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COMPACT COMPOSITION OPERATORS ON THE HARDY–ORLICZ AND WEIGHTED BERGMAN–ORLICZ SPACES ON THE BALL

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ABSTRACT. Using recent characterizations of the compactness of composition operators on the Hardy–Orlicz and Bergman–Orlicz spaces on the ball [3], [4], we first show that a composition operator which is compact on every Hardy–Orlicz (or Bergman–Orlicz) space has to be compact on H^∞. Then, although it is well-known that a map whose range is contained in some nice Korányi approach region induces a compact composition operator on $H^p(\mathbb{B}_N)$ or on $A^p_\alpha(\mathbb{B}_N)$, we prove that, for each Korányi region Γ, there exists a map $\phi : \mathbb{B}_N \to \Gamma$ such that C_ϕ is not compact on $H^\psi(\mathbb{B}_N)$, when ψ grows fast. Finally, we extend (and simplify the proof of) a result by K. Zhu for the classical weighted Bergman spaces, by showing that, under reasonable conditions, a composition operator C_ϕ is compact on the weighted Bergman–Orlicz space $A^\psi_\alpha(\mathbb{B}_N)$, if and only if

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|)^{N(\alpha)})}{\psi^{-1}(1/(1-|z|)^{N(\alpha)})} = 0.$$

In particular, we deduce that the compactness of composition operators on $A^\psi_\alpha(\mathbb{B}_N)$ does not depend on α anymore when the Orlicz function ψ grows fast.

KEYWORDS: *Carleson measure, composition operator, Hardy–Orlicz space, several complex variables, weighted Bergman–Orlicz space.*

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1. INTRODUCTION

Let $\mathbb{B}_N = \left\{ z = (z_1, \dots z_N) \in \mathbb{C}^N, \sum_{i=1}^{N} |z_i|^2 < 1 \right\}$ denote the open unit ball of $\mathbb{C}^N$. Given a holomorphic map $\phi : \mathbb{B}_N \to \mathbb{B}_N$, the composition operator C_ϕ

of symbol ϕ is defined by $C_\phi(f) = f \circ \phi$, for f holomorphic on $\mathbb{B}_N$. Composition operators have been extensively studied on common Banach spaces of analytic functions, in particular on the Hardy spaces $H^p(\mathbb{B}_N)$ and on the Bergman spaces $A^p(\mathbb{B}_N)$, $1 \leqslant p < \infty$. The continuity and compactness of these operators have been characterized in terms of Carleson measures [6]. In dimension one, the boundedness of C_ϕ for any $\phi : \mathbb{D} \to \mathbb{D}$ is a consequence of the Littlewood subordination principle [17]. In $\mathbb{C}^N$, $N > 1$, it is well-known that there exists some map $\phi : \mathbb{B}_N \to \mathbb{B}_N$ such that the associated composition operator is not bounded on $H^p(\mathbb{B}_N)$. Whatever the dimension, it appears that both boundedness and compactness of C_ϕ on $H^p(\mathbb{B}_N)$ (respectively $A^p(\mathbb{B}_N)$) are independent of p. On the other hand, every composition operator is obviously bounded on H^∞ and it is not difficult to check that C_ϕ is compact on H^∞ if and only if $\|\phi\|_\infty < 1$. Thus there is a "break" between H^∞ and $H^p(\mathbb{B}_N)$ (respectively $A^p(\mathbb{B}_N)$), for the compactness in dimension one, and even for the boundedness, when $N > 1$.

These observations first motivated P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza to study composition operators on Hardy–Orlicz spaces $H^\psi(\mathbb{D})$ (respectively Bergman–Orlicz spaces $A^\psi(\mathbb{D})$) of the disc [8], [9], [10], [11], and then the author of [3], [4] to look at these questions in $\mathbb{C}^N$. These spaces both provide an intermediate scale of spaces between H^∞ and $H^p(\mathbb{B}_N)$ (respectively $A^p(\mathbb{B}_N)$) and generalize the latter. In particular, in [11], the authors were interested in the question of whether there are Hardy–Orlicz spaces on which the compactness of C_ϕ is equivalent to that on H^∞. In fact, they answer this question in the negative, by proving Theorem 4.1 of [11], that, for every Hardy–Orlicz space $H^\psi(\mathbb{D})$, one can construct a surjective map $\phi : \mathbb{D} \to \mathbb{D}$ which induces a compact composition operator C_ϕ on $H^\psi(\mathbb{D})$. This result extends that obtained by B. MacCluer and J. Shapiro for $H^p(\mathbb{D})$ ([14], Example 3.12). The same problem in the Bergman–Orlicz case has not yet been completely solved. In several variables, the situation is much more surprizing, as we show in [3], [4] that there exist some Hardy–Orlicz and Bergman–Orlicz spaces, "close" enough to H^∞, on which every composition operator is bounded.

In this paper, we are mainly interested in the possibility to extend some known results about the compactness of composition operators on the classical Hardy or Bergman spaces, to the corresponding Orlicz spaces. We think that this study may outline some interesting phenomena and precise the link between the behavior of C_ϕ and that of ϕ.

First of all, we come back to the "break" between H^∞ and H^p, $1 \leqslant p < \infty$, for the compactness of C_ϕ. There is no difference between being compact for C_ϕ on one $H^p(\mathbb{B}_N)$ and on every $H^p(\mathbb{B}_N)$, while this property clearly depends on the Orlicz function ψ in $H^\psi(\mathbb{B}_N)$. Therefore, we can wonder if the above question answered by [11] was the good one; indeed, the study of C_ϕ on Hardy–Orlicz spaces arises the following question: what can we say about a composition operator which is compact on every Hardy–Orlicz space? It turns out that such an operator has to be compact on H^∞, which seems to us to be a positive result, because it somehow confirms that the Hardy–Orlicz spaces cover well the "gap" between every H^p and H^∞. This result also stands when we replace Hardy–Orlicz spaces by Bergman–Orlicz spaces.

Moreover, on the Hardy or Bergman spaces, the compactness (and boundedness) of composition operators is often handled in terms of geometric conditions, emphasizing the importance of the manner in which the symbol ϕ approaches the boundary of $\mathbb{B}_N$. To be precise, let us denote by $\Gamma(\zeta, a) \subset \mathbb{B}_N$, for $\zeta \in \mathbb{S}_N$ and $a > 1$, the Korányi approach region

$$\Gamma(\zeta, a) = \left\{ z \in \mathbb{B}_N,\ |1 - \langle z, \zeta \rangle| < \frac{a}{2}(1 - |z|^2) \right\}.$$

It is known ([13]) that if ϕ takes the unit ball into a Korányi region $\Gamma(\zeta, a)$ with a small enough angular opening a, then C_ϕ is compact on $H^p(\mathbb{B}_N)$ and on $A^p_\alpha(\mathbb{B}_N)$. When $N = 1$, the Korányi regions are just non-tangential approach regions. In this paper, we show that this result does not hold for Hardy–Orlicz spaces on $\mathbb{B}_N$ in general; for Bergman–Orlicz spaces, we obtain such a result in dimension one only.

In [14], the authors related the compactness of the composition operator C_ϕ on $H^p(\mathbb{D})$ or $A^p_\alpha(\mathbb{D})$ to the existence of angular derivative for ϕ at the boundary. We say that the *angular derivative* of ϕ exists at a point $\zeta \in \mathbb{T}$ if there exists $\omega \in \mathbb{T}$ such that

$$\frac{\phi(z) - \omega}{z - \zeta}$$

has a finite limit as ζ tends non-tangentially to ζ through $\mathbb{D}$. Julia–Caratheodory's theorem then asserts that the non-existence of an angular derivative for ϕ at some $\zeta \in \mathbb{T}$ is equivalent to

$$\lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0. \tag{1.1}$$

Shapiro and Taylor [18] pointed out that if C_ϕ is to be compact on $H^p(\mathbb{D})$, then ϕ cannot have an angular derivative at even a single point in $\mathbb{T}$, which may be written:

$$\lim_{|z| \to 1} \frac{1 - |z|}{1 - |\phi(z)|} = 0. \tag{1.2}$$

In [14], it is proved that (1.2) is not sufficient for the compactness of C_ϕ on the Hardy spaces of the unit disc in general, yet it is when ϕ is univalent. However, this condition is necessary and sufficient for C_ϕ to be compact on every weighted Bergman spaces of the disc. The last main goal of this paper is to extend some of these results to Hardy–Orlicz and Bergman–Orlicz spaces of the unit ball.

In several variables, we can also define the angular derivative of $\phi : \mathbb{B}_N \to \mathbb{B}_N$ at a point in $\mathbb{S}_N$ (see Definition 2.80 of [6]) and Julia–Caratheodory's theorem also holds in $\mathbb{B}_N$ ([6], Theorem 2.81 or [16], Theorem 8.5.6). Here, as we already said, the situation is complicated by the fact that some composition operators are not bounded on Hardy or Bergman spaces, and the fact that even the boundedness of C_ϕ on $A^p_\alpha(\mathbb{B}_N)$ depends on α. In [19], K. Zhu proves that C_ϕ is compact on $A^p_\alpha(\mathbb{B}_N)$ if and only if a condition similar to Condition (1.2) is satisfied, whenever C_ϕ is bounded on some $A^p_\beta(\mathbb{B}_N)$, for some $-1 < \beta < \alpha$. This assumption is somehow justified by the above observation and by Section 6 of [14], in which the authors show that, for any $\alpha > -1$ and any $0 < p < \infty$, there exists $\phi : \mathbb{B}_N \to \mathbb{B}_N$ with no angular derivative at any point of $\mathbb{S}_N$, such that C_ϕ is bounded on $A^p_\alpha(\mathbb{B}_N)$ but not compact. There even exists such a map ϕ such that C_ϕ is not bounded on $A^p_\alpha(\mathbb{B}_N)$. In the present paper, we generalize Zhu's result to the weighted Bergman–Orlicz spaces on the ball, by using recent characterizations of the boundedness and compactness of composition operators on these spaces [3]. We show that, if C_ϕ is bounded on some $A^\psi_\beta(\mathbb{B}_N)$, $-1 < \beta < \alpha$, then it is compact on $A^\psi_\alpha(\mathbb{B}_N)$ if and only if

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|)^{N(\alpha)})}{\psi^{-1}(1/(1-|z|)^{N(\alpha)})} = 0, \tag{1.3}$$

where $N(\alpha) = N + \alpha + 1$, under a mild and usual regularity condition on the Orlicz function ψ. Our proof is quite simple, while that of K. Zhu uses a Schur test in $H^2(\mathbb{B}_N)$ and the fact that the compactness of composition operators on $H^p(\mathbb{B}_N)$ does not depend on p. Combining this result with the automatic boundedness of every composition operator on $A^\psi_\alpha(\mathbb{B}_N)$ when ψ satisfies the Δ^2-condition ([3], Theorem 3.7), we get that the compactness on such $A^\psi_\alpha(\mathbb{B}_N)$ does not depend on α anymore. To be precise, C_ϕ is compact on $A^\psi_\alpha(\mathbb{B}_N)$ if and only if

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|))}{\psi^{-1}(1/(1-|z|))} = 0,$$

whenever ψ satisfies the Δ^2-condition.

We have to mention that condition (1.3) is, in any case, necessary. Moreover, the authors of [8] obtained such a result in dimension one, as announced in [12]. Their proof uses the characterization of the compactness of composition operators in terms of the Nevanlinna counting function and, for this reason, is more complicated.

We organize our paper as follows: a first preliminary part is devoted to the definitions and the statements of the already known results we need. The main part contains the proofs of the three most important results mentionned above.

NOTATIONS. Throughout this paper, we will denote by $d\sigma_N$ the normalized rotation-invariant positive Borel measure on the unit sphere $\mathbb{S}_N = \partial\mathbb{B}_N$, and by $dv_\alpha = c_\alpha(1-|z|^2)^\alpha dv$, $\alpha > -1$, the normalized weighted Lebesgue measure on the ball.

Given two points $z, w \in \mathbb{C}^N$, the euclidean inner product of z and w will be denoted by $\langle z, w\rangle$, that is $\langle z, w\rangle = \sum_{i=1}^{N} z_i \overline{w}_i$; the notation $|\cdot|$ will stand for the associated norm, as well as for the modulus of a complex number.

If $\alpha > -1$ is a real number, we will denote by $N(\alpha)$ the quantity $N + \alpha + 1$.

2. PRELIMINARIES

2.1. HARDY–ORLICZ AND BERGMAN–ORLICZ SPACES. DEFINITIONS. A strictly convex function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *Orlicz function* if $\psi(0) = 0$, ψ is continuous at 0 and $\psi(x)/x \xrightarrow[x\to+\infty]{} +\infty$. If $(\Omega, \mathbb{P})$ is a probability space, the Orlicz space $L^\psi(\Omega)$ associated with the Orlicz function ψ on $(\Omega, \mathbb{P})$ is the set of all (equivalence classes of) measurable functions f on Ω such that there exists some $C > 0$, such that $\int_\Omega \psi(|f|/C)\mathrm{d}\mathbb{P}$ is finite. $L^\psi(\Omega)$ is a vector space, which can be normed with the so-called Luxemburg norm defined by

$$\|f\|_\psi = \inf\Big\{C > 0,\ \int_\Omega \psi(|f|/C)\mathrm{d}\mathbb{P} \leqslant 1\Big\}.$$

It is well-known that $(L^\psi(\Omega), \|\cdot\|_\psi)$ is a Banach space (see [15]).

Taking $\Omega = \mathbb{S}_N$ and $\mathrm{d}\mathbb{P} = \mathrm{d}\sigma_N$, the Hardy–Orlicz space $H^\psi(\mathbb{B}_N)$ on $\mathbb{B}_N$ is the Banach space of analytic functions $f : \mathbb{B}_N \to \mathbb{C}$ such that $\|f\|_{H^\psi} := \sup_{0<r<1} \|f_r\|_\psi < \infty$, where $f_r \in L^\psi(\mathbb{S}_N)$ is defined by $f_r(z) = f(rz)$. Every function $f \in H^\psi(\mathbb{B}_N)$ admits a radial boundary limit f^* such that $\|f^*\|_\psi = \sup_{0<r<1} \|f_r\|_\psi < \infty$ ([4], Section 1.3). For simplicity, we will denote by $\|\cdot\|_\psi$ the norm on $H^\psi(\mathbb{B}_N)$, emphasizing that $H^\psi(\mathbb{B}_N)$ can be seen as a subspace of $L^\psi(\mathbb{S}_N)$.

With $\Omega = \mathbb{B}_N$ and $\mathrm{d}\mathbb{P} = \mathrm{d}v_\alpha$, $\alpha > -1$, the weighted Bergman–Orlicz space $A^\psi_\alpha(\mathbb{B}_N)$ is $L^\psi(\mathbb{B}_N) \cap H(\mathbb{B}_N)$, where $H(\mathbb{B}_N)$ stands for the vector space of analytic functions on the unit ball. $A^\psi_\alpha(\mathbb{B}_N)$ is a Banach space.

From the definitions, it is easy to verify that the following inclusions hold:

$$H^\infty \subset H^\psi(\mathbb{B}_N) \subset H^1(\mathbb{B}_N) \quad \text{and} \quad H^\infty \subset A^\psi_\alpha(\mathbb{B}_N) \subset A^1_\alpha(\mathbb{B}_N)$$

for every Orlicz function ψ. Moreover, if $\psi(x) = x^p$, for some $1 < p < \infty$ and for every $x \geqslant 0$, then $H^\psi(\mathbb{B}_N) = H^p(\mathbb{B}_N)$ and $A^\psi_\alpha(\mathbb{B}_N) = A^p_\alpha(\mathbb{B}_N)$.

2.2. FOUR CLASSES OF ORLICZ FUNCTIONS. Let ψ be an Orlicz function. In order to distinguish the Orlicz spaces and to get a significant scale of intermediate spaces between L^∞ and $L^p(\Omega)$, we define four classes of Orlicz functions. The first two conditions are regularity conditions.

(1) We say that ψ satisfies the ∇_0-condition if it satisfies one of the following two equivalent conditions:

(i) For any $B > 1$, there exists some constant $C_B \geqslant 1$, such that

$$\frac{\psi(Bx)}{\psi(x)} \leqslant \frac{\psi(C_B By)}{\psi(y)}$$

for any $x \leqslant y$ large enough;

(ii) For any $B > 1$, for any $n > 0$, there exists $C_{n,B} > 0$ such that

$$\frac{\psi(Bx)^n}{\psi(x)^n} \leqslant \frac{\psi(C_{n,B} y)}{\psi(By)}$$

for any $x \leqslant y$ large enough.

Let us notice that (ii) $\Rightarrow$ (i) is obvious since ψ is non-decreasing, while an easy induction allows to prove (i) $\Rightarrow$ (ii); the details are left to the reader.

If the constant C_B in (i) can be chosen independently of B, then ψ satisfies the *uniform* ∇_0-condition.

(2) The ∇_2-class consists of those Orlicz functions ψ such that there exist some $\beta > 1$ and some $x_0 > 0$, such that $\psi(\beta x) \geqslant 2\beta\psi(x)$, for $x \geqslant x_0$.

(3) The third condition is a condition of moderate growth: ψ satisfies the Δ_2-condition if there exist $x_0 > 0$ and a constant $K > 1$, such that $\psi(2x) \leqslant K\psi(x)$ for any $x \geqslant x_0$.

(4) The fourth condition is a condition of fast growth: ψ satisfies the Δ^2-condition if it satisfies one of the following equivalent conditions:

(i) There exist $C > 0$ and $x_0 > 0$, such that $\psi(x)^2 \leqslant \psi(Cx)$ for every $x \geqslant x_0$;

(ii) There exist $b > 1$, $C > 0$ and $x_0 > 0$ such that $\psi(x)^b \leqslant \psi(Cx)$, for every $x \geqslant x_0$;

(iii) For every $b > 1$, there exist $C_b > 0$ and $x_{0,b} > 0$ such that $\psi(x)^b \leqslant \psi(C_b x)$, for every $x \geqslant x_{0,b}$.

Finally, we mention that these conditions are not independent (see Proposition 4.7 of [8]):

PROPOSITION 2.1. *Let ψ be an Orlicz function.*

(i) *If ψ satisfies the uniform ∇_0-condition, then it satisfies the ∇_2-condition;*

(ii) *If ψ satisfies the Δ^2-condition, then it satisfies the uniform ∇_0-condition.*

For any $1 < p < \infty$, every function $x \longmapsto x^p$ is an Orlicz function which satisfies the uniform ∇_0-condition (so the ∇_2 and ∇_0-conditions too) and the Δ_2-condition. At the opposite side, for any $a > 0$ and $b \geqslant 1$, $x \longmapsto \mathrm{e}^{ax^b} - 1$ belongs to the Δ^2-class (and then to the uniform ∇_0-class), but not to the Δ_2-one. In addition, the Orlicz functions which can be written $x \to \exp(a(\ln(x+1))^b) - 1$ for $a > 0$ and $b \geqslant 1$, satisfy the ∇_2 and ∇_0-conditions, but do not belong to the Δ^2-class.

For a complete study of Orlicz spaces, we refer to [7] and [15]. We can also find precise and useful information in [8], such as other classes of Orlicz functions and their links with each other.

2.3. BACKGROUND RESULTS. All the results of the present paper are based on characterizations of the boundedness and compactness of composition operators on the Hardy–Orlicz and Bergman–Orlicz spaces [3], [4]. As I already said, these characterizations essentially depend on the manner in which the Orlicz function grows.

The characterizations of the boundedness and compactness of C_ϕ involve adapted Carleson measures, and then geometric notions. For $\zeta \in \mathbb{S}_N$ and $0 < h < 1$, let us denote by $S(\zeta,h)$ and $\mathcal{S}(\zeta,h)$ the non-isotropic "balls", respectively in $\mathbb{B}_N$ and $\overline{\mathbb{B}}_N$, defined by

$$S(\zeta,h) = \{z \in \mathbb{B}_N,\ |1-\langle z,\zeta\rangle| < h\} \quad \text{and} \quad \mathcal{S}(\zeta,h) = \{z \in \overline{\mathbb{B}}_N,\ |1-\langle z,\zeta\rangle| < h\}.$$

We say that a finite positive Borel measure μ on $\overline{\mathbb{B}}_N$ is a ψ-Carleson measure, ψ an Orlicz function, if

$$\mu(\mathcal{S}(\zeta,h)) = O_{h\to 0}\Big(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\Big),$$

uniformly in $\zeta \in \mathbb{S}_N$ and for some constant $A > 0$. μ is a *vanishing* ψ-Carleson measure if the above condition is satisfied for every $A > 0$ and with the big-Oh condition replaced by a little-oh condition.

A finite positive Borel measure μ on $\mathbb{B}_N$ is a (ψ,α)-Bergman–Carleson measure if

$$\mu(S(\zeta,h)) = O_{h\to 0}\Big(\frac{1}{\psi(A\psi^{-1}(1/h^{N(\alpha)}))}\Big),$$

uniformly in $\zeta \in \mathbb{S}_N$ and for some constant $A > 0$. μ is a *vanishing* (ψ,α)-Bergman–Carleson measure if the above condition is satisfied for every $A > 0$ and with the big-Oh condition replaced by a little-oh condition.

When ψ satisfies the Δ_2-condition, a (vanishing) ψ-Carleson measure (respectively (vanishing) (ψ,α)-Bergman–Carleson measure) is a (vanishing) Carleson measure (respectively (vanishing) Bergman–Carleson measure) (see Sections 3 of [3], [4]).

REMARK 2.2. (i) Note that, by the convexity of ψ, μ is a ψ-Carleson measure (respectively vanishing ψ-Carleson measure) if, for some $A > 0$ (respectively for every $A > 0$),

$$\mu(S(\zeta,h)) \leqslant \frac{1}{\psi(A\psi^{-1}(1/h^N))},$$

uniformly in $\zeta \in \mathbb{S}_N$ and for all h small enough. For the vanishing ψ-Carleson measure, this is also due to arbitrary $A > 0$. Of course, a similar observation holds for (vanishing) (ψ,α)-Bergman–Carleson measure.

(ii) Let us mention that the non-isotropics "balls" $S(\zeta,h)$ can be replaced by the usual *Carleson boxes* $W(\zeta,h)$ (see e.g. [16] for the definition); for convenience, we will just work with non-isotropic "balls".

For $\phi : \mathbb{B}_N \to \mathbb{B}_N$, we denote by μ_ϕ the pull-back measure of σ_N by the boundary limit ϕ^* of ϕ, and by $\mu_{\phi,\alpha}$ that of dv_α by ϕ. To be precise, for any $E \subset \overline{\mathbb{B}}_N$ (respectively $E \subset \mathbb{B}_N$),

$$\mu_\phi(E) = \sigma_N((\phi^*)^{-1}(E)) \text{ (respectively } \mu_{\phi,\alpha}(E) = v_\alpha(\phi^{-1}(E))).$$

RESULTS FOR HARDY–ORLICZ SPACES. (see Section 3 of [4]):
The main theorem is the following:

THEOREM 2.3. *Let ψ be an Orlicz function which satisfies the ∇_2-condition and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic.*

(i) *If ψ satisfies the uniform ∇_0-condition, then C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ is a ψ-Carleson measure.*

(ii) *If ψ satisfies the ∇_0-condition, then C_ϕ is compact from $H^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ is a vanishing ψ-Carleson measure.*

(iii) *If ψ satisfies the Δ_2-condition, then C_ϕ is bounded (respectively compact) from $H^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ is a Carleson measure (respectively a vanishing Carleson measure).*

(iv) *If ψ satisfies the Δ^2-condition, then C_ϕ is bounded on $H^\psi(\mathbb{B}_N)$.*

The first two points are contained in Theorem 3.2 of [4]; according to Theorem 3.35 of [6], the third point means that, if ψ satisfies the Δ_2-condition, then C_ϕ is bounded (respectively compact) on $H^\psi(\mathbb{B}_N)$ if and only if it is on $H^p(\mathbb{B}_N)$ (see Corollary 3.4 of [4]). The last point is Theorem 3.7 of [4].

Due to the non-separability of small Hardy–Orlicz spaces, Theorem 3.2 of [4] is not a direct consequence of Carleson-type embedding theorems obtained in Section 2 of [4]. However, if we follow the proofs of these embedding theorems directly for composition operators, by working on spheres of radius $0 < r < 1$, then we get the following characterizations of both boundedness and compactness of composition operators ([2], Theorem 3.30):

THEOREM 2.4. *Let ψ be an Orlicz function satisfying the ∇_2-condition and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic.*

(i) *If ψ satisfies the uniform ∇_0-condition, then C_ϕ is bounded on $H^\psi(\mathbb{B}_N)$ if and only if there exists some $A > 0$ such that*

$$\sup_{0<r<1} \mu_{\phi_r}(\mathcal{S}(\zeta, h)) = O_{h\to 0}\Big(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\Big) \tag{2.1}$$

uniformly in $\zeta \in \mathbb{S}_N$.

(ii) *If ψ satisfies the ∇_0-condition, then C_ϕ is compact on $H^\psi(\mathbb{B}_N)$ if and only if, for every $A > 0$,*

$$\sup_{0<r<1} \mu_{\phi_r}(\mathcal{S}(\zeta, h)) = o_{h\to 0}\Big(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\Big) \tag{2.2}$$

uniformly in $\zeta \in \mathbb{S}_N$.

In the further, we will see how these two characterizations are useful depending on the situations.

RESULTS FOR BERGMAN–ORLICZ SPACES. (see Section 3 of [3]):
The main result is similar to that stated in the previous paragraph:

THEOREM 2.5. *Let ψ be an Orlicz function, let $\alpha > -1$ and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic.*

(i) *If ψ satisfies the uniform ∇_0-condition, then C_ϕ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if $\mu_{\phi,\alpha}$ is a (ψ, α)-Bergman–Carleson measure.*

(ii) *If ψ satisfies the ∇_0-condition, then C_ϕ is compact from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if $\mu_{\phi,\alpha}$ is a vanishing (ψ, α)-Bergman–Carleson measure.*

(iii) *If ψ satisfies the Δ_2-condition, then C_ϕ is bounded (respectively compact) from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if $\mu_{\phi,\alpha}$ is a Bergman–Carleson measure (respectively a vanishing Bergman–Carleson measure).*

(iv) *If ψ satisfies the Δ^2-condition, then C_ϕ is bounded on $A_\alpha^\psi(\mathbb{B}_N)$.*

According to Theorem 3.37 of [6], the third point means that, if ψ satisfies the Δ_2-condition, then C_ϕ is bounded (respectively compact) on $A_\alpha^\psi(\mathbb{B}_N)$, if and only if C_ϕ is bounded (respectively compact) on $A_\alpha^p(\mathbb{B}_N)$.

3. MAIN RESULTS

3.1. COMPACTNESS OF C_ϕ ON ALL HARDY–ORLICZ OR BERGMAN–ORLICZ SPACES.

The following theorem completes both Theorem 2.3 and Theorem 2.5.

THEOREM 3.1. *Let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be a holomorphic map. The following assertions are equivalent:*

(i) *C_ϕ is compact on $H^\psi(\mathbb{B}_N)$, for every Orlicz function ψ;*

(ii) *for some $\alpha > -1$, C_ϕ is compact on $A_\alpha^\psi(\mathbb{B}_N)$, for every Orlicz function ψ;*

(iii) *C_ϕ is compact on $H^\infty(\mathbb{B}_N)$;*

(iv) *$\|\phi\|_\infty < 1$.*

Proof. It is well-known that C_ϕ is compact on $H^\infty(\mathbb{B}_N)$ if and only if $\|\phi\|_\infty < 1$. Using the fact ([3], Proposition 2.8 and [4], Proposition 2.11) that a composition operator is compact on $H^\psi(\mathbb{B}_N)$ (respectively $A_\alpha^\psi(\mathbb{B}_N)$) if and only if for every bounded sequence $(f_n)_n \subset H^\psi(\mathbb{B}_N)$, $\|f_n\|_\psi \leqslant 1$, (respectively $(f_n)_n \subset A_\alpha^\psi(\mathbb{B}_N)$, $\|f_n\|_{A_\alpha^\psi} \leqslant 1$) which tends to 0 uniformly on every compact subset of $\mathbb{B}_N$, $\|f_n \circ \phi\|_{H^\psi} \xrightarrow[n\to\infty]{} 0$ (respectively $\|f_n \circ \phi\|_{A_\alpha^\psi} \xrightarrow[n\to\infty]{} 0$), it is not difficult to show that if $\|\phi\|_\infty < 1$, then C_ϕ is compact on $H^\psi(\mathbb{B}_N)$ (respectively $A_\alpha^\psi(\mathbb{B}_N)$.)

It remains to prove (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (iv). We first deal with the proof of (i) $\Rightarrow$ (iv). We will use the necessary part of the second point of Theorem 2.4. Let us assume that ϕ induces a compact composition operator on every Hardy–Orlicz space. According to (2.2), this means that

$$\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h)))=o_{h\to 0}\Big(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\Big),$$

for every $A>0$ and every Orlicz function ψ, which is equivalent to

$$\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h)))\leqslant\frac{1}{\psi(A\psi^{-1}(1/h^N))}, \tag{3.1}$$

for every $A>0$, for every Orlicz function ψ and for h sufficiently small, according to Remark 2.2. We intend to show that

$$\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h)))=0,$$

for all $0<h\leqslant h_0$ and some $h_0\in(0,1)$. By contradiction, we suppose that

$$\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h)))\neq 0$$

for every $h>0$, since

$$h\longmapsto\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h)))$$

is a non-decreasing function on $(0,1)$. A straightforward computation shows that inequality (3.1) is satisfied for every $A>0$, for every Orlicz function ψ and for h small enough, if and only if we have, by putting $x=1/h$,

$$\frac{\psi^{-1}(x^N)}{\psi^{-1}(1/\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,1/x))))}\leqslant\frac{1}{A}, \tag{3.2}$$

for every $A>0$, for every Orlicz function ψ and for x large enough. The following lemma ensures that this cannot occur:

LEMMA 3.2. *Let $f,g:[0,+\infty[\to[0,+\infty[$ be two increasing functions which tend to $+\infty$ at $+\infty$. There exist $\delta>0$ and a continuous increasing concave function $\nu:[0,+\infty[\to[0,+\infty[$, with $\lim_{x\to+\infty}\nu(x)=+\infty$, such that $\nu(f(x))/\nu(g(x))\geqslant\delta>0$, for every x large enough.*

We assume for a while that this lemma has been proved, and we finish the proof of Theorem 3.1. With the notations of the lemma, we put

$$f(x)=x^N\quad\text{and}\quad g(x)=\frac{1}{\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,1/x)))}.$$

It is clear that $\lim_{x\to+\infty}f(x)=+\infty$; since C_ϕ is supposed to be compact on every $H^\psi(\mathbb{B}_N)$, it is in particular bounded on $H^p(\mathbb{B}_N)$ ([4], Corollary 3.5), then we have $g(x)\xrightarrow[x\to+\infty]{}+\infty$ (Theorem 2.3(iii)). Now, the above lemma provides a constant

$\delta > 0$ and a continuous increasing concave function ν, tending to infinity at infinity, such that

$$\frac{\nu(x^N)}{\nu\Big(\frac{1}{\sup\limits_{\zeta\in\mathbb{S}_N}(\mu_\phi(\mathcal{S}(\zeta,1/x)))}\Big)} \geqslant \delta > 0,$$

for every x large enough. It is not difficult to check that ν can be constructed such that $\psi = \nu^{-1}$ is an Orlicz function, i.e. such that $x/\nu(x) \xrightarrow[x\to+\infty]{} +\infty$ (that is what we do in the proof of Lemma 3.2 below). Therefore, we get a contradiction with condition (3.2), so we must have

$$\sup_{0<r<1}\sup_{\zeta\in\mathbb{S}_N}(\mu_{\phi_r}(\mathcal{S}(\zeta,h))) = 0,$$

for every $h > 0$ small enough. It follows that there exists some $0 < r_0 < 1$ such that

$$\sup_{0<r<1}\mu_{\phi_r}(\mathcal{C}(r_0,1)) = 0, \tag{3.3}$$

where $\mathcal{C}(r_0,1) = \{z\in\mathbb{B}_N,\ r_0 < |z| < 1\}$. We are going to show that

$$\phi^{-1}(\mathcal{C}(r_0,1)) = \emptyset,$$

which would give the result. Let $0 < r < 1$ and let us look at the set

$$\phi_r^{-1}(\mathcal{C}(r_0,1))\cap\mathbb{S}_N = \{\zeta\in\mathbb{S}_N,\ \phi_r(\zeta)\in\mathcal{C}(r_0,1)\}.$$

Condition (3.3) implies

$$\sigma_N(\phi_r^{-1}(\mathcal{C}(r_0,1))\cap\mathbb{S}_N) = 0.$$

Since ϕ_r is continuous on $\overline{\mathbb{B}}_N$, $\phi_r^{-1}(\mathcal{C}(r_0,1))\cap\mathbb{S}_N$ must be an open subset of $\mathbb{S}_N$ and then must be empty. So we proved that, for any $r\in(0,1)$,

$$\{\zeta\in\mathbb{S}_N,\ \phi_r(\zeta)\in\mathcal{C}(r_0,1)\} = \phi^{-1}(\mathcal{C}(r_0,1))\cap r\mathbb{S}_N = \emptyset,$$

where $r\mathbb{S}_N = \{z\in\overline{\mathbb{B}}_N,\ |z| = r\}$, hence

$$\phi^{-1}(\mathcal{C}(r_0,1)) = \bigcup_{0<r<1}(\phi^{-1}(\mathcal{C}(r_0,1))\cap r\mathbb{S}_N) = \emptyset.$$

The proof in the Bergman–Orlicz case is much easier. Proceeding as above and using the necessary part of the second point of Theorem 2.5, we get that condition $\mu_\phi(\mathcal{C}(r_0,1)) = 0$ must hold, for some $0 < r_0 < 1$. By continuity of the map ϕ on $\mathbb{B}_N$, $\phi^{-1}(\mathcal{C}(r_0,1))$ cannot be but empty.

To be complete, we have to prove Lemma 3.2:

Proof of Lemma 3.2. The proof is constructive. Let f and g be as in the statement of the lemma. We are going to build by induction a sequence $(a_n)_n$ which will be of interest in the construction of the desired function ν. We put $a_0 = 0$, $a_1 = 1$, and we deduce a_{n+2} from a_n and a_{n+1} in the following way: we define

$$b_{n+2} = \sup\{g(x),\ f(x)\leqslant a_{n+1}\} \quad\text{and}\quad a_{n+2} = \max\{b_{n+2}, a_{n+1} + (a_{n+1}-a_n)\}.$$

We observe that:

(i) if $f(x) \leqslant a_{n+1}$, then $g(x) \leqslant a_{n+2}$;
(ii) $a_{n+2} - a_{n+1} \geqslant a_{n+1} - a_n \geqslant 1$.

We now construct the concave function ν as a continuous affine one, whose derivative is equal to $\varepsilon_n = 1/\sqrt{n}(a_{n+1} - a_n)$ on the interval (a_n, a_{n+1}), and with $\nu(0) = 0$. Of course ν is increasing and then maps $[0, +\infty[$ into itself. Since ε_n is decreasing, because of (ii) above, ν is concave. In order to check that ν tends to infinity at infinity, we compute $\nu(a_n)$:

$$\nu(a_{n+1}) = \nu(a_n) + \varepsilon_n(a_{n+1} - a_n) = \nu(a_n) + \frac{1}{\sqrt{n}}.$$

Therefore $\nu(a_{n+1}) = \sum_{k=1}^{n+1} 1/\sqrt{k}$ which shows that $\lim_{x \to +\infty} \nu(x) = +\infty$, since $a_n \to +\infty$.

We now check that $(\nu \circ f(x))/(\nu \circ g(x))$ is bounded below by some constant $\delta > 0$, when x is large enough. Let $x \in [0, +\infty[$, and let n be an integer such that $a_n \leqslant f(x) \leqslant a_{n+1}$; we have $\nu(f(x)) \geqslant \nu(a_n)$. Using the first property of the sequence $(a_n)_n$ above, we get $\nu(g(x)) \leqslant \nu(a_{n+2})$. This yields, for $n \geqslant 1$,

$$\frac{\nu(f(x))}{\nu(g(x))} \geqslant \frac{\nu(a_n)}{\nu(a_{n+2})} = \frac{\sum_{k=1}^{n} 1/\sqrt{k}}{\sum_{k=1}^{n+2} 1/\sqrt{k}} \geqslant \delta > 0,$$

hence the proof of Lemma 3.2. ∎

The proof of Theorem 3.1 follows. ∎

3.2. KORÁNYI REGIONS AND COMPACTNESS OF C_ϕ ON THE HARDY–ORLICZ AND BERGMAN–ORLICZ SPACES. For $\zeta \in \mathbb{S}_N$ and $a > 1$, we recall that the Korányi approach region $\Gamma(\zeta, a)$ of angular opening a is defined by

$$\Gamma(\zeta, a) = \Big\{ z \in \mathbb{B}_N,\ |1 - \langle z, \zeta \rangle| < \frac{a}{2}(1 - |z|^2) \Big\}.$$

Theorem 6.4 of [6] and the third part of Theorem 2.3 yields the following result:

THEOREM 3.3. *Let ψ be an Orlicz function satisfying the $\Delta_2 \cap \nabla_2$-condition. Let also $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. We assume that $N > 1$ and we fix $b_N = (\cos(\pi/(2N)))^{-1}$.*

(i) *If $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, b_N)$, then C_ϕ is bounded on $H^\psi(\mathbb{B}_N)$.*

(ii) *If $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, b)$, for some $\zeta \in \mathbb{S}_N$ and for some $1 < b < b_N$, then C_ϕ is compact on $H^\psi(\mathbb{B}_N)$.*

(iii) *Both of the above results are sharp in the following sense: given $c > b_N$, there exists ϕ with $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, c)$, for some $\zeta \in \mathbb{S}_N$, and C_ϕ not bounded on $H^\psi(\mathbb{B}_N)$; there also exists some ϕ with $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, b_N)$, for some $\zeta \in \mathbb{S}_N$, and C_ϕ not compact on $H^\psi(\mathbb{B}_N)$.*

REMARK 3.4. (i) If $N = 1$, the first two points of the previous theorem are true if we put $b_1 = +\infty$ and $\Gamma(\zeta, +\infty) = \mathbb{D}$. Indeed, the first point is nothing but the continuity of every composition operator on the disc, and the second one is contained in Proposition 3.25 of [6] which says that, whenever $\phi(\mathbb{D})$ is contained in some nontangential approach region in $\mathbb{D}$, then C_ϕ is Hilbert–Schmidt in $H^2(\mathbb{D})$, hence compact on every $H^p(\mathbb{D})$, $1 \leqslant p < \infty$.

(ii) Following the proof of Theorem 3.3 of [5], it is not difficult to show that the boundedness or compactness of C_ϕ on $H^\psi(\mathbb{B}_N)$ implies that on $A^\psi_\alpha(\mathbb{B}_N)$, for any $\alpha > -1$, as soon as the Orlicz function ψ satisfies the Δ_2-condition. Thus, the first two points of the previous theorem also holds for Bergman–Orlicz spaces.

The following result shows that Theorem 3.3 does not hold as soon as the Orlicz function grows fast.

THEOREM 3.5. *Let ψ be an Orlicz function satisfying the Δ^2-condition. Then, for every $\zeta \in \mathbb{S}_N$ and every $b > 1$, there exists a holomorphic self-map ϕ taking $\mathbb{B}_N$ into $\Gamma(\zeta, b)$, such that C_ϕ is not compact on $H^\psi(\mathbb{B}_N)$.*

REMARK 3.6. Observe that there is no assumption on N.

Proof of Theorem 3.5. The proof will use the necessary part of the second point of Theorem 2.3. First of all, we recall that Δ^2-condition implies ∇_2-condition (see Subsection 2.2). We denote by e_1 the vector $(1, 0, \dots, 0)$ in $\mathbb{C}^N$. It is clearly sufficient to prove the theorem for $\zeta = e_1$. For any $b > 1$, we set

$$\beta = \frac{2\cos^{-1}(1/b)}{\pi} \tag{3.4}$$

in $(0, 1)$. We need a lemma whose proof is included in that of Theorem 6.4 of [6]:

LEMMA 3.7. *Let $b > 1$ and let β be defined by* (3.4). *There exists a holomorphic map $\phi : \mathbb{B}_N \to \mathbb{B}_N$, with $\phi(\mathbb{B}_N) \subset \Gamma(e_1, b)$, such that*

$$\sigma_N \phi^{-1}(\mathcal{S}(e_1, h)) \geqslant C h^{1/\beta}, \tag{3.5}$$

for some constant $C > 0$ depending only on ϕ and b.

Proof. Without going into details, we briefly give the ideas of the proof. It uses the deep Alexandrov's result which gives the existence of non-constant inner functions in $\mathbb{B}_N$ [1]. Therefore, we consider a function ϕ which can be written

$$\phi = (\kappa \circ \varphi, 0'),$$

where $0'$ is the $(n-1)$-tuple $(0, \dots, 0)$, φ is an inner function with $\varphi(0) = 0$, and where κ is a biholomorphic map from $\mathbb{D}$ onto the non-tangential approach region $\Gamma(1, b)$ in the disc, defined by

$$\Gamma(1, b) = \Big\{ z \in \mathbb{D}, |1 - z| < \frac{b}{2}(1 - |z|^2) \Big\}.$$

One can show that the lower-estimate (3.5) holds for this map ϕ, using the fact that inner functions φ which vanishes at 0 are measure preserving maps of $\mathbb{S}_N$ into $\mathbb{T}$ (see p. 405 of [16]) in the following sense:

$$\sigma_N((\varphi^*)^{-1}(E)) = \sigma_1(E),$$

for any Borel set E in $\mathbb{T}$. ∎

Let ϕ be as in the statement of the theorem. According to the necessary part of the second point of Theorem 2.3, the previous lemma ensures that, if we show that for any Orlicz function ψ satisfying the Δ^2-condition, for any $\beta \in (0,1)$, there exists some $A > 0$ such that

$$\frac{1}{\psi(A\psi^{-1}(1/h^N))} \leqslant h^{1/\beta}, \tag{3.6}$$

for every h small enough, then C_ϕ would not be compact on $H^\psi(\mathbb{B}_N)$. Now, if we set $y = \psi^{-1}(1/h^N)$, then an easy computation implies that (3.6) is equivalent to $\psi(y)^{1/N\beta} \leqslant \psi(Cy)$ for some constant $C > 0$. We conclude the proof by noticing that this latter condition is trivial if $0 < 1/N\beta \leqslant 1$, while it is nothing but Δ^2-condition if $1/N\beta > 1$ (see Subsection 2.2). ∎

REMARK 3.8. (i) Note that the $0 < 1/N\beta \leqslant 1$-setting, at the end of the previous proof, allows to extend the compactness part of the third point of Theorem 3.3 to any Orlicz functions satisfying the $\nabla_2 \cap \nabla_0$-condition. Anyway, this is a particular case of Corollary 3.5 of [4] which asserts that the compactness of C_ϕ on $H^\psi(\mathbb{B}_N)$ implies that on $H^p(\mathbb{B}_N)$ for such ψ.

(ii) When $N = 1$, the proof of Lemma 3.7 can be simplified: first, because the existence of a non-constant inner function in the unit disc is trivial, and then because it clearly suffices to take $\varphi(z) = z$, what just turns the proof of Lemma 3.7 into considering a biholomorphic map κ from $\mathbb{D}$ onto a non-tangential approach region.

The previous remark leads us to say some words about weighted Bergman–Orlicz spaces in dimension one. Indeed, we can adapt the proof of Lemma 3.7 to get the following result.

LEMMA 3.9. *Let $\alpha > -1$, let $b > 1$ and let β be defined by* (3.4). *There exists a holomorphic map $\phi : \mathbb{D} \to \mathbb{D}$, $\phi(\mathbb{D}) \subset \Gamma(1,b)$, such that*

$$v_\alpha(\phi^{-1}(S(1,h))) \geqslant Ch^{(2+\alpha)/\beta}, \tag{3.7}$$

for some constant $C > 0$ depending only on α, ϕ and b.

For the seek of completeness, we prefer to give some details of the proof of this lemma, in order to point out the slightly difference with that of Theorem 6.4, 3) of [6].

Proof. We consider a biholomorphic map κ from $\mathbb{D}$ onto

$$\Gamma(1,b) = \Big\{z \in \mathbb{D},\ |1-z| < \frac{b}{2}(1-|z|^2)\Big\},$$

for some $b > 1$. As it is explained in the proof of Theorem 6.4, 3) of [6], if $\beta \in (0,1)$ is defined by (3.4), then the function $g(z) := (1-\kappa(z))/((1-z)^\beta)$ is continuous and non-zero in $\overline{\mathbb{D}} \cap V$, where V is a closed disc (with non-empty interior) centered at 1, and $\kappa^{-1}(S(1,h)) \subset \overline{\mathbb{D}} \cap V$, for $h > 0$ sufficiently small. Then, for such h, we follow the computation at the end of the proof of Theorem 6.4, 3) of [6] to get

$$\kappa^{-1}(S(1,h)) \supset S(1, \widetilde{C}h^{1/\beta}),$$

for some constant $\widetilde{C} > 0$, depending only on κ and b. Therefore,

$$v_\alpha(\kappa^{-1}(S(1,h))) \geqslant Ch^{(2+\alpha)/\beta},$$

where $C > 0$ depends on α, κ and b. ∎

Now, it is sufficient to argue as at the end of the proof of Theorem 3.5 to get:

PROPOSITION 3.10. *Let $\alpha > -1$, let $b > 1$, let $\zeta \in \mathbb{S}_1$ and let ψ be an Orlicz function satisfying the Δ^2-condition. There exists a holomorphic map $\phi : \mathbb{D} \to \mathbb{D}$, with range contained in the non-tangential approach region $\Gamma(\zeta, b)$ such that the induced composition operator C_ϕ is not compact on $A^\psi_\alpha(\mathbb{D})$.*

The proof of the previous proposition does not work directly when $N > 1$, because we do not know if there exists a non-constant inner function which is measure-preserving from $\mathbb{B}_N$ to $\mathbb{D}$ in the following sense:

$$v_\alpha(\phi^{-1}(E)) = A_\alpha(E),$$

for any $E \subset \mathbb{D}$, where A_α is the weighted area measure in $\mathbb{D}$.

3.3. ANOTHER CHARACTERIZATION OF THE COMPACTNESS OF C_ϕ ON WEIGHTED BERGMAN–ORLICZ SPACES. The following result generalizes that obtained in [19] for classical Bergman spaces:

THEOREM 3.11. *Let $\alpha > -1$, let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic and let ψ be an Orlicz function which satisfies the ∇_0-condition. We assume that C_ϕ is bounded from $A^\psi_\beta(\mathbb{B}_N)$ into itself for some $-1 < \beta < \alpha$. Then C_ϕ is compact from $A^\psi_\alpha(\mathbb{B}_N)$ into itself if and only if*

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|)^{N(\alpha)})}{\psi^{-1}(1/(1-|z|)^{N(\alpha)})} = 0. \tag{3.8}$$

Proof. The proof of the necessary part is the same as that of Theorem 5.7 of [8], using Proposition 1.10 of [3]. We deal with the proof of the sufficiency of

(3.8). Without loss of generality, we assume that $\phi(0) = 0$. According to (2.2), it is sufficient to show that for every $B > 0$, there exists $h_0 \in (0,1)$, such that

$$\psi^{-1}\Big(\frac{1}{\mu_{\phi,\alpha}(S(\xi,h))}\Big) \geqslant B\psi^{-1}\Big(\frac{1}{h^{N(\alpha)}}\Big), \tag{3.9}$$

uniformly in $\xi \in \mathbb{S}_N$, and for any $0 < h < h_0$. Let α and β be as in the statement of the theorem. We have

$$\begin{aligned}
\mu_{\phi,\alpha}(S(\xi,h)) &= \int_{\phi^{-1}(S(\xi,h))} (1-|z|^2)^\alpha \mathrm{d}v(z) \\
&\leqslant 2^{\alpha-\beta} \sup_{z\in\phi^{-1}(S(\xi,h))} (1-|z|)^{\alpha-\beta} \int_{\phi^{-1}(S(\xi,h))} (1-|z|^2)^\beta \mathrm{d}v(z) \\
&= 2^{\alpha-\beta} \sup_{z\in\phi^{-1}(S(\xi,h))} (1-|z|)^{\alpha-\beta} \mu_{\phi,\beta}(S(\xi,h)) \\
&\leqslant 2^{\alpha-\beta} \sup_{z\in\phi^{-1}(S(\xi,h))} (1-|z|)^{\alpha-\beta} \frac{1}{\psi(C_\beta\psi^{-1}(1/h^{N(\beta)}))}, \qquad (3.10)
\end{aligned}$$

where the last inequality stands for some constant $C_\beta \geqslant 1$ and for h small enough, since C_ϕ is supposed to be bounded on $A^\psi_\beta(\mathbb{B}_N)$.

Now, since $\alpha - \beta > 0$, the hypothesis (3.8) is equivalent to the fact that, for any $A > 0$,

$$(1-|z|)^{\alpha-\beta} \leqslant \frac{1}{(\psi(A\psi^{-1}(1/(1-|\phi(z)|)^{N(\alpha)})))^{(\alpha-\beta)/N(\alpha)}},$$

whenever $|z|$ is close enough to 1. Moreover, observe that if $z \in \phi^{-1}(S(\xi,h))$, then

$$1-|z| \leqslant 1-|\phi(z)| \leqslant |1-\langle\phi(z),\xi\rangle| < h$$

so that, for any $A > 0$,

$$\sup_{z\in\phi^{-1}(S(\xi,h))} (1-|z|)^{\alpha-\beta} \leqslant \frac{1}{(\psi(A\psi^{-1}(1/h^{N(\alpha)})))^{(\alpha-\beta)/N(\alpha)}},$$

for any $h > 0$ small enough, using the fact that ψ is a non-decreasing function and that $\alpha - \beta > 0$. Thus, it follows from (3.10) that

$$\mu_{\phi,\alpha}(S(\xi,h)) \leqslant 2^{\alpha-\beta} \frac{1}{(\psi(A\psi^{-1}(1/h^{N(\alpha)})))^{(\alpha-\beta)/N(\alpha)}} \frac{1}{\psi(C_\beta\psi^{-1}(1/h^{N(\beta)}))},$$

for any $A > 0$ and h small enough. Using (3.9), the last inequality ensures that C_ϕ will be compact on $A^\psi_\alpha(\mathbb{B}_N)$ if, for any $B > 0$, there exists a constant $A > 0$ such that

$$\psi^{-1}((\psi(A\psi^{-1}(1/h^{N(\alpha)})))^{(\alpha-\beta)/N(\alpha)} \cdot \psi(C_\beta\psi^{-1}(1/h^{N(\beta)}))) \geqslant B\psi^{-1}(1/h^{N(\alpha)}), \tag{3.11}$$

for h small enough. Putting $x = \psi^{-1}(1/h^{N(\alpha)})$, (3.11) is equivalent to

$$\psi(Bx) \leqslant \psi(Ax)^{(\alpha-\beta)/N(\alpha)} \cdot \psi(C_\beta \psi^{-1}((\psi(x))^{N(\beta)/N(\alpha)}),$$

which is in turn satisfied, using the convexity of ψ and $C_\beta \geqslant 1$, if

$$\psi(Bx)^{N(\alpha)} \leqslant \psi(Ax)^{\alpha-\beta} \cdot \psi(x)^{N(\beta)},$$

for x large enough. Let us notice that this last inequality is equivalent to

$$\frac{\psi(Bx)^{N(\beta)/(\alpha-\beta)}}{\psi(x)^{N(\beta)/(\alpha-\beta)}} \leqslant \frac{\psi(Ax)}{\psi(Bx)},$$

for x large enough. Since ψ satisfies the ∇_0-condition (see Subsection 2.2), the proof is complete. ∎

REMARK 3.12. We mention that the proof of the necessary part of the previous theorem does not use the boundedness of C_ϕ on some "smaller" weighted Bergman–Orlicz space. Also, it is not necessary to assume that ψ satisfies the ∇_0-condition.

Since every composition operator is bounded on every $A^\psi_\alpha(\mathbb{B}_N)$ as soon as ψ satisfies the Δ^2-conditon, we have the following corollary:

COROLLARY 3.13. *Let $\alpha > -1$, let ψ be an Orlicz function satisfying the Δ^2-condition and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. Then C_ϕ is compact on $A^\psi_\alpha(\mathbb{B}_N)$ if and only if*

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|))}{\psi^{-1}(1/(1-|z|))} = 0.$$

Proof. It is sufficient to remark that we have

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|)^{N(\alpha)})}{\psi^{-1}(1/(1-|z|)^{N(\alpha)})} = 0 \iff \lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|))}{\psi^{-1}(1/(1-|z|))} = 0,$$

since ψ satisfies the Δ^2-condition. Indeed, it is easy to deduce from the third point of the definition of the Δ^2-condition that, if $a > 1$, then $\psi^{-1}(x^a) \leqslant C\psi^{-1}(x)$ for some constant $C > 0$ (which may depend on a) and for x large enough. ∎

This corollary highlights an important difference with the classical weighted Bergman case: when ψ satisfies the Δ^2-condition, the compactness (as well as the boundedness, Theorem 2.5) of composition operators on $A^\psi_\alpha(\mathbb{B}_N)$ does not depend on $\alpha > -1$.

Yet, this independency does not stand for $\alpha = -1$, i.e. for the Hardy–Orlicz spaces; indeed, it was shown ([8], Theorem 5.8) that there exists some Orlicz function ψ which satisfies the Δ^2-condition (to be precise, $\psi(x) = \mathrm{e}^{x^2} - 1$) such that there exists a holomorphic self-map of $\mathbb{D}$ inducing a compact operator on $A^\psi_\alpha(\mathbb{D})$, but not compact on $H^\psi(\mathbb{D})$.

Nevertheless, the same proof as that of the necessary part of Theorem 3.11 for the Hardy–Orlicz spaces yields:

PROPOSITION 3.14. *Let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic and let ψ be an Orlicz function. If C_ϕ is compact on $H^\psi(\mathbb{B}_N)$, then*

$$\lim_{|z|\to 1} \frac{\psi^{-1}(1/(1-|\phi(z)|))}{\psi^{-1}(1/(1-|z|))} = 0.$$

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