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A FAMILY OF NON-COCYCLE CONJUGATE E_0-SEMIGROUPS OBTAINED FROM BOUNDARY WEIGHT DOUBLES

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ABSTRACT. Let $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$ be states, and define unital q-positive maps ϕ and ψ by $\phi(A) = \rho(A)I_n$ and $\psi(D) = \rho'(D)I_{n'}$ for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. We show that if ν and η are type II Powers weights, then the boundary weight doubles (ϕ, ν) and (ψ, η) induce non-cocycle conjugate E_0-semigroups if ρ and ρ' have different eigenvalue lists. We then classify the q-corners and hyper maximal q-corners from ϕ to ψ, finding that if ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, where $\Lambda(1) \in B(L^2(0,\infty))$ is the operator of multiplication by e^{-x}, then the E_0-semigroups induced by (ϕ, ν) and (ψ, ν) are cocycle conjugate if and only if $n = n'$ and ϕ and ψ are conjugate.

KEYWORDS: *E_0-semigroup, completely positive map, q-positive map.*

MSC (2010): 46L57, 46L07.

1. INTRODUCTION

Let H be a separable Hilbert space, denoting its inner product by the symbol $(\cdot,\cdot)$ which is conjugate-linear in its first entry and linear in its second. An E_0-semigroup $\alpha = \{\alpha_t\}_{t\geqslant 0}$ is a semigroup of unital $*$-endomorphisms of $B(H)$ which is weakly continuous in t. E_0-semigroups are divided into three types, depending on the existence and structure of their units. More specifically, if α is an E_0-semigroup and there is a strongly continuous semigroup $U = \{U_t\}_{t\geqslant 0}$ of bounded operators acting on H such that $\alpha_t(A)U_t = U_t A$ for all $A \in B(H)$ and $t \geqslant 0$, then we say that U is a unit for α. An E_0-semigroup is said to be spatial if it has at least one unit, and a spatial E_0-semigroup is called completely spatial if, in essence, its units can reconstruct H. We say an E_0-semigroup α is type I if it is completely spatial and type II if it is spatial but not completely spatial. If α has no units, we say it is of type III. Every spatial E_0-semigroup α is assigned an index $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ which corresponds to the dimension of a particular Hilbert space associated to its units. The type I E_0-semigroups have been classified up to cocycle conjugacy by their index in [2]: If α is of type I_n (type I, index n) for $n \in \mathbb{N} \cup \{\infty\}$, then α is cocycle conjugate to the CCR/CAR flow of rank n, while if α is of type I_0, then it is a semigroup of $*$-automorphisms. Arveson's comprehensive book on E_0-semigroups [4] provides a detailed exploration of the index (Sections 2.5, 2.6, 3.6, and others), the CCR and CAR flows (Section 2.1), and other fundamental results in the theory of E_0-semigroups, along with many relatively recent results in the subject.

In contrast to the type I case, uncountably many examples of non-cocycle conjugate E_0-semigroups of types II and III are known (see, for example, [8], [7], [14], [13], [12], and [16]). Bhat's dilation theorem [5] and developments in the theory of CP-flows ([15] and [14]) have led to the introduction of boundary weight doubles and related cocycle conjugacy results for E_0-semigroups in [9]. A boundary weight double is a pair (ϕ, ν), where $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive (that is, $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geqslant 0$) and ν is a positive boundary weight over $L^2(0, \infty)$. If ϕ is unital and ν is normalized and unbounded (in which case we say ν is a type II Powers weight), then (ϕ, ν) induces a unital CP-flow whose Bhat minimal dilation is a type II_0 E_0-semigroup α^d. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital and q-positive and $U \in M_n(\mathbb{C})$ is unitary, then the map $\phi_U(A) = U^*\phi(UAU^*)U$ is also unital and q-positive. The relationship between ϕ and ϕ_U is analogous to the definition of conjugacy for E_0-semigroups. With this in mind, we say that q-positive maps $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are *conjugate* if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$. If ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, where $\Lambda(1) \in B(L^2(0, \infty))$ is defined by $(\Lambda(1)g)(x) = \mathrm{e}^{-x}g(x)$ for all $g \in L^2(0, \infty)$ and $x > 0$, then (ϕ, ν) and (ϕ_U, ν) induce cocycle conjugate E_0-semigroups (for details, see Proposition 2.11 of [10] and the discussion preceding it).

Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ are unital rank one q-positive maps, so for some states $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$, we have $\phi(A) = \rho(A)I_n$ and $\psi(D) = \rho'(D)I_{n'}$ for all $A \in M_n(\mathbb{C}), D \in M_{n'}(\mathbb{C})$. Let ν and η be type II Powers weights. We prove three main results. First, we find that if (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups, then ρ and ρ' have identical eigenvalue lists (Definition 2.13 and Proposition 3.4). We then find all q-corners and hyper maximal q-corners from ϕ to ψ (see Remark 3.3 and Theorems 3.8 and 3.9). With this result in hand, we complete the cocycle conjugacy comparison theory for E_0-semigroups α^d and β^d induced by (ϕ, ν) and (ψ, ν) in the case that ν is of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, finding that α^d and β^d are cocycle conjugate if and only if $n = n'$ and ϕ is conjugate to ψ (Theorem 3.10).

2. BACKGROUND

2.1. q-POSITIVE AND q-PURE MAPS. Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a linear map between unital C^*-algebras. For each $n \in \mathbb{N}$, define $\phi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ by

$$\phi_n \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{pmatrix}.$$

We say that ϕ is completely positive if ϕ_n is positive for all $n \in \mathbb{N}$. From the work of Choi [6] and Arveson [1], we know that every normal completely positive map $\phi : B(H) \to B(K)$ (H, K separable Hilbert spaces) can be written in the form

$$\phi(A) = \sum_{i=1}^{n} S_i A S_i^*$$

for some $n \in \mathbb{N} \cup \{\infty\}$ and bounded operators $S_i : H \to K$ which are linearly independent over $\ell_2(\mathbb{N})$.

We will be interested in a particular kind of completely positive map:

DEFINITION 2.1. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map with no negative eigenvalues. We say ϕ is *q-positive* (and write $\phi \geqslant_q 0$) if $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geqslant 0$.

Powers introduced the term "q-positive" in Definition 4.28 of [15] in order to describe boundary weight maps which give rise to CP-flows. Our definition of q-positivity serves the same purpose with regard to constructing unital CP-flows through boundary weight doubles, as we will see in Proposition 2.7. We make two observations in light of Definition 2.1. First, it is not uncommon for a completely positive map to have negative eigenvalues. Second, there is no "slowest rate of failure" for q-positivity: For every $s \geqslant 0$, there exists a linear map ϕ with no negative eigenvalues such that $\phi(I + t\phi)^{-1}$ ($t \geqslant 0$) is completely positive if and only if $t \leqslant s$. These observations are discussed in detail in Section 2.1 of [10].

There is a natural order structure for q-positive maps. If $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are q-positive, we say ϕ q-dominates ψ (i.e. $\phi \geqslant_q \psi$) if $\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1}$ is completely positive for all $t \geqslant 0$. It is not always true that $\phi \geqslant_q \lambda\phi$ if $\lambda \in (0, 1)$ (for a large family of counterexamples, see Theorem 6.11 of [9]). However, if ϕ is q-positive, then for every $s \geqslant 0$, we have $\phi \geqslant_q \phi(I + s\phi)^{-1} \geqslant_q 0$ ([9], Proposition 4.1). If these are the only nonzero q-subordinates of ϕ, we say ϕ is *q-pure*. The unital q-pure maps which are either rank one or invertible have been classified ([9], Proposition 5.2 and Theorem 6.11).

If ϕ is a unital q-positive map, then as $t \to \infty$, the maps $t\phi(I + t\phi)^{-1}$ converge to an idempotent completely positive map L_ϕ which has interesting properties (see Lemma 3.1 of [10]):

LEMMA 2.2. *Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive and $\|t\phi(I + t\phi)^{-1}\| < 1$ for all $t \geqslant 0$. Then the maps $t\phi(I + t\phi)^{-1}$ have a unique norm limit L_ϕ as $t \to \infty$, and L_ϕ is completely positive. Furthermore,*

(i) $\phi = \phi \circ L_\phi = L_\phi \circ \phi$,

(ii) $L_\phi^2 = L_\phi$,

(iii) range$(L_\phi) = $ range(ϕ), *and*
(iv) nullspace$(L_\phi) = $ nullspace(ϕ).

2.2. E_0-SEMIGROUPS AND CP-FLOWS. From a celebrated result of Wigner [17], we know that every one-parameter group $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $B(H)$ arises from a strongly continuous unitary group $\{V_t\}_{t\in\mathbb{R}}$ in the sense that $\alpha_t(A) = V_t A V_t^*$ for all $t \in \mathbb{R}$ and $A \in B(H)$.

DEFINITION 2.3. Let H be a separable Hilbert space. We say a family $\alpha = \{\alpha_t\}_{t\geqslant 0}$ of $*$-endomorphisms of $B(H)$ is an E_0-*semigroup* if:
(i) $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all $s, t \geqslant 0$ and $\alpha_0(A) = A$ for all $A \in B(H)$;
(ii) for each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in t;
(iii) $\alpha_t(I) = I$ for all $t \geqslant 0$.

We have two notions of equivalence for E_0-semigroups:

DEFINITION 2.4. Let α and β be E_0-semigroups acting on $B(H_1)$ and $B(H_2)$, respectively. We say α and β are *conjugate* if there is a $*$-isomorphism θ from $B(H_1)$ onto $B(H_2)$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for all $t \geqslant 0$.

We say α and β are *cocycle conjugate* if α is conjugate to β', where β' is an E_0-semigroup of $B(H_2)$ satisfying the following condition: For some strongly continuous family of unitaries $W = \{W_t\}_{t\geqslant 0}$ acting on H_2 and satisfying $W_t \beta_t(W_s) = W_{t+s}$ for all $s \geqslant 0$ and $t \geqslant 0$, we have $\beta'_t(A) = W_t \beta_t(A) W_t^*$ for all $A \in B(H_2)$ and $t \geqslant 0$.

Let K be a separable Hilbert space, and form $H = K \otimes L^2(0,\infty)$, which we identify with the space of K-valued measurable functions on $(0,\infty)$ which are square integrable. Let $U = \{U_t\}_{t\geqslant 0}$ be the right shift semigroup on H, so for all $t \geqslant 0$, $f \in H$, and $x > 0$, we have

$$(U_t f)(x) = f(x-t) \quad \text{if } x > t; \quad (U_t f)(x) = 0 \quad \text{if } x \leqslant t.$$

A strongly continuous semigroup $\alpha = \{\alpha_t\}_{t\geqslant 0}$ of completely positive contractions from $B(H)$ into itself is called a *CP-flow* over K if $\alpha_t(A)U_t = U_t A$ for all $A \in B(H)$ and $t \geqslant 0$. A result of Bhat in [5] shows that if α is unital, then it minimally dilates to a unique (up to conjugacy) E_0-semigroup α^d. We may naturally construct a CP-flow $\beta = \{\beta_t\}_{t\geqslant 0}$ over K using the right shift semigroup by defining

$$\beta_t(A) = U_t A U_t^*$$

for all $A \in B(H)$, $t \geqslant 0$. In fact, if α is any CP-flow over K, then α dominates β in the sense that $\alpha_t - \beta_t$ is completely positive for all $t \geqslant 0$.

Define $\Lambda : B(K) \to B(H)$ by

$$(\Lambda(A)f)(x) = \mathrm{e}^{-x} A f(x)$$

for all $A \in B(K)$, $f \in H$, and $x \in (0,\infty)$, and let $\mathfrak{A}(H)$ be the algebra

$$\mathfrak{A}(H) = \sqrt{I - \Lambda(I_K)}B(H)\sqrt{I - \Lambda(I_K)}.$$

We say a linear functional τ acting on $\mathfrak{A}(H)$ is a *boundary weight* (denoted $\tau \in \mathfrak{A}(H)_*$) if the functional ℓ defined on $B(H)$ by

$$\ell(A) = \tau\Big(\sqrt{I - \Lambda(I_K)}A\sqrt{I - \Lambda(I_K)}\Big)$$

satisfies $\ell \in B(H)_*$. Boundary weights were first defined in Definition 4.16 of [15], where their relationship to CP-flows was explored in depth. For an additional discussion of boundary weights and their properties, we refer the reader to Definition 1.10 of [11] and the remarks that follow it.

Every CP-flow over K corresponds to a *boundary weight map* $\rho \to \omega(\rho)$ from $B(K)_*$ to $\mathfrak{A}(H)_*$ ([15]). On the other hand, it is an extremely important and nontrivial fact that, under certain conditions, a map from $B(K)_*$ to $\mathfrak{A}(H)_*$ can induce a CP-flow (see Theorems 4.17, 4.23, and 4.27 of [15]):

THEOREM 2.5. *Let $\rho \to \omega(\rho)$ be a completely positive mapping from $B(K)_*$ into $\mathfrak{A}(H)_*$ satisfying $\omega(\rho)(I - \Lambda(I_K)) \leqslant \rho(I_K)$ for all positive ρ. Let $\{U_t\}_{t\geqslant 0}$ be the right shift semigroup acting on H. For each $t > 0$, define a truncated boundary weight map $\rho \in B(K)_* \to \omega_t(\rho) \in B(H)_*$ by*

$$\omega_t(\rho)(A) = \omega(\rho)(U_tU_t^*AU_tU_t^*)$$

for all $A \in B(H)$. If the maps

$$\widehat{\pi}_t := \omega_t(I + \widehat{\Lambda}\omega_t)^{-1}$$

are completely positive contractions from $B(K)_*$ into $B(H)_*$ for all $t > 0$, then $\rho \to \omega(\rho)$ is the boundary weight map of a CP*-flow over K. The* CP*-flow is unital if and only if $\omega(\rho)(I - \Lambda(I_K)) = \rho(I_K)$ for all $\rho \in B(K)_*$.*

If α is a CP-flow over $\mathbb{C}$, then we identify its boundary weight map $c \to \omega(c)$ with the single positive boundary weight $\omega := \omega(1)$, so ω has the form

$$\omega\Big(\sqrt{I - \Lambda(1)}A\sqrt{I - \Lambda(1)}\Big) = \sum_{i=1}^{k}(f_i, Af_i)$$

for some mutually orthogonal nonzero L^2-functions $\{f_i\}_{i=1}^k$ ($k \in \mathbb{N} \cup \{\infty\}$) with $\sum_{i=1}^{k} \|f_i\|^2 < \infty$. We call ω a *positive boundary weight over* $L^2(0,\infty)$, and, following the notation of [11], we write $\omega \in \mathfrak{A}(L^2(0,\infty))_*^+$. We say ω is bounded if there exists some $r > 0$ such that $|\omega(B)| \leqslant r\|B\|$ for all $B \in \mathfrak{A}(H)$. Otherwise, we say ω is unbounded. Suppose $\omega(I - \Lambda(1)) = 1$ (i.e. ω is *normalized*), so α is unital and therefore dilates to an E_0-semigroup α^d. Results from [15] show that α^d is of type I_k if ω is bounded but of type II_0 if ω is unbounded, leading us to make the following definition:

DEFINITION 2.6. A boundary weight $\nu \in \mathfrak{A}(L^2(0,\infty))_*$ is called a *Powers weight* if ν is positive and normalized. We say a Powers weight ν is *type I* if it is bounded and *type II* if it is unbounded.

We note that if ν is a type II Powers weight, then for the weights ν_t defined by $\nu_t(A) = \nu(U_tU_t^*AU_tU_t^*)$ for $A \in B(L^2(0,\infty))$ and $t > 0$, both $\nu_t(I)$ and $\nu_t(\Lambda(1))$ approach infinity as $t \to 0+$. We can combine unital q-positive maps with type II Powers weights to obtain E_0-semigroups (see Proposition 3.2 and Corollary 3.3 of [9]):

PROPOSITION 2.7. *Let $H = \mathbb{C}^n \otimes L^2(0,\infty)$. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital q-positive map, and let ν be a type* II *Powers weight. Let $\Omega_\nu : \mathfrak{A}(H) \to M_n(\mathbb{C})$ be the map that sends $A = (A_{ij}) \in M_n(\mathfrak{A}(L^2(0,\infty))) \cong \mathfrak{A}(H)$ to the matrix $(\nu(A_{ij})) \in M_n(\mathbb{C})$. The map $\rho \to \omega(\rho)$ from $M_n(\mathbb{C})^*$ into $\mathfrak{A}(H)_*$ defined by*

$$\omega(\rho)(A) = \rho(\phi(\Omega_\nu(A)))$$

is the boundary weight map of a unital CP*-flow α over $\mathbb{C}^n$ whose Bhat minimal dilation α^d is a type* II_0 *E_0-semigroup.*

In the notation of the previous proposition, we call α^d the E_0-semigroup induced by the *boundary weight double* (ϕ,ν). In order to compare E_0-semigroups induced by boundary weight doubles, we appeal to results of [15], where Powers defined corners between CP-semigroups and showed that two E_0-semigroups are cocycle conjugate if and only if only if there is a corner from one to the other ([15], Definition 3.7 and Lemma 3.8). Furthermore, if α and β are unital CP-flows which induce type II_0 E_0-semigroups α^d and β^d, then α^d and β^d are cocycle conjugate if and only if there is a hyper maximal flow corner from α to β ([15], Definition 4.53 and Theorem 4.56).

In [14], Powers defined q-corners and hyper maximal q-corners ([14], Definition 3.11) between Powers weights. As a consequence of Theorem 4.56 of [15], type II Powers weights ν and η induce cocycle conjugate E_0-semigroups if and only if there is a hyper maximal q-corner from ν to η. Powers also found a condition involving the trace density operators for ν and η which was necessary and sufficient for ν and η to induce cocycle conjugate E_0-semigroups ([14], Theorem 3.23). Motivated by the above results, we define corners, q-corners, and hyper maximal q-corners in an analogous context ([9], Definitions 3.4 and 4.4):

DEFINITION 2.8. Suppose $\phi : B(H_1) \to B(K_1)$ and $\psi : B(H_2) \to B(K_2)$ are normal completely positive maps. Write each $A \in B(H_1 \oplus H_2)$ as $A = (A_{ij})$, where $A_{ij} \in B(H_j, H_i)$ for each $i,j = 1,2$. We say a linear map $\gamma : B(H_2,H_1) \to B(K_2,K_1)$ is a *corner* from α to β if $\Theta : B(H_1 \oplus H_2) \to B(K_1 \oplus K_2)$ defined by

$$\Theta\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \psi(A_{22}) \end{pmatrix}$$

is normal and completely positive.

Suppose $H_1 = K_1 = \mathbb{C}^n$ and $H_2 = K_2 = \mathbb{C}^m$. We say $\gamma : M_{n,m}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$ is a *q-corner* from ϕ to ψ if $\Theta \geqslant_q 0$. A q-corner γ is *hyper maximal* if, whenever

$$\Theta \geqslant_q \Theta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi' \end{pmatrix} \geqslant_q 0;$$

we have $\Theta = \Theta'$.

Hyper maximal q-corners between unital q-positive maps ϕ and ψ allow us to compare E_0-semigroups induced by (ϕ, ν) and (ψ, ν) if ν is a particular kind of type II Powers weight ([9], Proposition 4.6):

PROPOSITION 2.9. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be unital q-positive maps, and let ν be a type* II *Powers weight of the form*

$$\nu\Big(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}\Big) = (f, Bf).$$

The boundary weight doubles (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups if and only if there is a hyper maximal q-corner from ϕ to ψ.

From [9], we know that a unital rank one map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive if and only if it has the form $\phi(A) = \rho(A)I$ for a state $\rho \in M_n(\mathbb{C})^*$, and that ϕ is q-pure if and only if ρ is *faithful*. We also have the following comparison result ([9], Theorem 5.4), which we will extend in this paper to all unital rank one q-positive maps (Theorem 3.10):

THEOREM 2.10. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be rank one unital q-pure maps, so for some faithful states $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$, we have*

$$\phi(A) = \rho(A)I_n \quad \textit{and} \quad \psi(D) = \rho'(D)I_{n'}$$

for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. Let ν be a type II *Powers weight of the form*

$$\nu\Big(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}\Big) = (f, Bf).$$

The E_0-semigroups induced by (ϕ, ν) and (ψ, ν) are cocycle conjugate if and only if $n = n'$ and for some unitary $U \in M_n(\mathbb{C})$ we have $\rho'(A) = \rho(UAU^*)$ for all $A \in M_n(\mathbb{C})$.

2.3. CONJUGACY FOR q-POSITIVE MAPS. We will consider equivalence classes of q-positive maps up to a relation we call conjugacy. More specifically, if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital q-positive map and $U \in M_n(\mathbb{C})$ is any unitary matrix, the map $\phi_U(A) := U^*\phi(UAU^*)U$ is also unital and q-positive. We have the following definition from [10]:

DEFINITION 2.11. Let $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be q-positive maps. We say ϕ is *conjugate* to ψ if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$.

Conjugacy is clearly an equivalence relation, and its definition is analogous to that of conjugacy for E_0-semigroups. Indeed, since every $*$-isomorphism of $M_n(\mathbb{C})$ is implemented by unitary conjugation, it follows that two q-positive maps $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are conjugate if and only if $\theta \circ \phi = \psi \circ \theta$ for some $*$-isomorphism θ of $M_n(\mathbb{C})$. If ν is a type II Powers weight of the form

$$\nu\Big(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}\Big) = (f, Bf),$$

then conjugacy between unital q-positive maps ϕ and ψ is always a sufficient condition for (ϕ, ν) and (ψ, ν) to induce cocycle conjugate E_0-semigroups. To see this, we note that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital and q-positive, then the map $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\gamma(A) = \phi(AU^*)U$ is a hyper maximal q-corner from ϕ to ϕ_U (for details, see the discussion preceding Proposition 2.11 of [10]), whereby Proposition 2.9 gives us:

PROPOSITION 2.12. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and q-positive, and suppose ψ is conjugate to ϕ. If ν is a type* II *Powers weight of the form*

$$\nu\Big(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}\Big) = (f, Bf),$$

then (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups.

In the case that ϕ and ψ are unital rank one q-pure maps and ν is a type II Powers weight of the form $\nu(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}) = (f, Bf)$, Theorem 2.10 states that conjugacy between ϕ and ψ is both necessary and sufficient for (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups.

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital rank one q-positive map, so $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$. It is well-known that we can write ρ in the form

$$\rho(A) = \sum_{i=1}^{k\leqslant n} \lambda_i (g_i, Ag_i), \tag{2.1}$$

for some mutually orthogonal unit vectors $\{g_i\}_{i=1}^k \subset \mathbb{C}^n$ and some positive numbers $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ such that $\sum_{i=1}^k \lambda_i = 1$. With the conditions of the previous sentence satisfied, the number k and the monotonically decreasing set $\{\lambda_i\}_{i=1}^k$ are unique.

DEFINITION 2.13. Assume the notation of the previous paragraph. We call $\{\lambda_i\}_{i=1}^k$ the *eigenvalue list* for ρ.

We should note that our definition differs from a previous definition of eigenvalue list in the literature (for example, [3]) in that our eigenvalue lists do not include zeros.

Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{C}^n$. If ρ has the form (2.1) and $U \in M_n(\mathbb{C})$ is any unitary matrix such that $Ue_i = g_i$ for all $i = 1, \dots, k$, then

$$\rho(UAU^*) = \sum_{i=1}^k \lambda_i(g_i, UAU^* g_i) = \sum_{i=1}^k \lambda_i(U^* g_i, AU^* g_i) = \sum_{i=1}^k \lambda_i(e_i, Ae_i)$$

and

$$\phi_U(A) = U^*\phi(UAU^*)U = U^*\Big[\Big(\sum_{i=1}^k \lambda_i(e_i, Ae_i)\Big)I\Big]U = \Big(\sum_{i=1}^k \lambda_i a_{ii}\Big)I \tag{2.2}$$

for all $A \in M_n(\mathbb{C})$. We will use this fact repeatedly.

3. OUR RESULTS

We begin with the following observation:

LEMMA 3.1. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital q-positive maps, and let ν and η be type* II *Powers weights. If the boundary weight doubles (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups, there is a corner γ from L_ϕ to L_ψ such that $\|\gamma\| = 1$.*

Proof. This is a slight generalization of Lemma 5.3 of [9] (where ϕ and ψ were assumed to have rank one and be q-pure), but its proof is identical. The exact same argument as in the proof of Lemma 5.3 shows that there is a corner γ from $\lim_{t\to 0^+} \nu_t(\Lambda(1))\phi(I + \nu_t(\Lambda(1))\phi)^{-1}$ to $\lim_{t\to 0^+} \eta_t(\Lambda(1))\psi(I + \eta_t(\Lambda(1))\psi)^{-1}$ (if the limits exist) such that $\|\gamma\| = 1$. We observe that the former limit is L_ϕ and the latter limit is L_ψ. Indeed, the values $\{\nu_t(\Lambda(1))\}_{t>0}$ and $\{\eta_t(\Lambda(1))\}_{t>0}$ are monotonically decreasing in t, and since ν and η are unbounded, we have

$$\lim_{t\to 0+} \nu_t(\Lambda(1)) = \lim_{t\to 0+} \eta_t(\Lambda(1)) = \infty. \quad \blacksquare$$

We have the following lemma (a consequence of Lemma 3.5 of [9]):

LEMMA 3.2. *Let $\phi : M_n(\mathbb{C}) \to M_r(\mathbb{C})$, $\psi : M_{n'}(\mathbb{C}) \to M_{r'}(\mathbb{C})$ be completely positive maps, so for some $k, k' \in \mathbb{N}$ and sets of linearly independent matrices $\{S_i\}_{i=1}^k \subset M_{r,n}(\mathbb{C})$ and $\{T_i\}_{i=1}^{k'} \subset M_{r',n'}(\mathbb{C})$, we have*

$$\phi(A) = \sum_{i=1}^k S_i A S_i^*, \quad \psi(D) = \sum_{i=1}^{k'} T_i A T_i^* \tag{3.1}$$

for all $A \in M_n(\mathbb{C})$, $D \in M_{n'}(\mathbb{C})$.

A linear map $\gamma : M_{n,n'}(\mathbb{C}) \to M_{r,r'}(\mathbb{C})$ is a corner from ϕ to ψ if and only if, for some $C = (c_{ij}) \in M_{k,k'}(\mathbb{C})$ with $\|C\| \leqslant 1$, we have, for all $B \in M_{n,n'}(\mathbb{C})$,

$$\gamma(B) = \sum_{i=1}^k \sum_{j=1}^{k'} c_{ij} S_i B T_j^*.$$

REMARK 3.3. Suppose γ is a q-corner from ϕ to ψ. Let $U \in M_n(\mathbb{C})$ and $V \in M_{n'}(\mathbb{C})$ be arbitrary unitary matrices, and let

$$\vartheta = \begin{pmatrix} \phi & \gamma \\ \gamma^* & \psi \end{pmatrix} \geqslant_q 0.$$

For the unitary matrix

$$Z = \begin{pmatrix} U & 0_{n,n'} \\ 0_{n',n} & V \end{pmatrix} \in M_{n+n'}(\mathbb{C}),$$

we have $\vartheta_Z \geqslant_q 0$ (since $\vartheta \geqslant_q 0$), where

$$\vartheta_Z \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \phi_U(A) & U^*\gamma(UBV^*)V \\ V^*\gamma^*(VCU^*)U & \psi_V(D) \end{pmatrix}.$$

Therefore, $B \to U^*\gamma(UBV^*)V$ is a q-corner from ϕ_U to ψ_V. By Proposition 4.5 of [9], we know that if $\Phi : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C})$ is a linear map, then $\vartheta \geqslant_q \Phi \geqslant_q 0$ if and only if $\vartheta_Z \geqslant_q \Phi_Z \geqslant_q 0$. It follows that γ is a hyper maximal q-corner from ϕ to ψ if and only if $B \to U^*\gamma(UBV^*)V$ is a hyper maximal q-corner from ϕ_U to ψ_V. The same argument gives us a bijection between norm one corners from ϕ to ψ and norm one corners from ϕ_U to ψ_V.

PROPOSITION 3.4. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital rank one q-positive maps, so for some states $\ell \in M_n(\mathbb{C})^*$ and $\ell' \in M_{n'}(\mathbb{C})^*$ with eigenvalue lists $\{\lambda_i\}_{i=1}^k$ and $\{\mu_i\}_{i=1}^{k'}$, respectively, we have*

$$\phi(A) = \ell(A)I_n, \quad \psi(D) = \ell'(D)I_{n'}$$

for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. Let ν and η be type II *Powers weights.*

If the boundary weight doubles (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups α^d and β^d, then $k = k'$ and $\lambda_i = \mu_i$ for all $i = 1, \dots, k$.

Proof. Our proof is similar to the proof of Theorem 5.4 of [9]. Suppose α^d and β^d are cocycle conjugate. For some unitaries $U \in M_n(\mathbb{C})$ and $V \in M_{n'}(\mathbb{C})$, we have

$$\phi_U(A) = \Big(\sum_{i=1}^{k} \lambda_i a_{ii}\Big) I_n, \quad \psi_V(D) = \Big(\sum_{i=1}^{k'} \mu_i d_{ii}\Big) I_{n'}$$

for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. Let $\{e_i\}_{i=1}^n$ and $\{e_i'\}_{i=1}^{n'}$ be the standard bases for $\mathbb{C}^n$ and $\mathbb{C}^{n'}$, respectively, and let $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$ be the functionals

$$\rho(A) = \sum_{i=1}^{k} \lambda_i e_i^* A e_i = \sum_{i=1}^{k} \lambda_i a_{ii}, \quad \rho'(D) = \sum_{i=1}^{k'} \mu_i e_i'^* D e_i' = \sum_{i=1}^{k'} \mu_i d_{ii}, \tag{3.2}$$

so $\phi_U(A) = \rho(A)I_n$ and $\psi_V(D) = \rho'(D)I_{n'}$ for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. Note that $L_\phi = \phi$ and $L_\psi = \psi$, so by Lemma 3.1, there is a norm one corner from

ϕ to ψ. Therefore, by Remark 3.3, there is a norm one corner γ from ϕ_U to ψ_V, so the map $\Theta : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C})$ defined by

$$\Theta\begin{pmatrix} A_{n,n} & B_{n,n'} \\ C_{n',n} & D_{n',n'} \end{pmatrix} = \begin{pmatrix} \rho(A)I_n & \gamma(B) \\ \gamma^*(C) & \rho'(D)I_{n'} \end{pmatrix}$$

is completely positive.

Since $\|\gamma\| = 1$, there is some $X \in M_{n,n'}(\mathbb{C})$ with $\|X\| = 1$ and some unit vector $g \in \mathbb{C}^{n'}$ such that $\|\gamma(X)g\|^2 = (\gamma(X)g, \gamma(X)g) = 1$. Let $\tau \in M_{n,n'}(\mathbb{C})^*$ be the functional defined by

$$\tau(B) = (\gamma(X)g, \gamma(B)g).$$

Letting

$$S = \begin{pmatrix} \gamma(X)g & 0_{n,1} \\ 0_{n',1} & g \end{pmatrix} \in M_{n+n',2}(\mathbb{C}),$$

we observe that

$$\begin{pmatrix} \rho(A) & \tau(B) \\ \tau^*(C) & \rho'(D) \end{pmatrix} = S^*\Theta\begin{pmatrix} A & B \\ C & D \end{pmatrix} S \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{n+n'}(\mathbb{C}),$$

hence τ is a corner from ρ to ρ'. Note that $\|\tau\| = \tau(X) = 1$.

Let $D_\lambda \in M_k(\mathbb{C})$ and $D_\mu \in M_{k'}(\mathbb{C})$ be the diagonal matrices whose ii entries are $\sqrt{\lambda_i}$ and $\sqrt{\mu_i}$, respectively. Since τ is a corner from ρ to ρ', equation (3.2) and Lemma 3.2 imply that τ has the form $\tau(B) = \sum_{i,j} c_{ij}\sqrt{\lambda_i\mu_j}(e_i, Be'_j)$ for some $C = (c_{ij}) \in M_{k,k'}(\mathbb{C})$ such that $\|C\| \leqslant 1$. For each $B \in M_{n,n'}(\mathbb{C})$, let $\widetilde{B} \in M_{k',k}(\mathbb{C})$ be the top left $k' \times k$ minor of B^{T}, observing that

$$\tau(B) = \sum_{i=1}^{k}\sum_{j=1}^{k'} c_{ij}\sqrt{\lambda_i\mu_j}b_{ij} = \operatorname{tr}(CD_\mu\widetilde{B}D_\lambda) = \operatorname{tr}(CD_\mu(D_\lambda(\widetilde{B})^*)^*).$$

Let $M = \widetilde{X} \in M_{k',k}(\mathbb{C})$. Applying the Cauchy–Schwarz inequality to the inner product $\langle A, B\rangle = \operatorname{tr}(BA^*)$ on $M_{k,k'}(\mathbb{C})$, we see

$$\begin{aligned} 1 = |\tau(X)|^2 &= |\operatorname{tr}(CD_\mu(D_\lambda M^*)^*)|^2 = |\langle D_\lambda M^*, CD_\mu\rangle|^2 \\ &\leqslant \|CD_\mu\|_{\mathrm{tr}}^2\|D_\lambda M^*\|_{\mathrm{tr}}^2 = \operatorname{tr}(D_\mu C^*CD_\mu)\operatorname{tr}(D_\lambda M^*MD_\lambda) \\ &\leqslant \operatorname{tr}(D_\mu I_{k'}D_\mu)\operatorname{tr}(D_\lambda I_kD_\lambda) = \Big(\sum_{i=1}^{k'}\mu_i\Big)\Big(\sum_{i=1}^{k}\lambda_i\Big) = 1 * 1 = 1. \end{aligned} \tag{3.3}$$

Since equality holds in Cauchy–Schwarz, it follows that for some $m \in \mathbb{C}$,

$$mCD_\mu = D_\lambda M^*, \tag{3.4}$$

where $|m| = 1$ since $\|CD_\mu\|_{\mathrm{tr}} = \|D_\lambda M^*\|_{\mathrm{tr}} = 1$. In fact, $m = 1$ since $\tau(X) = 1$.

Since equality holds in (3.3) and the trace map is faithful, we have $C^*C = I_{k'}$ and $M^*M = I_k$. Note that

$$\min\{k, k'\} \geqslant \operatorname{rank}(C) = k', \quad \min\{k, k'\} \geqslant \operatorname{rank}(M) = k,$$

hence $k = k'$ and the previous sentence shows that C and M are unitary. Therefore, from (3.4) we have

$$D_\mu = C^* D_\lambda M^* = C^* M^* (M D_\lambda M^*),$$

whereby uniqueness of the right polar decomposition for the invertible positive matrix D_μ implies $D_\mu = M D_\lambda M^*$. Since the eigenvalues of D_μ and D_λ are listed in decreasing order, we have $D_\mu = D_\lambda$, hence $\lambda_i = \mu_i$ for all $i = 1, \ldots, k$. ∎

REMARK 3.5. Let $\phi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be a unital rank one q-pure map, and suppose γ is a nonzero q-corner from ϕ to ϕ, so the map $\Theta : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ below is unital and q-positive:

$$\Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \phi(A) & \gamma(B) \\ \gamma^*(C) & \phi(D) \end{pmatrix}.$$

Applying Lemma 2.2 to Θ yields the idempotent completely positive map

$$L_\Theta = \begin{pmatrix} \lim\limits_{t\to\infty} t\phi(I + t\phi)^{-1} & \lim\limits_{t\to\infty} t\gamma(I + t\gamma)^{-1} \\ \lim\limits_{t\to\infty} t\gamma^*(I + t\gamma^*)^{-1} & \lim\limits_{t\to\infty} t\phi(I + t\phi)^{-1} \end{pmatrix} = \begin{pmatrix} \phi & \sigma \\ \sigma^* & \phi \end{pmatrix},$$

so $\sigma := \lim\limits_{t\to\infty} t\gamma(I + t\gamma)^{-1}$ is a corner from ϕ to ϕ satisfying $\sigma^2 = \sigma$. We note that $\|\sigma\| = 1$. Indeed, since $\sigma^2 = \sigma$ and $\text{range}(\sigma) = \text{range}(\gamma) \supsetneq \{0\}$, we have $\|\sigma\| \geqslant 1$, while the fact that σ is a corner between norm one completely positive maps implies $\|\sigma\| \leqslant 1$, hence $\|\sigma\| = 1$. The following lemma gives us the form of σ:

LEMMA 3.6. *Let $\phi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be a unital q-positive map of the form $\phi(A) = \rho(A)I$. Assume ρ is a faithful state of the form*

$$\rho(A) = \sum_{i=1}^{k} \mu_i a_{ii},$$

where $\mu_1, \ldots, \mu_k$ are positive numbers and $\sum\limits_{i=1}^{k} \mu_i = 1$. Let D_μ be the diagonal matrix with ii entries $\sqrt{\mu_i}$ for $i = 1, \ldots, k$, so $\Omega := (D_\mu)^2$ is the trace density matrix for ρ.

Let $\sigma : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be a nonzero linear map such that $\sigma^2 = \sigma$. Then σ is a corner from ϕ to ϕ if, and only if, for some unitary $X \in M_k(\mathbb{C})$ that commutes with Ω, we have

$$\sigma(B) = \text{tr}(X^* B \Omega) X$$

for all $B \in M_k(\mathbb{C})$.

Proof. For the forward direction, suppose that σ is a nonzero corner from ϕ to ϕ and $\sigma^2 = \sigma$, so the map $\Theta : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ defined below is completely positive:

$$\Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \phi(A) & \sigma(B) \\ \sigma^*(C) & \phi(D) \end{pmatrix}.$$

Note that $\|\sigma\| = 1$ by Remark 3.5. We first show that σ has rank one. If $\text{rank}(\sigma) \geqslant 2$, then there is a non-zero non-invertible element $A \in \text{range}(\sigma)$. Scaling A if necessary, we may assume $\|A\| = 1$. Let P be the orthogonal projection onto the range of A, so $PA = A$ and $A^* = A^*P$. Since $P \neq I$ and ρ is faithful, we have $\phi(P) = \rho(A)I = aI$ for some $a < 1$. We note that

$$\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & A \\ A^* & I \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} P & PA \\ A^*P & I \end{pmatrix} = \begin{pmatrix} P & A \\ A^* & I \end{pmatrix} \geqslant 0,$$

so by complete positivity of Θ and the fact that $\sigma^2 = \sigma$, we have

$$\begin{pmatrix} \phi(P) & \sigma(A) \\ \sigma^*(A^*) & \phi(I) \end{pmatrix} = \begin{pmatrix} aI & A \\ A^* & I \end{pmatrix} \geqslant 0,$$

which is impossible since $a < 1$ and $\|A\| = 1$. This shows that not only does σ have rank one, but that every non-zero element of its range is invertible. In other words, for some linear functional $\tau \in M_k(\mathbb{C})^*$ and some invertible matrix $X \in M_k(\mathbb{C})$ with $\|X\| = 1$, we have $\sigma(B) = \tau(B)X$ for all $B \in M_k(\mathbb{C})$. Since σ fixes its range and $\|\sigma\| = 1$, we have $\|\tau\| = \tau(X) = 1$.

Let $g \in \mathbb{C}^k$ be a unit vector such that $\|Xg\| = 1$. We observe that τ is merely the functional $\tau(B) = (\sigma(X)g, \sigma(B)g)$ for all $B \in M_k(\mathbb{C})$, and an argument analogous to the one given in the proof of Proposition 3.4 shows that τ is a corner from ρ to ρ. By Lemma 3.2, there is some $C \in M_k(\mathbb{C})$ with $\|C\| \leqslant 1$ such that

$$\tau(B) = \sum_{i,j=1}^{k} c_{ij}\sqrt{\mu_i\mu_j}(e_i, Be_j) = \text{tr}(CD_\mu B^{\text{T}} D_\mu)$$

for all $A \in M_k(\mathbb{C})$. By the above equation and the fact that $\tau(X) = 1$, we may use the exact same Cauchy–Schwarz argument as in the proof of Proposition 3.4 to conclude that C and X^{T} are unitary and that

$$D_\mu = C^* D_\mu (X^{\text{T}})^* = C^*(X^{\text{T}})^*(X^{\text{T}} D_\mu (X^{\text{T}})^*).$$

Uniqueness of the polar decomposition for the invertible positive matrix D_μ gives us $C^*(X^{\text{T}})^* = I$ and $X^{\text{T}} D_\mu (X^{\text{T}})^* = D_\mu$, where the transpose of the last equality is $X^* D_\mu X = D_\mu$. Therefore, $C = (X^*)^{\text{T}}$ and X commutes with Ω, so for all $B \in M_k(\mathbb{C})$ we have

$$\tau(B) = \text{tr}\left((X^*)^{\text{T}} D_\mu B^{\text{T}} D_\mu\right) = \text{tr}(D_\mu B D_\mu X^*) = \text{tr}(X^* B \Omega)$$

and $\sigma(B) = \tau(B)X = \text{tr}(X^* B \Omega)X$.

Now assume the hypotheses of the backward direction and define $\tau \in M_k(\mathbb{C})^*$ by $\tau(B) = \text{tr}(X^* B \Omega)$, noting that $\sigma^2 = \sigma$ and $\sigma(B) = \tau(B)X$ for all $B \in M_k(\mathbb{C})$. Let $\eta, \eta' : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ be the maps

$$\eta\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A)I & \tau(B)X \\ \tau^*(C)X^* & \rho(D)I \end{pmatrix}, \quad \eta'\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A)I & \tau(B)I \\ \tau^*(C)I & \rho(D)I \end{pmatrix}.$$

Define $\Upsilon : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ by

$$\Upsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}.$$

Note that Υ and Υ^{-1} are completely positive, $\Upsilon \circ \eta = \eta'$, and $\Upsilon^{-1} \circ \eta' = \eta$. Therefore, η is completely positive if and only if η' is completely positive. Since a complex matrix $(m_{ij}) \in M_r(\mathbb{C})$ ($r \in \mathbb{N}$) is positive if and only if $(m_{ij} I_n) \in M_r(M_n(\mathbb{C})) = M_{rn}(\mathbb{C})$ is positive for every $n \in \mathbb{N}$, it follows that η' is completely positive if and only if η'' below is completely positive:

$$\eta'' \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A) & \tau(B) \\ \tau^*(C) & \rho(D) \end{pmatrix}.$$

Thus, η is completely positive if and only if η'' is. In other words, σ is a corner from ϕ to ϕ if and only if τ is a corner from ρ to ρ. But for all $B \in M_k(\mathbb{C})$, we have

$$\tau(B) = \sum_{i,j=1}^{k} c_{ij} \sqrt{\mu_i \mu_j} (e_i, B e_j)$$

for the unitary matrix $C = (X^*)^{\mathrm{T}}$, so τ is a corner from ρ to ρ by Lemma 3.2. ∎

We will make use of the following standard result regarding completely positive maps, providing a proof here for the sake of completeness:

LEMMA 3.7. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive map. If $\phi(E) = 0$ for a projection E, then $\phi(A) = \phi(FAF)$ for all $A \in M_n(\mathbb{C})$, where $F = I - E$.*

Proof. We know from [6] and [1] that ϕ can be written $\phi(A) = \sum_{i=1}^{p} S_i A S_i^*$ for some $p \leqslant n^2$ and $\{S_i\}_{i=1}^{p} \subset M_n(\mathbb{C})$. If $\phi(E) = 0$ for a projection E, then

$$0 = S_i E S_i^* = S_i E E S_i^* = (S_i E)(S_i E)^*$$

for all i, so $S_i E = E S_i^* = 0$ for all i. Therefore, $\phi(EAE) = \phi(EAF) = \phi(FAE) = 0$ for every $A \in M_n(\mathbb{C})$. Letting $F = I - E$, we observe that for every $A \in M_n(\mathbb{C})$,

$$\phi(A) = \phi(EAE + EAF + FAE + FAF) = \phi(FAF). \quad ∎$$

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital rank one q-positive maps. We ask two very important questions: Is there a q-corner from ϕ to ψ ? If so, can we find all such q-corners, and, even further, determine which q-corners are hyper maximal? The following two theorems give us a complete answer to both questions when ϕ and ψ are implemented by diagonal states. This suffices, since for any unital rank one q-positive maps ϕ and ψ, there are always unitaries $U \in M_n(\mathbb{C})$ and $V \in M_{n'}(\mathbb{C})$ such that ϕ_U and ψ_V are implemented by diagonal states, where Remark 3.3 tells us exactly how to transform the q-corners and hyper maximal q-corners from ϕ_U to ψ_V into those from ϕ to ψ.

THEOREM 3.8. *Let $\{\mu_i\}_{i=1}^{k}$ and $\{r_i\}_{i=1}^{k'}$ be monotonically decreasing sequences of strictly positive numbers such that $\sum_{i=1}^{k} \mu_k = \sum_{i=1}^{k'} r_i = 1$. Define unital q-positive maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ (where $n \geqslant k$ and $n' \geqslant k'$) by*

$$\phi(A) = \Big(\sum_{i=1}^{k} \mu_i a_{ii}\Big) I_n \quad \text{and} \quad \psi(D) = \Big(\sum_{i=1}^{k'} r_i d_{ii}\Big) I_{n'} \tag{3.5}$$

for all $A = (a_{ij}) \in M_n(\mathbb{C})$ and $D = (d_{ij}) \in M_{n'}(\mathbb{C})$. Let $\Omega \in M_k(\mathbb{C})$ be the trace density matrix for the faithful state $\rho \in M_k(\mathbb{C})^*$ defined by $\rho(A) = \sum_{i=1}^{k} \mu_i a_{ii}$.

If there is a nonzero q-corner from ϕ to ψ, then $k = k'$ and $\mu_i = r_i$ for all $i = 1, \ldots, k$. In that case, a linear map $\gamma : M_{n,n'}(\mathbb{C}) \to M_{n,n'}(\mathbb{C})$ is a q-corner from ϕ to ψ if and only if: for some unitary $X \in M_k(\mathbb{C})$ that commutes with Ω, some contraction $E \in M_{n-k,n'-k}(\mathbb{C})$, and some $\lambda \in \mathbb{C}$ with $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$, we have

$$\gamma \begin{pmatrix} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{pmatrix} = \lambda \ \mathrm{tr}(X^* B_{k,k} \Omega) \begin{pmatrix} X & 0_{k,n'-k} \\ 0_{n-k,k} & E \end{pmatrix}$$

for all

$$\begin{pmatrix} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{pmatrix} \in M_{n,n'}(\mathbb{C}).$$

Proof. Suppose that γ is a nonzero q-corner from ϕ to ψ, so $\vartheta : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C})$ below is q-positive:

$$\vartheta \begin{pmatrix} A_{n,n} & B_{n,n'} \\ C_{n',n} & D_{n',n'} \end{pmatrix} = \begin{pmatrix} \phi(A_{n,n}) & \gamma(B_{n,n'}) \\ \gamma^*(C_{n',n}) & \psi(D_{n',n'}) \end{pmatrix}.$$

We observe that

$$L_\vartheta \begin{pmatrix} A_{n,n} & B_{n,n'} \\ C_{n',n} & D_{n',n'} \end{pmatrix} = \begin{pmatrix} \phi(A_{n,n}) & \sigma(B_{n,n'}) \\ \sigma^*(C_{n',n}) & \psi(D_{n',n'}) \end{pmatrix},$$

where by Lemma 2.2, the map $\sigma := \lim_{t\to\infty} t\gamma(I + t\gamma)^{-1}$ is a corner of norm one from ϕ to ψ satisfying $\sigma^2 = \sigma$, $\mathrm{range}(\sigma) = \mathrm{range}(\gamma)$, and $\gamma \circ \sigma = \sigma \circ \gamma = \gamma$. Since $\|\sigma\| = 1$, the proof of Proposition 3.4 implies $k = k'$ and $r_i = \mu_i$ for all $i = 1, \ldots, k$.

We observe that $L_\vartheta(P) = 0$ for the projection

$$P = \Big(\sum_{i=k+1}^{n} e_{ii} + \sum_{i=n+k+1}^{n+n'} e_{ii}\Big) \in M_{n+n'}(\mathbb{C}).$$

Therefore, $L_\vartheta(A) = L_\vartheta((I - P)A(I - P))$ for all $A \in M_{n+n'}(\mathbb{C})$ by Lemma 3.7. In particular, σ satisfies

$$\sigma \begin{pmatrix} 0_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{pmatrix} \equiv 0.$$

In other words, σ depends only on its top left $k \times k$ minor, so for some $\widetilde{\sigma} : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ and some maps ℓ_i from $M_k(\mathbb{C})$ into the appropriate matrix spaces, we have

$$\sigma \begin{pmatrix} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{pmatrix} = \begin{pmatrix} \widetilde{\sigma}(B_{k,k}) & \ell_1(B_{k,k}) \\ \ell_2(B_{k,k}) & \ell_3(B_{k,k}) \end{pmatrix}.$$

From the facts $\sigma^2 = \sigma$ and $\|\sigma\| = 1$, it follows that $\widetilde{\sigma}^2 = \widetilde{\sigma}$ and $\|\widetilde{\sigma}\| = 1$.

Let $\widetilde{\phi} : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be the map

$$\widetilde{\phi}(A) = \rho(A) I_k = \Big(\sum_{i=1}^{k} \mu_i a_{ii} \Big) I_k$$

for all $A = (a_{ij}) \in M_k(\mathbb{C})$. Define $\Theta : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ by

$$\Theta \begin{pmatrix} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{pmatrix} = \begin{pmatrix} \widetilde{\phi}(A_{k,k}) & \widetilde{\sigma}(B_{k,k}) \\ \widetilde{\sigma}^*(C_{k,k}) & \widetilde{\phi}(D_{k,k}) \end{pmatrix},$$

and let

$$S = \begin{pmatrix} I_{k,k} & 0_{k,n-k} & 0_{k,k} & 0_{k,n'-k} \\ 0_{k,k} & 0_{k,n-k} & I_{k,k} & 0_{k,n'-k} \end{pmatrix} \in M_{2k,n+n'}(\mathbb{C}).$$

Note that

$$\Theta(N) = S L_{\vartheta}(S^* N S) S^*$$

for all $N \in M_{2k}(\mathbb{C})$, so Θ is completely positive. Therefore, $\widetilde{\sigma}$ is a norm one corner from $\widetilde{\phi}$ to $\widetilde{\phi}$. Since $\|\widetilde{\sigma}\| = 1$ and $\widetilde{\sigma}^2 = \widetilde{\sigma}$, Lemma 3.6 implies that for some unitary $X \in M_k(\mathbb{C})$ that commutes with Ω, we have

$$\widetilde{\sigma}(B) = \mathrm{tr}(X^* B \Omega) X \tag{3.6}$$

for all $B \in M_k(\mathbb{C})$. For simplicity of notation in what follows, let $\tau \in M_k(\mathbb{C})^*$ be the functional $\tau(B) = \mathrm{tr}(X^* B \Omega)$.

We claim that $\ell_1 = \ell_2 \equiv 0$. For this, let

$$M = \begin{pmatrix} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{pmatrix} \in M_{n,n'}(\mathbb{C})$$

be arbitrary. We will suppress the subscripts for B, Q, W, and Y for the remainder of the proof. From (3.6) and the fact that $\sigma^2(M) = \sigma(M)$, we have

$$\ell_i(B) = \ell_i(\widetilde{\sigma}(B)) = \ell_i(\tau(B)X) = \tau(B)\ell_i(X) \tag{3.7}$$

for $i = 1, 2, 3$. Since σ is a contraction, it follows that

$$1 \geqslant \left\| \sigma \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} X & \ell_1(X) \\ \ell_2(X) & \ell_3(X) \end{pmatrix} \right\|.$$

But X is unitary, so the line above implies that $\ell_1(X) = \ell_2(X) = 0$, hence $\ell_1 = \ell_2 \equiv 0$ by (3.7). Let $E = \ell_3(X) \in M_{n-k,n'-k}(\mathbb{C})$, noting that $\|E\| \leqslant 1$ since σ is a contraction. Therefore, σ has the form

$$\sigma \begin{pmatrix} B & W \\ Q & Y \end{pmatrix} = \tau(B) \begin{pmatrix} X & 0_{k,n'-k} \\ 0_{n-k,k} & E \end{pmatrix}.$$

Since $\gamma = \gamma \circ \sigma$ and

$$\mathrm{range}(\gamma) = \mathrm{range}(\sigma) = \Big\{ c \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix} : c \in \mathbb{C} \Big\},$$

we have

$$\gamma \begin{pmatrix} B & W \\ Q & Y \end{pmatrix} = \gamma\Big(\sigma \begin{pmatrix} B & W \\ Q & Y \end{pmatrix}\Big) = \gamma \Big(\tau(B) \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix}\Big)$$
$$= \tau(B)\, \gamma \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix} = \tau(B) \Big[\lambda \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix}\Big] = \lambda \tau(B) \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix}$$

for some $\lambda \in \mathbb{C}$. Since γ is a nonzero q-corner between unital completely positive maps and is thus necessarily a contraction with no negative eigenvalues, we have $\lambda \nleq 0$ and $|\lambda| \leqslant 1$.

In summary: we have proved that if γ is a nonzero q-corner, then it is of the form

$$\gamma \begin{pmatrix} B & W \\ Q & Y \end{pmatrix} = \lambda \ \mathrm{tr}(X^* B \Omega) \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix}$$

for some $\lambda \nleq 0$ with $|\lambda| \leqslant 1$, where X and E satisfy the conditions stated in the theorem. To complete the proof, we show that such a map γ is a q-corner if and only if $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$.

Straightforward computations show that for all $t \geqslant 0$,

$$(I + t\gamma)^{-1} \begin{pmatrix} B & W \\ Q & Y \end{pmatrix} = \begin{pmatrix} B - \frac{t\lambda\tau(B)}{1+t\lambda} X & W \\ Q & Y - \frac{t\lambda\tau(B)}{1+t\lambda} E \end{pmatrix}, \text{ and}$$

$$\gamma(I + t\gamma)^{-1} \begin{pmatrix} B & W \\ Q & Y \end{pmatrix} = \begin{pmatrix} \frac{\lambda\tau(B)}{1+t\lambda} X & 0 \\ 0 & \frac{\lambda\tau(B)}{1+t\lambda} E \end{pmatrix} = \frac{1}{1+t\lambda} \gamma \begin{pmatrix} B & W \\ Q & Y \end{pmatrix}.$$

For each $t \geqslant 0$, define maps $\Theta_t : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$, $L_t : M_{2k}(\mathbb{C}) \to M_{n+n'-2k}(\mathbb{C})$, and $\Upsilon_t : M_{2k}(\mathbb{C}) \to M_{n+n'-2k}(\mathbb{C})$ by

$$\Theta_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t}\rho(A) I_{k,k} & \frac{\lambda}{1+t\lambda}\tau(B) X \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C) X^* & \frac{1}{1+t}\rho(D) I_{k,k} \end{pmatrix}, \tag{3.8}$$

$$L_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t}\rho(A) EE^* & \frac{\lambda}{1+t\lambda}\tau(B) E \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C) E^* & \frac{1}{1+t}\rho(D) I_{n'-k,n'-k} \end{pmatrix}, \text{ and} \tag{3.9}$$

$$\Upsilon_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = L_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} \frac{1}{1+t}\rho(A)(I_{n-k,n-k} - EE^*) & 0_{n-k,n'-k} \\ 0_{n'-k,n-k} & 0_{n'-k,n'-k} \end{pmatrix}. \tag{3.10}$$

Let

$$T = \begin{pmatrix} 0_{n-k,k} & I_{n-k,n-k} & 0_{n-k,k} & 0_{n-k,n'-k} \\ 0_{n'-k,k} & 0_{n'-k,n-k} & 0_{n'-k,k} & I_{n'-k,n'-k} \end{pmatrix} \in M_{n+n'-2k,n+n'}(\mathbb{C}),$$

and let $M \in M_{n+n'}(\mathbb{C})$ be arbitrary, writing

$$M = \begin{pmatrix} A_{k,k} & q_{k,n-k} & B_{k,k} & r_{k,n'-k} \\ s_{n-k,k} & t_{n-k,n-k} & u_{n-k,k} & v_{n-k,n'-k} \\ C_{k,k} & w_{k,n-k} & D_{k,k} & c_{k,n'-k} \\ d_{n'-k,k} & e_{n'-k,n-k} & f_{n'-k,k} & g_{n'-k,n'-k}, \end{pmatrix}, \quad \text{so } SMS^* = \begin{pmatrix} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{pmatrix}. \tag{3.11}$$

For every $t \geqslant 0$, $\vartheta(I + t\vartheta)^{-1}(M)$ is equal to the quantity below:

$$\begin{pmatrix} \frac{1}{1+t}\rho(A)I_{k,k} & 0_{k,n-k} & \frac{\lambda}{1+t\lambda}\tau(B)X & 0_{k,n'-k} \\ 0_{n-k,k} & \frac{1}{1+t}\rho(A)I_{n-k,n-k} & 0_{n-k,k} & \frac{\lambda}{1+t\lambda}\tau(B)E \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)X^* & 0_{k,n-k} & \frac{1}{1+t}\rho(D)I_{k,k} & 0_{k,n'-k} \\ 0_{n'-k,k} & \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)E^* & 0_{n'-k,k} & \frac{1}{1+t}\rho(D)I_{n'-k,n'-k} \end{pmatrix}.$$

In other words,

$$\vartheta(I + t\vartheta)^{-1}(M) = S^*\Theta_t(SMS^*)S + T^*\Upsilon_t(SMS^*)T. \tag{3.12}$$

Note also that for all $N \in M_{2k}(\mathbb{C})$,

$$\Theta_t(N) = S(\vartheta(I + t\vartheta)^{-1}(S^*NS))S^*, \quad \Upsilon_t(N) = T(\vartheta(I + t\vartheta)^{-1}(S^*NS))T^*. \tag{3.13}$$

It follows from (3.12) and (3.13) that ϑ is q-positive if and only if Θ_t and Υ_t are completely positive for all $t \geqslant 0$.

We may easily argue as in the proof of Lemma 3.6 to conclude that Θ_t is completely positive for all $t \geqslant 0$ if and only if the maps $\eta''_t : M_{2k}(\mathbb{C}) \to M_2(\mathbb{C})$ below are completely positive for all $t \geqslant 0$:

$$\eta''_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t}\rho(A) & \frac{\lambda}{1+t\lambda}\tau(B) \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C) & \frac{1}{1+t}\rho(D) \end{pmatrix}.$$

Recall that in the proof of Lemma 3.6, we showed that τ is a corner from ρ to ρ. Since $\|\rho\| = \|\tau\| = 1$, it follows from Lemma 3.2 that $c\tau$ is a corner from ρ to ρ if and only if $|c| \leqslant 1$. Since

$$(1+t)\eta''_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A) & \frac{\lambda(1+t)}{1+t\lambda}\tau(B) \\ \frac{\overline{\lambda}(1+t)}{1+\overline{\lambda}}\tau^*(C) & \rho(D) \end{pmatrix},$$

we see that η''_t is completely positive for all $t \geqslant 0$ if and only if

$$\left|\frac{\lambda(1+t)}{1+t\lambda}\right| \leqslant 1 \quad (\text{where we already know } \lambda \nleq 0 \text{ and } |\lambda| \leqslant 1)$$

for all $t \geqslant 0$. Squaring both sides of the above equation and then cross multiplying gives us

$$|\lambda|^2(1 + 2t + t^2) \leqslant 1 + 2t\,\mathrm{Re}(\lambda) + |\lambda|^2t^2, \quad (\lambda \nleq 0, \ |\lambda| \leqslant 1)$$

which is equivalent to

$$|\lambda|^2 \leqslant \frac{1 + 2t\,\mathrm{Re}(\lambda)}{1 + 2t} \quad (\lambda \nleq 0, \ |\lambda| \leqslant 1) \tag{3.14}$$

for all nonnegative t. Note that if $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$, then $\mathrm{Re}(\lambda) \leqslant 1$ and equation (3.14) holds for $t \geqslant 0$. On the other hand, suppose that λ is any complex number that satisfies (3.14) for all $t \geqslant 0$. We conclude immediately that $\mathrm{Re}(\lambda) > 0$, whereby the fact that $|\lambda| \leqslant 1$ implies $\mathrm{Re}(\lambda) \in (0,1]$. A computation shows that the net $\{\frac{1+2t\,\mathrm{Re}(\lambda)}{1+2t}\}_{t\geqslant 0}$ is monotonically decreasing and converges to $\mathrm{Re}(\lambda)$, hence $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$ by (3.14). We have now shown that η_t'' (and thus Θ_t) is completely positive for all $t \geqslant 0$ if and only if $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$. Therefore, if $|\lambda|^2 > \mathrm{Re}(\lambda)$ then (3.13) implies that ϑ is not q-positive, which is to say that γ is not a q-corner from ϕ to ψ.

Suppose that $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$. Then from above, the maps $\{\Theta_t\}_{t\geqslant 0}$ are all completely positive. Let

$$G = \begin{pmatrix} E & 0_{n-k,n'-k} \\ 0_{n'-k,n'-k} & I_{n'-k} \end{pmatrix} \in M_{n+n'-2k,2n'-2k}(\mathbb{C}).$$

We observe that

$$(1+t)L_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = G \begin{pmatrix} \rho(A)I_{n'-k} & \frac{\lambda(1+t)}{1+t\lambda}\tau(B)I_{n'-k} \\ \frac{\bar{\lambda}(1+t)}{1+t\bar{\lambda}}\tau^*(C)I_{n'-k} & \rho(D)I_{n'-k} \end{pmatrix} G^*,$$

where we have already shown that the map in the middle is completely positive since $|\lambda|^2 \leqslant \mathrm{Re}(\lambda)$. Thus, L_t is completely positive for every $t \geqslant 0$. Also, $\Upsilon_t - L_t$ has the form

$$(\Upsilon_t - L_t) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A)(I_{n-k} - EE^*) & 0_{n-k,n'-k} \\ 0_{n'-k,n-k} & 0_{n'-k,n'-k} \end{pmatrix},$$

where the right hand side is completely positive since $\|E\| \leqslant 1$. Therefore, the maps $\{\Upsilon_t\}_{t\geqslant 0}$ are all completely positive, so (3.12) implies that $\vartheta(I + t\vartheta)^{-1}$ is completely positive for all $t \geqslant 0$, hence γ is a q-corner from ϕ to ψ. ∎

THEOREM 3.9. *Assume the notation of the previous theorem, and suppose that* $k = k'$ *and* $\mu_i = r_i$ *for all* $i = 1, \dots, k$. *A* q*-corner* $\gamma : M_{n,n'}(\mathbb{C}) \to M_{n,n'}(\mathbb{C})$ *from* ϕ *to* ψ *is hyper maximal if and only if* $n = n'$, $0 < |\lambda|^2 = \mathrm{Re}(\lambda)$, *and* E *is unitary.*

Proof. We first show that γ is not hyper maximal if $n \neq n'$, regardless of the assumptions for λ or E. If $n > n'$, then $EE^* \in M_{n-k}(\mathbb{C})$ is a positive contraction of rank at most $n' - k$, so $EE^* \neq I_{n-k}$.

Define $\phi' : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\phi'(R) = \phi(R) \begin{pmatrix} I_{k,k} & 0_{k,n-k} \\ 0_{n-k,k} & EE^* \end{pmatrix},$$

observing that $\phi'(I + t\phi')^{-1} = \frac{1}{1+t}\phi'$ for all $\geqslant 0$. Define $\vartheta' : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C})$ by

$$\vartheta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi \end{pmatrix},$$

noting that ϑ' has no negative eigenvalues. Define maps $\{\Theta_t\}_{t\geqslant 0}$, $\{L_t\}_{t\geqslant 0}$, and $\{\Upsilon_t\}_{t\geqslant 0}$ as in equations (3.8), (3.9), and (3.10). Writing each $M \in M_{n+n'}(\mathbb{C})$ in the form (3.11), we see that $\vartheta'(I+t\vartheta')^{-1}(M)$ is equal to

$$\begin{pmatrix} \frac{1}{1+t}\rho(A)I_{k,k} & 0_{k,n-k} & \frac{\lambda}{1+t\lambda}\tau(B)X & 0_{k,n'-k} \\ 0_{n-k,k} & \frac{1}{1+t}\rho(A)EE^* & 0_{n-k,k} & \frac{\lambda}{1+t\lambda}\tau(B)E \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)X^* & 0_{k,n-k} & \frac{t}{1+t}\rho(D)I_{k,k} & 0_{k,n'-k} \\ 0_{n'-k,k} & \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)E^* & 0_{n'-k,k} & \frac{1}{1+t}\rho(D)I_{n'-k,n'-k} \end{pmatrix}.$$

In other words,

$$\vartheta'(I+t\vartheta')^{-1}(M) = S^*\Theta_t(SMS^*)S + T^*L_t(SMS^*)T \tag{3.15}$$

for every $t \geqslant 0$ and $M \in M_{n+n'}(\mathbb{C})$, hence ϑ' is q-positive. By (3.12) and (3.15), we have

$$(\vartheta(I+t\vartheta)^{-1} - \vartheta'(I+t\vartheta')^{-1})(M) = T^*((\Upsilon_t - L_t)(SMS^*))T.$$

Since $\Upsilon_t - L_t$ is completely positive for all $t \geqslant 0$ (as shown in the previous proof), the above equation implies that $\vartheta \geqslant_q \vartheta'$. However, $\vartheta \neq \vartheta'$ since $EE^* \lneq I_{n-k}$, so γ is not hyper maximal.

If $n < n'$, then since $E^*E \lneq I_{n'-k}$, we may replace $\{L_t\}_{t=0}^{\infty}$ with the maps $\{R_t\}_{t=0}^{\infty}$ below and argue analogously (this time cutting down ψ using E^*E) to show that γ is not hyper maximal:

$$R_t\begin{pmatrix} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t}\rho(A)I_{n-k} & \frac{\lambda}{1+t\lambda}\tau(B)E \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)E^* & \frac{1}{1+t}\rho(D)E^*E \end{pmatrix}.$$

Of course, if $n = n'$ but E is not unitary, then $EE^* \lneq I_{n-k}$, and the same argument given in the case that $n > n'$ shows that γ is not hyper maximal.

Therefore, we may suppose for the remainder of the proof that $n = n'$ and E is unitary. Note that $\phi = \psi$ since $n = n'$. For some $a \in (0,1]$, we have $|\lambda|^2 = a\,\mathrm{Re}(\lambda)$. We first show that γ is not hyper maximal if $a \neq 1$. We claim that the map $\vartheta'' : M_{2n}(\mathbb{C}) \to M_{2n}(\mathbb{C})$ defined by

$$\vartheta''\begin{pmatrix} A_{n,n} & B_{n,n} \\ C_{n,n} & D_{n,n} \end{pmatrix} = \begin{pmatrix} a\phi(A_{n,n}) & \gamma(B_{n,n}) \\ \gamma^*(C_{n,n}) & a\phi(D_{n,n}) \end{pmatrix}$$

satisfies $\vartheta'' \geqslant_q 0$. For each $t \geqslant 0$, let $\eta_t^{(a)} : M_{2k}(\mathbb{C}) \to M_2(\mathbb{C})$ be the map

$$\eta_t^{(a)}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{a}{1+at}\rho(A) & \frac{\lambda}{1+t\lambda}\tau(B) \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C) & \frac{a}{1+at}\rho(D) \end{pmatrix}.$$

It is routine to check that since τ is a corner from ρ to ρ, the condition $|\lambda|^2 = a\,\mathrm{Re}(\lambda)$ implies that $\frac{\lambda}{1+t\lambda}\tau$ is a corner from $\frac{a}{1+at}\rho$ to $\frac{a}{1+at}\rho$ for every $t \geqslant 0$, so $\eta_t^{(a)}$

is completely positive for all $t \geqslant 0$. Defining $\Theta_t^{(a)}$ and $\Upsilon_t^{(a)}$ for each $t \geqslant 0$ by

$$\Theta_t^{(a)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{a}{1+at}\rho(A)I_k & \frac{\lambda}{1+t\lambda}\tau(B)X \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)X^* & \frac{a}{1+at}\rho(D)I_k \end{pmatrix}, \text{ and}$$

$$\Upsilon_t^{(a)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = G \begin{pmatrix} \frac{a}{1+at}\rho(A)I_{n-k} & \frac{\lambda}{1+t\lambda}\tau(B)I_{n-k} \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)I_{n-k} & \frac{a}{1+at}\rho(D)I_{n-k} \end{pmatrix} G^*$$

$$= \begin{pmatrix} \frac{a}{1+at}\rho(A)I_{n-k} & \frac{\lambda}{1+t\lambda}\tau(B)E \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)E^* & \frac{a}{1+at}\rho(D)I_{n-k} \end{pmatrix},$$

we observe that the maps $\{\Theta_t^{(a)}\}_{t\geqslant 0}$ and $\{\Upsilon_t^{(a)}\}_{t\geqslant 0}$ are all completely positive since $\eta_t^{(a)}$ is completely positive for all $t \geqslant 0$. Note that

$$(a\phi)(I + ta\phi)^{-1} = \frac{a}{1+at}\phi$$

for all $t \geqslant 0$, so for every $M \in M_{2n}(\mathbb{C})$, we have

$$\vartheta''(I + t\vartheta'')^{-1}(M) = S^*(\Theta_t^{(a)}(SMS^*))S + T^*(\Upsilon_t^{(a)}(SMS^*))T.$$

Therefore, $\vartheta'' \geqslant_q 0$, and trivially $\vartheta \geqslant_q \vartheta''$. If $a \neq 1$, then $\vartheta'' \neq \vartheta$, hence γ is not hyper maximal. To finish the proof, it suffices to show that γ is hyper maximal if $a = 1$ (of course, maintaining our assumption that E is unitary).

Suppose $a = 1$, and let ϕ' be any q-subordinate of ϕ such that

$$\chi := \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \phi \end{pmatrix} \geqslant_q 0.$$

If $L_{\phi'}(I) \neq I$, then $L_{\phi'}(I) = R \lneq I$ for some positive $R \in M_n(\mathbb{C})$. Recall that $\sigma = \lim_{t\to\infty} t\gamma(I + t\gamma)^{-1}$, so applying Lemma 2.2 to χ gives us

$$L_\chi = \begin{pmatrix} L_{\phi'} & \sigma \\ \sigma^* & \phi \end{pmatrix}.$$

Letting Z be the unitary matrix

$$Z = \begin{pmatrix} X & 0_{k,n-k} \\ 0_{n-k,k} & E \end{pmatrix} \in M_n(\mathbb{C}),$$

we observe that $\sigma(Z) = Z$, so by complete positivity of L_χ,

$$0 \leqslant L_\chi \begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix} = \begin{pmatrix} R & Z \\ Z^* & I \end{pmatrix}. \tag{3.16}$$

Since $R \lneq I$, we have $(f, Rf) < 1$ for some unit vector $f \in \mathbb{C}^n$. A quick calculation shows that

$$\left\langle \begin{pmatrix} f \\ -Z^*f \end{pmatrix}, \begin{pmatrix} R & Z \\ Z^* & I \end{pmatrix} \begin{pmatrix} f \\ -Z^*f \end{pmatrix} \right\rangle = (f, Rf) - 1 < 0,$$

contradicting (3.16).

Therefore, $L_{\phi'}(I) = I$. Since $\phi \geqslant_q \phi'$, it follows that $L_\phi - L_{\phi'}$ is completely positive, so

$$\|L_\phi - L_{\phi'}\| = \|L_\phi(I) - L_{\phi'}(I)\| = 0,$$

hence $L_{\phi'}(A) = L_\phi(A) = \phi(A) = \ell(A)I$ for the state $\ell \in M_n(\mathbb{C})^*$ defined by $\ell(A) = \sum_{i=1}^{k} \mu_i a_{kk}$. But $\mathrm{range}(\phi') = \mathrm{range}(L_{\phi'}) = \{cI : c \in \mathbb{C}\}$ and $\phi' = \phi' \circ L_{\phi'}$, so $\phi'(I) = rI$ for some $r \leqslant 1$ and

$$\phi'(A) = \phi'(L_{\phi'}(A)) = \phi(\ell(A)I) = \ell(A)\phi'(I) = r\ell(A)I = r\phi(A)$$

for all $A \in M_n(\mathbb{C})$.

We claim that $r = 1$. To prove this, we define $V_t : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ for each $t \geqslant 0$ by

$$\begin{aligned} V_t\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= S\Big(\chi(I+t\chi)^{-1}\Big[S^*\begin{pmatrix} A & B \\ C & D \end{pmatrix}S\Big]\Big)S^* \\ &= \begin{pmatrix} \frac{r}{1+rt}\rho(A)I_k & \frac{\lambda}{1+t\lambda}\tau(B)X \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}\tau^*(C)X^* & \frac{1}{1+t}\rho(D)I_k \end{pmatrix}. \end{aligned}$$

Since $\chi \geqslant_q 0$, each V_t is completely positive. Therefore,

$$0 \leqslant \begin{pmatrix} X^* & 0 \\ 0 & I \end{pmatrix}\Big[V_t\begin{pmatrix} I & X \\ X^* & I \end{pmatrix}\Big]\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \frac{r}{1+rt}I & \frac{\lambda}{1+t\lambda}I \\ \frac{\overline{\lambda}}{1+t\overline{\lambda}}I & \frac{1}{1+t}I \end{pmatrix},$$

hence

$$\frac{r}{(1+rt)(1+t)} \geqslant \frac{|\lambda|^2}{|1+t\lambda|^2} = \frac{\mathrm{Re}(\lambda)}{1+(t^2+2t)\,\mathrm{Re}(\lambda)}$$

for all $t \geqslant 0$. This is equivalent to

$$r \geqslant \frac{(1+t)\,\mathrm{Re}(\lambda)}{1+t\,\mathrm{Re}(\lambda)} \tag{3.17}$$

for all $t \geqslant 0$. We take the limit as $t \to \infty$ in (3.17) and observe $r \geqslant 1$. Since $r \leqslant 1$ we have $r = 1$, so $\phi' = \phi$.

We have shown that if

$$\begin{pmatrix} \phi & \gamma \\ \gamma^* & \phi \end{pmatrix} \geqslant_q \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \phi \end{pmatrix} \geqslant_q 0,$$

then $\phi = \phi'$. An analogous argument shows that if

$$\begin{pmatrix} \phi & \gamma \\ \gamma^* & \phi \end{pmatrix} \geqslant_q \begin{pmatrix} \phi & \gamma \\ \gamma^* & \phi' \end{pmatrix} \geqslant_q 0,$$

then $\phi = \phi'$. Therefore, γ is hyper maximal. ∎

We are now ready to prove the following:

THEOREM 3.10. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be rank one unital q-positive maps, and let ν be a type* II *Powers weight of the form*

$$\nu\Big(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}\Big) = (f, Bf).$$

The E_0-semigroups induced by (ϕ,ν) and (ψ,ν) are cocycle conjugate if and only if $n = n'$ and ϕ is conjugate to ψ.

Proof. The backward direction follows trivially from Proposition 2.12. For the forward direction, suppose (ϕ,ν) and (ψ,ν) induce cocycle conjugate E_0-semigroups α^d and β^d. For some sets $\{\mu_i\}_{i=1}^k$ and $\{r_i\}_{i=1}^{k'}$ satisfying the conditions of Theorem 3.8 and some unitaries $U \in M_n(\mathbb{C})$ and $V \in M_{n'}(\mathbb{C})$, ϕ_U and ψ_V have the form of (3.5). Let α_U^d and β_V^d be the E_0-semigroups induced by (ϕ_U,ν) and (ψ_V,ν), respectively. Since $\alpha_U^d \simeq \alpha^d$ and $\beta_V^d \simeq \beta^d \simeq \alpha^d$, we have $\alpha_U^d \simeq \beta_V^d$, so by Proposition 2.9, there is a hyper maximal q-corner from ϕ_U to ψ_V. Theorems 3.8 and 3.9 imply that $n = n'$, $k = k'$, and $\mu_i = r_i$ for all $i = 1,\dots,k$. In other words, $\phi_U = \psi_V$. Therefore, $\phi = \psi_{(VU^*)}$, so ϕ and ψ are conjugate. ∎

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