similarly for $M_x M_y^*$. ∎

REMARK 4.11. Corollary 4.10 shows that $C^*(\mathcal{H}_{\mathcal{E}})$ contains all the rank-1 projections corresponding to the functions $\{v_x\}$. Since the span of this set is dense in $\mathcal{H}_{\mathcal{E}}$, this implies that $C^*(\mathcal{H}_{\mathcal{E}})$ contains all finite-rank operators, and hence all the compact operators (since the compact operators are obtained by closing the space of finite-rank operators). Thus Corollary 4.10 shows that $C^*(\mathcal{H}_{\mathcal{E}})$ is quite large.

REMARK 4.12. Let us introduce the normalized functions

$$u_x := \frac{v_x}{\|v_x\|_{\mathcal{E}}} \quad \text{and} \quad d_x := \frac{\delta_x}{\|\delta_x\|_{\mathcal{E}}} \tag{4.17}$$

and the corresponding rank-1 projections onto the spans of these elements:

$$U_x := |u_x\rangle\langle u_x| = \operatorname{proj}\operatorname{span} u_x = \frac{1}{c(x)R(x)} M_x^* M_x = (\mathbb{P}[x \to o]) M_x^* M_x, \text{ and} \tag{4.18}$$

$$D_x := |d_x\rangle\langle d_x| = \operatorname{proj}\operatorname{span} d_x = \frac{1}{c(x)R(x)} M_x M_x^* = (\mathbb{P}[x \to o]) M_x M_x^*. \tag{4.19}$$

Then one has two systems of orthonormal projections satisfying the relations:

$$U_x U_y = \frac{\langle v_x, v_y\rangle_{\mathcal{E}}}{\sqrt{R(x)R(y)}} |u_x\rangle\langle u_y|, \qquad U_x D_y = \langle u_x, d_y\rangle_{\mathcal{E}} \; |u_x\rangle\langle d_y|,$$

$$D_x U_y = \langle d_x, u_y\rangle_{\mathcal{E}} \; |d_x\rangle\langle u_y|, \qquad D_x D_y = \frac{\langle \delta_x, \delta_y\rangle_{\mathcal{E}}}{\sqrt{c(x)c(y)}} |d_x\rangle\langle d_y|.$$

Moreover, one also has

$$\bigvee_{x\in X} \operatorname{ran} U_x = \mathcal{H}_{\mathcal{E}}, \quad \text{and} \quad \bigvee_{x\in X} \operatorname{ran} D_x = \mathcal{F}in, \tag{4.20}$$

where $\bigvee$ indicates that one takes the closed linear span.

Theorem 4.13 gives a necessary and sufficient condition for determining whether or not an operator is bounded. In the statement and proof, the ordering is as defined by (3.4). It will also be helpful to keep in mind that

$$D_f V D_{\overline{f}} = \begin{bmatrix} f(x_1) & 0 & 0 & \dots \\ 0 & f(x_2) & 0 & \dots \\ 0 & 0 & f(x_3) & \dots \\ \vdots & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} V_{x_1x_1} & V_{x_1x_2} & V_{x_1x_3} & \dots \\ V_{x_2x_1} & V_{x_2x_2} & V_{x_2x_3} & \dots \\ V_{x_3x_1} & V_{x_3x_2} & V_{x_3x_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \overline{f(x_1)} & 0 & 0 & \dots \\ 0 & \overline{f(x_2)} & 0 & \dots \\ 0 & 0 & \overline{f(x_3)} & \dots \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} f(x_1)V_{x_1x_1}\overline{f(x_1)} & f(x_1)V_{x_1x_2}\overline{f(x_2)} & f(x_1)V_{x_1x_3}\overline{f(x_3)} & \dots \\ f(x_2)V_{x_2x_1}\overline{f(x_1)} & f(x_2)V_{x_2x_2}\overline{f(x_2)} & f(x_2)V_{x_2x_3}\overline{f(x_3)} & \dots \\ f(x_3)V_{x_3x_1}\overline{f(x_1)} & f(x_3)V_{x_3x_2}\overline{f(x_2)} & f(x_3)V_{x_3x_3}\overline{f(x_3)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{4.21}$$

as in (4.7), and that V_F and D_F are the *finite* submatrices of V and D obtained by taking only the rows and columns corresponding to those vertices x which lie in the finite subset $F \subseteq X$. The limit of the filter $\{T_F\}_{F\subseteq X}$ of the operators defined in (4.22) will be computed in Corollary 4.18.

THEOREM 4.13. *The multiplication operator $M = M_f$ is bounded on $\mathcal{H}_\mathcal{E}$ if and only if the family of operators*

$$T_F = V_F^{1/2} D_F V_F^{-1/2} \tag{4.22}$$

is uniformly bounded, i.e., there exists a constant $b < \infty$ such that $\sup_F \|T_F\|_{\mathbb{V}\to\mathbb{V}} \leqslant b$, as F ranges over all finite subsets of X. Here V_F is the truncation of (4.3) *with entries $(V_{xy})_{x,y\in F}$, and D_F is the truncated diagonal operator with entries $(f(x)\delta_{x,y})_{x,y\in F}$.*

In the case when these equivalent conditions are satisfied,

$$\|M_f\|_{\mathcal{H}_\mathcal{E}\to\mathcal{H}_\mathcal{E}} = \sup_F \|T_F\|_{\mathbb{V}\to\mathbb{V}} \leqslant b, \tag{4.23}$$

where the sup *is taken over all finite subsets $F \subseteq X$.*

Proof. From Theorem 3.11, we know that M is bounded if and only if $s_f(x, y)$ in (3.11) is semidefinite, and this inequality can be written in terms of matrices as $b^2V - DVD \geqslant 0$, with respect to the ordering (3.4); see Lemma 3.4. This transforms a difficult condition (positive semidefiniteness) into an easier condition to check:

$$b^2\langle \xi, V_F\xi\rangle_\mathbb{V} - \langle \xi, D_F V_F D_F \xi\rangle_\mathbb{V} \geqslant 0, \quad \forall \xi \in \mathbb{V}. \tag{4.24}$$

Note that V is psd (essentially by definition):

$$\sum_{x,y\in F} \overline{\xi}_x \xi_y V_{x,y} = \sum_{x,y\in F} \overline{\xi}_x \xi_y \langle v_x, v_y\rangle_\mathcal{E} = \Big\langle \sum_{x\in F} \xi_x v_x, \sum_{y\in F} \xi_y v_y \Big\rangle_\mathcal{E} = \Big\| \sum_{x\in F} \xi_x v_x \Big\|_\mathcal{E}^2 \geqslant 0,$$

and so we have $V = (V^{1/2})^2$ by Lemma 3.3. Then (4.24) gives

$$\|V_F^{1/2} D_F \xi\|_\mathbb{V}^2 = \|T_F V_F^{1/2}\xi\|_\mathbb{V}^2 \leqslant b^2 \|V_F^{1/2}\xi\|_\mathbb{V}^2 \quad \forall \xi \in \mathbb{V}. \tag{4.25}$$

Thus there is a bounded operator sending $V_F^{1/2}\xi$ to $V_F^{1/2}D_F\xi$, for any $\xi \in \mathbb{V}$. Less grandiosely, this means there is an $n \times n$ matrix T_F satisfying

$$T_F V_F^{1/2} = V_F^{1/2} D_F, \quad \text{and} \quad \|T_F\|_{\mathbb{V}\to\mathbb{V}} \leqslant b. \tag{4.26}$$

From (4.26), it is clear that T_F is given by (4.22), and the independence of b from F follows by the Uniform Boundedness principle. ∎

REMARK 4.14. Note that T_F is not self-adjoint for general finite F (even in the case when f is $\mathbb{R}$-valued) because

$$(V_F^{1/2} D_F V_F^{-1/2})^* = V_F^{-1/2} D_F V_F^{1/2}.$$

However, one can still compute the operator norm of T_F as the square root of the largest eigenvalue of $T_F^* T_F$.

REMARK 4.15. Even in the case when $M_z = M_{\delta_z}$, it may be very difficult to use (4.22) to compute $\|M_z\|$, and preferable to use Theorem 4.4 instead. In this situation, one has only

$$T_F = V_F^{1/2} (\delta_{x,z}\delta_{y,z}) V_F^{-1/2},$$

but it is even difficult to compute the entries of $V_F^{1/2}$ and $V_F^{-1/2}$.

Our next goal is to compute the limit of the filter $\{T_F\}_{F\subseteq X}$ in Corollary 4.18, where the ordering is the usual partial order of set containment on the finite sets F. However, this will require some futher discussion of $\mathbb{V}$ from Definition 4.5.

DEFINITION 4.16. Given a finite subset $F \subseteq X$, define P_F to be the projection to the subspace spanned by $\{v_x : x \in F\}$.

The purpose of J in the following lemma is that it serves to intertwine M_f with a more computable operator, see (4.30) in the corollary below, and also (4.31). Recall that $\mathbb{V}$ is defined in Definition 4.2 and discussed in Remark 4.3.

LEMMA 4.17. *A unitary equivalence between* $\mathbb{V}$ *and* $\mathcal{H}_\mathcal{E}$ *is given by the operator*

$$J : \mathcal{H}_\mathcal{E} \to \mathbb{V} \quad \text{by} \quad Ju = V^{1/2}\xi, \quad \text{for } u = \sum_{x\in X} \xi_x v_x, \tag{4.27}$$

where convergence of the sum in (4.27) *is with respect to* $\mathcal{E}$.

Proof. Let $u, w \in \operatorname{span}\{v_x\}_{x\in F}$ be given by

$$u = \sum_{x\in F} \xi_x v_x \quad \text{and} \quad w = \sum_{x\in F} \eta_x v_x, \tag{4.28}$$

where F is some finite subset of X. Then (4.4) gives

$$\langle u, w\rangle_\mathcal{E} = \Big\langle \sum_{x\in F} \xi_x v_x, \sum_{y\in X} \eta_y v_y \Big\rangle_\mathcal{E} = \sum_{x,y\in F} \overline{\xi}_x \eta_y \langle v_x, v_y\rangle_\mathcal{E} = \langle \xi, \eta\rangle_\mathbb{V}. \tag{4.29}$$

Now for general $u, w \in \mathcal{H}_\mathcal{E}$, let P_F be as in Definition 4.16, and compute

$$\langle u, w\rangle_\mathcal{E} = \lim_{F\to X} \langle P_F u, P_F w\rangle_\mathcal{E} = \lim_{F\to X} \langle V_F^{1/2}\xi, V_F^{1/2}\eta\rangle_\mathbb{V} = \langle V^{1/2}\xi, V^{1/2}\eta\rangle_\mathbb{V},$$

where the middle equality comes by (4.29). ∎

COROLLARY 4.18. *Let* T_F *be defined as in* (4.22)*, and let* J *be defined as in* (4.27)*. In the case when the equivalent conditions of Theorem* 4.13 *are satisfied, one has*

$$T = \lim_{F\to X} T_F = J M_f^* J^*, \quad \text{and} \quad T^* = \lim_{F\to X} T_F^* = J M_f J^*, \tag{4.30}$$

where M_f^* *is the adjoint with respect to* $\mathcal{E}$*,* T^* *is the adjoint with respect to* $\mathbb{V}$*, and the limit is taken in the strong operator topology. Thus,* $M_f^* \cong \lim_{F\to X} T_F$.

Proof. To see (4.30), first pick a finite $F \subseteq X$ and with P_F as in Definition 4.16,

$$\begin{aligned} \|M_f^* P_F u\|_\mathcal{E}^2 &= \langle P_F u, M_f M_f^* P_F u\rangle_\mathcal{E} = \sum_{x,y\in F} \overline{\xi}_x \xi_y f(x)\overline{f(y)} V_{xy} \\ &= \langle \xi, \overline{D_F} V_F D_F \xi\rangle_\mathbb{V} = \|V_F^{1/2} D_F \xi\|_\mathbb{V}^2. \end{aligned}$$

However, (4.22) means that $T_F V_F^{1/2} = V_F^{1/2} D_F$, and so the computation continues with

$$\|M_f^* P_F u\|_\mathcal{E}^2 = \|V_F^{1/2} D_F \xi\|_\mathbb{V}^2 = \|T_F V_F^{1/2} \xi\|_\mathbb{V}^2.$$

Now let $F \to X$ on both sides, and the proof follows by Theorem 4.13. ∎

Consequently, one has a commutative square

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{E}} & \xrightarrow{M_f^*} & \mathcal{H}_{\mathcal{E}} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathbb{V} & \xrightarrow{T} & \mathbb{V} \end{array} \tag{4.31}$$

In light of Remark 4.15, it will be helpful to have a condition which is only sufficient to ensure the boundedness of M_f (not necessary), but is much easier to check.

THEOREM 4.19. *The operator $M = M_f$ satisfies*

$$\|M_f\|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} \leqslant \sum_{x \in X} |f(x)| \sqrt{c(x) R(x)} = \sum_{x \in X} \frac{|f(x)|}{\sqrt{\mathbb{P}[x \to o]}}, \tag{4.32}$$

and is hence a bounded operator on $\mathcal{H}_{\mathcal{E}}$ whenever the right side of (4.32) *converges.*

Proof. For $F \subseteq X$ finite, let $f|_F = f \chi_F$ be the restriction of f to F. Then Theorem 4.4 and (4.12) give

$$M_{f|_F} = \sum_{x \in F} f(x) M_x = \sum_{x \in F} f(x) |\delta_x\rangle\langle v_x|, \tag{4.33}$$

where the summation is finite, so that $M_{f|_F}$ is clearly bounded. Now we show that $M_{f|_F}$ converges to M_f in norm, as $F \to X$. Since

$$\begin{aligned} \|M_{f|_F}\|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} &\leqslant \sum_{x \in F} |f(x)| \, \| |\delta_x\rangle\langle v_x| \|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} = \sum_{x \in F} |f(x)| \, \|\delta_x\|_{\mathcal{E}} \|v_x\|_{\mathcal{E}} \\ &= \sum_{x \in F} |f(x)| \sqrt{c(x) R(x)}, \end{aligned}$$

we have (4.32). Moreover, when the right side of (4.32) converges, then for any $\varepsilon > 0$ there exists an F_0 such that

$$\sum_{x \in X \setminus F_0} |f(x)| \sqrt{c(x) R(x)} < \varepsilon,$$

which shows that $\lim_{F \to X} \|M_{f|_F} - M_f\|_{\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}} = 0$, and (2.9) completes the proof. ∎

One result appearing in the proof of Theorem 4.19 will be helpful on its own.

COROLLARY 4.20. *If M_f satisfies* (4.32), *then $M_{f|_F}$ converges to M_f in norm, where $M_{f|_F}$ is as in* (4.33). *In particular,* (4.32) *implies*

$$M_f = \sum_{x \in x} f(x) M_x = \sum_{x \in X} f(x) |\delta_x\rangle\langle v_x|, \tag{4.34}$$

where the sum converges in the norm operator topology.

It seems doubtful that $M_{f|_F}$ converges to M_f in norm, in general. However, we do have a partial result in this direction, in Theorem 4.22.

LEMMA 4.21. *Let* $\{F_n\}_{n=1}^{\infty}$ *be an exhaustion of* X*, and define* P_n *to be the projection to* $\operatorname{span}\{v_x : x \in F_n\}$*, for each* $n \in \mathbb{N}$*. If* M_f *is bounded, then for each* n*, there is an* $m = m_n$ *with*

$$P_n M_f P_n = P_n M_{f_{m_n}} P_n, \tag{4.35}$$

where $f_k = f|_{F_k} = f\chi_{F_k}$ *is the restriction of* f *to* F_k*.*

Proof. Since the energy kernel has dense span in $\mathcal{H}_{\mathcal{E}}$, we can apply the Gram–Schmidt algorithm to obtain an onb $\{\varepsilon_x\}_{x\in X}$. (This is carried out in more detail in Section 3.1 of [22]. Note that $v_o = v_{x_0}$ is *not* included in the enumeration.) Thus we can write

$$\begin{aligned} P_n M_f P_n &= \sum_x \sum_{y \leqslant x} \sum_{z \leqslant y} f(x) |\varepsilon_y\rangle\langle\varepsilon_y|\, |\delta_x\rangle\langle v_x|\, |\varepsilon_z\rangle\langle\varepsilon_z| \\ &= \sum_x \sum_{y \leqslant x} \sum_{z \leqslant y} f(x) \langle\varepsilon_y, \delta_x\rangle_{\mathcal{E}} \langle v_x, \varepsilon_z\rangle_{\mathcal{E}} |\varepsilon_y\rangle\langle\varepsilon_z|. \end{aligned} \tag{4.36}$$

However, for all n, there exists an $m \geqslant n$ (which we write as m_n to emphasize the dependence on n) such that, for $x \in F_n$ and $y, z \in F_{m_n}^C$, one has

$$\langle\varepsilon_y, \delta_x\rangle_{\mathcal{E}} = \langle v_x, \varepsilon_z\rangle_{\mathcal{E}} = 0.$$

This essentially follows from the finite range of c and the nature of the Gram–Schmidt algorithm and shows that the sum in (4.36) is finite. ∎

THEOREM 4.22. *Let* $\{F_n\}_{n=1}^{\infty}$ *be an exhaustion of* X*, and for a fixed* $f : X \to \mathbb{C}$*, and let* $f_n = f|_{F_n} = f\chi_{F_n}$ *be the restriction of* f *to* F_n*. If* M_f *is bounded, then* $P_n M_{f_{m_n}} P_n$ *converges to* M_f *in the strong operator topology, where* $M_{f_{m_n}}$ *is a finite-dimensional suboperator of* M_f *as in Lemma* 4.21*.*

Proof. Note that $P_n M P_n$ converges strongly to M whenever M is a bounded operator, by general operator theory. Then by Lemma 4.21, the right side of (4.35) also converges to M_f in the strong operator topology. ∎

COROLLARY 4.23. *If* M_f *is bounded, then the range of* M_f *lies in* $\mathcal{F}in$*.*

Proof. Since $M_x = |\delta_x\rangle\langle v_x|$ by (4.12), and $\operatorname{ran} M_x \subseteq \mathbb{C}\delta_x$, this follows immediately from Theorem 4.22. ∎

5. BOUNDED FUNCTIONS OF FINITE ENERGY

In the preceding section, we considered the functions f for which M_f is a bounded operator. In this section, we consider the algebra of bounded functions f in $\mathcal{H}_{\mathcal{E}}$. Neither of these spaces of operators is contained in the other, as illustrated in the examples of Section 6.

DEFINITION 5.1. For $u \in \mathcal{H}_{\mathcal{E}}$, denote $\|u\|_\infty := \sup_{x \in G} |u(x) - u(o)|$, and say u is *bounded* if and only if $\|u\|_\infty < \infty$.

In [22], we give two ways of constructing a Gel'fand triple $\mathcal{S}_G \subseteq \mathcal{H}_{\mathcal{E}} \subseteq \mathcal{S}'_G$ for the energy space. Here $\mathcal{S}_G$ is a dense subspace of $\mathcal{H}$ which should be thought of as a space of test functions. Indeed, $\mathcal{S}_G$ is equipped with a strictly finer "test function topology" given by a countable system of seminorms; this yields a Fréchet topology which is strictly finer than the norm topology on $\mathcal{H}_{\mathcal{E}}$. Then $\mathcal{S}'_G$ is the dual of $\mathcal{S}_G$ with respect to this finer (Fréchet) topology, so that one obtains a strict containment $\mathcal{H} \subsetneq \mathcal{S}'_G$. In fact, it is possible to chose the seminorms in such a way that the inclusion map of $\mathcal{S}_G$ into $\mathcal{H}$ is a nuclear operator.

To make all this concrete, let us briefly describe the two constructions given in [22].

(i) Fix an enumeration of the vertices, and apply the Gram–Schmidt procedure (as in the proof of Lemma 4.21) to $\{v_{x_n}\}_{n=1}^\infty$ to obtain an orthonormal basis $\{\varepsilon_n\}_{n=1}^\infty$. Then define $\mathcal{S}_G = \bigcap_{p \in \mathbb{N}} \{s : \|s\|_p < \infty\}$, where the Fréchet *p-seminorm* of $s = \sum_{n \in \mathbb{N}} s_n \varepsilon_n$ is given by

$$\|s\|_p := \Big(\sum_{n \in \mathbb{N}} n^p |s_n|^2 \Big)^{1/2}, \quad s \in \mathcal{S}_G, p \in \mathbb{N}. \tag{5.1}$$

(ii) In the case where Δ is an unbounded operator on $\mathcal{H}_{\mathcal{E}}$, let $\mathbb{\Delta}$ be a self-adjoint extension of Δ and define $\mathcal{S}_G := \operatorname{dom}(\mathbb{\Delta}^\infty) = \bigcap_{p=1}^\infty \operatorname{dom}(\mathbb{\Delta}^p)$, with Fréchet p-seminorms $\|u\|_p := \|\mathbb{\Delta}^p u\|_{\mathcal{E}}$. (Details regarding the precise domain of Δ and $\mathbb{\Delta}$ in this context may be found in [22].)

Either way, it turns out that $\mathcal{S}'_G$ is large enough to support a nice probability measure, even though $\mathcal{H}$ is not. This allows one to establish an isometric embedding of $\mathcal{H}_{\mathcal{E}}$ into the Hilbert space $L^2(\mathcal{S}'_G, \mathbb{P})$, where $\mathbb{P}$ is a Gaussian probability measure on $\mathcal{S}'_G$.

THEOREM 5.2 (Wiener embedding ([22], Theorem 5.2)). *The Wiener transform $\mathcal{W} : \mathcal{H}_{\mathcal{E}} \to L^2(\mathcal{S}'_G, \mathbb{P})$ defined by*

$$\mathcal{W} : v \mapsto \widetilde{v}, \quad \widetilde{v}(\xi) := \langle v, \xi \rangle_{\mathcal{W}}, \tag{5.2}$$

is an isometry. (The inner product in (5.2) *is the extension of the energy inner product given by the Gel'fand triple; if $\xi \in \mathcal{H}_{\mathcal{E}}$, then $\langle v, \xi \rangle_{\mathcal{W}} = \langle v, \xi \rangle_{\mathcal{E}}$. Formulas for computing $\langle v, \xi \rangle_{\mathcal{W}}$ for general $\xi \in \mathcal{S}'_G$ are given in Theorem* 4.3 *and Lemma* 4.4 *of* [22].*) The extended reproducing kernel $\{\widetilde{v}_x\}_{x \in G}$ is a system of Gaussian random variables from which one can obtain the free effective resistance (see Definition* 2.14*) by*

$$R^{\mathrm{F}}(x,y) = \mathcal{E}(v_x - v_y) = \mathbb{E}_\xi((\widetilde{v}_x - \widetilde{v}_y)^2). \tag{5.3}$$

Moreover, for any $u, v \in \mathcal{H}_\mathcal{E}$, the energy inner product extends directly as

$$\langle u, v\rangle_\mathcal{E} = \mathbb{E}_\xi(\overline{\widetilde{u}}\widetilde{v}) = \int_{\mathcal{S}'_G} \overline{\widetilde{u}}\widetilde{v}\, d\mathbb{P}. \tag{5.4}$$

REMARK 5.3. The Wiener transform gives a representation of the Hilbert space $\mathcal{H}_\mathcal{E}$ as an L^2 space of functions on a probability "sample space" $(\mathcal{S}'_G, \mathbb{P})$. This is useful in many ways.

(i) While direct computation in $\mathcal{H}_\mathcal{E}$ is typically difficult (when solving equations, for example), passing to the transform allows us instead to convert geometric problems in $\mathcal{H}_\mathcal{E}$ into manipulation of functions on $\mathcal{S}'_G$ or on a subspace of it.

(ii) As we show in this section, problems involving bounded operators in $\mathcal{H}_\mathcal{E}$ can be subtle. The Wiener transform immediately offers a maximal abelian algebra of bounded operators, viz., multiplication by L^∞ functions on $\mathcal{S}'_G$. (The multiplication operator on $\mathcal{H}_\mathcal{E}$ "before the transform", discussed in Section 3–Section 4, should not be confused with those in $L^2(\mathcal{S}'_G, \mathbb{P})$ "after the transform".)

DEFINITION 5.4. Denote the collection of bounded functions of finite energy by

$$\mathcal{A}_\mathcal{E} := \{u \in \mathcal{H}_\mathcal{E} \,\vdots\, u \text{ is bounded}\}. \tag{5.5}$$

Define multiplication on $\mathcal{A}_\mathcal{E}$ by the pointwise product

$$(u_1u_2)(x) := u_1(x)u_2(x), \tag{5.6}$$

and a norm on $\mathcal{A}_\mathcal{E}$ by

$$\|u\|_\mathcal{A} := \|u\|_\infty + \|u\|_\mathcal{E}. \tag{5.7}$$

LEMMA 5.5. *$(\mathcal{A}_\mathcal{E}, \|\cdot\|_\mathcal{A})$ is a Banach algebra.*

Proof. It is obvious that u_1u_2 is bounded; one checks that $u_1u_2 \in \operatorname{dom}\mathcal{E}$ by directly computing:

$$\begin{aligned}
\|u_1u_2\|_\mathcal{E}^2 &= \frac{1}{2}\sum_{x,y} c_{xy}|u_1u_2(x) - u_1u_2(y)|^2 \\
&= \frac{1}{2}\sum_{x,y} c_{xy}|(u_1(x) - u_1(y))u_2(x) + (u_2(x) - u_2(y))u_1(x)|^2 \\
&\leqslant \frac{1}{2}\sum_{x,y} c_{xy}(|u_1(x) - u_1(y)||u_2(x)| + |u_2(x) - u_2(y)||u_1(x)|)^2
\end{aligned}$$

$$= \frac{1}{2}\sum_{x,y} c_{xy}|u_1(x) - u_1(y)|^2|u_2(x)|^2 + \frac{1}{2}\sum_{x,y} c_{xy}|u_2(x) - u_2(y)|^2|u_1(x)|^2$$
$$+ \frac{1}{2}\sum_{x,y} c_{xy}|u_1(x)||u_2(x)||u_1(x) - u_1(y)||u_2(x) - u_2(y)|$$
$$\leqslant \|u_2^2\|_\infty \|u_1\|_{\mathcal{E}}^2 + 2\|u_1\|_\infty \|u_2\|_\infty |\langle u_1, u_2\rangle_{\mathcal{E}}| + \|u_1^2\|_\infty \|u_2\|_{\mathcal{E}}^2,$$

which is clearly finite. This estimate also implies that $(u, v) \mapsto \|uv\|_{\mathcal{A}}$ is a closed linear functional on the product space $\mathcal{A}_{\mathcal{E}} \times \mathcal{A}_{\mathcal{E}}$. The closed graph theorem then implies that it is continuous, i.e.,

$$\|uv\|_{\mathcal{A}} \leqslant C\|u\|_{\mathcal{A}}\|v\|_{\mathcal{A}}, \quad \text{for all } u, v \in \mathcal{A}_{\mathcal{E}}.$$

It is a standard argument that one can then find an equivalent norm for which the same inequality holds with $C = 1$; see [23], for example. ∎

DEFINITION 5.6. By the *Gel'fand space* of a Banach algebra $\mathcal{A}$, we mean the spectrum $\operatorname{spec}(\mathcal{A})$ realized as either the collection of maximal ideals of $\mathcal{A}$ or as the collection of multiplicative linear functionals on $\mathcal{A}$. See [6], [7].

Let $\zeta \in \operatorname{spec}(\mathcal{A}_{\mathcal{E}})$ denote a multiplicative linear functional on $\mathcal{A}_{\mathcal{E}}$, so that $\ker \zeta$ is a maximal ideal of $\mathcal{A}_{\mathcal{E}}$, and let $\Phi_{\mathcal{A}} : \mathcal{A}_{\mathcal{E}} \to C(\operatorname{spec}(\mathcal{A}_{\mathcal{E}}))$ denote the Gel'fand transform, so that $\Phi_{\mathcal{A}}(v)(\zeta) := \zeta(v)$.

There is a norm equivalent to the one given in (5.7) with respect to which $\mathcal{A}_{\mathcal{E}}$ becomes a Banach algebra (see [23], e.g.), and we are concerned with the Gel'fand space of this one.

LEMMA 5.7. *As a Banach algebra, $\mathcal{A}_{\mathcal{E}}$ is isometrically isomorphic to $C(\operatorname{spec}(\mathcal{A}_{\mathcal{E}}))$.*

Proof. We need to show that $\ker \Phi_{\mathcal{A}} = 0$. This is equivalent to showing that $\mathcal{A}_{\mathcal{E}}$ is *semisimple*, i.e., that the intersection of all the maximal ideals is 0. It therefore suffices to show that an intersection of a subcollection of the maximal ideals is 0. Let L_x denote the multiplicative linear functional defined by $L_x u := u(x)$. Since $L_x u = \langle u, v_x\rangle_{\mathcal{E}}$ under convention (3.1), and $\{v_x\}$ is dense in $\mathcal{H}_{\mathcal{E}}$ (and therefore total), it follows that $\bigcap \ker L_x = 0$. ∎

DEFINITION 5.8. Recall from Definition 2.9 that $\operatorname{span}\{\delta_x\}$ is the collection of functions of finite support; see also the first paragraph of Section 3. If we complete $\operatorname{span}\{\delta_x\}$ in the sup norm, we obtain the collection of bounded functions on G, and if we complete in $\mathcal{E}$, we obtain $\mathcal{F}in$. Therefore, the closure of $\operatorname{span}\{\delta_x\}$ in the norm of $\mathcal{A}_{\mathcal{E}}$ is

$$\mathcal{A}_{\mathcal{F}in} := \mathcal{F}in \cap \mathcal{A}_{\mathcal{E}}. \tag{5.8}$$

LEMMA 5.9. *$\mathcal{A}_{\mathcal{F}in}$ is a closed ideal in $\mathcal{A}_{\mathcal{E}}$.*

Proof. Fix $x \in G$ and let $\delta_x \in \mathcal{A}_{\mathcal{F}in}$ be the characteristic function of $\{x\}$ as defined in Definition 2.9. Take any finite set $F \subseteq G$ and any linear combination

$f = \sum_{x \in F} \xi_x \delta_x$. Since $v \cdot \delta_x = v(x)\delta_x$, one has $v \cdot f = \sum_{x \in F} \xi_x v(x)\delta_x$, which is clearly supported in F again. This shows that the collection of all finitely supported functions on G is an ideal.

Now for $f \in \mathcal{A}_{\mathcal{F}in}$, take $\{f_n\}$ where each f_n has finite support and $\|f - f_n\|_{\mathcal{A}} \to 0$. This is possible in view of Definition 5.8. Since $v \cdot f_n \in \operatorname{span}\{\delta_x\}$ by the first part,

$$\|(v \cdot f) - (v \cdot f_n)\|_{\mathcal{A}} = \|v \cdot (f - f_n)\|_{\mathcal{A}} \leqslant \|v\|_{\mathcal{A}} \|f - f_n\|_{\mathcal{A}} \to 0, \tag{5.9}$$

shows $v \cdot f \in \mathcal{A}_{\mathcal{F}in}$ (by Definition 5.8 again). ∎

DEFINITION 5.10. Since $\mathcal{A}_{\mathcal{F}in}$ is a closed ideal, it is standard that

$$\mathcal{A}_{\mathcal{H}arm} := \mathcal{A}_{\mathcal{E}} / \mathcal{A}_{\mathcal{F}in} \tag{5.10}$$

is an algebra, and in fact a Banach algebra under the usual norm

$$\|[u]\|_{\mathcal{H}arm} := \inf\{\|u + f\|_{\mathcal{A}} \,\vdots\, f \in \mathcal{B}_{\mathcal{F}in}\}. \tag{5.11}$$

THEOREM 5.11. *The Gel'fand space of $\mathcal{A}_{\mathcal{H}arm}$ contains* $\operatorname{bd} G$*, and there is an isometric embedding $\mathcal{A}_{\mathcal{H}arm} \hookrightarrow C(\operatorname{bd} G)$.*

Proof. Recall from Corollary 4.5 of [22] that for $v_x \in \mathcal{H}_{\mathcal{E}}$, one defines $\widetilde{v}_x \in L^2(\mathcal{S}'_G, \mathbb{P})$ by

$$\widetilde{v}_x(\xi) = \langle v_x, \xi \rangle_{\mathcal{G}} = \lim_{n \to \infty} \langle v_{x,n}, \xi \rangle_{\mathcal{E}}, \tag{5.12}$$

where $\{v_{x,n}\}_{n \in \mathbb{N}}$ is any sequence in $\mathcal{S}_G$ converging to v_x, and that with this extension, harmonic functions in $\mathcal{H}_{\mathcal{E}}$ have the boundary representation

$$h(x) = \int_{\mathcal{S}'_G} \widetilde{v}_x(\xi) \frac{\partial \widetilde{h}}{\partial n}(\xi) \, d\mathbb{P}(\xi) + h(o). \tag{5.13}$$

(Full details on this notation may be found in [22] or [19].) It follows immediately from this representation that if $\widetilde{h} = 0$ on $\operatorname{bd} G$, then $h = 0$ everywhere on G. (Here, $\widetilde{h} = 0$ on $\operatorname{bd} G$ means $\lim_{n \to \infty} h(x_n) = k$ for some $k \in \mathbb{C}$ and any sequence $\{x_n\}$ with $\lim_{n \to \infty} x_n = \infty$.)

Given any $\beta \in \operatorname{bd} G$, the evaluation $\chi_\beta(u) := \widetilde{u}(\beta)$ defines a multiplicative linear functional on $\mathcal{A}_{\mathcal{H}arm}$, so that $\operatorname{bd} G$ is contained in the Gel'fand space of $\mathcal{A}_{\mathcal{H}arm}$. ∎

THEOREM 5.12. *If $\mathcal{H}arm = 0$, then the Gel'fand space of $\mathcal{A}_{\mathcal{E}}$ is $G \cup \{\infty\}$.*

Proof. Let $\chi \in \operatorname{spec}(\mathcal{A}_{\mathcal{E}})$ and apply it to both sides of $v \cdot \delta_x = v(x)\delta_x$ (the left side is a product in $\mathcal{A}_{\mathcal{E}}$ and the right side is a scalar multiple of δ_x) to obtain $\chi(v) \cdot \chi(\delta_x) = v(x)\chi(\delta_x)$, and hence

$$\chi(\delta_x) \cdot (\chi(v) - v(x)) = 0, \quad \forall x \in G, \forall v \in \mathcal{A}_{\mathcal{E}}. \tag{5.14}$$

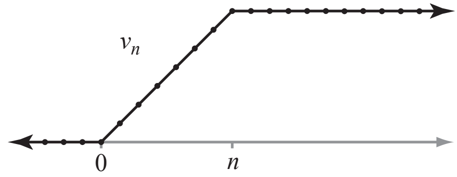


FIGURE 1. The energy kernel element v_n on the integer network $(\mathbb{Z}, \mathbf{1})$.

This implies (i) $\chi(\delta_x) = 0$ for all x, or else (ii) $\exists y \in G$ for which $\chi(\delta_y) \neq 0$. Since $\mathcal{H}arm = 0$, Theorem 2.12 implies that χ is determined by its action on $\{\delta_x\}_{x \in G}$. Thus, only the zero functional satisfies $\chi(\delta_x) = 0$ for all $x \in G$, and we may safely ignore case (i). For case (ii), it follows that $\chi(\delta_x) = 0$ for all $x \neq y$, so $\chi(v) = v(y)$ by (5.14). This shows that χ corresponds to evaluation at the vertex y; note that the uniqueness of y for which $\chi(\delta_y) \neq 0$ is implicit.

Observe that $C(G)$ is not unital, because the constant function $\mathbf{1} \simeq 0$ in $\mathcal{H}_\mathcal{E}$. We unitalize $\mathcal{A}_\mathcal{E}$ in the usual way:

$$\widetilde{\mathcal{A}_\mathcal{E}} = \mathcal{A}_\mathcal{E} \times \mathbb{C} \quad \text{with} \quad (a_1, \lambda_1)(a_1, \lambda_1) := (a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2). \tag{5.15}$$

The unit in this new algebra is then $(0, 1)$. By standard theory, this corresponds to taking the one-point compactification of G. ∎

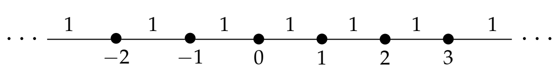
Roughly speaking, taking the one-point compactification of G corresponds to conjoining the single multiplicative linear functional "evaluation at ∞" to $\mathcal{A}_\mathcal{E}$. It is known from [3] that when $\mathcal{H}arm = 0$, $u(x)$ tends to a common value along $\mathbb{P}$-a.e. path to ∞, for any $u \in \mathcal{H}_\mathcal{E}$.

CONJECTURE 5.13. We conjecture that the converses of Theorem 5.11 and of Theorem 5.12 both hold. In other words, we expect that $\mathcal{A}_{\mathcal{H}arm} \cong C(\operatorname{bd} G)$, and that if $\mathcal{H}arm \neq 0$, then the Gel'fand space of $\mathcal{A}_\mathcal{E}$ contains at least two elements that don't correspond to any vertex of G.

6. EXAMPLES

The following example shows that even though v_x is a bounded function on any network (Lemma 2.16), the corresponding multiplication operator may not be bounded. This highlights the disparity between $C^*(\mathcal{H}_\mathcal{E})$ from Definition 4.5 and $\mathcal{A}_\mathcal{E}$ from Definition 5.4.

Consider the integer network with unit conductances $(\mathbb{Z}, \mathbf{1})$:



We label the vertex x_n by "n" to simplify notation. Then if (4.32) held, Corollary 4.20 would give

$$M_{v_n} = |\delta_1\rangle\langle v_1| + 2|\delta_2\rangle\langle v_2| + \cdots + n|\delta_n\rangle\langle v_n| + n|\delta_{n+1}\rangle\langle v_{n+1}| + \cdots,$$

for each fixed n. The operator norm corresponding to one of these terms is

$$\|n|\delta_{n+k}\rangle\langle v_{n+k}|\| = n\|\delta_{n+k}\|_{\mathcal{E}}\|v_{n+k}\|_{\mathcal{E}} = n\sqrt{2}\sqrt{n+k} \xrightarrow{k\to\infty} \infty,$$

so clearly (4.32) cannot hold.

Checking Theorem 4.13 directly is harder; one must compute

$$\|M_{v_n}\|_{\mathcal{H}_{\mathcal{E}}\to\mathcal{H}_{\mathcal{E}}} = \sup_F \|V_F^{1/2} D_F V_F^{-1/2}\|_{\ell^2\to\ell^2},$$

where the latter is the operator norm on $\ell^2(F)$ and F ranges over all finite subsets of X. For our purposes, it will suffice to consider sets F of the form $F = \{1, 2, \dots, n\}$. The matrix for V_F is then

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & & & \\ 1 & 2 & 2 & 2 & \dots & & & \\ 1 & 2 & 3 & 3 & \dots & & & \\ 1 & 2 & 3 & 4 & \dots & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & \dots & n-2 & n-2 & n-2 \\ & & & & \dots & n-2 & n-1 & n-1 \\ & & & & \dots & n-2 & n-1 & n \end{bmatrix},$$

but $V^{-1/2}$ is a complicated even for small F. For example, for $V_F = \left[\begin{smallmatrix} 1&1&1\\1&2&2\\1&2&3 \end{smallmatrix}\right]$, one has

$$V_F^{1/2} = \begin{bmatrix} [\![-8-28\gamma+49\gamma^3, 3]\!] & [\![8-28\gamma+49\gamma^3, 2]\!] & [\![1+7\gamma-49\gamma^2+49\gamma^3, 2]\!] \\ [\![8-28\gamma+49\gamma^3, 2]\!] & [\![13-21\gamma-49\gamma^2+49\gamma^3, 3]\!] & 1+[\![-8+98\gamma^2+49\gamma^3, 2]\!] \\ [\![1+7\gamma-49\gamma^2+49\gamma^3, 2]\!] & [\![41-49\gamma-49\gamma^2+49\gamma^3, 2]\!] & [\![97-105\gamma-49\gamma^2+49\gamma^3, 3]\!] \end{bmatrix},$$

where $[\![p(\gamma), k]\!]$ is the root of the polynomial $p(\gamma)$ closest to the number k.

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