J. OPERATOR THEORY
69:1(2013), 161–193
doi: 10.7900/jot.2010aug22.1949

THE L^p-FOURIER TRANSFORM ON LOCALLY COMPACT QUANTUM GROUPS

MARTIJN CASPERS

Communicated by Şerban Strătilă

ABSTRACT. Using interpolation properties of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra, we define an L^p-Fourier transform $1 \leqslant p \leqslant 2$ on locally compact quantum groups. We show that the Fourier transform determines a distinguished choice for the interpolation parameter as introduced by Izumi. We define a convolution product in the L^p-setting and show that the Fourier transform turns the convolution product into a product.

KEYWORDS: *Locally compact quantum groups, Fourier transform, interpolation spaces.*

MSC (2010): 20G42, 20N99, 47N99.

1. INTRODUCTION

The Fourier transform is one of the most powerful tools coming from abstract harmonic analysis. Many classical applications, in particular in the direction of L^p-spaces, can be found in for example [6]. Here we extend this tool by giving a definition of a Fourier transform on the non-commutative L^p-spaces associated with a locally compact quantum group. This gives a link between quantum groups and non-commutative measure theory.

Recall that the Fourier transform on locally compact abelian groups can be defined in an L^p-setting for p any real number between 1 and 2. This is done in the following way. Let G be a locally compact group and let $\widehat{G}$ be its Pontrjagin dual. For an L^1-function f on G, we define its Fourier transform $\widehat{f}$ to be the function on $\widehat{G}$, which is defined by

$$\widehat{f}(\pi) = \int f(x)\pi(x)\mathrm{d}_l x, \quad \pi \in \widehat{G}. \tag{1.1}$$

Then $\widehat{f}$ is a continuous function on $\widehat{G}$ vanishing at infinity. So we can consider this transform as a bounded map $\mathcal{F}_1 : L^1(G) \to L^\infty(\widehat{G})$. The Plancherel theorem

yields that if f is moreover an L^2-function on G, then $\widehat{f}$ is an L^2-function on $\widehat{G}$ and this map extends to a unitary map $\mathcal{F}_2 : L^2(G) \to L^2(\widehat{G})$.

It is known that the Fourier transform can be generalized in an L^p-setting by means of the Riesz–Thorin theorem, see [1]. The statement of this theorem directly implies the following. For any p, with $1 \leqslant p \leqslant 2$, the linear map $L^1(G) \cap L^2(G) \to L^2(\widehat{G}) \cap L^\infty(\widehat{G}) : f \mapsto \widehat{f}$ extends uniquely to a bounded map

$$\mathcal{F}_p : L^p(G) \to L^q(\widehat{G}), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This map $\mathcal{F}_p$ is known as the L^p-Fourier transform.

Quantum groups have been around for quite some time and have been studied in many different guises. From the 80'ies onwards quantum groups are studied in an operator algebraic approach. In particular, a satisfactory C^*-algebraic definition of a compact quantum group has been given by Woronowicz [21].

Around 2000 locally compact quantum groups were introduced by Kustermans and Vaes [13], [14], see also [19]. Their definitions include a C^*-algebraic one and a von Neumann algebraic one. Since their introduction many aspects of abstract harmonic analysis have been given a suitable extension for quantum groups. In particular, the Pontrjagin duality theorem has been generalized to locally compact quantum groups. In fact, this was one of the main motivations for their definition. So every locally compact quantum group admits a (Pontrjagin) dual quantum group such that the double dual is isomorphic to the originial quantum group.

Since we now have a von Neumann algebraic interpretation for quantum groups at hand, it is natural to ask if the L^p-Fourier transform can be defined in this context. This is for two reasons. First of all, this framework studies quantum groups in a measurable setting which appeals to a more general interest: what links can be found between on one hand non-commutative measure spaces, in particular non-commutative L^p-spaces, and on the other hand the theory of quantum groups. The L^p-Fourier transform studied in the present paper establishes such a link. Secondly, the existence of a Pontrjagin dual is always guaranteed in the Kustermans–Vaes setting. This is an essential ingredient for defining Fourier transforms.

The L^1- and L^2-Fourier transform already appear in the present theory of quantum groups. In fact, they are implicitly used to define duals of quantum groups. Let us comment on this.

First of all, the L^2-Fourier transform is implicitly used in the construction of the Pontrjagin dual of a (von Neumann algebraic) quantum group. For the classical case of a locally compact abelian group G, let $(L^\infty(G), \Delta_G)$ be the usual quantum group associated with it. Its dual is given by $(\mathcal{L}(G), \widehat{\Delta}_G)$, where $\mathcal{L}(G)$ is the group von Neumann algebra of G. This structure is spatially isomorphic to $(L^\infty(\widehat{G}), \Delta_{\widehat{G}})$ by means of the L^2-Fourier transform. That is $L^\infty(\widehat{G}) = \mathcal{F}_2 \mathcal{L}(G) \mathcal{F}_2^{-1}$ and similarly the coproduct and other concepts translate.

Secondly, in a paper by Van Daele [20] he explains how the Fourier transform should be interpreted on the algebraic level of quantum groups. In his concluding remarks he suggests to study this transform in the operator algebraic framework. Here, this investigation is carried out. We take Van Daele's definition, which agrees with the classical transform (1.1), as a starting point for defining an L^2-Fourier transform in the operator algebraic framework.

Finally, an operator algebraic interpretation of the Fourier transform can be found in [10]. The main ideas for our L^2-Fourier transform first appear here. However, the suggested Fourier transform ([10], Definition 3) is well-defined only if the Haar weights of a quantum group are states, i.e. if the quantum group is compact. In the more general situation, one has to give a more careful definition, which we work out in Section 5.

The present paper is related to a collection of papers studying module structures of L^p-spaces associated to the Fourier algebra of locally compact groups [3], [4], [5]. These papers are based on the theory of non-commutative L^p-spaces associated to arbitrary (not necessarily semi-finite) von Neumann algebras, which we recall below.

When dealing with these spaces, we are confronted with the following obstruction. For classical L^p-spaces associated with a measure space X, there is a clear understanding of the intersections of L^p-spaces by means of disjunction of sets. So $L^p(X) \cap L^{p'}(X)$ gives the intersection of $L^p(X)$ and $L^{p'}(X)$. For non-commutative L^p-spaces it is more difficult to find the intersection of two such spaces. In fact, there is a choice which determines the intersection and which depends on a *complex interpolation parameter* $z \in \mathbb{C}$. In [5] the parameter $z = -1/2$ is used, whereas [3] focuses on the case $z = 0$ in order to define module actions. In the final remarks of [4], it is questioned which parameter would fit best for quantum groups.

One of the results of the present paper is that to define an L^p-Fourier transform, one is obliged to choose the parameter $z = -1/2$. We also determine intersections of the L^1- and L^2-space and of the L^2- and L^∞-space associated with a von Neumann algebra for this parameter, which are natural spaces.

STRUCTURE OF THE PAPER. In Section 2 we recall the definition of non-commutative L^p-spaces and introduce the complex interpolation parameter $z \in \mathbb{C}$. We specialize the theory for $z = -1/2$ and introduce short hand notation. In Sections 3–6, we only work with L^p-spaces with respect to this parameter. The justification for this specialization is given in the final chapter.

As indicated, the study of the intersections of L^p-spaces becomes more intricate in the non-commutative setting. In Section 3 we determine the intersections of L^1- and L^2-space and of the L^2- and L^∞-space associated with a von Neumann algebra. These intersections turn out to be well-known spaces in the theory of quantum groups. This gives a confirmation that our choice for the interpolation

parameter made at the beginning is a natural one. Moreover, it gives the necessary ammunition to apply the reiteration theorem. We warn the reader that the contents of Section 3 are relatively technical and if one is more interested in Fourier theory on quantum groups, one can skip Section 3 at first reading.

Section 4 recalls the definition of a locally compact quantum group as given by Kustermans and Vaes. We give the von Neumann algebraic definition.

In Section 5 we define the L^p-Fourier transform. We start with the L^1- and L^2-theory and then obtain the L^p-Fourier transform ($1 \leqslant p \leqslant 2$) through the complex interpolation method, a method similar to the Riesz–Thorin theorem mentioned in the introduction.

In Section 6 we define a convolution product in the L^p-setting and show that the Fourier transform turns the convolution product into a product.

Finally, in Section 7, we prove that the interpolation parameter used in Sections 3–6 is distinguished. That is, we prove that given the L^2-Fourier transform, there is only one choice for the interpolation parameter that allows an L^p-Fourier transform. This justifies our choice for this parameter made in the beginning.

NOTATIONS AND CONVENTIONS. Throughout this paper, let M be a von Neumann algebra and φ a normal, semi-finite, faithful weight on M. We adopt the standard notations from [15]. So $\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$, $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$. We denote ∇, J, σ for the modular operator, modular conjugation and modular automorphism group associated with φ. We denote $\mathcal{T}_\varphi$ for the Tomita algebra defined by

$$\mathcal{T}_\varphi = \{x \in M : x \text{ is analytic with respect to } \sigma \text{ and } \forall z \in \mathbb{C} : \sigma_z(x) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\}.$$

Let $(\mathcal{H}, \pi, \Lambda)$ be the GNS-representation of M with respect to φ. Note that M can be considered as acting on $\mathcal{H}$ and therefore we omit the map π if possible. For $x \in \mathfrak{m}_\varphi$ and for $a \in M$ analytic with respect to σ, we have $ax \in \mathfrak{m}_\varphi$ and $xa \in \mathfrak{m}_\varphi$ and

$$\varphi(ax) = \varphi(x\sigma_{-\mathrm{i}}a).$$

For a subset $A \subseteq M$, we denote A^+ for the positive elements in A. Similarly, M_*^+ denotes the space of positive normal functionals on M. Let $\omega \in M_*$. We donote $\overline{\omega} \in M_*$ for the functional defined by $\overline{\omega}(x) = \overline{\omega(x^*)}, x \in M$. For $y \in M$, we denote $y\omega$ and ωy for the normal functionals defined respectively by $(y\omega)(x) = \omega(xy)$ and $(\omega y)(x) = \omega(yx)$ with $x \in M$. Inner products on a Hilbert spaces are linear in the first entry and anti-linear in the second. Suppose that M acts on a Hilbert space $\mathcal{H}$ and let $\xi, \eta \in \mathcal{H}$. We denote $\omega_{\xi,\eta}$ for the normal functional defined by $\omega_{\xi,\eta}(x) = \langle x\xi, \eta \rangle$. The character ι will always stand for the identity homomorphism. If x is a preclosed operator, we use $[x]$ for its closure.

2. NON-COMMUTATIVE L^p-SPACES

To any von Neumann algebra M, there is a way to associate a non-commutative L^p-space to it. In fact there are many ways to do this. If M is semi-finite, i.e. it admits a normal, semi-finite, faithful trace τ, then one can define $L^p(M)$ as the space of closed, densely defined operators x affiliated with M for which if $|x| = \int_{[0,\infty)} \lambda \mathrm{d}E_\lambda$ is the spectral decomposition of $|x|$, then

$$\|x\|_p := \Big(\sup_{n\in\mathbb{N}} \tau\Big(\int_{[0,n]} \lambda^p \mathrm{d}E_\lambda\Big)\Big)^{1/p} < \infty.$$

If M is abelian, we recover the classical spaces $L^p(X)$ for a certain measure space X.

Since the introduction of Tomita–Takesaki theory, L^p-spaces have been defined for von Neumann algebras that are not semi-finite. Definitions of non-commutative L^p-spaces have been given by Haagerup [7], [16], Hilsum [8], Terp [17] and Izumi [9]. The definitions can be shown te be equivalent. That is, the L^p-spaces obtained by the various definitions are isometrically isomorphic Banach spaces. For a good introduction to this theory, we refer to [16], where a comparison of Haagerup's definition and Hilsum's definition is made.

Here we mainly use Izumi's definition [9] which is abstract in nature. He defines L^p-spaces associated with M by means of the complex interpolation method; a method that admits a property that is reminiscent of the Riesz–Thorin theorem, see the introduction. It is for this reason that Izumi's definition is the most suitable context to work in.

A drawback of this context is that the more concrete approach of the other definitions is absent. Whenever it feels appropriate we comment on this.

2.1. THE COMPLEX INTERPOLATION METHOD.

We recall the complex interpolation method as explained in Section 4.1 of [1].

DEFINITION 2.1. Let E_0, E_1 be Banach spaces. The couple (E_0, E_1) is called a *compatible couple (of Banach spaces)* if E_0 and E_1 are continuously embedded into a Banach space E.

Note that we suppress E in the notation (E_0, E_1). We can consider the spaces $E_0 \cap E_1$ and $E_0 + E_1$ interpreted within E and equip them with norms

$$\begin{aligned} \|x\|_{E_0\cap E_1} &= \max\{\|x\|_{E_0}, \|x\|_{E_1}\}, && x \in E_0 \cap E_1, \\ \|x\|_{E_0+E_1} &= \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x_0 + x_1 = x\}, && x \in E_0 + E_1, \end{aligned}$$

which make them Banach spaces. In that case we can consider E_0 and E_1 as subspaces of $E_0 + E_1$.

DEFINITION 2.2. A morphism between two compatible couples (E_0, E_1) and (F_0, F_1) is a bounded map $T : E_0 + E_1 \to F_0 + F_1$ such that for any $j \in \{0, 1\}$, $T(E_j) \subseteq F_j$ and the restriction $T : E_j \to F_j$ is bounded.

REMARK 2.3. Let (E_0, E_1) and (F_0, F_1) be compatible couples. If $T_0 : E_0 \to F_0, T_1 : E_1 \to F_1$ are bounded maps such that T_0 and T_1 agree on $E_0 \cap E_1$, then we call T_0 and T_1 compatible morphisms. In this case, there is a unique bounded map $T : E_0 + E_1 \to F_0 + F_1$. This gives a way to find morphisms of compatible couples.

Now we describe the complex interpolation method. Let (E_0, E_1) be a compatible couple. Let $\mathcal{S} = \{z \in \mathbb{C} : 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$ and let $\mathcal{S}^\circ$ denote its interior. Let $\mathcal{G}(E_0, E_1)$ be the set of functions $f : \mathcal{S} \to E_0 + E_1$ such that:

(1) f is bounded and continuous on $\mathcal{S}$ and analytic on $\mathcal{S}^\circ$.

(2) For $t \in \mathbb{R}, j \in \{0, 1\}$, $f(\mathrm{i}t + j) \in E_j$ and $t \mapsto f(\mathrm{i}t + j)$ is continuous and bounded with respect to the norm on E_j.

(3) For $j \in \{0, 1\}$, $\|f(\mathrm{i}t + j)\|_{E_j} \to 0$ as $t \to \infty$.

Note that at this point our notation is different from [1] and [9], where $\mathcal{G}$ is denoted by $\mathcal{F}$, which we reserve for the Fourier transform. For $f \in \mathcal{G}(E_0, E_1)$, we define a norm

$$|||f||| = \max\{\sup \|f(\mathrm{i}t)\|_{E_0}, \sup \|f(\mathrm{i}t + 1)\|_{E_1}\}.$$

Let $\theta \in [0, 1]$. We define $(E_0, E_1)_{[\theta]} \subseteq E$ to be the space $\{f(\theta) : f \in \mathcal{G}(E_0, E_1)\}$ with norm

$$\|x\|_{[\theta]} = \inf\{|||f||| : f(\theta) = x, f \in \mathcal{G}(E_0, E_1)\}.$$

With this norm, $(E_0, E_1)_{[\theta]}$ is a Banach space ([1], Theorem 4.1.2).

DEFINITION 2.4. The assignment from compatible couples of Banach spaces to Banach spaces which is given by $C_\theta : (E_0, E_1) \to (E_0, E_1)_{[\theta]}$ is called the *complex interpolation method* (at parameter $\theta \in [0, 1]$). $(E_0, E_1)_{[\theta]}$ is called a *complex interpolation space*.

The following Riesz–Thorin-like theorem plays a central role in the present paper. It gives the functorial property of the complex interpolation method.

THEOREM 2.5 ([1], Theorem 4.1.2). *Let $\theta \in [0, 1]$. Let T be a morphism between compatible couples (E_0, E_1) and (F_0, F_1). Then, it restricts to a bounded linear map $T : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$. The norm is bounded by $\|T\| \leqslant \|T : E_0 \to F_0\|^{1-\theta} \|T : E_1 \to F_1\|^\theta$.*

If we let C_θ of Definition 2.4 act on the morphisms $T : (E_0, E_1) \to (F_0, F_1)$ of compatible couples by assigning its restriction $T : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ to it, we see that C_θ is a functor.

REMARK 2.6. Using the notation of Remark 2.3, the compatible morphisms T_0, T_1 give rise to a morphism $C_\theta(T) : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ on the interpolation spaces with norm $\|C_\theta(T)\| \leqslant \|T_0\|^{1-\theta} \|T_1\|^\theta$.

We will need the following useful fact, see Theorem 4.2.2 of [1].

LEMMA 2.7. *Let (E_0, E_1) be a compatible couple and $\theta \in [0,1]$. $E_0 \cap E_1$ is dense in $(E_0, E_1)_{[\theta]}$.*

2.2. IZUMI'S L^p-SPACES. In [17] Terp shows that the L^p-spaces as introduced by Hilsum can be obtained by applying the complex interpolation method to a specific compatible couple (M, M_*), see Theorem 36 of [17]. Izumi [9] realized that there is more than one way to turn (M, M_*) into a compatible couple in order to obtain the L^p-spaces through interpolation. His idea is to *define* non-commutative L^p-spaces as complex interpolation spaces of certain compatible structures. It is this definition which we recall here.

Here, we present the general picture. However, in the larger part of the present paper, we only work with the complex interpolation parameters $z = -1/2$ and $z = 1/2$ (we introduce the parameter in a minute). We will specialize the theory for these parameters in Sections 2.3 and 2.4 and introduce short hand notation there. The more general theory is used in Section 7, where we prove that there is in principle only one interpolation parameter that allows an L^p-Fourier transform, namely $z = -1/2$.

Fix a von Neumann algebra M with normal, semi-finite, faithful weight φ. The following construction of L^p-spaces can be found in [9].

DEFINITION 2.8. For $z \in \mathbb{C}$, we put

$$L_{(z)} = \{x \in M : \exists \varphi_x^{(z)} \in M_* \text{ such that } \forall a, b \in \mathcal{T}_\varphi : \varphi_x^{(z)}(a^*b) = \langle xJ\nabla^{\overline{z}}\Lambda(a) : J\nabla^{-z}\Lambda(b)\rangle\}.$$

The number $z \in \mathbb{C}$ will be called the complex interpolation parameter.

REMARK 2.9. We will mainly be dealing with the cases $z = -1/2$ and $z = 1/2$. Note that if φ is a state, then for any $x \in M$, we see that for $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned}\langle xJ\nabla^{-1/2}\Lambda(a) : J\nabla^{1/2}\Lambda(b)\rangle &= \langle xJ\nabla^{1/2}\Lambda(\sigma_{\mathrm{i}}(a)) : J\nabla^{1/2}\Lambda(b)\rangle \\ &= \varphi(bx\sigma_{-\mathrm{i}}(a^*)) = \varphi(a^*bx),\end{aligned}$$

and hence $L_{(-1/2)} = M$ and $\varphi_x^{(-1/2)} = x\varphi$. Similarly, $L_{(1/2)} = M$ and $\varphi_x^{(1/2)} = \varphi x$.

The following proposition implies that there are plenty of elements contained in $L_{(z)}$.

PROPOSITION 2.10 ([9], Propostion 2.3). *$\mathcal{T}_\varphi^2 = \{ab : a, b \in \mathcal{T}_\varphi\}$ is contained in $L_{(z)}$.*

We are now able to construct Izumi's L^p-spaces using the complex interpolation method. First, we define a compatible couple. For $x \in L_{(z)}$, we define a norm:

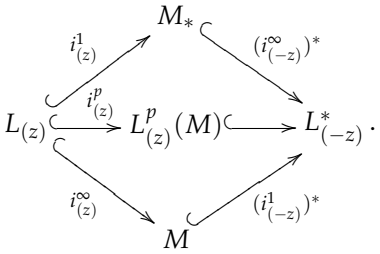
$$\|x\|_{L_{(z)}} = \max\{\|x\|, \|\varphi_x^{(z)}\|\}.$$

We define norm-decreasing injections:

$$i_{(z)}^{\infty} : L_{(z)} \to M : x \mapsto x; \quad i_{(z)}^{1} : L_{(z)} \to M_* : x \mapsto \varphi_x^{(z)}.$$

Using the duals of the maps, we obtain the following diagram. Note that $(i^{\infty}_{(-z)})^* : M^* \to L^*_{(-z)}$ is restricted to M_*.

(2.1)



Now, Theorem 2.5 of [9] yields that the outer rectangle of (2.1) commutes. This turns (M, M_*) into a compatible couple of Banach spaces.

DEFINITION 2.11. For $p \in (1, \infty)$, we define $L^p_{(z)}(M)$ to be the *complex interpolation space* $(M, M_*)_{[1/p]}$. We set $L^1_{(z)}(M) = M_*$ and $L^{\infty}_{(z)}(M) = M$.

By Lemma 2.7, $L_{(z)}$ can be embedded in $L^p_{(z)}(M)$. This map is denoted by $\imath^p_{(z)}$. Note that by definition of the complex interplation method $L^p_{(z)}(M)$ is a linear subspace of $L^*_{(-z)}$.

NOTATION 2.12. The map $i^{\infty}_{(z)} : L_{(z)} \to M$ is basically the inclusion of a subspace. Therefore, it is convenient to omit the map $i^{\infty}_{(z)}$ in our notation if the norms of the spaces do not play a role in the statement. Similarly, we do not introduce notation for the inclusion of $L^p_{(z)}(M)$ in $L^*_{(-z)}$, where $p \in (1, \infty)$.

A priori one could think that $L^p_{(z)}(M)$ and $L^p_{(z')}(M)$ with $z \neq z'$, are different as Banach spaces. However, Izumi proves that they are isometrically isomorphic. Terp [17] considers the case $z = 0$. The main result of [17] is that $L^p_{(0)}(M)$ is isometrically isomorphic to the L^p-spaces by Hilsum [8]. We will come back to this in Section 2.4.

THEOREM 2.13 ([9], Theorem 3.8). *For $z, z' \in \mathbb{C}$, there is an isometric isomorphism*

$$U_{p,(z',z)} : L^p_{(z)}(M) \to L^p_{(z')}(M), \quad p \in (1, \infty),$$

such that for $a \in \mathcal{T}^2_{\varphi}$,

$$U_{p,(z',z)}(i^p_{(z)}(a)) = i^p_{(z')}(\sigma_{\mathrm{i}(r'-r)/p-(s'-s)}(a)), \tag{2.2}$$

where $z = r + \mathrm{i}s$ and $z' = r' + \mathrm{i}s'$, $r, r', s, s' \in \mathbb{R}$.

We emphasize, that although the L^p-spaces appearing in (2.1) are isomorphic for different complex interpolation parameters, the intersections defined by

this figure may be different. In any case, by Corollary 2.13 of [9],

$$(i^1_{(-z)})^*(L_{(z)}) = (i^1_{(-z)})^*(M) \cap (i^\infty_{(-z)})^*(M_*), \tag{2.3}$$

i.e. if one consideres $L_{(z)}$, M, M_* as subspaces of $L^*_{(-z)}$, then $L_{(z)} = M \cap M_*$.

2.3. SPECIALIZATIONS FOR THE COMPLEX INTERPOLATION PARAMETERS. In the present paper we will mainly work with the parameter $z = -1/2$. In order to study these spaces also the parameter $z = 1/2$ will play a role. In this section, we specialize the theory for these parameters. The following proposition shows that $L_{(-1/2)}$ and $L_{(1/2)}$ can be described by a condition that is in generally more easy to check. If φ is a state it reduces to Remark 2.9.

PROPOSITION 2.14. *We have the following alternative descriptions:*

(i) *Let* $L = \{x \in \mathfrak{n}_\varphi : \exists\, {}_x\varphi \in M_* \text{ such that } \forall y \in \mathfrak{n}_\varphi :\ {}_x\varphi(y^*) = \varphi(y^*x)\}$. *Then,* $L = L_{(-1/2)}$.

(ii) *Let* $R = \{x \in \mathfrak{n}^*_\varphi : \exists \varphi_x \in M_* \text{ such that } \forall y \in \mathfrak{n}_\varphi :\ \varphi_x(y) = \varphi(xy)\}$. *Then,* $R = L_{(1/2)}$.

Proof. We only give the proof of (i), since (ii) can be proved similarly. We first prove $\subseteq$. For $x \in L$, $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned} {}_x\varphi(a^*b) = \varphi(a^*bx) = \varphi(bx\sigma_{-\mathrm{i}}(a^*)) &= \langle x\Lambda(\sigma_{-\mathrm{i}}(a^*)), \Lambda(b^*)\rangle \\ &= \langle x\nabla J\nabla^{1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle = \langle xJ\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle. \end{aligned}$$

Hence $x \in L_{(-1/2)}$ and ${}_x\varphi = \varphi_x^{(-1/2)}$.

To prove $\supseteq$, we first prove that $M\mathcal{T}^2_\varphi \subseteq L_{(-1/2)}$. Indeed, let $x \in M$ and let $c, d \in \mathcal{T}_\varphi$. The functional $M \ni y \mapsto \varphi(\sigma_{\mathrm{i}}(d)yxc)$ is normal. Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned} \langle xcdJ\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle &= \langle \Lambda(xcd\sigma_{-\mathrm{i}}(a^*)), \Lambda(b^*)\rangle \\ &= \varphi(bxcd\sigma_{-\mathrm{i}}(a^*)) = \varphi(\sigma_{\mathrm{i}}(d)a^*bxc). \end{aligned}$$

Hence, $xcd \in L_{(-1/2)}$.

Next, we prove that $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$. Take $x \in L_{(-1/2)}$ and let $(e_j)_{j\in J}$ be a bounded net in $\mathcal{T}_\varphi$ such that $\sigma_i(e_j)$ is bounded and such that $e_j \to 1$ σ-weakly, see Lemma 9 of [17]. Then, $xe_j \to x$ σ-weakly. Furthermore,

$$\|\Lambda(xe_j)\|^2 = \varphi(e_j^*x^*xe_j) = \varphi^{(-1/2)}_{xe_j\sigma_{-\mathrm{i}}(e_j^*)}(x^*) \leqslant \|\varphi^{(-1/2)}_{xe_j\sigma_{-\mathrm{i}}(e_j^*)}\|\|x\|, \tag{2.4}$$

where the second equality is due to the previous paragraph. By Proposition 2.6 of [9],

$$\varphi^{(-1/2)}_{xe_j\sigma_{-\mathrm{i}}(e_j^*)} = \varphi_x^{(-1/2)}\sigma_{\mathrm{i}}(e_j)e_j^*, \tag{2.5}$$

where for $\omega \in M_*, y \in M$, ωy is the normal functional defined by $(\omega y)(a) = \omega(ya), a \in M$. From (2.4) and (2.5) it follows that $(\Lambda(xe_j))_{j\in J}$ is a bounded net.

Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$\langle \Lambda(xe_j), \Lambda(ab) \rangle = \varphi(b^*a^*xe_j) = \varphi(a^*xe_j\sigma_{-\mathrm{i}}(b^*)) \to \varphi(a^*x\sigma_{-\mathrm{i}}(b^*)).$$

Since $(\Lambda(xe_j))_{j\in J}$ is bounded, this proves that $(\Lambda(xe_j))_{j\in J}$ is weakly convergent. Since Λ is σ-weak/weak closed, this implies that $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$. So $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$.

To finish the proof, let again $x \in L_{(-1/2)}$ and let $a, b \in \mathcal{T}_\varphi$. We prove that $\varphi_x^{(-1/2)}((ab)^*) = \langle \Lambda(x), \Lambda(ab) \rangle$. The proposition then follows since by Lemma 8.2, $\mathcal{T}_\varphi^2$ is a σ-weak/weak-core for Λ. The proposition follows from:

$$\begin{aligned}\langle \Lambda(x), \Lambda(ab) \rangle &= \varphi(b^*a^*x) = \varphi(a^*x\sigma_{-\mathrm{i}}(b^*)) \\ &= \langle xJ\nabla^{-1/2}\Lambda(b), J\nabla^{1/2}\Lambda(a^*) \rangle = \varphi_x^{(-1/2)}(b^*a^*). \quad \blacksquare\end{aligned}$$

In particular, it follows from Proposition 2.14 that for $y \in \mathfrak{n}_\varphi$,

$$ {}_x\varphi(y^*) = \varphi(y^*x), \quad x \in L; \quad \varphi_x(y) = \varphi(xy), x \in R. \tag{2.6}$$

We emphasize that one has to be careful that (2.6) does not make sense for every $x, y \in M$. Also, (2.6) justifies why (2.1) is also called the left injection for $z = -1/2$ and the right injection for $z = 1/2$.

Part of the next corollary is already proved in [9]. Using the alternative descriptions of Proposition 2.14, it is easy to prove the remaining statements.

COROLLARY 2.15. *We have inclusions $M\mathcal{T}_\varphi^2 \subseteq L$, $\mathcal{T}_\varphi^2 M \subseteq R$, $\mathcal{T}_\varphi^2 \subseteq L \cap R$, $L\mathcal{T}_\varphi \subseteq L$, $\mathcal{T}_\varphi R \subseteq R$, $ML \subseteq L$ and $RM \subseteq R$. Moreover, $R = \{x^* : x \in L\}$ and for $x \in L$, $\varphi_{x^*} = \overline{x}\varphi$.*

Proof. The first inclusion has already been proved in the proof of Proposition 2.14. Here we have proved that for $x \in M, a, b \in \mathcal{T}_\varphi$,

$$ {}_{xab}\varphi(z) = \varphi(\sigma_{\mathrm{i}}(b)\, z\, xa), \quad z \in M.$$

Similarly, one can prove that for $x, z \in M, a, b \in \mathcal{T}_\varphi, y_l \in L, y_r \in R$,

$$\begin{array}{lll} \varphi_{abx}(z) = \varphi(bx\, z\, \sigma_{-\mathrm{i}}(a)); & \varphi_{ab}(z) = \varphi(b\, z\, \sigma_{-\mathrm{i}}(a)); & {}_{ab}\varphi(z) = \varphi(\sigma_{\mathrm{i}}(b)\, z\, a); \\ {}_{y_l a}\varphi(z) = {}_{y_l}\varphi(\sigma_{\mathrm{i}}(a) z); & \varphi_{ay_r}(z) = \varphi_{y_r}^{(1/2)}(z\sigma_{-\mathrm{i}}(a)); & {}_{xy_l}\varphi(z) = {}_{y_l}\varphi(zx); \\ \varphi_{y_r x}(z) = \varphi_{y_r}(xz); & \varphi_{x^*} = \overline{x}\varphi. \quad \blacksquare & \end{array}$$

Since we are mainly dealing with complex interpolation parameter $z = -1/2$ and $z = 1/2$, it is more convenient to adapt our notation.

NOTATION 2.16. We use the following short hand notations. For $p \in [1, \infty]$,

$$\begin{array}{llll} L^p(M)_{\mathrm{left}} = L^p_{(-1/2)}(M), & L = L_{(-1/2)}, & l^p = i^p_{(-1/2)}, & {}_x\varphi = \varphi_x^{(-1/2)} \text{ for } x \in L, \\ L^p(M)_{\mathrm{right}} = L^p_{(1/2)}(M), & R = L_{(1/2)}, & r^p = i^p_{(1/2)}, & \varphi_x = \varphi_x^{(1/2)} \text{ for } x \in R. \end{array}$$

Recall that by definition $M_* = L^1(M)_{\mathrm{left}}$ and $M = L^\infty(M)_{\mathrm{left}}$. From now on, we consider M_* and M as subspaces of R^* by means of the respective maps r_∞^* and

r_1^* and it is convenient to omit these maps in the notation. So the identifications of M_* and M in R^* are given by the pairings:

$$\langle \omega, y \rangle_{R^*,R} = \omega(y), \quad \omega \in M_*, y \in R, \tag{2.7}$$

$$\langle x, y \rangle_{R^*,R} = \varphi_y(x), \quad x \in M, y \in R. \tag{2.8}$$

The norm on L will be denoted by $\| \cdot \|_L$.

2.4. COMPARISON WITH HILSUM'S L^p-SPACES. Here, we recall the definition of non-commutative L^p-spaces given in [8], see also [17]. We need these spaces for two reasons.

First of all, many of the objects we introduce are constructed by means of Theorem 2.5. For that reason the structures are abstract in nature. The advantage of the Hilsum approach is that it is much more concrete. Hence, also the objects defined in Section 5 have a more concrete meaning when they are considered in the Hilsum setting.

Secondly, a non-commutative L^2-space associated with a von Neumann algebra M with weight φ can be identified with the GNS-space $\mathcal{H}$ of the weight. In Theorem 23 of [17] this identification is given for Hilsum's definition. In [9], Izumi does not explicitly keep track of an isomorphism between $L^2(M)_{\text{left}}$ with $\mathcal{H}$. Here we make this isomorphism explicit. This is useful for the L^p-Fourier transform. In particular, Corollary 5.5 relies heavily on this identification.

We refer to the original paper [8] for Hilsum's L^p-spaces. The following is also nicely summarized in Sections III and IV of [16]. Fix a normal, semi-finite, faithful weight ϕ on the commutant M'. Let σ^ϕ be its modular automorphism group.

DEFINITION 2.17. A closed, densely defined operator x on $\mathcal{H}$ is called γ-*homogeneous*, with $\gamma \in \mathbb{R}$ if the following skew commutation relation holds

$$ax \subseteq x\sigma^\phi_{i\gamma}(a), \quad \text{for all } a \in M' \text{ analytic with respect to } \sigma^\phi.$$

The following theorem requires the spatial derivative [2], [15]. We will not recall this construction, but rather cite its properties. For a good introduction, we refer to Section III of [17]. The spatial derivative construction gives a passage between M_* and the (-1)-homogeneous operators. The following theorem can be found under the given references in [16]. It can be derived from Theorem 13 of [2].

THEOREM 2.18 ([16], Theorem 29, Definition 33 and Corollary 34). *Let x be a closed, densely defined operator with polar decomposition $x = u|x|$. Let $p \in [1, \infty]$. The following are equivalent:*

(i) *x is $(-1/p)$-homogeneous;*

(ii) *$u \in M$ and $|x|^p$ is (-1)-homogeneous;*

(iii) *$u \in M$ and there is a normal, semi-finite weight ψ on M such that $|x|^p$ equals the spatial derivative $\mathrm{d}\psi/\mathrm{d}\phi$.*

DEFINITION 2.19. Let $p \in [1, \infty)$. The Hilsum L^p-space $L^p(\phi)$ is defined as the space of closed, densely defined operators x on the GNS-space $\mathcal{H}$ of φ such that if $x = u|x|$ is the polar decomposition, then $|x|^p$ is the spatial derivative of a positive $\omega \in M_*$ and $u \in M$. It carries the norm $\|x\|_p = (\omega(1))^{1/p}$. We set $L^\infty(M) = M$.

In particular, every operator in $L^p(\phi)$ is closed, densely defined and $(-1/p)$-homogeneous. This includes $p = \infty$. By Theorem 2.18, the spatial derivative gives an isometric isomorphism between M_* and $L^1(\phi)$.

We introduce notation for the distinguished spatial derivative

$$d = \frac{\mathrm{d}\varphi}{\mathrm{d}\phi}.$$

d is a strictly positive, self-adjoint operator acting on the GNS-space $\mathcal{H}$. We need the fact that it implements the modular automorphism group of φ and ϕ, i.e.

$$\sigma_t(x) = d^{\mathrm{i}t} x d^{-\mathrm{i}t}, \quad x \in M; \quad \sigma_t^{\phi}(x) = d^{-\mathrm{i}t} y d^{\mathrm{i}t}, \quad y \in M'.$$

Using this, one can prove that d is (-1)-homogeneous, see Lemma 22 of [17]. The operator d forms a handy tool to find elements of $L^p(\phi)$.

LEMMA 2.20 ([17], Theorem 26). *Let $p \in [2, \infty]$ and let $x \in \mathfrak{n}_\varphi$. Then, $xd^{1/p}$ is preclosed and its closure $[xd^{1/p}]$ is in $L^p(\phi)$. Moreover, there is an isometric isomorphism $\mathcal{P} : \mathcal{H} \to L^2(\phi)$ given by $[xd^{1/2}] \mapsto \Lambda(x)$.*

We use this result to prove the following.

PROPOSITION 2.21. *Let $p \in [1, \infty]$.*

(i) *Let $a, b \in \mathcal{T}_\varphi$. Then, $abd^{1/p}$ is preclosed and its closure $[abd^{1/p}]$ is in $L^p(\phi)$.*

(ii) *There is an isometric isomorphism $\Phi_p : L^p(\phi) \to L^p(M)_{\mathrm{left}}$ such that*

$$\Phi_p : [abd^{1/p}] \mapsto l^p(ab), \quad a, b \in \mathcal{T}_\varphi.$$

(iii) *There is a unitary map $U_l : L^2(M)_{\mathrm{left}} \to \mathcal{H}$ determined by*

$$U_l : l^2(a) \mapsto \Lambda(a), \quad a \in \mathcal{T}_\varphi^2.$$

(iv) *More general, there is a unitary map $U_{(z)} : L^2_{(z)}(M) \to \mathcal{H}$ determined by*

$$U_{(z)} : i^2_{(z)}(a) \mapsto \Lambda(\sigma_{-\mathrm{i}(z/2+1/4)}(a)), \quad a \in \mathcal{T}_\varphi^2.$$

Proof. (i) First note that using Lemma 22 of [17] for the first inclusion, Lemma 2.20 for the third and Theorem 4 (3) of [8] for the last,

$$\begin{aligned} abd^{1/p} &\subseteq d^{2/p}\sigma_{2\mathrm{i}/p}(ab)d^{2/p} \\ &\subseteq d^{2/p}\sigma_{2\mathrm{i}/p}(a) \cdot [\sigma_{2\mathrm{i}/p}(b)d^{2/p}] \in L^{p/2}(\phi) \cdot L^{p/2}(\phi) \subseteq L^p(\phi). \end{aligned}$$

Hence,

$$(abd^{1/p})^* \supseteq (d^{2/p}\sigma_{2\mathrm{i}/p}(a) \cdot [\sigma_{2\mathrm{i}/p}(b)d^{2/p}])^* \in L^p(\phi). \tag{2.9}$$

So that $(abd^{1/p})^*$ is densely defined. Hence $abd^{1/p}$ is preclosed and by the proof of Theorem 4 (1) of [8], $(abd^{1/p})^* = (d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}])^*$, hence $[abd^{1/p}]$ $= d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}]$.

(ii) It is argued in the remarks following Proposition 2.4 of [9] that (2.1) for $z = 0$ equals the compatible couple as considered in [17]. First note that by equation (50) of [17],

$$[abd^{1/p}] = d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}] = \mu_p(\sigma_{2i/p}(ab)),$$

where μ_p is the embedding of $L_{(0)}$ in $L^p(\phi)$, see Theorem 27 of [17]. The main result of [17] is that $L^p(\phi)$ is isometrically isomorphic to $L^p_{(0)}(M)$. The isomorphism is given by the map $\nu_p : L^p(\phi) \to L^p_{(0)}(M)$ of Theorem 30 in [17]. Moreover, we see that $\nu_p\mu_p = (i^1_{(0)})^* i^\infty_{(0)}$ by commutativity of equation (55) of [17]. In turn we have $(i^1_{(0)})^* i^\infty_{(0)} = i^p_{(0)}$ by commutativity of (2.1). Hence, we have an isometric isomorphism $L^p(\phi) \to L^p_{(0)}(M)$ for which

$$[abd^{1/p}] \mapsto \nu_p([abd^{1/p}]) = \nu_p\mu_p(\sigma_{2i/p}(ab)) = i^p_{(0)}(\sigma_{2i/p}(ab)).$$

We conclude the proof by applying the isometric isomorphism $U_{(-1/2,0)}$ of Theorem 2.13, so that we get an isometric isomorphism

$$\Phi_p : L^p(\phi) \to L^p_{(-1/2)}(M) = L^p(M)_{\text{left}},$$

such that

$$\Phi_p : [abd^{1/p}] \mapsto U_{(-1/2,0)} i^p_{(0)}(\sigma_{2i/p}(ab)) = i^p_{(-1/2)}(ab) = l^p(ab), \quad a, b \in \mathcal{T}_\varphi.$$

(iii) This follows from (ii) by applying Lemma 2.20 and the fact that $\Lambda(\mathcal{T}^2_\varphi)$ is dense in $\mathcal{H}$. So $U_l = \Phi_p^{-1}\mathcal{P}^{-1}$.

(iv) $U_{(z)} = U_l U_{2,(-1/2,z)}$. ∎

Recall that $L^2(M)_{\text{left}}$ is by definition a subspace of R^*. Therefore, we can pair elements of $L^2(M)_{\text{left}}$ with elements of R.

PROPOSITION 2.22. *For $\xi \in \mathcal{H}$, $y \in R$,*

$$\langle U_l^*\xi, y\rangle_{R^*,R} = \langle \xi, \Lambda(y^*)\rangle.$$

Proof. First assume that $\xi = \Lambda(x) = U_l l_2(x)$, $x \in L$. Using the commutativity of (2.1) in the second equality,

$$\begin{aligned}\langle U_l^*\xi, y\rangle_{R^*,R} = \langle l_2(x), y\rangle_{R^*,R} &= \langle l_1(x), y\rangle_{R^*,R} \\ &\overset{(2.7)}{=} ({}_x\varphi)(y) \overset{(2.6)}{=} \varphi(yx) = \langle \Lambda(x), \Lambda(y^*)\rangle = \langle \xi, \Lambda(y^*)\rangle.\end{aligned}$$

The proposition follows by the fact that $\Lambda(\mathcal{T}^2_\varphi) \subseteq \Lambda(L)$ is dense in $\mathcal{H}$. ∎

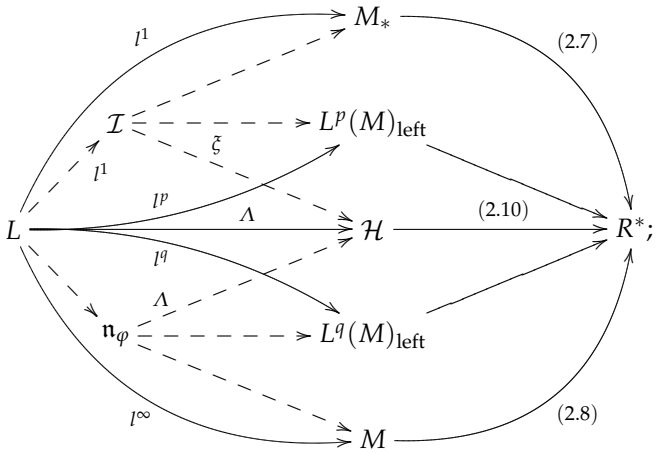
NOTATION 2.23. From now on, we will identify $\mathcal{H}$ and $L^2(M)_{\text{left}}$ and consider it as a subspace of R^*. The identification is given via the unitary U_l. Under this identification the map l_2 becomes the GNS-map Λ, see Proposition 2.21. By Proposition 2.22 we see that $\mathcal{H}$ is identified as a subspace of R^* by means of the pairing

$$\langle \xi, y \rangle_{R^*,R} = \langle \xi, \Lambda(y^*) \rangle \quad \xi \in \mathcal{H}, y \in R. \tag{2.10}$$

3. INTERSECTIONS OF L^p-SPACES

As indicated in the Section 2, the intersections of the various L^p-spaces depend on the interpolation parameter z of Definition 2.8. Here we study the intersections of $L^1(M)_{\text{left}}$ and $L^2(M)_{\text{left}}$, as well as the intersections of $L^2(M)_{\text{left}}$ and $L^\infty(M)_{\text{left}}$. The spaces turn out to be natural and well-known in the theory of locally compact quantum groups. We use the intersections in order to apply the reiteration theorem, see [1].

NOTATION 3.1. In this section, any interpolation space should be understood with respect to the diagram in (2.1) for the parameter $z = -1/2$. Recall that we introduced short hand notation for this diagram in Notations 2.16 and 2.23. Moreover, we *identified* $M_*, \mathcal{H}$ and M as subspaces of R^* by means of the pairings (2.7), (2.10) and (2.8). Similarly, any intersection of two such spaces should be understood as an intersection within R^*. The notation can be summarized by means of the non-dotted arrows in the following diagram. The dotted part of the diagram is the main topic of the present section.



3.1. THE INTERSECTION OF M_* AND $\mathcal{H}$. The following set defines the intersection of M_* and $\mathcal{H}$.

DEFINITION 3.2. We set:

$$\mathcal{I} = \{\omega \in M_* : \Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi \text{ is bounded}\}.$$

By the Riesz theorem, for every $\omega \in \mathcal{I}$, there exists a $\xi(\omega) \in \mathcal{H}$ such that $\langle \xi(\omega), \Lambda(x)\rangle = \omega(x^*)$.

THEOREM 3.3. *We have $\mathcal{I} = \mathcal{H} \cap M_*$, where the equality should be interpreted within R^*, see Notation* 3.1. *Within R^*, $\omega \in \mathcal{I}$ equals $\xi(\omega) \in \mathcal{H}$.*

Proof. We first prove $\supseteq$. Let $\xi \in \mathcal{H}$ and $\omega \in M_*$ be such that $\xi = \omega$ in R^*. For $y \in R$,

$$\omega(y) \overset{(2.7)}{=} \langle \omega, y\rangle_{R^*,R} = \langle \xi, y\rangle_{R^*,R} \overset{(2.10)}{=} \langle \xi, \Lambda(y^*)\rangle.$$

L contains $\mathcal{T}_\varphi^2$. Moreover, $\mathcal{T}_\varphi^2$ is a σ-strong*/norm core for Λ, see Lemma 8.2. Hence, it follows that $\omega \in \mathcal{I}$.

To prove $\subseteq$, let $\omega \in \mathcal{I}$. For $y \in R$,

$$\langle \xi(\omega), y\rangle_{R^*,R} \overset{(2.10)}{=} \langle \xi(\omega), \Lambda(y^*)\rangle = \omega(y) \overset{(2.7)}{=} \langle \omega, y\rangle_{R^*,R}.$$

Hence, $\xi(\omega) = \omega$ in R^*. ∎

Note that $(M_*, \mathcal{H})$ forms a compatible couple. As explained in Section 2.1, the intersection of these two spaces carries a natural norm for which it is a Banach space. So, for $\omega \in \mathcal{I}$ we define

$$\|\omega\|_{\mathcal{I}} = \max\{\|\omega\|, \|\xi(\omega)\|\}.$$

PROPOSITION 3.4. *The map $k : L \to \mathcal{I} : x \mapsto {}_x\varphi$ is injective, norm-decreasing and has dense range. In fact, $k(\mathcal{T}_\varphi^2)$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}$.*

Proof. Suppose that $x \in L$ and ${}_x\varphi = 0$, then $0 = ({}_x\varphi)(x^*) = \varphi(x^*x)$. So $x = 0$ and hence k is injective. For $x \in L$, $\|{}_x\varphi\| \leqslant \|x\|_L$ and

$$\|\xi({}_x\varphi)\| = \|\Lambda(x)\| = \|{}_x\varphi(x^*)\|^{1/2} \leqslant \|{}_x\varphi\|^{1/2}\|x^*\|^{1/2} \leqslant \|x\|_L,$$

so that k is norm-decreasing. Now we prove that the range of k is dense in $\mathcal{I}$. We identify $\mathcal{I}$ with the subspace $\{(\omega, \xi(\omega)) : \omega \in \mathcal{I}\} \subseteq M_* \times \mathcal{H}$. We equip $M_* \times \mathcal{H}$ with the norm $\|(\omega, \xi)\|_{\max} = \max\{\|\omega\|, \|\xi\|\}$. The norm coincides with $\|\cdot\|_{\mathcal{I}}$ on $\mathcal{I}$. The dual of $(M_* \times \mathcal{H}, \|\cdot\|_{\max})$ can be identified with $(M \times \mathcal{H}^*, \|\cdot\|_{\text{sum}})$, where $\|(x, \xi)\|_{\text{sum}} = \|x\| + \|\xi\|$. Let $N \subseteq M \times \mathcal{H}^*$ be the space of all (y, η) such that $\langle(\omega, \xi(\omega)), (y, \eta)\rangle_{M_*\times\mathcal{H}, M\times\mathcal{H}^*} = 0$ for all $\omega \in \mathcal{I}$. The dual of $\mathcal{I}$ is given by $(M \times \mathcal{H})/N$ equipped with the quotient norm.

Now, let $(y, \eta) \in M \times \mathcal{H}$ be such that

$$\langle({}_x\varphi, \Lambda(x)), (y, \eta)\rangle_{M_*\times\mathcal{H}, M\times\mathcal{H}^*} = ({}_x\varphi)(y) + \langle \Lambda(x), \eta\rangle = 0$$

for all $x \in \mathcal{T}_\varphi^2$. The proof is finished if we can show that $(y, \eta) \in N$. In order to do this, let $(e_j)_{j\in J}$ be a net as in Lemma 8.1. Put $a_j = \sigma_{-\mathrm{i}/2}(e_j)$. By the assumptions on (y, η), for $x \in \mathcal{T}_\varphi$,

$$({}_{xa_j}\varphi)(y) = -\langle \Lambda(xa_j), \eta\rangle. \tag{3.1}$$

For the left hand side we find by Corollary 2.15,

$$(xa_j\varphi)(y) = \varphi(\sigma_{\mathrm{i}}(a_j)yx) = \langle \Lambda(x), \Lambda(y^*\sigma_{\mathrm{i}}(a_j)^*)\rangle, \tag{3.2}$$

where the first equality follows from Proposition 2.3 of [9]. For the right hand side of (3.1) we find

$$\langle \Lambda(xa_j), \eta\rangle = \langle J\sigma_{-\mathrm{i}/2}(a_j^*)J\Lambda(x), \eta\rangle = \langle \Lambda(x), J\sigma_{\mathrm{i}/2}(a_j)J\eta\rangle. \tag{3.3}$$

Hence (3.1) together with (3.2) and (3.3) yield

$$\Lambda(y^*\sigma_{\mathrm{i}}(a_j)^*) = -J\sigma_{\mathrm{i}/2}(a_j)J\eta.$$

Hence, since $\sigma_{\mathrm{i}/2}(a_j) = e_j \to 1$ strongly, $\Lambda(y^*\sigma_{\mathrm{i}}(a_j)^*) \to -\eta$ weakly. For $\omega \in \mathcal{I}$,

$$\begin{aligned}\langle \xi(\omega), \eta\rangle &= -\lim_{j\in J}\langle \xi(\omega), \Lambda(y^*\sigma_{\mathrm{i}}(a_j)^*)\rangle \\ &= -\lim_{j\in J}\omega(\sigma_{\mathrm{i}}(a_j)y) = -\lim_{j\in J}\omega(\sigma_{\mathrm{i}/2}(e_j)y) = -\omega(y).\end{aligned}$$

Thus $(y, \eta) \in N$. ∎

3.2. THE INTERSECTION OF $\mathcal{H}$ AND M. It turns out that $\mathfrak{n}_\varphi$ is the intersection of M and $\mathcal{H}$.

THEOREM 3.5. *We have $\mathfrak{n}_\varphi = \mathcal{H} \cap M$, where the equality should be interpreted within R^*, see Notation 3.1. Within R^*, $x \in \mathfrak{n}_\varphi$ equals $\Lambda(x) \in \mathcal{H}$. Moreover, let $\overline{L}$ be the closure of $l^\infty(L)$ in M. Then $\mathfrak{n}_\varphi = \mathcal{H} \cap \overline{L}$.*

Proof. First we prove that $\mathfrak{n}_\varphi = \mathcal{H} \cap M$ in R^*. For $x \in \mathfrak{n}_\varphi, y \in R$,

$$\langle \Lambda(x), y\rangle_{R^*,R} \overset{(2.10)}{=} \langle \Lambda(x), \Lambda(y^*)\rangle = \varphi(yx) \overset{(2.6)}{=} \varphi_y(x) \overset{(2.8)}{=} \langle x, y\rangle_{R^*,R},$$

so $\Lambda(x) = x$ in R^*. Hence the inclusion $\subseteq$ follows.

Now let $x \in M$, $\xi \in \mathcal{H}$ be such that $x = \xi$ in R^*. For $y \in R$,

$$\varphi_y(x) \overset{(2.8)}{=} \langle x, y\rangle_{R^*,R} = \langle \xi, y\rangle_{R^*,R} \overset{(2.10)}{=} \langle \xi, \Lambda(y^*)\rangle.$$

For $a \in \mathcal{T}_\varphi^2$, $y \in \mathfrak{n}_\varphi$, using Corollary 2.15 for the third, fourth and fifth equality,

$$\begin{aligned}\langle \Lambda(xa), \Lambda(y)\rangle &= \varphi(y^*xa) =_a \varphi(y^*x) = \varphi_{\sigma_{\mathrm{i}}(a)}(y^*x) \\ &= \varphi_{\sigma_{\mathrm{i}}(a)y^*}(x) = \langle \xi, \Lambda(y\sigma_{\mathrm{i}}(a)^*)\rangle = \langle J\sigma_{\mathrm{i}/2}(a)^*J\xi, \Lambda(y)\rangle.\end{aligned}$$

So for $a \in \mathcal{T}_\varphi^2$, $\Lambda(xa) = J\sigma_{\mathrm{i}/2}(a)^*J\xi$. Let $(e_j)_{j\in J}$ be a net as in Lemma 8.1. Put $a_j = e_j^2$. Then $xa_j \to x$ σ-weakly. Furthermore, $J\pi(\sigma_{\mathrm{i}/2}(a_j)^*)J\xi \to \xi$ weakly, hence $\Lambda(xa_j)$ converges weakly. Since Λ is σ-weak/weak closed, $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$ and $\xi = \Lambda(x)$. This proves $\supseteq$.

Recall the complex interpolation method from Definition 2.4. Recall that in this section every interpolation space should be interpreted with respect to (2.1) for parameter $z{=}{-}1/2$. Note that Theorem 4.2.2 of [1] gives the second equality in

$$\mathcal{H} = (M, M_*)_{[1/2]} = (\overline{L}, M_*)_{[1/2]} \subseteq \overline{L} + M_*. \tag{3.4}$$

We now prove that

$$M \cap (\overline{L} + M_*) = \overline{L}. \tag{3.5}$$

Take any $s \in M \cap (\overline{L} + M_*) \subseteq R^*$. Since $s \in \overline{L} + M_*$, we can choose representatives $x \in \overline{L}, \omega \in M_*$ such that $s = x + \omega$ in R^*. Since $s \in M$, we can find a representative $y \in M$ such that $s = y$ in R^*. Then $\omega = y - x$ is both in M_* and M, and hence by (2.3) in $M_* \cap M = L$. Hence we see that $s = x + \omega \in \overline{L} + L = \overline{L}$. This proves $\subseteq$, the other inclusion is trivial.

Now, (3.4) and (3.5) imply:

$$\mathcal{H} \cap M = \mathcal{H} \cap M \cap (\overline{L} + M_*) = \mathcal{H} \cap \overline{L}. \quad \blacksquare$$

Again, we introduce the norm on an intersection of a compatible couple as in Section 2.1. For $x \in \mathfrak{n}_\varphi$, we put

$$\|x\|_{\mathfrak{n}_\varphi} = \max\{\|x\|, \|\Lambda(x)\|\}.$$

Again, we can prove a density result similar to Propostion 3.4

PROPOSITION 3.6. *The map $k' : L \to \mathfrak{n}_\varphi : x \mapsto x$ is injective, norm-decreasing and has dense range.*

Proof. The non-trivial part is that $k'(L)$ is dense in $\mathfrak{n}_\varphi$ with respect to $\|\cdot\|_{\mathfrak{n}_\varphi}$. To prove this, we identify $\mathfrak{n}_\varphi$ with the subspace $\{(x, \Lambda(x)) : x \in \mathfrak{n}_\varphi\} \subseteq M \times \mathcal{H}$. For $(x, \xi) \in M \times \mathcal{H}$, we set $\|(x, \xi)\|_{\max} = \max\{\|x\|, \|\xi\|\}$. So $\|\cdot\|_{\max}$ coincides with $\|\cdot\|_{\mathfrak{n}_\varphi}$ on $\mathfrak{n}_\varphi$. The dual of $(M \times \mathcal{H}, \|\cdot\|_{\max})$ is given by $(M^* \times \mathcal{H}^*, \|\cdot\|_{\text{sum}})$, where $\|(\theta, \xi)\|_{\text{sum}} = \|\theta\| + \|\xi\|$.

Let $(\theta, \xi) \in M^* \times \mathcal{H}^*$ be such that for all $x \in L$,

$$\theta(x) + \langle \Lambda(x), \xi \rangle = 0. \tag{3.6}$$

We must prove that (3.6) holds for all $x \in \mathfrak{n}_\varphi$. The proof proceeds in several steps.

Claim 1. There exists an $\omega \in M_*$ such that for $x \in L \cap R$, $\omega(x) = \theta(x)$.

Proof of the claim. From Corollary 2.15 it follows that

$$\overline{L \cap R} (= \overline{l^\infty(L) \cap r^\infty(R)})$$

is a C^*-algebra. Here and in the rest of this proof the closure has to be interpreted within M. Let $(u_j)_{j \in J}$ be an approximate unit for the C^*-algebra $\overline{L \cap R}$. We may assume that $u_j \in (L \cap R)^+$. Set $\omega_j(x) = -\langle x\Lambda(u_j), \xi \rangle, x \in M$. So $\omega_j \in M_*$. Moreover, by (3.6) and Corollary 2.15,

$$\omega_j(x) = -\langle x\Lambda(u_j), \xi \rangle = \theta(xu_j).$$

Let ρ be a representation of $\overline{L \cap R}$ on a Hilbert space $\mathcal{H}_\rho$ such that $\theta(x) = \langle \rho(x)\xi, \eta \rangle$ for certain vectors $\xi, \eta \in \mathcal{H}_\rho$. Then $\omega_j(x) = \langle \rho(x)\rho(u_j)\xi, \eta \rangle$. Since $\rho(u_j) \to 1$ strongly, $\|\omega_j|_{\overline{L \cap R}} - \theta|_{\overline{L \cap R}}\| \to 0$. $L \cap R (\supseteq \mathcal{T}_\varphi^2)$ is σ-weakly, hence strongly dense in M so that by Kaplansky's density theorem $\|\omega_j\| = \|\omega_j|_{L \cap R}\|$. Hence $(\omega_j)_{j \in J}$ is a Cauchy net in M_*. Let $\omega \in M_*$ be its limit. This proves the first claim.

Claim 2. For $x \in \mathfrak{n}_\varphi$, we find $\omega(x) = -\langle \Lambda(x), \xi \rangle$.

Proof of the claim. Note that $L \cap R$ is a σ-weak/weak core for Λ. Indeed, $\mathcal{T}_\varphi^2$ is contained in $L \cap R$ so that we can apply Lemma 8.2.

Now, if $x \in L \cap R$, the claim follows by the first claim and the properties of θ, i.e. $\omega(x) = \theta(x) = -\langle \Lambda(x), \xi \rangle$. Let $x \in \mathfrak{n}_\varphi$ and, by the previous paragraph, let $(x_i)_{i \in I}$ be a net in $L \cap R$ converging σ-weakly to x such that $\Lambda(x_i) \to \Lambda(x)$ weakly. Then, we arrive at the following equation:

$$\omega(x) = \lim_{i \in I} \omega(x_i) = -\lim_{i \in I} \langle \Lambda(x_i), \xi \rangle = -\langle \Lambda(x), \xi \rangle. \tag{3.7}$$

This proves the second claim.

Claim 3. $\overline{L \cap R} = \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} = \overline{L \cap R}$, where the closures are interpreted with respect to the norm on M.

Proof of the claim. Note that by Proposition 2.14, $\overline{L \cap R} \subseteq \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*}$. By Theorem 3.5, we see that $\mathfrak{n}_\varphi \subseteq \overline{L}$. Since $R = \{x^* : x \in L\}$, see Corollary 2.15, we also have $\mathfrak{n}_\varphi^* \subseteq \overline{R}$. Hence,

$$\overline{L \cap R} \subseteq \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} \subseteq \overline{L} \cap \overline{R} \quad \text{(closures in } M\text{)}.$$

The inclusions are in fact equalities. Indeed, let $x \in \overline{L} \cap \overline{R}$ be positive. Let x_n and y_n be sequences in L, respectively R, converging in norm to x. Then, by Corollary 2.15, $y_n x_n \in LR \subseteq L \cap R$. $y_n x_n$ is norm convergent to x^2. So $x^2 \in \overline{L \cap R}$, hence $x \in \overline{L \cap R}$. From Corollary 2.15 it follows that $\overline{L \cap R}$ and $\overline{L \cap R}$ are C^*-algebras. Hence, $\overline{L \cap R} = \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} = \overline{L \cap R}$.

Claim 4. Equation (3.6) holds for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

Proof of the claim. Let $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and by the third claim, let $x_n \in L \cap R$ be a sequence converging in norm to x. Then, using the first claim in the second equality and the third claim in the fourth equality,

$$\theta(x) = \lim_{n \to \infty} \theta(x_n) = \lim_{n \to \infty} \omega(x_n) = \omega(x) = -\langle \Lambda(x), \xi \rangle.$$

Hence (3.6) follows for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

We are coming to the proof of the proposition. Let $x \in \mathfrak{n}_\varphi$ and let $x = u|x|$ be its polar decomposition. Since (3.6) holds for $y \in L$, we find for $y \in L$ that $uy \in L$ by Corollary 2.15 and,

$$(\theta u)(y) + \langle \Lambda(y), u^* \xi \rangle = 0, \tag{3.8}$$

where we defined $\theta u \in M$ by $(\theta u)(a) = \theta(ua), a \in M$. If we apply the arguments in the previous paragraphs to the pair $(\theta u, u^* \xi)$, we see that actually (3.8) holds for all $y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. In particular, putting $y = |x|$, the required equation (3.6) follows. ∎

3.3. REITERATION. Here we apply the reiteration theorem, see Theorems 4.6.1 of [1], for the complex interpolation method to obtain $L^p(M)_{\text{left}}$, $p \in (1,2]$ as an interpolation space of $\mathcal{H}$ and M_*. Similarly, $L^p(M)_{\text{left}}$, $p \in [2,\infty)$ as an interpolation space of $\mathcal{H}$ and M. Recall that in this section every intersection and interpolation is understood with respect to (2.1) for the parameter $z = -1/2$.

THEOREM 3.7. *We have the following interpolation properties:*
(i) *For* $p \in (1,2]$, $(\mathcal{H}, M_*)_{[(2/p)-1]} = L^p(M)_{\text{left}}$.
(ii) *For* $q \in [2,\infty)$, $(\mathcal{H}, M)_{[1-(2/q)]} = (M, \mathcal{H})_{[2/q]} = L^q(M)_{\text{left}}$.

Proof. (i) Recall that $\overline{L} = \overline{l^\infty(L)}$ denotes the closure of L in M. Recall from (2.3) that $M_* \cap M = L$. By Theorem 4.2.2 (b) of [1] we get the first equality of:

$$(\overline{L}, M_*)_{[1/2]} = (M, M_*)_{[1/2]} = L^2(M)_{\text{left}} \simeq \mathcal{H}. \tag{3.9}$$

The latter isomorphism is the identification in Notation 2.23. On the other hand, we find:

$$(\overline{L}, M_*)_{[1]} = (M, M_*)_{[1]}. \tag{3.10}$$

Since $l^1(L)$ is dense in M_*, see Proposition 2.4 of [9], we find that (3.10) in turn equals M_*, by Proposition 4.2.2 of [1].

Note that the following three density assumptions are satisfied:
(i) $l^1(L)$ is dense in M_*, see Proposition 2.4 of [9].
(ii) $l^\infty(L)$ is dense in $\overline{L}$ (trivial).
(iii) $l^1(L)$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}$ by Proposition 3.4. Moreover $\mathcal{I}$ is the intersection of M_* and $\mathcal{H}$ in R^*, see Theorem 3.3.

Hence, we have checked the assumptions of the reiteration theorem Theorems 4.6.1 of [1] which is used in the third equality,

$$\begin{aligned} L^p(M)_{\text{left}} &= (M, M_*)_{[1/p]} = (\overline{L}, M_*)_{[1/p]} \\ &= ((\overline{L}, M_*)_{[1/2]}, (\overline{L}, M_*)_{[1]})_{[2/p)-1]} = (\mathcal{H}, M_*)_{[2/p)-1]}, \end{aligned}$$

(here the second equality follows again by Theorem 4.2.2 of [1]).

(ii) Completely analogously, using Theorem 3.5 and Proposition 3.6, one proves that

$$L^q(M)_{\text{left}} = (\mathcal{H}, M)_{[1-(2/q)]},$$

which in turn equals $(M, \mathcal{H})_{[2/q]}$ by Theorem 4.2.1 of [1]. ∎

4. LOCALLY COMPACT QUANTUM GROUPS

We now recall the Kustermans–Vaes definition of a locally compact quantum groups, see [13] and [14]. Since we will be dealing with non-commutative L^p-spaces, we stick to the von Neumann algebra setting. For an introduction to the theory of locally compact quantum groups we refer to [12] or [18], where the

results below are summarized. See also [19] were a simple von Neumann algebraic approach to quantum groups is presented.

4.1. VON NEUMANN ALGEBRAIC QUANTUM GROUPS.

DEFINITION 4.1. A locally compact quantum group (M, Δ) consists of the following data:

(i) A von Neumann algebra M.

(ii) A unital, normal $*$-homomorphism $\Delta : M \to M \otimes M$ satisfying the coassociativity relation $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$, where $\iota : M \to M$ is the identity.

(iii) Two normal, semi-finite, faithful weights φ, ψ on M so that

$$\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1), \quad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\varphi^+ \quad \text{(left invariance)};$$
$$\psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1), \quad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\psi^+ \quad \text{(right invariance)};$$

where φ is the left Haar weight and ψ the right Haar weight.

Note that we suppress the Haar weights in the notation. The triple $(\mathcal{H}, \pi, \Lambda)$ denotes the GNS-construction with respect to the left Haar weight φ. We may assume that M acts on the GNS-space $\mathcal{H}$.

In order to reflect to the classical situation of a locally compact group, we include the following example.

EXAMPLE 4.2. Let G be a locally compact group. Consider $M = L^\infty(G)$ and define the coproduct $\Delta_G : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \simeq L^\infty(G \times G)$ by putting

$$(\Delta_G(f))(x, y) = f(xy),$$

where φ and ψ are given by integrating against the left and right Haar weight respectively. In this way $(L^\infty(G), \Delta_G)$ is a locally compact quantum group.

4.2. MULTIPLICATIVE UNITARY. There exists a unique unitary operator

$$W \in B(\mathcal{H} \otimes \mathcal{H}),$$

defined by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)), \quad a, b \in \mathfrak{n}_\varphi,$$

where W is known as the multiplicative unitary. It satisfies the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ in $B(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$. Furthermore, $\Delta(x) = W^*(1 \otimes x)W, x \in M$.

4.3. THE DUAL QUANTUM GROUP. In [13], [14], it is proved that there exists a dual locally compact quantum group $(\widehat{M}, \widehat{\Delta})$, so that $(\widehat{\widehat{M}}, \widehat{\widehat{\Delta}}) = (M, \Delta)$. The dual left and right Haar weight are denoted by $\widehat{\varphi}$ and $\widehat{\psi}$. Similarly, all other dual objects will be denoted by a hat. By construction,

$$\widehat{M} = \overline{\{(\omega \otimes \iota)(W) : \omega \in B(\mathcal{H})_*\}}^{\sigma-\text{strong}^*}.$$

Furthermore, $\widehat{W} = \Sigma W^* \Sigma$, where Σ denotes the flip on $\mathcal{H} \otimes \mathcal{H}$. This implies that $W \in M \otimes \widehat{M}$ and

$$M = \overline{\{(\iota \otimes \omega)(W) : \omega \in B(\mathcal{H})_*\}}^{\sigma-\text{strong}^*}.$$

The dual coproduct can be given by the dualized formula

$$\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}, x \in \widehat{M}.$$

For $\omega \in M_*$, we use the standard notation

$$\lambda(\omega) = (\omega \otimes \iota)(W).$$

Finally, we introduce the dual left Haar weight $\widehat{\varphi}$. Recall from Definition 3.2, that we let $\mathcal{I}$ be the set of $\omega \in M_*$, such that $\Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi$ extends to a bounded functional on $\mathcal{H}$. By the Riesz theorem, for every $\omega \in \mathcal{I}$, there is a unique vector denoted by $\xi(\omega) \in \mathcal{H}$ such that $\omega(x^*) = \langle \Lambda(x), \xi(\omega) \rangle, x \in \mathfrak{n}_\varphi$.

DEFINITION 4.3. The dual left Haar weight $\widehat{\varphi}$ is defined to be the unique normal, semi-finite, faithful weight on $\widehat{M}$, with GNS-construction $(\mathcal{H}, \iota, \widehat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a σ-strong*/norm core for $\widehat{\Lambda}$ and $\widehat{\Lambda}(\lambda(\omega)) = \xi(\omega), \omega \in \mathcal{I}$.

Since we do not need it, we merely mention that there also exists a dual right Haar weight. The following example gives the dual structure in the classical situation.

EXAMPLE 4.4. Let $G \to B(L^2(G)) : x \mapsto \lambda_x$ be the left regular representation. For (M, Δ_G) as in Example 4.2, one finds that $\mathcal{H} \otimes \mathcal{H} = L^2(G) \otimes L^2(G) \simeq L^2(G \times G)$ and

$$Wf(x, y) = f(x, x^{-1}y).$$

For $f \in L^1(G)$, let ω_f be the functional on $L^\infty(G)$ defined by

$$\omega_f(g) = \int_G f(x)g(x)\mathrm{d}_l x.$$

Then,

$$\lambda(\omega_f) = (\omega_f \otimes \iota)(W) = \int_G f(x)\lambda_x \mathrm{d}_l x,$$

where the integral is in the σ-strong* topology. So λ is the left regular representation. We find that $\widehat{M}$ is given by the group von Neumann algebra $\widehat{M} = \mathcal{L}(G)$.

For completeness, we mention that $\widehat{\Delta}(\lambda_x) = \lambda_x \otimes \lambda_x$. The dual left Haar weight is given by the Plancherel weight [15]. For a continous, compactly supported function f on G, one finds $\widehat{\varphi}(\lambda(f)) = f(e)$, where e is the identity of G.

If G is abelian, conjugation with the L^2-Fourier transform shows that this structure is isomorphic to $(L^\infty(\widehat{G}), \Delta_{\widehat{G}})$.

5. FOURIER THEORY

In this section we define an L^p-Fourier transform. Our strategy is similar to the one defining the classical L^p-Fourier transform on locally compact abelian groups. We first define an L^1- and L^2-Fourier transform and show that they form a compatible pair of morphisms, see Remark 2.3. Then we apply the complex interpolation method to get a bounded L^p-Fourier transform for $p \in [1,2]$,

$$\mathcal{F}_p : L^p(M)_{\text{left}} \to L^q(M)_{\text{left}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The crucial property of the complex interpolation method is the one given by Theorem 2.5. The theorem gives the non-commutative analogue of the Riesz–Thorin theorem as mentioned in the introduction. It is for this reason that we have approached L^p-spaces from the perspective of interpolation spaces and that we have used Izumi's definition.

NOTATION 5.1. From now on, let (M, Δ) be a locally compact quantum group with left Haar weight φ. $(\widehat{M}, \widehat{\Delta})$ is the Pontrjagin dual. In this section all L^p-spaces we encounter are "left" L^p-spaces which are defined with respect to the (dual) left Haar weight. More precisely, we stick to Notation 3.1. We equip the objects introduced in Section 2 with a hat if they are associated with the dual quantum group. So we get $\widehat{L}, \widehat{R}, L^p(\widehat{M})_{\text{left}}, \ldots$ Recall that by construction $\widehat{\mathcal{H}} = \mathcal{H}$.

THEOREM 5.2 (L^1- and L^2-Fourier transform). *We can define compatible Fourier transforms in the following way:*

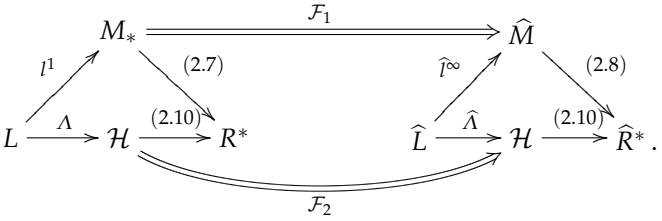
(i) *There exists a unique unitary map $\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$, which is determined by:*

$$\Lambda(x) \mapsto \widehat{\Lambda}(\lambda({}_x\varphi)), \quad x \in L. \tag{5.1}$$

(ii) *There exists a bounded map $\mathcal{F}_1 : M_* \to \widehat{M} : \omega \mapsto \lambda(\omega)$. Moreover, $\|\mathcal{F}_1\| = 1$.*

(iii) *$\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$ and $\mathcal{F}_1 : M_* \to \widehat{M}$ are compatible in the sense of Remark 2.3, i.e. the following diagram commutes:*

(5.2)



Proof. (i) By Proposition 3.3, we see that for $x \in L$, we have ${}_x\varphi \in \mathcal{I}$. By definition of $\widehat{\Lambda}$, we have $\widehat{\Lambda}(\lambda({}_x\varphi)) = \xi({}_x\varphi)$. Since by (2.6),

$${}_x\varphi(y^*) = \varphi(y^*x) = \langle \Lambda(x), \Lambda(y) \rangle, \quad y \in \mathfrak{n}_\varphi,$$

we see that by definition of $\xi({}_x\varphi)$, we have $\xi({}_x\varphi) = \Lambda(x)$. So (5.1) is the identity map. Since $\mathcal{T}_\varphi^2 \subseteq L$ and $\Lambda(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{H}$, see the much stronger result of Lemma 8.2, this determines a map on $\mathcal{H}$.

(ii) The norm bound follows, since:

$$\|\lambda(\omega)\| = \|(\omega \otimes \iota)(W)\| \leqslant \|(\omega \otimes \iota)\|\|W\| \leqslant \|\omega\|.$$

(iii) For $y \in \widehat{R}, x \in L$, we find:

$$\begin{aligned}\langle \mathcal{F}_2\Lambda(x), y\rangle_{\widehat{R}^*,\widehat{R}} &= \langle \widehat{\Lambda}(\lambda({}_x\varphi)), y\rangle_{\widehat{R}^*,\widehat{R}} \overset{(2.10)}{=} \langle \widehat{\Lambda}(\lambda({}_x\varphi)), \widehat{\Lambda}(y^*)\rangle \\ &= \widehat{\varphi}(y\lambda({}_x\varphi)) \overset{(2.6)}{=} \widehat{\varphi}_y(\lambda({}_x\varphi)) \overset{(2.8)}{=} \langle \lambda({}_x\varphi), y\rangle_{\widehat{R}^*,\widehat{R}} = \langle \mathcal{F}_1({}_x\varphi), y\rangle_{\widehat{R}^*,\widehat{R}},\end{aligned}$$

which proves the commutativity of the diagram. ∎

REMARK 5.3. Kahng [10] defines an operator algebraic Fourier transform and in principle the idea behind Theorem 5.2 can also be found here. However, Definition 3 of [10] has to be given a more careful interpretation, since, if φ is not a state, the expression $(\varphi \otimes \iota)(W(a \otimes 1)), a \in \widehat{\lambda}(\widehat{\mathcal{I}})$, is in general undefined. In case φ is a state, our definition of $\mathcal{F}_2$ equals Kahng's by Remark 2.9.

We comment on the classical situation. As is shown in Example 4.4, the Fourier transform is implicitly used to define the dual quantum group of a classical abelian group. It is for this reason that the L^2-Fourier transform $\mathcal{F}_2$ trivializes on the level of GNS-spaces. We work this out in the next example.

EXAMPLE 5.4. Let G be a locally compact abelian group. For $f \in L^1(G)$, let ω_f be the normal functional on $L^\infty(G)$ given by $\omega_f(g) = \int_G f(x)g(x)\mathrm{d}_l x$. Then,

$$\lambda((\omega_f)) = (\omega_f \otimes \iota)(W) = \int_G f(x)\lambda_x \mathrm{d}_l x \in \mathcal{L}(G),$$

where $x \mapsto \lambda_x$ is the left-regular representation. On the other hand, using the direct integral decomposition $L^\infty(\widehat{G}) = \int_{\widehat{G}}^{\oplus} \mathbb{C}\mathrm{d}\pi$, we find for (1.1),

$$\widehat{f} = \int_{\widehat{G}}^{\oplus} \int_G f(x)\pi(x)\mathrm{d}x\,\mathrm{d}\pi = \int_G f(x) \int_{\widehat{G}}^{\oplus} \pi(x)\mathrm{d}\pi\,\mathrm{d}x.$$

The left regular representation $x \mapsto \lambda_x$ is unitarily equivalent to $\int_{\widehat{G}}^{\oplus} \pi\mathrm{d}\pi$, where the intertwiner is given by the (classical) L^2-Fourier transform. Since the dual quantum group associated to a classical group is given by conjugating $L^\infty(\widehat{G})$ with the classical L^2-Fourier transform, see Example 4.4, the identification of the dual GNS-space $L^2(\widehat{G})$ with the GNS-space $L^2(G) = \mathcal{H}$ is given by applying the classical (inverse) L^2-Fourier transform. Hence, using these identifications,

we see that the transform defined in (1.1) is the quantum group analogue of the transform of Theorem 5.2.

Note that it is due to the identifications $L^2(M)_{\text{left}}$ with $\mathcal{H}$ and $L^2(\widehat{M})_{\text{left}}$ with $\widehat{\mathcal{H}} = \mathcal{H}$ that the L^2-Fourier transform becomes the identity map. If we had not made these identifications the map would be less trivial. It is for this reason that we have choosen to write *unitary map* in the first statement of Theorem 5.2 instead of *identity map*.

Note that moreover, our transform coincides with the definition given in Definition 1.3 of [20]. To comment on this, suppose that (M, Δ) is compact, i.e. φ is a state. Let $A \subseteq M$ be the Hopf algebra of the underlying algebraic quantum group. We mention that A is the Hopf algebra of matrix coefficients of irreducible, unitary corepresentations of M and refer to [18] for more explanation. Let $\widehat{A}$ be its dual, which is the space of linear functionals on A of the form $\varphi(\,\cdot\, x)$, where $x \in A$, ([20], Theorem 1.2). Van Daele defines the transform by

$$A \to \widehat{A} : x \mapsto \varphi(\,\cdot\, x), \quad x \in A.$$

On the other hand, by Remark 2.9, the normal functional ${}_x\varphi$, with $x \in L$ is given by $x\varphi$, since we assumed that φ is a state. Here $x\varphi \in M_*$ is defined by $(x\varphi)(y) = \varphi(yx), y \in M$ (we use this notation to distinguish it from the algebraic map $\varphi(\,\cdot\, x)$). So the Fourier transform is defined by:

$$x \mapsto \lambda(x\varphi), \quad x \in M.$$

Since in the transition of compact algebraic quantum groups to compact von Neumann algebraic quantum groups, the element $\varphi(\,\cdot\, x) \in \widehat{A}$ corresponds to $\lambda(x\varphi) \in \widehat{M}$, this shows the correspondence. Here, we refer to [18] for compact algebraic quantum groups and their relations to locally compact quantum groups.

By Pontrjagin duality, one also find dual Fourier transforms. At every point in Theorem 5.2 where one of the objects $L, R, M, \varphi, \Lambda, \lambda, \mathcal{F}_2$ or $\mathcal{F}_1$ appears, one should replace the object by the same object equipped with a hat and vise versa. In that way, we get a dual L^2-Fourier transform $\widehat{\mathcal{F}_2} : \mathcal{H} \to \mathcal{H}$, determined by

$$\widehat{\Lambda}(x) \mapsto \Lambda(\widehat{\lambda}({}_x\widehat{\varphi})), \quad x \in \widehat{L}.$$

Since this map is on the GNS-level given by the identity, we automatically find the following corollary.

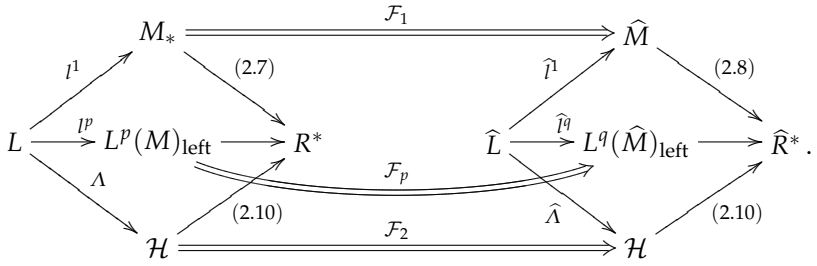
COROLLARY 5.5. *We have* $\mathcal{F}_2^{-1} = \widehat{\mathcal{F}_2}$.

In the following theorem, we apply the complex interpolation method to define L^p-Fourier transforms.

THEOREM 5.6. *Let* $p \in [1, 2]$ *and set* q *by* $1/p + 1/q = 1$. *There exists a unique bounded linear map* $\mathcal{F}_p : L^p(M)_{\text{left}} \to L^q(\widehat{M})_{\text{left}}$ *such that* $\mathcal{F}_p$ *is compatible with* $\mathcal{F}_1$

and $\mathcal{F}_2$ in the sense of Remark 2.3, i.e. the following diagram commutes:

(5.3)



Moreover, $\|\mathcal{F}_p\| \leqslant 1$.

Proof. We can apply the complex interpolation method with parameter $\theta = 2/p - 1 = 1 - 2/q$ to the pairs $(\mathcal{H}, M_*)$ and $(\mathcal{H}, \widehat{M})$ which are compatible couples as in (5.2). By Theorem 3.7 the corresponding interpolation spaces are respectively $L^p(M)_{\text{left}}$ and $L^q(\widehat{M})_{\text{left}}$.

Since by Theorem 5.6, $\mathcal{F}_1 : M_* \to \widehat{M}$ and $\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$ are compatible, with respect to diagram (5.2), we can use Remark 2.3 to obtain a map $\mathcal{F}_p : L^p(M)_{\text{left}} \to L^q(\widehat{M})_{\text{left}}$ with the desired properties. ∎

We conclude this section by giving the Fourier transform explicitly in terms of Hilsum's L^p-spaces. We omit the proof and merely give a few comments. The result relies on some technicalities involving Hilsum's L^p-spaces, which was not our focus. The theorem is not needed for the subsequent sections.

THEOREM 5.7. *Let $p \in [1,2]$ and set q by $1/p + 1/q = 1$. Fix a normal, semi-finite, faithful weight ϕ on M' and $\widehat{\phi}$ on $\widehat{M}'$. Set the corresponding spatial derivatives $d = \mathrm{d}\varphi/\mathrm{d}\phi$ and $\widehat{d} = \mathrm{d}\widehat{\varphi}/\mathrm{d}\widehat{\phi}$. Then*

$$\widehat{\Phi}_q^{-1}\mathcal{F}_p\Phi_p : L^p(\phi) \to L^q(\widehat{\phi}) : [ad^{1/p}] \mapsto [\lambda({}_a\varphi)\widehat{d}^{1/q}], \quad a \in \mathcal{T}_\varphi^2.$$

Note that in Theorem 5.7, we see that $[ad^{1/p}]$ is in $L^p(\phi)$ by Proposition 2.21. Moreover, since ${}_a\varphi \in \mathcal{I}$, we see that $\lambda({}_a\varphi) \in \mathfrak{n}_{\widehat{\varphi}}$. Therefore, $[\lambda({}_a\varphi)\widehat{d}^{\,1/q}]$ is in $L^q(\widehat{\phi})$ by Lemma 2.20. The theorem follows by a careful analyis of (5.3) involving Proposition 2.21. The proof then relies on the following fact. For $x \in \mathfrak{n}_{\widehat{\varphi}}$, one can consider x as an element of $L^q(\widehat{M})_{\text{left}}$, see Section 3, and one can prove that $\widehat{\Phi}_q x = [x\widehat{d}^{\,1/q}]$.

6. CONVOLUTION PRODUCT

We define convolutions of elements in $L^1(M)_{\text{left}} = M_*$ with elements in $L^p(M)_{\text{left}}$. We prove that the Fourier transform transfers the convolution product into a product on the dual quantum group.

NOTATION 6.1. We keep the notation as in Section 5, c.f. Notation 5.1.

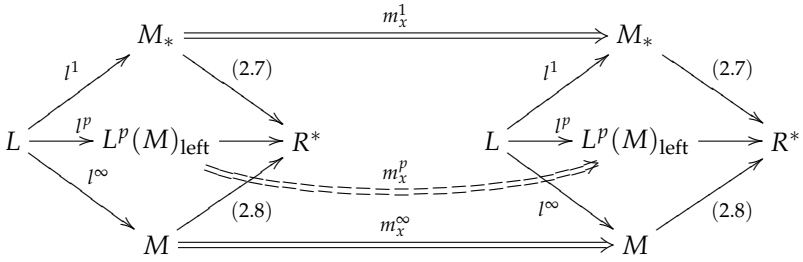
Note that since Izumi's L^p-spaces are defined by means of complex interpolation, there is a priori no multiplication on these spaces. Therefore, we extend the multiplication of M to the L^p-setting in the following proposition. This seems to be the most natural definition of a multiplication in the L^p-setting. The proof of the following proposition is completely similar to the one of Theorems 5.2(iii) and 5.6.

PROPOSITION 6.2. *We extend the product of M to the L^p-setting.*
(i) *Let $x \in M$. The maps*

$$m_x^\infty : M \to M : y \mapsto xy, \quad m_x^1 : M_* \to M_* : \omega \mapsto x\omega,$$

are compatible in the sense of Remark 2.3*, i.e. the non-dotted arrows in the following diagram commute:*

(6.1)



(ii) *Let $p \in (1, \infty)$. There is a unique bounded map $m_x^p : L^p(M)_{\text{left}} \to L^p(M)_{\text{left}}$ that is compatible with m_x^∞ and m_x^1, i.e. the dotted arrow in* (6.1) *makes the diagram commutative.*

DEFINITION 6.3. Let $x \in M$ and let $y \in L^p(M)_{\text{left}}$. We will write xy for $m_x^p(y)$.

For $\omega_1, \omega_2 \in M_*$, we define the convolution product,

$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta.$$

This product is well-known in the theory of l.c. quantum groups. We show that it is possible to extend it to the L^p-setting for $p \in [1, 2]$. Moreover, the convolution product is turned into the product of Definition 6.3 by the Fourier transfrom.

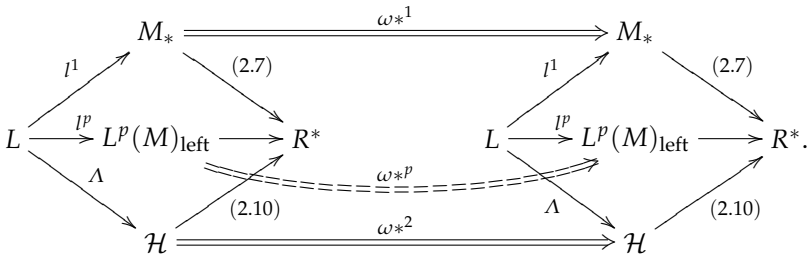
THEOREM 6.4. *Let $p \in [1, 2]$ and set $q \in [2, \infty]$ by $1/p + 1/q = 1$.*
(i) *Let $x \in L$ and let $\omega \in M_*$. Then,*

$$\omega * ({}_x\varphi) \in \mathcal{I} \quad \text{and} \quad \xi(\omega * ({}_x\varphi)) = \lambda(\omega)\Lambda(x).$$

(ii) *Let $\omega \in M_*$. We denote $\omega *^2$ for the bounded operator $\lambda(\omega) : \mathcal{H} \to \mathcal{H}$. Furthermore, we define $\omega *^1 : M_* \to M_* : \theta \mapsto \omega * \theta$. Then, $\omega *^1$ and $\omega *^2$ are compatible, i.e.*

the non-dotted arrows in following diagram commute:

(6.2)



(iii) *There is a unique bounded operator* $\omega*^p : L^p(M)_{\text{left}} \to L^p(M)_{\text{left}}$ *that is compatible with* $\omega*^1$ *and* $\omega*^2$, *i.e.* (6.2) *commutes.*

(iv) *For* $\omega \in M_*$, $a \in L^p(M)_{\text{left}}$,

$$\mathcal{F}_1(\omega)\mathcal{F}_p(a) = \mathcal{F}_p(\omega *^p a),$$

where the left hand side uses Definition 6.3 *for* $L^q(\widehat{M})_{\text{left}}$.

Proof. Let $\theta \in \widehat{\mathcal{I}}$ and put $y = \widehat{\lambda}(\theta) = (\iota \otimes \theta)(W^*)$. Now, (ii) follows from,

$$\begin{aligned}(\omega * ({}_x\varphi))(y^*) &= (\omega \otimes ({}_x\varphi))\Delta((\iota \otimes \theta)(W^*)^*) = (\omega \otimes ({}_x\varphi) \otimes \overline{\theta})(W_{13}W_{23}) \\ &= \overline{\theta((\,(\omega \otimes \iota)(W)((_x\varphi) \otimes \iota)(W)\,)^*)} \\ &= \langle \widehat{\Lambda}((\omega \otimes \iota)(W)((_x\varphi) \otimes \iota)(W)), \widehat{\xi}(\theta)\rangle \\ &= \langle (\omega \otimes \iota)(W)\xi({}_x\varphi), \Lambda(y)\rangle = \langle \lambda(\omega)\Lambda(x), \Lambda(y)\rangle,\end{aligned}$$

and the fact that $\{\Lambda((\iota \otimes \theta)(W^*)) : \theta \in \widehat{\mathcal{I}}\}$ is dense in $\mathcal{H}$.

The compatibility in (ii) follows directly from (i) using Theorem 3.3. (iii) follows by applying Theorem 3.7 to (ii). (iv) For $\omega_1, \omega_2 \in M_*$, note that

$$\begin{aligned}\mathcal{F}_1(\omega_1 * \omega_2) &= (\omega_1 \otimes \omega_2 \otimes \iota)(\Delta \otimes \iota)(W) \\ &= (\omega_1 \otimes \omega_2 \otimes \iota)W_{13}W_{23} = (\omega_1 \otimes \iota)(W)(\omega_2 \otimes \iota)(W).\end{aligned}$$

For $x \in L, \omega \in M_*$,

$$\mathcal{F}_p(\omega *^p l^p(x)) = \mathcal{F}_1(\omega * ({}_x\varphi)) = \mathcal{F}_1(\omega)\mathcal{F}_1({}_x\varphi) = \mathcal{F}_1(\omega)\mathcal{F}_p(l^p(x)).$$

Here, the first and last equality follows from commutativity of (5.3), (6.1) and (6.2). Since the range of l^p is dense in $L^p(M)_{\text{left}}$, see Lemma 2.7, (iv) follows. ∎

7. A DISTINGUISHED CHOICE FOR THE INTERPOLATION PARAMETER

Recall that in Sections 3 to 6 we considered the compatible couple (M, M_*) for the interpolation parameter $z = -1/2$, see Definition 2.8. For this parameter one is able to define an L^p-Fourier transform. In this section we show that the

real part of the parameter is distinguished. More precisely, we investigate the example of $(M, \Delta) = SU_q(2)$ and show that given the fact that

$$\mathcal{F}_1 : M_* \to \widehat{M} : \omega \mapsto (\omega \otimes \iota)(W), \tag{7.1}$$

is the L^1-Fourier transform the only interpolation parameters z that allows a passage to an L^p-Fourier transform are $z = -1/2 + \mathrm{i}t$, where $t \in \mathbb{R}$.

The importance of this result is strengthened by the final remark of [4]. For classical, locally compact groups there is an approximation property called Reiter's property (P_p), where $p \in [1, \infty)$. The definition assumes the existence of a net of functions in $L^p(G)$ satisfying the approximation axiom of Definition 1.2 of [4]. Daws and Runde show that (P_1) and (P_2) can be defined for quantum groups as well and they use them to study (co-)amenability properties of quantum groups.

In the final remark of [4], Daws and Runde mention that it remains to be seen if there is a property (P_p) for any $p \in [1, \infty)$. In particular, they mention that it remains unclear how the L^p-space associated with a quantum group should be turned into an L^1-module. In [5] this is done using Izumi's L^p-spaces for the complex interpolation parameter $z = -1/2$, whereas in [3] a similar, but not identical construction was used for the parameter $z = 0$. We believe that the Fourier transform indicates that the most natural choice would be $z = -1/2$.

From now on we let (M, Δ) be the quantum group $SU_q(2)$, see [21], [22], [11]. See also [12] for a concise introduction. We recall its most important properties.

We set $\mathcal{H} = L^2(\mathbb{N}) \otimes L^2(\mathbb{T})$. Let $(e_i)_{i \in \mathbb{N}}$ be the canonical orthonormal basis of $L^2(\mathbb{N})$ and let $(f_k)_{k \in \mathbb{Z}}$ be the canonical orthonormal basis for $L^2(\mathbb{Z})$ (so $f_k = \zeta^k$, where ζ is the identity function on the complex unit circle $\mathbb{T}$). Define operators α, γ given by:

$$\alpha\, e_i \otimes f_k = \sqrt{1 - q^{2i}} e_{i-1} \otimes f_k, \quad \gamma\, e_i \otimes f_k = q^i e_i \otimes f_{k+1}. \tag{7.2}$$

Then, M is the von Neumann algebra generated by α and γ, i.e.

$$M = B(L^2(\mathbb{N})) \otimes L^\infty(\mathbb{T}) \simeq L^\infty(\mathbb{T}, B(L^2(\mathbb{N}))).$$

For $x = x(t) \in L^\infty(\mathbb{T}, B(L^2(\mathbb{N})))$, both the left and right Haar weight are given by the state

$$\varphi(x) = \frac{(1 - q^2)}{2\pi} \int_{\mathbb{T}} \sum_{i=0}^{\infty} q^{2i} \langle x(t) e_i, e_i \rangle \mathrm{d}t.$$

Next, we need Peter–Weyl theory for $SU_q(2)$. Recall [11] that for every $l \in (1/2)\mathbb{N}$, there exists a unique irreducible corepresentation $t^{(l)} \in M \otimes M_{2l+1}(\mathbb{C})$. In fact, these are all the irreducible corepresentations of (M, Δ) and we have a Peter–Weyl decomposition

$$W \simeq \bigoplus_{l \in (1/2)\mathbb{N}} t^{(l)} \otimes 1_{2l+1} \quad \Big(\quad \in M \otimes \bigoplus_{l \in \mathbb{N}} M_{2l+1}(\mathbb{C}) \otimes M_{2l+1}(\mathbb{C}) \quad \Big).$$

So every corepresentation $t^{(l)}$ appears $2l+1$ times in the multiplicative unitary.

Let $g^{(l)}_{-l}, g^{(l)}_{-l+1}, \dots, g^{(l)}_{l}$ denote the standard basis vectors of $\mathbb{C}^{2l+1}$. Let us denote $t^{(l)}_{i,j}$ for the matrix elements $(\iota \otimes \omega_{g^{(l)}_j, g^{(l)}_i})(t^{(l)})$. For every $l \in (1/2)\mathbb{N}$, there exists a unique strictly positive operator $Q^{(l)}$ such that we have orthogonality relations between matrix coefficients:

$$\varphi((t^{(l)}_{i,j})^* t^{(l')}_{i',j'}) = \delta_{l,l'}\delta_{j,j'}\langle Q^{(l)} g^{(l)}_i, g^{(l)}_{i'}\rangle. \tag{7.3}$$

In fact, with respect to the basis $g^{(l)}_i$, the matrix $Q^{(l)}$ is diagonal. It follows that $\widehat{M} \simeq \bigoplus_{l\in(1/2)\mathbb{N}} M_{2l+1}(\mathbb{C})$. We put $Q = \bigoplus_{l\in(1/2)\mathbb{N}} Q^{(l)}$, so that Q is affiliated with $\widehat{M}$. Moreover,

$$\widehat{\sigma}_t(x) = Q^{-\mathrm{i}t} x Q^{\mathrm{i}t}, \quad x \in \widehat{M}. \tag{7.4}$$

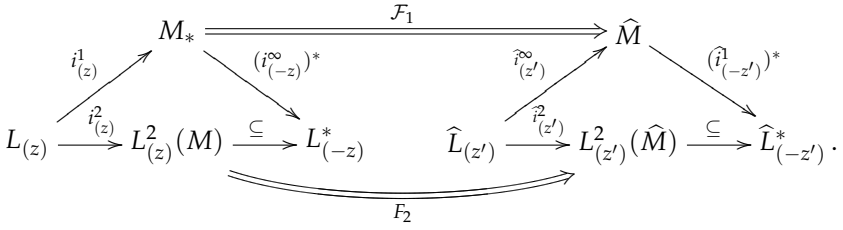
Finally, the following two matrix coefficients will play an essential role in the proof of the main theorem of this section. It follows from Chapter 4 of [11] that for $n \in \mathbb{N}$,

$$t^{(n/2)}_{n/2,n/2} = \alpha^n, \quad t^{(n/2)}_{-n/2,-n/2} = (\alpha^*)^n.$$

Recall the notational conventions from Section 2.

THEOREM 7.1. *Consider $(M,\Delta) = SU_q(2)$ and let $z, z' \in \mathbb{C}$. Let $\mathcal{F}_1 : M_* \to \widehat{M}$ be defined as in* (7.1)*. Suppose that there is bounded map $F_2 : L^2_{(z)}(M) \to L^2_{(z')}(\widehat{M})$ making the following diagram commutative:*

(7.5)



Then, $z = -1/2 + \mathrm{i}t$ for some $t \in \mathbb{R}$.

Proof. We will prove that F_2 is unbounded unless $z = -1/2 + \mathrm{i}t$ for some $t \in \mathbb{R}$. We need three preparations.

Firstly, the modular automorphism group of φ is given by

$$\sigma_t(x) = (\gamma\gamma^*)^{\mathrm{i}t} x (\gamma\gamma^*)^{-\mathrm{i}t}.$$

Hence, it follows from (7.2) that $\alpha \in \mathcal{T}_\varphi$ and

$$\sigma_z(\alpha) = q^{-2\mathrm{i}z}\alpha, \quad \sigma_z(\alpha^*) = q^{2\mathrm{i}z}\alpha^*.$$

Secondly, for $a, b \in \mathcal{T}_\varphi, x \in \mathcal{T}_\varphi^2, z \in \mathbb{C}$, we find

$$\begin{aligned}\varphi_x^{(z)}(a^*b) &= \langle xJ\nabla^{\overline{z}}\Lambda(a), J\nabla^{-z}\Lambda(b)\rangle \\ &= \langle \nabla^{z+(1/2)}x\nabla^{-z-(1/2)}\nabla J\nabla^{1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle \\ &= \langle \sigma_{-\mathrm{i}(z+(1/2))}(x)\Lambda(\sigma_{-\mathrm{i}}(a^*)), \Lambda(b^*)\rangle = \varphi(b\sigma_{-\mathrm{i}(z+(1/2))}(x)\sigma_{-\mathrm{i}}(a^*)) \\ &= \varphi(a^*b\sigma_{-\mathrm{i}(z+(1/2))}(x)).\end{aligned}$$

So we conclude that $\varphi_x^{(z)} = \sigma_{-\mathrm{i}(z+(1/2))}(x)\varphi$.

Thirdly, we identify $\widehat{M}$ with $\bigoplus_{l\in(1/2)\mathbb{N}} M_{2l+1}(\mathbb{C})$. Let $e_{n/2,n/2}^{(n/2)}$ (and $e_{-n/2,-n/2}^{(n/2)}$) be the element of $\widehat{M}$, with matrix elements equal to zero everywhere, except for the summand with index $n/2$, where it has a 1 on the upper left (respectively lower right) corner. Using the Peter–Weyl orthogonality relations, we see that for every $n \in \mathbb{N}$:

$$(\varphi\otimes\iota)(W(\alpha^n\otimes 1)) = \bigoplus_{l\in(1/2)\mathbb{N}} (\varphi\otimes\iota)(t^{(l)}(t_{-n/2,-n/2}^{(n/2)*}\otimes 1)) = \varphi((\alpha^*)^n\alpha^n)e_{-n/2,-n/2}^{(n/2)},$$

$$(\varphi\otimes\iota)(W((\alpha^*)^n\otimes 1)) = \bigoplus_{l\in\mathbb{N}} (\varphi\otimes\iota)(t^{(l)}(t_{n/2,n/2}^{(n/2)*}\otimes 1)) = \varphi(\alpha^n(\alpha^*)^n)e_{n/2,n/2}^{(n/2)},$$

$$\varphi((\alpha^*)^n\alpha^n) = \frac{(1-q^2)q^{2n}}{1-q^{2n+2}}, \quad \varphi(\alpha^n(\alpha^*)^n) = \frac{(1-q^2)}{1-q^{2n+2}}. \tag{7.6}$$

By (7.4) and the fact that Q is diagonal we see that (7.6) are in $\mathcal{T}_{\widehat{\varphi}}^2$ and

$$\widehat{\sigma}_z(e_{-n/2,-n/2}^{(n/2)}) = e_{-n/2,-n/2}^{(n/2)}, \quad \widehat{\sigma}_z(e_{n/2,n/2}^{(n/2)}) = e_{n/2,n/2}^{(n/2)}. \tag{7.7}$$

Now we prove that F_2 must be unbounded by proving that the map

$$U_{(z')}F_2U_{(z)}^* : \mathcal{H} \to \mathcal{H},$$

is unbounded. Recall that $U_{(z)}$ was defined in Proposition 2.21. Then

$$\begin{aligned}F_2U_{(z)}^*\Lambda(\sigma_{-\mathrm{i}(z+1/2)/2}(\alpha^n)) &= F_2i_{(z)}^2(\alpha^n) = (\widehat{i}_{(-z')}^1)^*F_1i_{(z)}^1(\alpha^n) \\ &= (\widehat{i}_{(-z')}^1)^*(\varphi_{\alpha^n}^{(z)}\otimes\iota)(W) \\ &= (\widehat{i}_{(-z')}^1)^*(\varphi\otimes\iota)(W(\sigma_{-\mathrm{i}(z+1/2)}(\alpha^n)\otimes 1)) \\ &= q^{-2n(z+1/2)}(\widehat{i}_{(-z')}^1)^*(\varphi\otimes\iota)(W(\alpha^n\otimes 1)).\end{aligned}$$

Since $(\varphi\otimes\iota)(W(\alpha^n\otimes 1)) \in \mathcal{T}_{\widehat{\varphi}}^2$, we see that by commutativity of the right triangle in (7.5),

$$F_2U_{(z)}^*\Lambda(\sigma_{-\mathrm{i}(z+1/2)/2}(\alpha^n)) = q^{-2n(z+1/2)}(\widehat{i}_{(z')}^2)(\varphi\otimes\iota)(W(\alpha^n\otimes 1)).$$

Hence,

$$\begin{aligned} U_{(z')}F_2U^*_{(z)}\Lambda(\sigma_{-\mathrm{i}(z+1/2)/2}(\alpha^n)) &= q^{-2n(z+1/2)}\widehat{\Lambda}(\widehat{\sigma}_{-\mathrm{i}(z'+1/2)/2}(\varphi\otimes\iota)(W(\alpha^n\otimes 1))) \\ &\overset{(7.6),(7.7)}{=} q^{-2n(z+1/2)}\widehat{\Lambda}((\varphi\otimes\iota)(W(\alpha^n\otimes 1))) \\ &= q^{-2n(z+1/2)}\xi(\alpha^n\varphi) = q^{-2n(z+1/2)}\Lambda(\alpha^n). \end{aligned}$$

Hence,

$$U_{(z')}F_2U^*_{(z)} : \Lambda(\alpha^n) = q^{n(z+1/2)}\Lambda(\sigma_{-\mathrm{i}(z+1/2)/2}(\alpha^n)) \mapsto q^{-n(z+1/2)}\Lambda(\alpha^n),$$

which is unbounded in case $\mathrm{Re}(z) > -1/2$.

By a similar computation, one finds that

$$U_{(z')}F_2U^*_{(z)} : \Lambda((\alpha^*)^n) \mapsto q^{n(z+1/2)}\Lambda((\alpha^*)^n).$$

In this case we see that $U_{(z')}F_2U^*_{(z)}$ is unbounded for $\mathrm{Re}(z) < -1/2$. ∎

REMARK 7.2. By Pontrjagin duality also the dual statement holds. So in order to get a proper Fourier theory on quantum groups with Fourier transforms and inverse Fourier transforms, one is obliged to take the interpolation parameter on both M and $\widehat{M}$ to be $-1/2$.

8. APPENDIX A

We have not found an explicit proof of the following lemmas' in the literature. For completeness, we prove them here. Here M is a von Neumann algebra with normal, semi-finite, faithful weight φ.

The following lemma is a variant of Lemma 9 of [17].

LEMMA 8.1. *Let $\delta > 0$. There exists a net $(e_j)_{j\in J}$ in $\mathcal{T}_\varphi$ such that:*

(i) $\|\sigma_z(e_j)\| \leqslant e^{\delta \mathrm{Im}(z)^2}$;
(ii) $e_j \to 1$ *strongly; and*
(iii) $\sigma_{\mathrm{i}/2}(e_j) \to 1$ *σ-weakly.*

Proof. Let $(f_j)_{j\in J}$ and $(e_j)_{j\in J}$ be nets as in Lemma 9 of [17]. This lemma proves already that $(e_j)_{j\in J}$ satisfies (i) and (ii). Now, (iii) follows, since for $\xi \in \mathcal{H}$,

$$\begin{aligned} \langle\sigma_{\mathrm{i}/2}(e_j)\xi,\xi\rangle &= \omega_{\xi,\xi}\Big(\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\delta(t-(\mathrm{i}/2))^2}\sigma_t(f_j)\mathrm{d}t\Big) = \Big(\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\delta(t-(\mathrm{i}/2))^2}(\omega_{\xi,\xi}\circ\sigma_t)\mathrm{d}t\Big)(f_j) \\ &\to \Big(\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\delta(t-(\mathrm{i}/2))^2}(\omega_{\xi,\xi}\circ\sigma_t)\mathrm{d}t\Big)(1) \\ &= \sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\delta(t-(\mathrm{i}/2))^2}\mathrm{d}t\langle\xi,\xi\rangle = \langle\xi,\xi\rangle, \end{aligned}$$

where the last equality is obtained by means of the residue formula for meromorphic functions. So $\sigma^{\varphi}_{i/2}(e_k)$ is bounded and converges weakly, hence σ-weakly to 1. ∎

LEMMA 8.2. *$\mathcal{T}^2_\varphi$ is a σ-strong*/norm core for Λ.*

Proof. It is enough to prove that $\mathcal{T}^2_\varphi$ is a σ-weak/weak core for Λ, since the σ-weak/weak continuous functionals on the graph of Λ equal the σ-strong*/norm continuous functionals. It is not too hard to show that $\mathfrak{n}_\varphi \cap \mathfrak{n}^*_\varphi$ is a σ-weak/weak core for Λ. Now, let $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}^*_\varphi$. Put

$$x_n = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(nt)^2} \sigma_t(x) \mathrm{d}t,$$

where the integral is taken in the σ-strong* sense. By standard techniques (c.f. the proof of Lemma 9 of [17]), $x_n \in \mathcal{T}_\varphi$ and x_n converges σ-weakly to x. Moreover, using the the fact that Λ is σ-strong*/norm closed, we obtain

$$\Lambda(x_n) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(nt)^2} \nabla^{\mathrm{i}t} \Lambda(x) \mathrm{d}t \to \Lambda(x) \quad \text{weakly},$$

where the integral is a Bochner integral, c.f. Chapter VI, Lemma 2.4 of [15]. Hence $\mathcal{T}_\varphi$ is a core for Λ. Now, let $x \in \mathcal{T}_\varphi$ and let $(e_j)_{j \in J}$ be a net in $\mathcal{T}_\varphi$ such that $e_j \to 1$ σ-weakly, c.f. Lemma 8.1. Then, $e_j x \in \mathcal{T}^2_\varphi$ and $e_j x \to x$ σ-weakly and $\Lambda(e_j x) = e_j \Lambda(x) \to \Lambda(x)$ weakly. ∎

Acknowledgements. The author thanks Erik Koelink for useful discussions and also many thanks to the referee for his comments and suggestions.

REFERENCES

[1] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces. An introduction*, Grundlehren Math. Wiss., vol. 223. Springer-Verlag, Berlin-New York 1976.

[2] A. CONNES, On the spatial theory of von Neumann algebras, *J. Funct. Anal.* **35** (1980), 153–164.

[3] M. DAWS, Representing multipliers of the Fourier algebra on non-commutative L^p spaces, *Canad. J. Math.* **63**(2011), 798–825.

[4] M. DAWS, V. RUNDE, Reiter's properties (P_1) and (P_2) for locally compact quantum groups, *Math. Anal. Appl.* **2** (2010), 352–365.

[5] B. FORREST, H.H. LEE, E. SAMEI, Projectivity of modules over Fourier algebras, *Proc. Lond. Math. Soc.* **102**(2011), 697–730.

[6] L. GRAFAKOS, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Upper Saddle River, NJ 2004.

[7] U. HAAGERUP, L^p-spaces associated with an arbitrary von Neumann algebra, in *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille 1977)*, Colloq. Internat. CNRS, vol. 274, CNRS, Paris 1979, pp. 175–184.

[8] M. HILSUM, Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, *J. Funct. Anal.* **40**(1981), 151–169.

[9] H. IZUMI, Constructions of non-commutative L^p-spaces with a complex parameter arising from modular actions, *Internat. J. Math.* **8**(1997), 1029–1066.

[10] B.J. KAHNG, Fourier transform on locally compact quantum groups, *J. Operator Theory* **64** (2010), 69–87.

[11] A. KLIMYK, K. SCHMÜDGEN, *Quantum Groups and their Representations*, Text Monographs Phys., Springer-Verlag, Berlin 1997.

[12] J. KUSTERMANS, Locally compact quantum groups, in *Quantum Independent Increment Processes*. I, Lecture Notes in Math., vol. 1865, Springer, Berlin 2005, pp. 99–180.

[13] J. KUSTERMANS, S. VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* **33**(2000), 837–934.

[14] J. KUSTERMANS, S. VAES, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92**(2003), 68–92.

[15] M. TAKESAKI, *Theory of Operator Algebras*. II, Encyclopaedia Math. Sci., vol. 125, Springer-Verlag, Berlin 2003.

[16] M. TERP, L^p spaces associated with von Neumann algebras. Notes, Report No. 3a + 3b, Københavns Univ., Matematiske Inst., Københavns 1981.

[17] M. TERP, Interpolation spaces between a von Neumann algebra and its predual, *J. Operator Theory* **8**(1982), 327–360.

[18] T. TIMMERMANN, *An invitation to Quantum Groups and Duality*, From Hopf algebras to multiplicative unitaries and beyond, EMS Textbooks in Math., European Math. Soc., Zürich 2008.

[19] A. VAN DAELE, Locally compact quantum groups. A von Neumann algebra approach, http://arxiv.org/abs/math/0602212 (2006).

[20] A. VAN DAELE, The Fourier transform in quantum group theory, in *New Techniques in Hopf Algebras and Graded Ring Yheory*, K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels 2007, pp. 187–196.

[21] S.L. WORONOWICZ, Compact matrix pseudogroups , *Commun. Math. Phys.* **111**(1987), 613–665.

[22] S.L. WORONOWICZ, Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. RIMS Kyoto University* **23**(1987), 117–181.

MARTIJN CASPERS, IMAPP, RADBOUD UNIVERSITEIT, NIJMEGEN, 6525 AJ, THE NETHERLANDS

E-mail address: martijn.caspers@univ-fcomte.fr *and* caspers@math.ru.nl

Received August 22, 2010; revised May 12, 2011; posted on February 8, 2013.