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RADON–NIKODYM THEOREM IN QUASI $*$-ALGEBRAS

C. LA RUSSA and S. TRIOLO

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ABSTRACT. In this paper some properties of continuous representable linear functionals on a quasi $*$-algebra are investigated. Moreover we give properties of operators acting on a Hilbert algebra, whose role will reveal to be crucial for proving a Radon–Nikodym type theorem for positive linear functionals.

KEYWORDS: *Quasi $*$-algebra, Radon–Nikodym theorem.*

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1. INTRODUCTION AND PRELIMINARIES

As it is well-known, the classical Radon–Nikodym theorem states that if μ and λ are bounded positive measures on a σ-algebra $\mathcal{M}$ in a set X and λ is absolutely continuous with respect to μ (i.e. if $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\lambda(E) = 0$), then there exists $h \in L^1(\mu)$ such that

$$\lambda(E) = \int_E h \mathrm{d}\mu.$$

We may look at the r.h.s. as to a linear functional defined on the (normed) quasi $*$-algebra $(L^1(\mu), L^\infty(\mu))$. Also the l.h.s. can be thought as the value of a functional on L^∞, due to the fact that

$$\lambda(E) = \int_X \chi_E \mathrm{d}\lambda.$$

Since $L^\infty(\mu)$ is an abelian von Neumann algebra, in a natural way, the question arises, whether one can find generalizations of the Radon–Nikodym theorem in the contest of more general topological $*$-algebras.

In 1973 Pedersen and Takesaki [10] obtained a Radon–Nikodym theorem for von Neumann algebras. The case of arbitrary $*$-algebras was considered first by Gudder in [6]. He proved that if a positive linear functional ψ on a $*$-algebra $\mathfrak{A}$ with identity e, is strongly absolutely continuous with respect to a positive linear functional φ on $\mathfrak{A}$ (that is, the map $\lambda_\varphi(x) \mapsto \lambda_\psi(x)$ is closable) then there exists a positive self-adjoint operator H on $\mathcal{H}_\varphi$ such that

$$\psi(x) = \langle H\lambda_\varphi(x)|H\lambda_\varphi(e)\rangle, \quad \forall\, x \in \mathfrak{A}, \tag{1.1}$$

where λ_φ is a linear map from $\mathfrak{A}$ into the Hilbert space $\mathcal{H}_\varphi$ defined via GNS representation. Some counterexamples show that one cannot hope for a Radon–Nikodym theorem with arbitrary functionals. In 1983 Inoue [7] obtained a similar result for two positive invariant sesquilinear forms Φ and Ψ on a $*$-algebra $\mathfrak{A}$, under the hypothesis that Ψ is strongly Φ-absolutely continuous. Inoue's result will be the starting point of our analysis. Some of these results have also been extended to partial $*$-algebras (we refer to [1] for a detailed discussion).

In this paper we consider the following situation: given a locally convex $*$-algebra $\mathfrak{A}_0[\tau]$, if by $\mathfrak{A}$ we denote its completion, then the pair $(\mathfrak{A}, \mathfrak{A}_0)$ is a locally convex *quasi $*$-algebra*. Certain linear functionals on $\mathfrak{A}$, called *representable*, allow a generalized version of the GNS construction [12]. If φ is representable and ψ is a positive linear functional absolutely continuous with respect to φ (in a sense that will be specified later), it is natural to pose the question as to whether the representation of ψ as in (1.1) can be improved so to have a form which is more reminiscent of the classical one for functions. This means, in other words: Does there exist a *positive* element $h \in \mathfrak{A}$ such that

$$\psi(a^*a) = \varphi(a^*ha), \quad \forall\, a \in \mathfrak{A}_0? \tag{1.2}$$

The paper is organized as follows. After some basic preliminaries, we provide in Section 2 a series of properties of continuous representable linear functionals on a locally convex quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ and of the associated positive sesquilinear forms.

In Section 3 we give a sufficient condition for the representation (1.2) to hold. To this aim, we first give some properties of operators acting on a Hilbert algebra, whose role will reveal to be crucial for proving our variant of the Radon–Nikodym theorem.

In what follows we recall some definitions and facts needed in the sequel.

DEFINITION 1.1. Let $\mathfrak{A}$ be a complex vector space and $\mathfrak{A}_0$ a $*$-algebra contained in $\mathfrak{A}$. $\mathfrak{A}$ is said a *quasi $*$-algebra with distinguished $*$-algebra* $\mathfrak{A}_0$ (or, simply, over $\mathfrak{A}_0$) if

(i) the left multiplication ax and the right multiplication xa of an element a of $\mathfrak{A}$ and an element x of $\mathfrak{A}_0$ which extend the multiplication of $\mathfrak{A}_0$ are always defined and bilinear;

(ii) $x_1(x_2a) = (x_1x_2)a$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$;

(iii) an involution $*$ which extends the involution of $\mathfrak{A}_0$ is defined in $\mathfrak{A}$ with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$.

A quasi $*$-algebra will be denoted by $(\mathfrak{A}, \mathfrak{A}_0)$.

DEFINITION 1.2. A quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said a *topological (respectively locally convex) quasi $*$-algebra*, if $\mathfrak{A}$ is endowed with a topology τ which makes of it a topological (respectively locally convex) space with the properties:

(i) the involution $a \to a^*$ is continuous;

(ii) for every $x \in \mathfrak{A}_0$ the maps $a \mapsto ax$, $a \mapsto xa$ are continuous from $\mathfrak{A}[\tau]$ into itself;

(iii) $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\tau]$.

Now let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$.

We recall that the set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial $*$-algebra ([1]) with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \restriction \mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, if and only if $X_2 \mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with respect to the usual operations.

DEFINITION 1.3. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi $*$-algebra (with identity e) and $\mathcal{D}_\pi$ a dense domain in a Hilbert space $\mathcal{H}_\pi$.

A linear map π from $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that:

(i) $\pi(a^*) = \pi(a)^\dagger$, $\forall\, a \in \mathfrak{A}$,

(ii) for $a \in \mathfrak{A}$, $x \in \mathfrak{A}_0$, $\pi(a) \,\square\, \pi(x)$ is well defined and

$$\pi(ax) = \pi(a) \,\square\, \pi(x),$$

is called a *$*$-representation of* $\mathfrak{A}$. Moreover, if

(iii) $\pi(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$,

then π is said to be a *$*$-representation of the quasi $*$-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$.

If π is a $*$-representation of $(\mathfrak{A}, \mathfrak{A}_0)$, then the *closure* $\widetilde{\pi}$ of π is defined, for each $x \in \mathfrak{A}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widetilde{\mathcal{D}}_\pi$, which is the completion of $\mathcal{D}_\pi$ under the *graph topology* t_π [11] defined by the seminorms $\xi \in \mathcal{D}_\pi \to \|\pi(a)\xi\|$, $a \in \mathfrak{A}$. If $\pi = \widetilde{\pi}$ the representation is said to be *closed*. The representation π is said to be *ultra-cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\mathcal{D}_\pi = \pi(\mathfrak{A}_0)\xi_0$, while is said to be *cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathfrak{A}_0)\xi_0$ is dense in $\mathcal{D}_\pi$ with respect to t_π.

The following proposition, proved in [12], extends the GNS construction to quasi $*$-algebras.

PROPOSITION 1.4. *Let ω be a linear functional on $\mathfrak{A}$ satisfying the following requirements:*

(L1) $\omega(a^* a) \geqslant 0$, *for all* $a \in \mathfrak{A}_0$;

(L2) $\omega(b^* x^* a) = \overline{\omega(a^* x b)}$, $\forall\, a, b \in \mathfrak{A}_0$, $x \in \mathfrak{A}$;

(L3) $\forall\, x \in \mathfrak{A}$ *there exists* $\gamma_x > 0$ *such that* $|\omega(x^*a)| \leqslant \gamma_x\, \omega(a^*a)^{1/2}$.
Then there exists a triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ *with the properties:*

- π_ω *is an ultra-cyclic* $*$*-representation of* $\mathfrak{A}$ *with ultra-cyclic vector* ξ_ω*;*
- λ_ω *is a linear map of* $\mathfrak{A}$ *into* $\mathcal{H}_\omega$ *with* $\lambda_\omega(\mathfrak{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(e)$ *and* $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, *for every* $x \in \mathfrak{A}$, $a \in \mathfrak{A}_0$*;*
- $\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$, *for every* $x \in \mathfrak{A}$*;*
- $\pi_{\omega_0} = \pi_\omega \restriction_{\mathfrak{A}_0}$, *with* $\omega_0 = \omega \restriction_{\mathfrak{A}_0}$*;*
- $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, $x \in \mathfrak{A}$, $a \in \mathfrak{A}_0$.

For shortness, a linear functional ω on $\mathfrak{A}$ satisfying $(L1)$–$(L3)$ will be called a *representable* functional on $\mathfrak{A}$. If ω is representable, $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ will be called, as usual, the GNS *construction for* ω.

2. REPRESENTABLE FUNCTIONALS AND POSITIVE SESQUILINEAR FORMS ASSOCIATED TO THEM

Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi $*$-algebra. Put

$$\mathfrak{A}_0^+ = \Big\{ \sum_{i=1}^{n} x_i^* x_i : x_i \in \mathfrak{A}_0, n \in \mathbb{N} \Big\}.$$

This set (which is a cone) is usually called the set of positive elements of $\mathfrak{A}_0$. We define $\mathfrak{A}^+ = \overline{\mathfrak{A}_0^+}^{\tau}$, the τ-closure of $\mathfrak{A}_0^+$ in $\mathfrak{A}$.

DEFINITION 2.1. A linear functional ω on $\mathfrak{A}$ is said to be *positive* if $\omega(x) \geqslant 0$ for every $x \in \mathfrak{A}^+$.

We recall that if $(\mathfrak{A}, \mathfrak{A}_0)$ is a locally convex quasi $*$-algebra with topology τ on $\mathfrak{A}$ and $D(\Phi)$ is a subspace of $\mathfrak{A}$, a *positive sesquilinear form* Φ is a map $\Phi : D(\Phi) \times D(\Phi) \to \mathbb{C}$ with the properties:

(i) $\Phi(a,a) \geqslant 0$, $\forall\, a \in \mathcal{D}(\Phi)$;
(ii) $\Phi(a + \lambda b, c) = \Phi(a,c) + \lambda\Phi(b,c)$, $\forall\, a,b,c \in \mathcal{D}(\Phi), \forall\, \lambda \in \mathbb{C}$;
(iii) $\Phi(a, b + \lambda c) = \Phi(a,b) + \overline{\lambda}\Phi(a,c)$, $\forall\, a,b,c \in \mathcal{D}(\Phi), \forall\, \lambda \in \mathbb{C}$.

Note that for positive sesquilinear forms the Cauchy–Schwarz inequality holds:

$$|\Phi(a,b)|^2 \leqslant \Phi(a,a)\Phi(b,b), \quad \forall\, a,b \in \mathcal{D}(\Phi).$$

We denote by $N(\Phi)$ the null subspace of Φ; i.e.,

$$N(\Phi) = \{a \in D(\Phi) : \Phi(a,a) = 0\}.$$

The quotient space $D(\Phi)/N(\Phi)$ can be made into a pre-Hilbert space by defining

$$\langle \lambda_\Phi(a) | \lambda_\Phi(b) \rangle_\Phi = \Phi(a,b), \quad a,b \in D(\Phi) \quad \text{and} \quad \|\lambda_\Phi(a)\|_\Phi = \Phi(a,a)^{1/2}, \quad a \in D(\Phi),$$

where $\lambda_\Phi(a)$ denotes the coset corresponding to $a \in D(\Phi)$. For shortness we put $\lambda_\Phi(D(\Phi)) := D(\Phi)/N(\Phi)$. We observe that the completion of $\lambda_\Phi(D(\Phi))[\|\cdot\|_\Phi]$ is a Hilbert space, which we denote by $\mathcal{H}_\Phi$.

DEFINITION 2.2. Let Φ be defined as above on $D(\Phi) \times D(\Phi)$.
The form Φ is said to be *closed* [5] if for a net $\{a_\delta\}_{\delta\in\Delta}$ in $D(\Phi)$, one has:

$$\begin{aligned} &\{a_\delta\}_{\delta\in\Delta} \xrightarrow{\tau} a, \quad \text{and} \quad \Phi(a_\delta - a_\gamma, a_\delta - a_\gamma) \to 0 \\ &\Rightarrow a \in D(\Phi) \quad \text{and} \quad \Phi(a_\delta, a_\delta) \to \Phi(a, a). \end{aligned}$$

Assume the following two conditions:

(C1) Let $a_\delta, b \in D(\Phi)$, $\delta \in \Delta$. If $a_\delta \xrightarrow{\tau} a$ and $\Phi(a_\delta - b, a_\delta - b) \to 0$, then $a \in D(\Phi)$ and $\Phi(a - b, a - b) = 0$.

(C2) Let $\{a_\delta\}_{\delta\in\Delta}$ be a net of elements of $D(\Phi)$ such that the infinite sum $\sum\limits_\delta \Phi(a_\delta, a_\delta)$ is convergent, then the infinite sum $\sum\limits_{\delta\in\Delta} a_\delta$ converges with respect to τ to an element $a \in \mathfrak{A}$.

Under these conditions, that establish a sort of *compatibility* of Φ with the topology τ of $\mathfrak{A}$, the following lemma holds.

LEMMA 2.3 ([3]). *The space $\lambda_\Phi(D(\Phi))[\|\cdot\|_\Phi]$ is complete if and only if Φ is closed.*

Now we give two examples, one from the classical theory of integration and the other from the noncommutative integration theory.

EXAMPLE 2.4. If $\mathcal{M}(E)$ is the algebra of Lebesgue measurable functions on a Lebesgue measurable subset E of $\mathbb{R}$, the conditions (C1) and (C2) given above, are satisfied when Φ is defined on $\mathcal{M}(E) \times \mathcal{M}(E)$ by

$$\Phi(f, g) = \int_E f(x)\overline{g(x)}\mathrm{d}x.$$

In fact, (C2) is nothing but an abstract extension of Beppo Levi's theorem. It is clear that in this case $D(\Phi) = \mathcal{L}^2(E)$.

EXAMPLE 2.5. Let $\mathcal{M}$ be a von Neumann algebra and ω a normal faithful semifinite trace defined on $\mathcal{M}_+$ the set of all positive elements of $\mathcal{M}$. Following Nelson [8], we define for $\epsilon, \delta > 0$, $N(\epsilon, \delta) = \{A \in \mathcal{M} :$ for some projection $P \in \mathcal{M}, \|AP\| \leqslant \varepsilon$ and $\omega(P^\perp) \leqslant \delta\}$. We give $\mathcal{M}$ the translation-invariant topology τ in which the $N(\epsilon, \delta)$'s from a fundamental system of neighborhoods of 0. We denote by $\mathfrak{A}$ the space obtained by completion of $\mathcal{M}$ with respect to τ. It is known that $\mathfrak{A}$ is a $*$-algebra of operators affiliated with $\mathcal{M}$, called the $*$-algebra of strictly ω-measurable operators. One denotes by $\mathcal{L}^2$ the completion of the ideal $\mathcal{J}_2 = \{X \in \mathcal{M} : \omega(|X|^2) < \infty\}$ with respect to the norm $\|X\|_2 = \omega(|X|^2)^{1/2}$. We consider

$$\Phi_\omega(X, Y) = \omega(Y^* X), \quad X, Y \in \mathcal{D}(\Phi_\omega) = \mathcal{L}^2.$$

It is easily shown with the usual techniques of (noncommutative) integration that Φ_ω enjoys the conditions (C1) and (C2); (see [3] and [4]).

DEFINITION 2.6. Let Φ be a positive sesquilinear form defined on $\mathfrak{A}_0 \times \mathfrak{A}_0$. We say that Φ is closable, if for a net $\{x_\delta\}_{\delta\in\Delta}$ in $\mathfrak{A}_0$, one has ([5]):

$$x_\delta \xrightarrow{\tau} 0 \text{ and } \Phi(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0 \Rightarrow \Phi(x_\delta, x_\delta) \to 0.$$

We recall that if Φ is closable, then it can be extended to a positive sesquilinear form $\overline{\Phi}$ defined on $D(\overline{\Phi}) \times D(\overline{\Phi})$, where

$$D(\overline{\Phi}) = \{a \in \mathfrak{A} : \exists \ \{x_\delta\}_{\delta\in\Delta} \subset \mathfrak{A}_0, x_\delta \xrightarrow{\tau} a, \text{ and } \Phi(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0\},$$

by

$$\overline{\Phi}(a,a) = \lim_{\delta\in\Delta} \Phi(x_\delta, x_\delta).$$

This definition extends in obvious way to pairs (a,b) with $a, b \in D(\overline{\Phi})$.

By Lemma 2.3, the space $\lambda_{\overline{\Phi}}(D(\overline{\Phi}))[\| \cdot \|_{\overline{\Phi}}]$ is complete.

If ω is a positive linear functional on $\mathfrak{A}$, we can define a positive sesquilinear form Φ_ω on $\mathfrak{A}_0 \times \mathfrak{A}_0$ by

$$\Phi_\omega(a,b) = \omega(b^* a).$$

Note that Φ_ω is *invariant*, i.e. $\Phi_\omega(ab,c) = \Phi_\omega(b, a^* c)$.

Even if ω is continuous, Φ_ω may fail to be continuous, due to the fact that the multiplication in $\mathfrak{A}_0$ is only *separately* continuous.

The following proposition may be proved in a standard way.

PROPOSITION 2.7. *Let ω be a positive continuous linear functional on $\mathfrak{A}_0$. Then Φ_ω is closable.*

Therefore, if ω is a positive continuous linear functional on $\mathfrak{A}_0$, then the closure $\overline{\Phi}_\omega$ of Φ_ω is well defined. Moreover $\overline{\Phi}_\omega$ is a closed positive sesquilinear form on $D(\overline{\Phi}_\omega) \times D(\overline{\Phi}_\omega)$.

LEMMA 2.8. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi $*$-algebra and ω a positive continuous linear functional on $\mathfrak{A}$. Then ω satisfies the conditions* (L1), (L2), (L3)*, and therefore it is representable.*

Proof. Since ω is a positive continuous linear functional on $\mathfrak{A}_0$, (i.e. the condition (L1) is satisfied), it is hermitian on $\mathfrak{A}_0$.

By the continuity of ω and of the involution, it follows that ω is hermitian and positive on $\mathfrak{A}$. Thus (L2) holds.

The positive sesquilinear form $\Phi_\omega(x,y) := \omega(x,y)$ is everywhere defined on $\mathfrak{A}_0$. Hence for every $x, a \in \mathfrak{A}_0$, by the Cauchy–Schwarz inequality we have

$$|\omega(x^* a)| = |\Phi_\omega(a,x)| \leqslant \Phi_\omega(x,x)^{1/2} \Phi_\omega(a,a)^{1/2}.$$

If $x \in \mathfrak{A}$ there exists a net $\{x_\delta\}_{\delta\in\Delta} \subset \mathfrak{A}_0$ such that $x_\delta \xrightarrow{\tau} x$. By the continuity of ω and by Proposition 2.7 we have

$$|\omega(x^*a)| = \lim_{\delta\in\Delta} |\omega(x_\delta^* a)| \leqslant \lim_{\delta\in\Delta} \Phi_\omega(x_\delta, x_\delta)^{1/2}\Phi_\omega(a,a)^{1/2} = \overline{\Phi}_\omega(x,x)^{1/2}\Phi_\omega(a,a)^{1/2}.$$

Thus (L3) also holds. By Proposition 1.4, it follows that ω is representable. ∎

Assume that $\mathfrak{A}[\tau]$ is a topological $*$-algebra, $\mathfrak{A}_0$ a dense $*$-subalgebra of $\mathfrak{A}$. In addition, let us suppose that $\mathfrak{A}_0$ carries a norm $\|\cdot\|_0$ which makes of $\mathfrak{A}_0$ a C^*-algebra. Then what has been said above, applies also in this situation, but we get something more. In particular if ω is a positive linear functional on $\mathfrak{A}_0$, then it is $\|\cdot\|_0$-continuous. Moreover assume that Φ_ω is closable. Then:

(i) $\mathcal{D}(\overline{\Phi}_\omega)$ is a left $\mathfrak{A}_0$-module ($a \in \mathfrak{A}_0$ and $x \in \mathcal{D}(\overline{\Phi}_\omega) \Rightarrow ax \in \mathcal{D}(\overline{\Phi}_\omega)$);

(ii) if ω is a trace (i.e. for each $a, b \in \mathfrak{A}_0, \omega(ab) = \omega(ba)$), then $(\mathcal{D}(\overline{\Phi}_\omega), \mathfrak{A}_0)$ is a quasi $*$-algebra and by continuity of left and right multiplication (Definition 1.1(i)) $\omega(ax) = \omega(xa)$, for each $x \in \mathfrak{A}$ and $a \in \mathfrak{A}_0$.

Moreover if Φ_ω satisfies (C2), then $\mathcal{H}_\omega = D(\overline{\Phi}_\omega)/N_\omega$.

DEFINITION 2.9. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi $*$-algebra.

(i) If φ, ψ are positive linear functionals on $\mathfrak{A}_0$, we say that ψ is *φ-absolutely continuous* if:

(a) $\varphi(x^*x) = 0$, $x \in \mathfrak{A}_0$, implies $\psi(x^*x) = 0$;

(b) $\lambda_\varphi(x) \mapsto \lambda_\psi(x)$, $x \in \mathfrak{A}_0$ is a closable linear map of the pre-Hilbert space $\lambda_\omega(\mathfrak{A}_0)$ into the Hilbert space $\mathcal{H}_\psi$.

(ii) If ω is a representable linear functional on $\mathfrak{A}$ and ψ is a positive linear functional on $\mathfrak{A}_0$, we say that ψ is *ω-absolutely continuous* if ψ is ω_0-absolutely continuous, where $\omega_0 := \omega \restriction \mathfrak{A}_0$.

By Corollary 9.2.9 of [1] we have

THEOREM 2.10. *Let φ and ψ be positive linear functionals on $\mathfrak{A}_0$. Assume that ψ is φ-absolutely continuous. Then there exists a positive self-adjoint operator H in $\mathcal{H}_\phi$ such that: $\lambda_\varphi(\mathfrak{A}_0) \subset D(H)$; $\psi(x^*x) = \langle H\lambda_\varphi(x)|H\lambda_\varphi(x)\rangle$ for every $x \in \mathfrak{A}_0$.*

3. A RADON–NIKODYM TYPE THEOREM

Theorem 2.10 provides an abstract version of Radon–Nikodym theorem for $*$-algebras.

Let now $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi $*$-algebra, ω a representable linear functional on $(\mathfrak{A}, \mathfrak{A}_0)$ and ψ a ω-absolutely continuous positive linear functional on $\mathfrak{A}_0$. An application of Theorem 2.10 yields the representation

$$\psi(x^*x) = \langle H\lambda_\omega(x)|H\lambda_\omega(x)\rangle,$$

for every $x \in \mathfrak{A}_0$.

We will now prove, that, under certain conditions, the operator H is an operator of multiplication. In other words, there exists $a \in \mathfrak{A}$ such that $H\lambda_\omega(x) = \lambda_\omega(a)\lambda_\omega(x)$, for every $x \in \mathfrak{A}_0$. This will be derived as a consequence of some properties of operators acting on Hilbert algebras, since, if ω is a trace, $\lambda_\omega(\mathfrak{A}_0)$ is a Hilbert algebra, in the following sense.

A Hilbert algebra ([9], Section 11.7) is a $*$-algebra $\mathfrak{A}_0$ which is also a pre-Hilbert space with inner product $\langle\cdot|\cdot\rangle$ such that:

(i) The map $b \mapsto ab$ is continuous with respect to the norm defined by the inner product.
(ii) $\langle a|b\rangle = \langle b^*|a^*\rangle$ for all $a, b \in \mathfrak{A}_0$.
(iii) $\mathfrak{A}_0^2$ is total in $\mathfrak{A}_0$.

Let $\mathcal{H}(\mathfrak{A}_0)$ denote the Hilbert space which is the completion of $\mathfrak{A}_0$ with respect to the norm defined by the inner product. The involution of $\mathfrak{A}_0$ extends to the whole of $\mathcal{H}(\mathfrak{A}_0)$, since (ii) implies that $*$ is isometric. Then $(\mathcal{H}(\mathfrak{A}_0), \mathfrak{A}_0)$ is a Banach quasi $*$-algebra.

LEMMA 3.1. *Let T be a positive self-adjoint operator on $\mathcal{H}(\mathfrak{A}_0)$ defined on the dense domain $D(T) \supset \mathfrak{A}_0$. Assume that $D(T)$ has the properties:*
(i) $\xi \in D(T) \Rightarrow \xi^* \in D(T)$;
(ii) $\xi \in D(T)$, $x \in \mathfrak{A}_0 \Rightarrow \xi x \in \mathcal{D}(T)$.
Then, T has the properties:
(a) $T(\xi x) = T(\xi)x$, $\forall\, \xi \in D(T)$, $x \in \mathfrak{A}_0$,
(b) $T(\xi^*) = (T\xi)^*$, $\forall\, \xi \in D(T)$,
if and only if the sesquilinear form

$$\Phi_T(\xi, \eta) := \langle T\xi|\eta\rangle, \quad \xi, \eta \in D(T)$$

has the property

$$\Phi_T(\xi x, y) = \Phi_T(x, \xi^* y), \quad \forall\, \xi \in D(T),\, x, y \in \mathfrak{A}_0.$$

Proof. If T has the properties (a), (b) then for every $\xi \in D(T)$, $x, y \in \mathfrak{A}_0$ we have

$$\Phi_T(\xi x, y) = \langle T(\xi)x|y\rangle = \langle x|T(\xi^*)y\rangle = \langle T(x)|\xi^* y\rangle = \Phi_T(x, \xi^* y).$$

Vice-versa, for each $y, \xi \in \mathfrak{A}_0$ we have

$$\begin{aligned}\langle T(\xi x) - T(\xi)x|y\rangle &= \langle T(\xi x)|y\rangle - \langle T(\xi)x|y\rangle = \Phi_T(\xi x, y) - \langle T(\xi)|yx^*\rangle \\ &= \Phi_T(\xi x, y) - \Phi_T(\xi, yx^*) = \Phi_T(\xi x, y) - \Phi_T(\xi x, y) = 0.\end{aligned}$$

Therefore $T(\xi x) - T(\xi)x = 0$ for each $x, \xi \in \mathfrak{A}_0$, hence by continuity we have $T(\xi x) = T(\xi)x$, $\forall\, \xi \in D(T)$, $x \in \mathfrak{A}_0$. Moreover

$$\langle T(\xi)x|y\rangle = \langle T(\xi x)|y\rangle = \Phi_T(\xi x, y) = \Phi_T(x, \xi^* y) = \langle Tx|\xi^* y\rangle = \langle x|T(\xi^*)y\rangle. \quad \blacksquare$$

THEOREM 3.2. *Let Φ be a sesquilinear form on $\mathfrak{A}_0 \times \mathfrak{A}_0$. The following statements are equivalent:*

(i) *Φ enjoys the following properties:*
(i.a) $\Phi(x,x) \geqslant 0,\ \forall\, x \in \mathfrak{A}_0$;
(i.b) $\Phi(y^*,x^*) = \Phi(x,y),\ \forall\, x,y \in \mathfrak{A}_0$;
(i.c) $\Phi(xy,z) = \Phi(y,x^*z),\ \forall\, x,y,z \in \mathfrak{A}_0$;
(i.d) $\forall\, y \in \mathfrak{A}_0,\ \exists\, \gamma_y > 0 :\ |\Phi(x,y)| \leqslant \gamma_y \|x\|,\ \forall\, x \in \mathfrak{A}_0$.

(ii) *There exists a symmetric operator T with $D(T) = \mathfrak{A}_0$ such that:*
(ii.a) $\Phi(x,y) = \langle x|Ty\rangle,\ \forall\, x,y \in \mathfrak{A}_0$;
(ii.b) $\langle Tx|x\rangle \geqslant 0,\ \forall\, x \in \mathfrak{A}_0$;
(ii.c) $(Tx)^* = Tx^*,\ \forall\, x \in \mathfrak{A}_0$;
(ii.d) $T(xy) = (Tx)y,\ \forall\, x,y \in \mathfrak{A}_0$.

If, in addition, $\mathfrak{A}_0$ has a unit e then T is a multiplication operator; i.e., there exists $h \in \mathcal{H}(\mathfrak{A}_0)$, commuting with every $x \in \mathfrak{A}_0$, such that

$$Tx = hx, \quad \forall\, x \in \mathfrak{A}_0.$$

Proof. If T has the properties (ii.a)–(ii.d), then for each $x,y,z \in \mathfrak{A}_0$:

- $\Phi(x,x) \geqslant \langle x|Tx\rangle = \langle Tx|x\rangle \geqslant 0$;
- $\Phi(y^*,x^*) = \langle y^*|Tx^*\rangle = \langle (Tx^*)^*|y)\rangle = \langle Tx|y\rangle = \Phi(x,y)$;
- $\Phi(xy,z) = \langle xy|Tz\rangle = \langle Txy|z\rangle = \langle T(x)y|z\rangle = \langle y|T(x)^*z\rangle = \langle y|T(x^*)z\rangle = \langle y|T(x^*z)\rangle = \Phi(y,x^*z)$;
- by the continuity of T, there exists $C > 0$ such that

$$\|Ty\| \leqslant C\|y\|,$$

for each $y \in \mathfrak{A}_0$.

Hence $|\Phi(x,y)| = \Phi(x,Ty) \leqslant \|Ty\|\|x\| \leqslant C\|y\|\|x\|$.

Suppose now, that Φ enjoys the properties (i.a)–(i.d).
Note that (i.d) implies that Φ is bounded; therefore, by Riesz's theorem there exists a symmetric operator T with $D(T) = \mathfrak{A}_0$ such that $\Phi(x,y) = \langle x|Ty\rangle$ for $x,y \in \mathfrak{A}_0$.

Moreover (i.a) implies that $\langle Tx|x\rangle \geqslant 0$, for each $x \in \mathfrak{A}_0$. The conditions (ii.c) and (ii.d) follow by Lemma 3.1. ∎

REMARK 3.3. The operator T commutes with the operator $J : x \in \mathfrak{A}_0 \to x^* \in \mathfrak{A}_0$ which is a conjugation. This implies that T has self-adjoint extensions.

THEOREM 3.4. *Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi $*$-algebra, ω a rapresentable linear functional on $\mathfrak{A}$ and f an ω-absolutely continuous linear functional on $\mathfrak{A}_0$. If f and ω are traces such that $\overline{\Phi}_\omega$ satisfies the conditions* (C1), (C2) *and* (i.d) *then there exist a positive element $h \in \mathfrak{A}$ such that*

$$f(a^*a) = \omega(a^*ha), \quad \forall\, a \in \mathfrak{A}_0.$$

Proof. If f is continuous on $\mathfrak{A}_0$, we can construct the closure $\overline{\Phi}_f$ of Φ_f. Of course it is a closed positive sesquilinear form on $D(\overline{\Phi}_f) \times D(\overline{\Phi}_f)$ such that for each a, b in $\mathfrak{A}_0$ we have

$$\overline{\Phi}_f(\lambda_\omega(a), \lambda_\omega(b)) = f(b^*a).$$

If $\lambda_\omega(a) = 0$ for $a \in \mathfrak{A}_0$, then $\overline{\Phi}_f(\lambda_\omega(a), \lambda_\omega(a)) = f(a^*a) = 0$, therefore $\omega(a^*a) = 0$. Since ω is a trace, $\lambda_\omega(\mathfrak{A}_0)$ is a Hilbert algebra with unit.

Since $\overline{\Phi}_f$ satisfies conditions (i.a)–(i.d) of Theorem 3.2, with respect to the inner product defined by ω on $\lambda_\omega(\mathfrak{A}_0)$, then there exists a positive operator of multiplication T such that

$$\overline{\Phi}_f(\lambda_\omega(a), \lambda_\omega(b)) = \langle \lambda_\omega(a) | T\lambda_\omega(b) \rangle, \quad \forall\, a, b \in \mathfrak{A}_0.$$

Let $\widehat{T}$ be a self-adjoint positive extension of T. Then, we may consider $H = \widehat{T}^{1/2}$. Thus $f(b^*a) = \langle H\lambda_\omega(a) | H\lambda_\omega(b) \rangle$ for each $a, b \in \mathfrak{A}_0$. Therefore there exists $\xi \in \mathcal{H}_\omega = \mathcal{H}(\lambda_\omega(\mathfrak{A}_0))$ such that

$$\overline{\Phi}_f(\lambda_\omega(a), \lambda_\omega(b)) = \langle H\lambda_\omega(a) | H\lambda_\omega(b) \rangle = \langle \lambda_\omega(a) | \xi\lambda_\omega(b) \rangle = f(b^*a).$$

Hence,

$$H\lambda_\omega(b) \in \mathcal{D}((H \restriction \lambda_\omega(\mathfrak{A}_0))^*) \quad \text{and} \quad H\lambda_\omega(\mathfrak{A}_0) \subseteq \mathcal{D}((H \restriction \lambda_\omega(\mathfrak{A}_0))^*).$$

But $\overline{\Phi}_\omega$ satisfies (C1) and (C2), whence $\mathcal{H}_\omega = D(\overline{\Phi}_\omega)/N_\omega = \lambda_\omega(\mathfrak{A}_0)$. Therefore, there exists $h \in \mathfrak{A}$ such that

$$f(a^*a) = \langle \lambda_\omega(a) | \lambda_\omega(h)\lambda_\omega(a) \rangle = \omega(a^*ha). \quad \blacksquare$$

REMARK 3.5. It is easily seen that the condition (1.d) holds if and only if $H\lambda_\omega(\mathfrak{A}_0) \subset D(H)$. A rather strong, but almost obvious condition is the following:

$\forall\, b \in \mathfrak{A}_0$, there exists $\gamma_b > 0$ such that $|f(b^*a)| \leqslant \gamma_b \omega(b^*b)^{1/2}$.
Note that this condition is fulfilled if f is w-dominated i.e. there exists an $M > 0$ such that $f(x^*x) \leqslant M\omega(x^*x)$, for each $x \in \mathfrak{A}_0$.

EXAMPLE 3.6. The conditions (C1), (C2) and (1.d) given in the previous theorem are satisfied in the case of commutative and noncommutative integration, considered in Examples 2.4 and 2.5.

It is important to notice that our version of the Radon–Nikodym theorem includes the classical statement for measure spaces and as in that case, when we consider integrals as functionals on some space $L^\infty(X, \mathrm{d}\mu)$ the Radon–Nikodym derivative lives in a large space (in this example $L^1(X, \mathrm{d}\mu)$). Moreover in the noncommutative case, the Radon–Nikodym derivatives for functionals on a von Neumann algebra $\mathcal{M}$ with a normal finite faithful trace is a closed, densely defined operator affiliated with $\mathcal{M}$.

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C. LA RUSSA, DIP. DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI PALERMO, I-90123 PALERMO, ITALY
E-mail address: larussa@math.unipa.it

S. TRIOLO, DIP. DI ENERGIA, ING. DELL'INFORMAZIONE E MODELLI MATEMATICI (DEIM), UNIVERSITÀ DI PALERMO, VIA DELLE SCIENZE, I-90123 PALERMO, ITALY
E-mail address: salvatore.triolo@unipa.it

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