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# A FUNCTIONAL MODEL FOR PURE $\Gamma$-CONTRACTIONS

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ABSTRACT. A pair of commuting operators $(S,P)$ defined on a Hilbert space $\mathcal{H}$ for which the closed symmetrized bidisc $\Gamma = \{(z_1+z_2, z_1z_2) : |z_1| \leqslant 1, |z_2| \leqslant 1\} \subseteq \mathbb{C}^2$ is a spectral set is called a $\Gamma$-contraction in the literature. A $\Gamma$-contraction $(S,P)$ is said to be pure if $P$ is a pure contraction, i.e., $P^{*n} \to 0$ strongly as $n \to \infty$. Here we construct a functional model and produce a set of unitary invariants for a pure $\Gamma$-contraction. The key ingredient in these constructions is an operator, which is the unique solution of the operator equation $S - S^*P = D_P X D_P$, where $X \in \mathcal{B}(\mathcal{D}_P)$, and is called the fundamental operator of the $\Gamma$-contraction $(S,P)$. We also discuss some important properties of the fundamental operator.

KEYWORDS: *Symmetrized bidisc, fundamental operator, functional model, unitary invariants.*

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## 1. INTRODUCTION AND PRELIMINARIES

The closed symmetrized bidisc $\Gamma$ is polynomially convex. Thus, a pair of commuting bounded operators $(S,P)$ is a $\Gamma$-contraction if and only if $\|p(S,P)\| \leqslant \|p\|_{\infty,\Gamma}$, for any polynomial $p$. The $\Gamma$-contractions were introduced by Agler and Young in [3] and have been thoroughly studied in [4], [7] and [15]. An understanding of this family of operator pairs has led to the solution of a special case of the spectral Nevanlina–Pick problem [5], [8], which is one of the problems that arise in $H^\infty$ control theory [19]. Also they play a pivotal role in the study of complex geometry of the set $\Gamma$ (see [6], [9]).

Spectral sets and complete spectral sets for a bounded operator $T$ on a Hilbert space $\mathcal{H}$ or for a tuple of bounded operators have been well-studied for long and several important results are known (see [14], [17], [22]). Dilation theory for an operator or a tuple of operators is well-studied too and has made some rapid progress in the last twenty years through Arveson [12], Popescu [23], [24], Muller and Vasilescu [21], Pott [25] and others.

Sz.-Nagy and Foias developed the model theory for a contraction [26]. They found the minimal unitary dilation of a contraction and it has become a powerful tool for studying an arbitrary contraction. By von Neumann's inequality, an operator $T$ is a contraction if and only if $\|p(T)\| \leqslant \|p\|_{\infty,\mathbb{D}}$ for all polynomials $p$, $\mathbb{D}$ being the open unit disc in the complex plane. This property itself is very beautiful and so is the concept of spectral set of an operator. A compact subset $X$ of $\mathbb{C}$ is called a spectral set for an operator $T$ if

$$\|\pi(T)\| \leqslant \sup_{z\in X} \|\pi(z)\| = \|\pi\|_{\infty,X},$$

for all rational functions $\pi$ with poles off $X$. If the above inequality holds for matrix valued rational functions $\pi$, then $X$ is called a complete spectral set for the operator $T$. Moreover, $T$ is said to have a normal $\partial X$-dilation if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a normal operator $N$ on $\mathcal{K}$ with $\sigma(N) \subseteq \partial X$ such that

$$\pi(T) = P_{\mathcal{H}}\pi(N)|_{\mathcal{H}},$$

for all rational functions $\pi$ with poles off $X$. It is a remarkable consequence of Arveson's extension theorem that $X$ is a complete spectral set for $T$ if and only if $T$ has a normal $\partial X$-dilation. Rephrased in this language, the Sz.-Nagy dilation theorem says that if $\mathbb{D}$ is a spectral set for $T$ then $T$ has a normal $\partial\mathbb{D}$-dilation. For $T$ to have a normal $\partial X$-dilation it is necessary that $X$ be a spectral set for $T$. Sufficiency has been investigated for many domains in $\mathbb{C}$ and several interesting results are known including success of such a dilation on an annulus ([1]) and its failure in triply connected domains ([2], [18]). When $(T_1, T_2)$ is a commuting pair of operators for which $\mathbb{D}^2$ is a spectral set, Ando's theorem provides a simultaneous commuting unitary dilation of $(T_1, T_2)$. Such classically beautiful concepts led Agler and Young to the following definitions.

DEFINITION 1.1. A commuting pair $(S, P)$ is called a *$\Gamma$-unitary* if $S$ and $P$ are normal operators and the joint spectrum $\sigma(S, P)$ of $(S, P)$ is contained in the distinguished boundary $b\Gamma$ defined by

$$b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = |z_2| = 1\} \subseteq \Gamma.$$

DEFINITION 1.2. A commuting pair $(\widetilde{S}, \widetilde{P})$ on $\mathcal{N}$ is said to be a *$\Gamma$-unitary extension* of a $\Gamma$-contraction $(S, P)$ on $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{N}$, $(\widetilde{S}, \widetilde{P})$ is a $\Gamma$-unitary, $\mathcal{H}$ is a common invariant subspace of both $\widetilde{S}$ and $\widetilde{P}$ and $\widetilde{S}|_{\mathcal{H}} = S, \widetilde{P}|_{\mathcal{H}} = P$.

DEFINITION 1.3. A commuting pair $(S, P)$ is called a *$\Gamma$-isometry* if it has a $\Gamma$-unitary extension. A commuting pair $(S, P)$ is a *$\Gamma$-co-isometry* if $(S^*, P^*)$ is a $\Gamma$-isometry.

DEFINITION 1.4. Let $(S, P)$ be a $\Gamma$-contraction on $\mathcal{H}$. A pair of commuting operators $(T, V)$ acting on a Hilbert space $\mathcal{N} \supseteq \mathcal{H}$ is called a *$\Gamma$-isometric dilation* of $(S, P)$ if $(T, V)$ is a $\Gamma$-isometry, $\mathcal{H}$ is a co-invariant subspace of both $T$ and $V$

and $T^*|_{\mathcal{H}} = S^*, V^*|_{\mathcal{H}} = P^*$. Moreover, the dilation will be called minimal if

$$\mathcal{N} = \overline{\text{span}}\{V^n h : h \in \mathcal{H} \text{ and } n = 0, 1, 2, \dots\}.$$

Thus $(T, V)$ is a $\Gamma$-isometric dilation of a $\Gamma$-contraction $(S, P)$ if and only if $(T^*, V^*)$ is a $\Gamma$-co-isometric extension of $(S^*, P^*)$.

A $\Gamma$-contraction $(S, P)$ acting on a Hilbert space $\mathcal{H}$ is said to be pure if $P$ is a pure contraction, i.e., $P^{*n} \to 0$ strongly as $n \to \infty$. The class of pure $\Gamma$-contractions plays a pivotal role in deciphering the structure of a class of $\Gamma$-contractions. In Theorem 2.8 of [7], Agler and Young proved that every $\Gamma$-contraction $(S, P)$ acting on a Hilbert space $\mathcal{H}$ can be decomposed into two parts $(S_1, P_1)$ and $(S_2, P_2)$ of which $(S_1, P_1)$ is a $\Gamma$-unitary and $(S_2, P_2)$ is a $\Gamma$-contraction with $P$ being a completely non-unitary contraction. This shows an analogy with the decomposition of a single contraction. Indeed, if $\mathcal{H}_1$ is the maximal subspace of $\mathcal{H}$ which reduces $P$ and on which $P$ is unitary, then $\mathcal{H}_1$ reduces $S$ as well and $(S_1, P_1)$ is same as $(S|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})$. Also both $S$ and $P$ are reduced by the subspace $\mathcal{H}_2$, the orthocomplement of $\mathcal{H}_1$ in $\mathcal{H}$, and $(S_2, P_2)$ is same as $(S|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})$. The functional model and unitary invariants we produce here give a good vision of those $\Gamma$-contractions $(S, P)$ for which the part $(S_2, P_2)$ described above is a pure $\Gamma$-contraction.

The program that Sz.-Nagy and Foias carried out for a contraction had two parts. The dilation was the first part which was followed by a functional model and a complete unitary invariant. For a $\Gamma$-contraction, the first part of that program was carried out in [7] by Agler and Young. The second half is the content of this article.

For a contraction $P$ defined on a Hilbert space $\mathcal{H}$, let $\Lambda_P$ be the set of all complex numbers for which the operator $I - zP^*$ is invertible. For $z \in \Lambda_P$, the characteristic function of $P$ is defined as

$$\Theta_P(z) = [-P + zD_{P^*}(I - zP^*)^{-1}D_P]|_{\mathcal{D}_P}. \tag{1.1}$$

Here the operators $D_P$ and $D_{P^*}$ are the defect operators $(I - P^*P)^{1/2}$ and $(I - PP^*)^{1/2}$ respectively. By virtue of the relation $PD_P = D_{P^*}P$ ([26], Section I.3), $\Theta_P(z)$ maps $\mathcal{D}_P = \overline{\text{Ran}}D_P$ into $\mathcal{D}_{P^*} = \overline{\text{Ran}}D_{P^*}$ for every $z$ in $\Lambda_P$.

For a pair of commuting bounded operators $S, P$ on a Hilbert space $\mathcal{H}$ with $\|P\| \leqslant 1$, we introduced in [15] the notion of the fundamental equation. For the pair $S, P$ it is defined as

$$S - S^*P = D_P X D_P, \quad X \in \mathcal{B}(\mathcal{D}_P), \tag{1.2}$$

and the same for the pair $S^*, P^*$ is

$$S^* - SP^* = D_{P^*} Y D_{P^*}, \quad Y \in \mathcal{B}(\mathcal{D}_{P^*}). \tag{1.3}$$

In the same paper we also proved the existence and uniqueness of solutions of such equations when $(S, P)$ is a $\Gamma$-contraction ([15], Theorem 4.2). The unique solution was named the fundamental operator of the $\Gamma$-contraction because it led us to a new characterization for $\Gamma$-contractions ([15], Theorem 4.4).

In Section 2, we discuss some interesting properties of the fundamental operator. In Section 3, we construct a functional model for a pure $\Gamma$-contraction $(S,P)$ and this is the main content of this paper. The fundamental operator $F_*$ of $(S^*,P^*)$ is taken as the key ingredient in that construction. In Section 4, we produce a set of unitary invariants for pure $\Gamma$-contractions. For the unitary equivalence of two pure $\Gamma$-contractions $(S,P)$ and $(S_1,P_1)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ respectively, we produce here a set of unitary invariants which consists of two things mainly. The first one demands the coincidence of the characteristic functions of $P$ and $P_1$. The second condition is the unitary equivalence of the fundamental operators $F_*$ and $F_{*1}$ of $(S^*,P^*)$ and $(S_1^*,P_1^*)$ by the same unitary from $\mathcal{D}_{P^*}$ to $\mathcal{D}_{P_1^*}$ that is involved in establishing the coincidence of the characteristic functions of $P$ and $P_1$.

## 2. AUTOMORPHISMS AND THE FUNDAMENTAL OPERATOR

For a $\Gamma$-contraction $(S,P)$ we find out an explicit form of the fundamental operator of $\tau(S,P)$, where $\tau$ is an automorphism of the open symmetrized bidisc

$$G = \{(z_1+z_2, z_1z_2) : \ |z_1|<1, |z_2|<1\}.$$

It is well-known, see [10] and [20], that any automorphism $\tau$ of $G$ is given as follows:

$$\tau(z_1+z_2,z_1z_2)=\tau_m(z_1+z_2,z_1z_2)=(m(z_1)+m(z_2),m(z_1)m(z_2)), \quad z_1,z_2\in\mathbb{D}, \tag{2.1}$$

where $m$ is an automorphism of the disc $\mathbb{D}$. Recall that the joint spectrum $\sigma(S,P)$ of a $\Gamma$-contraction $(S,P)$ is contained in $\Gamma$. Thus if $\tau$ is a $\mathbb{C}^2$-valued holomorphic map in a neighbourhood $\mathbf{N}(\Gamma)$ of $\Gamma$ mapping $\Gamma$ into itself, then by functional calculus (see [27]), $(S_\tau,P_\tau) := \tau(S,P)$ is well defined as a pair of commuting bounded operators.

LEMMA 2.1. *For $(S,P)$ and $\tau$ as above, $(S_\tau,P_\tau)$ is a $\Gamma$-contraction.*

*Proof.* We show that $\Gamma$ is a spectral set of $(S_\tau,P_\tau)$. Let $f$ be a polynomial over $\mathbb{C}$ in two variables. Then

$$\|f(S_\tau,P_\tau)\| = \|f\circ\tau(S,P)\| \leqslant \|f\circ\tau\|_{\infty,\Gamma} = \sup_{z\in\Gamma}|f(\tau(z))| \leqslant \|f\|_{\infty,\Gamma},$$

since $\tau(z)\in\Gamma$ for all $z\in\Gamma$ and hence $(S_\tau,P_\tau)$ is a $\Gamma$-contraction. ∎

The following is the main result of this section.

THEOREM 2.2. *Let $(S,P)$ be a $\Gamma$-contraction defined on a Hilbert space $\mathcal{H}$ and let $\tau$ be an automorphism of $G$. Let $\tau=\tau_m$ as in* (2.1) *and $m$ be given by $m(z)=\beta(z-a)/(1-\bar{a}z)$ for some $a\in\mathbb{D}$ and $\beta\in\mathbb{T}$. Let $F$ and $F_\tau$ be the fundamental* *operators of $(S,P)$ and $(S_\tau,P_\tau)$ respectively. Then there is a unitary $U:\mathcal{D}_{P_\tau}\to\mathcal{D}_P$ such that*

$$F_\tau = U^*((1+|a|^2)-\overline{a}F-aF^*)^{-1/2}\beta(F+a^2F^*-2a)((1+|a|^2)-\overline{a}F-aF^*)^{-1/2}U.$$

*Proof.* We have

$$\begin{aligned}\tau(s,p)=\tau(z_1+z_2,z_1z_2) &= \Big(\beta\Big(\frac{z_1-a}{1-\overline{a}z_1}+\frac{z_2-a}{1-\overline{a}z_2}\Big),\beta^2\frac{(z_1-a)(z_2-a)}{(1-\overline{a}z_1)(1-\overline{a}z_2)}\Big)\\ &= \Big(\beta\frac{(z_1+z_2)-2\overline{a}z_1z_2+|a|^2(z_1+z_2)-2a}{1-\overline{a}(z_1+z_2)+\overline{a}^2z_1z_2},\beta^2\frac{z_1z_2-a(z_1+z_2)+a^2}{1-\overline{a}(z_1+z_2)+\overline{a}^2z_1z_2}\Big)\\ &= \Big(\beta\frac{(1+|a|^2)s-2\overline{a}p-2a}{1-\overline{a}s+\overline{a}^2p},\beta^2\frac{p-as+a^2}{1-\overline{a}s+\overline{a}^2p}\Big).\end{aligned}$$

It is obvious that $\tau$ can be defined on the open set $\Gamma_a=\{(z_1+z_2,z_1z_2):\ |z_1|<1/|a|\,,|z_2|<1/|a|\}$, which contains $\Gamma$. Clearly

$$\begin{aligned}(S_\tau,P_\tau)&=\tau(S,P)\\ &=(\beta((1+|a|^2)S-2\overline{a}P-2a)(I-\overline{a}S+\overline{a}^2P)^{-1},\beta^2(P-aS+a^2)(I-\overline{a}S+\overline{a}^2P)^{-1}).\end{aligned}$$

Here

$$\begin{aligned}D_{P_\tau}^2 &= (I-P_\tau^*P_\tau)\\ &= I-(I-aS^*+a^2P^*)^{-1}(P^*-\overline{a}S^*+\overline{a}^2)(P-aS+a^2)(I-\overline{a}S+\overline{a}^2P)^{-1}\\ &= (I-aS^*+a^2P^*)^{-1}[(I-aS^*+a^2P^*)(I-\overline{a}S+\overline{a}^2P)\\ &\quad -(P^*-\overline{a}S^*+\overline{a}^2)(P-aS+a^2)](I-\overline{a}S+\overline{a}^2P)^{-1}\\ &= (I-aS^*+a^2P^*)^{-1}[-\overline{a}(1-|a|^2)(S-S^*P)-a(1-|a|^2)(S^*-P^*S)\\ &\quad (1-|a|^4)(I-P^*P)](I-\overline{a}S+\overline{a}^2P)^{-1}\\ &= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}[(1+|a|^2)(I-P^*P)\\ &\quad -\overline{a}(S-S^*P)-a(S^*-P^*S)](I-\overline{a}S+\overline{a}^2P)^{-1}\\ &= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}[(1+|a|^2)D_P^2-\overline{a}D_PFD_P-aD_PF^*D_P]\\ &\quad (I-\overline{a}S+\overline{a}^2P)^{-1},\quad (\text{since } S-S^*P=D_PFD_P)\\ &= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}D_P[(1+|a|^2)-\overline{a}F-aF^*]D_P(I-\overline{a}S+\overline{a}^2P)^{-1}.\end{aligned}$$

Now we show that the operator $(1+|a|^2)-\overline{a}F-aF^*$ defined on $\mathcal{D}_P$ is invertible. Since $F\in\mathcal{B}(\mathcal{D}_P)$, it is enough to show that $(1+|a|^2)-\overline{a}F-aF^*$ is bounded below, i.e.,

$$\inf_{\|x\|\leqslant 1}\langle((1+|a|^2)-\overline{a}F-aF^*)x,x\rangle>0,$$

or equivalently

$$\sup_{\|x\|\leqslant 1}|\overline{a}\langle Fx,x\rangle+a\langle F^*x,x\rangle|<(1+|a|^2).$$

Since the numerical radius of $F$ is not greater than 1,

$$\sup_{\|x\|\leqslant 1} |\bar{a}\langle Fx,x\rangle + a\langle F^*x,x\rangle| \leqslant 2|a| < (1+|a|^2)$$

as $1+|a|^2-2|a| = (1-|a|)^2 > 0$ for $a \in \mathbb{D}$ and consequently the operator $(1+|a|^2-\bar{a}F-aF^*)$ is invertible.

Let $X = (1-|a|^2)^{1/2}[(1+|a|^2)-\bar{a}F-aF^*]^{1/2}D_P(I-\bar{a}S+\bar{a}^2P^*)^{-1}$. Then $X$ is an operator from $\mathcal{H}$ to $\mathcal{D}_P$. Also $D_{P_\tau}^2 = X^*X$ and $\overline{\text{Ran}}X = \mathcal{D}_P$ as $(1+|a|^2)-\bar{a}F-aF^*$ is invertible. Now define

$$U : \mathcal{D}_{P_\tau} \to \overline{\text{Ran}}X = \mathcal{D}_P$$
$$D_{P_\tau}h \mapsto Xh.$$

Clearly $U$ is onto. Moreover,

$$\|UD_{P_\tau}h\|^2 = \|Xh\|^2 = \langle X^*Xh,h\rangle = \langle D_{P_\tau}^2 h,h\rangle = \|D_{P_\tau}h\|^2.$$

So $U$ is a surjective isometry i.e., a unitary. Also

$$\begin{aligned}
S_\tau - S_\tau^* P_\tau &= \beta[((1+|a|^2)S-2\bar{a}P-2a)(I-\bar{a}S+\bar{a}^2P)^{-1} - (I-aS^*+a^2P^*)^{-1} \\
&\quad ((1+|a|^2)S^*-2aP^*-2\bar{a})(P-aS+a^2)(I-\bar{a}S+\bar{a}^2P)^{-1})] \\
&= (I-aS^*+a^2P^*)^{-1}\beta[(I-aS^*+a^2P^*)((1+|a|^2)S-2\bar{a}P-2a) \\
&\quad -((1+|a|^2)S^*-2aP^*-2\bar{a})(P-aS+a^2)](I-\bar{a}S+\bar{a}^2P)^{-1} \\
&= (I-aS^*+a^2P^*)^{-1}\beta[(1-|a|^2)(S-S^*P)+2a^2(S^*-P^*S)-a^2(1+|a|^2) \\
&\quad (S^*-P^*S)-2a(I-P^*P)+2a|a|^2(I-P^*P)](I-\bar{a}S+\bar{a}^2P)^{-1} \\
&= (I-aS^*+a^2P^*)^{-1}\beta[(1-|a|^2)(S-S^*P)+a^2(1-|a|^2)(S^*-P^*S) \\
&\quad -2a(1-|a|^2)(I-P^*P)](I-\bar{a}S+\bar{a}^2P)^{-1} \\
&= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}\beta[(S-S^*P)+a^2(S^*-P^*S)-2a(I-P^*P)] \\
&\quad (I-\bar{a}S+\bar{a}^2P)^{-1} \\
&= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}\beta[D_PFD_P+a^2D_PF^*D_P-2aD_P^2] \\
&\quad (I-\bar{a}S+\bar{a}^2P)^{-1}, \quad (\text{since } S-S^*P = D_PFD_P) \\
&= (1-|a|^2)(I-aS^*+a^2P^*)^{-1}\beta D_P[F+a^2F^*-2a]D_P(I-\bar{a}S+\bar{a}^2P)^{-1} \\
&= X^*[((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}\beta(F+a^2F^*-2a) \\
&\quad ((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}]X \\
&= D_{P_\tau}U^*[((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}\beta(F+a^2F^*-2a) \\
&\quad ((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}]UD_{P_\tau}.
\end{aligned}$$

Again since $S_\tau - S_\tau^*P_\tau = D_{P_\tau}F_\tau D_{P_\tau}$ and $F_\tau$ is unique, we have

$$F_\tau = U^*((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}\beta(F+a^2F^*-2a)((1+|a|^2)-\bar{a}F-aF^*)^{-1/2}U. \quad ∎$$

Here is an interesting result which relates the fundamental operator of a $\Gamma$-contraction $(S, P)$ with that of $(S^*, P^*)$.

PROPOSITION 2.3. *Let $(S, P)$ be a $\Gamma$-contraction on $\mathcal{H}$ and let $F, F_*$ be the fundamental operators of $(S, P)$ and $(S^*, P^*)$ respectively. Then $PF = F_*{}^*P|_{\mathcal{D}_P}$.*

*Proof.* Since $F \in \mathcal{B}(\mathcal{D}_P)$ and $F_* \in \mathcal{B}(\mathcal{D}_{P^*})$, both $PF$ and $F_*{}^*P|_{\mathcal{D}_P}$ are in $\mathcal{B}(\mathcal{D}_P, \mathcal{D}_{P^*})$. For $D_P h \in \mathcal{D}_P$ and $D_{P^*}h' \in \mathcal{D}_{P^*}$, we have

$$\begin{aligned}
\langle PFD_P h, D_{P^*}h' \rangle &= \langle D_{P^*}PFD_P h, h' \rangle \\
&= \langle PD_P FD_P h, h' \rangle, \quad (\text{since } PD_P = D_{P^*}P) \\
&= \langle P(S - S^*P)h, h' \rangle, \quad (\text{since } S - S^*P = D_P F D_P) \\
&= \langle (PS - PS^*P)h, h' \rangle = \langle (SP - PS^*P)h, h' \rangle = \langle (S - PS^*)Ph, h' \rangle \\
&= \langle D_{P^*}F_*{}^*D_{P^*}Ph, h' \rangle, \quad (\text{since } S^* - SP^* = D_{P^*}F_*D_{P^*}) \\
&= \langle F_*{}^*PD_P h, D_{P^*}h' \rangle.
\end{aligned}$$

Hence $PF = F_*{}^*P|_{\mathcal{D}_P}$. ∎

## 3. FUNCTIONAL MODEL

In [26], Sz.-Nagy and Foias showed that every pure contraction $P$ defined on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the operator $\mathbb{P} = P_{\mathbb{H}_P}(M_z \otimes I)|_{\mathcal{D}_{P^*}}$ on the Hilbert space $\mathbb{H}_P = (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \ominus M_{\Theta_P}(H^2(\mathbb{D}) \otimes \mathcal{D}_P)$, where $M_z$ is the multiplication operator on $H^2(\mathbb{D})$ and $M_{\Theta_P}$ is the multiplication operator from $H^2(\mathbb{D}) \otimes \mathcal{D}_P$ into $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ corresponding to the multiplier $\Theta_P$, which is the characteristic function of $P$ defined in Section 1. This is known as Sz.-Nagy–Foias model for a pure contraction. Here analogously we produce a model for a pure $\Gamma$-contraction.

THEOREM 3.1. *Every pure $\Gamma$-contraction $(S, P)$ defined on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the pair $(S_1, P_1)$ on the Hilbert space $\mathbb{H}_P = (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \ominus M_{\Theta_P}(H^2(\mathbb{D}) \otimes \mathcal{D}_P)$ defined as $S_1 = P_{\mathbb{H}_P}(I \otimes F_*{}^* + M_z \otimes F_*)|_{\mathbb{H}_P}$ and $P_1 = P_{\mathbb{H}_P}(M_z \otimes I)|_{\mathbb{H}_P}$.*

REMARK 3.2. It is interesting to see here that the model space for a pure $\Gamma$-contraction $(S, P)$ is same as that of $P$ and the model operator for $P$ is the same given in Sz.-Nagy-Foias model.

To prove the above theorem, we define an operator $W$ in the following way:

$$\begin{aligned}
W : \mathcal{H} &\to H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*} \\
h &\mapsto \sum_{n=0}^{\infty} z^n \otimes D_{P^*}P^{*n}h.
\end{aligned}$$

It is obvious that $W$ embeds $\mathcal{H}$ isometrically inside $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ (see proof of Theorem 4.6 of [15]) and its adjoint $L : H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*} \to \mathcal{H}$ is given by

$$L(f \otimes \xi) = f(P)D_{P^*}\xi, \quad \text{for all } f \in \mathbb{C}[z], \text{ and } \xi \in \mathcal{D}_{P^*}.$$

Here we mention an interesting and well-known property of the operator $L$ which we use to prove the above theorem.

LEMMA 3.3. *For a pure contraction $P$, the identity*

$$L^*L + M_{\Theta_P}M_{\Theta_P}^* = I_{H^2(\mathbb{D})\otimes\mathcal{D}_{P^*}}$$

*holds.*

*Proof.* As observed by Arveson in the proof of Theorem 1.2 in [11], the operator $L$ satisfies the identity

$$L(k_z \otimes \xi) = (I - \bar{z}P)^{-1}D_{P^*}\xi \quad \text{for } z \in \mathbb{D}, \xi \in \mathcal{D}_{P^*},$$

where $k_z(w) = (1 - \langle w, z\rangle)^{-1}$. Therefore, for $z, w$ in $\mathbb{D}$ and $\xi, \eta$ in $\mathcal{D}_{P^*}$, we obtain that

$$\begin{aligned}
&\langle (L^*L + M_{\Theta_P}M_{\Theta_P}^*)k_z \otimes \xi, k_w \otimes \eta\rangle \\
&\quad = \langle L(k_z \otimes \xi), L(k_w \otimes \eta)\rangle + \langle M_{\Theta_P}^*(k_z \otimes \xi), M_{\Theta_P}^*(k_w \otimes \eta)\rangle \\
&\quad = \langle (I - \bar{z}P)^{-1}D_{P^*}\xi, (I - \overline{w}P)^{-1}D_{P^*}\eta\rangle + \langle k_z \otimes \Theta_P(z)^*\xi, k_w \otimes \Theta_P(w)^*\eta\rangle \\
&\quad = \langle D_{P^*}(I - wP^*)^{-1}(I - \bar{z}P)^{-1}D_P^*\xi, \eta\rangle + \langle k_z, k_w\rangle\langle \Theta_P(w)\Theta_P(z)^*\xi, \eta\rangle \\
&\quad = \langle k_z \otimes \xi, k_w \otimes \eta\rangle.
\end{aligned}$$

The last equality follows from the following well-known identity,

$$1 - \Theta_P(w)\Theta_P(z)^* = (1 - w\bar{z})D_{P^*}(1 - wP^*)^{-1}(1 - \bar{z}P)^{-1}D_{P^*},$$

where $\Theta_P$ is the characteristic function of $P$. Using the fact that the vectors $k_z$ forms a total set in $H^2(\mathbb{D})$, the assertion follows. ∎

*Proof of Theorem* 3.1. It is evident from Lemma 3.3 that

$$L^*(\mathcal{H}) = W(\mathcal{H}) = \mathbb{H}_P = (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \ominus M_{\Theta_P}(H^2(\mathbb{D}) \otimes \mathcal{D}_P).$$

Let $T = I \otimes F_*^* + M_z \otimes F_*$ and $V = M_z \otimes I$. For a basis vector $z^n \otimes \xi$ of $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ and $h \in \mathcal{H}$ we have

$$\langle L(z^n \otimes \xi), h\rangle = \left\langle z^n \otimes \xi, \sum_{k=0}^{\infty} z^k \otimes D_{P^*}P^{*k}h \right\rangle = \langle \xi, D_{P^*}P^{*n}h\rangle = \langle P^n D_{P^*}\xi, h\rangle.$$

This implies that

$$L(z^n \otimes \xi) = P^n D_{P^*}\xi, \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Therefore

$$\langle L(M_z \otimes I)(z^n \otimes \xi), h\rangle = \left\langle z^{n+1} \otimes \xi, \sum_{k=0}^{\infty} z^k \otimes D_{P^*} P^{*k} h \right\rangle$$
$$= \langle \xi, D_{P^*} P^{*n+1} h\rangle = \langle P^{n+1} D_{P^*} \xi, h\rangle.$$

Consequently, $LV = PL$ on vectors of the form $z^n \otimes \xi$ which span $H^2 \otimes \mathcal{D}_{P^*}$ and hence

$$LV = PL.$$

Therefore $V^*$ leaves the range of $L^*$ (isometric copy of $\mathcal{H}$) invariant and $V^*|_{L^*\mathcal{H}} = L^*P^*L$ which is the copy of the operator $P^*$ on range of $L^*$. Also

$$LT(z^n \otimes \xi) = L(I \otimes F_*^* + M_z \otimes F_*)(z^n \otimes \xi) = L(I \otimes F_*^*)(z^n \otimes \xi) + L(M_z \otimes F_*)(z^n \otimes \xi)$$
$$= L(z^n \otimes F_*^* \xi) + L(z^{n+1} \otimes F_* \xi) = P^n D_{P^*} F_*^* \xi + P^{n+1} D_{P^*} F_* \xi.$$

Again $SL(z^n \otimes \xi) = SP^n D_{P^*} \xi$. Therefore for showing $LT = SL$, it is enough to show that

$$P^n D_{P^*} F_*^* + P^{n+1} D_{P^*} F_* = SP^n D_{P^*} = P^n S D_{P^*} \quad \text{i.e., } D_{P^*} F_*^* + P D_{P^*} F_* = S D_{P^*}.$$

Let $H = D_{P^*} F_*^* + P D_{P^*} F_* - S D_{P^*}$. Then $H$ is defined from $\mathcal{D}_{P^*} \to \mathcal{H}$. Since $F_*$ is a solution of (1.3), we have

$$HD_{P^*} = D_{P^*} F_*^* D_{P^*} + P D_{P^*} F_* D_{P^*} - S D_{P^*}^2 = (S - PS^*) + P(S^* - SP^*) - S(I - PP^*) = 0.$$

Hence $H = 0$. So we have

$$D_{P^*} F_*^* + P D_{P^*} F_* = S D_{P^*}$$

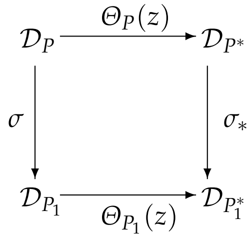
and therefore

$$L(I \otimes F_*^* + M_z \otimes F_*) = SL.$$

This shows that $T^*$ leaves $L^*(\mathcal{H})$ invariant as well as $T^*|_{L^*(\mathcal{H})} = L^*S^*L$. Thus $\mathbb{H}_P$ is co-invariant under $I \otimes F_*^* + M_z \otimes F_*$ and $M_z \otimes I$. Hence $\mathbb{H}_P$ is a model space and $P_{\mathbb{H}_P}(I \otimes {F_*}^* + M_z \otimes F_*)|_{\mathbb{H}_P}$ and $P_{\mathbb{H}_P}(M_z \otimes I)|_{\mathbb{H}_P}$ are model operators for $S$ and $P$ respectively. ∎

## 4. A SET OF UNITARY INVARIANTS FOR PURE $\Gamma$-CONTRACTIONS

The characteristic function of a contraction is a classical complete unitary invariant devised by Sz.-Nagy and Foias [26]. In [23], Popescu gave the characteristic function for an infinite sequence of non-commuting operators. The same for a commuting contractive tuple of operators was invented by Bhattacharyya, Eschmeier and Sarkar [13]. Popescu's characteristic function for a non-commuting tuple, when specialized to a commuting one, gives the same function. Given two contractions $P$ and $P_1$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$, the characteristic functions of $P$ and $P_1$ are said to coincide if there are unitary operators $\sigma : \mathcal{D}_P \to \mathcal{D}_{P_1}$ and $\sigma_* : \mathcal{D}_{P^*} \to \mathcal{D}_{P_1^*}$ such that the following diagram commutes for all $z \in \mathbb{D}$:




The following result is due to Sz.-Nagy and Foias.

THEOREM 4.1. *Two completely non-unitary contractions are unitarily equivalent if and only if their characteristic functions coincide.*

Let $(S, P)$ and $(S_1, P_1)$ be two pure $\Gamma$-contractions on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ respectively. As we mentioned in Section 1, the complete unitary invariant that we shall produce has two contents namely the equivalence of the fundamental operators of $(S^*, P^*)$ and $(S_1^*, P_1^*)$ and the coincidence of the characteristic functions of $P$ and $P_1$.

PROPOSITION 4.2. *If two $\Gamma$-contractions $(S, P)$ and $(S_1, P_1)$ defined on $\mathcal{H}$ and $\mathcal{H}_1$ respectively are unitarily equivalent then so are their fundamental operators $F$ and $F_1$.*

*Proof.* Let $U : \mathcal{H} \to \mathcal{H}_1$ be a unitary such that $US = S_1U$ and $UP = P_1U$. Then clearly $UP^* = P_1^*U$ and consequently

$$UD_P^2 = U(I - P^*P) = (U - P_1^*UP) = (U - P_1^*P_1U) = D_{P_1}^2U,$$

which implies that $UD_P = D_{P_1}U$. Let $V = U|_{\mathcal{D}_P}$. Then $V \in \mathcal{B}(\mathcal{D}_P, \mathcal{D}_{P_1})$ and $VD_P = D_{P_1}V$. Now

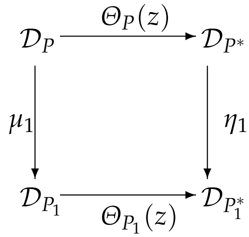
$$D_{P_1}VFV^*D_{P_1} = VD_PFD_PV^* = V(S - S^*P)V^* = S_1 - S_1^*P_1 = D_{P_1}F_1D_{P_1}.$$

Thus $F_1 = VFV^*$ and the proof is complete. ∎

The next result is a partial converse to the previous proposition for pure $\Gamma$-contractions.

PROPOSITION 4.3. *Let $(S, P)$ and $(S_1, P_1)$ be two pure $\Gamma$-contractions on $\mathcal{H}$ and $\mathcal{H}_1$ respectively such that the characteristic functions of $P$ and $P_1$ coincide. Also suppose that the fundamental operators $F_*$ of $(S^*, P^*)$ and $F_{1*}$ of $(S_1^*, P_1^*)$ are unitarily equivalent by the unitary from $\mathcal{D}_{P^*}$ and $\mathcal{D}_{P_1^*}$ that establishes the coincidence of the characteristic functions of $P$ and $P_1$. Then $(S, P)$ and $(S_1, P_1)$ are unitarily equivalent.*

*Proof.* Let $\mu_1 : \mathcal{D}_P \to \mathcal{D}_{P_1}$ and $\eta_1 : \mathcal{D}_{P^*} \to \mathcal{D}_{P_1^*}$ be unitaries such that the following diagram




commutes for all $z \in \mathbb{D}$ and $\eta_1 F_* = F_{1*}\eta_1$. Let us define

$$\eta = (I \otimes \eta_1) : H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*} \to H^2(\mathbb{D}) \otimes \mathcal{D}_{P_1^*}.$$

Since $\eta_1 \Theta_P = \Theta_{P_1}\mu_1$, we have for any $f \in H^2(\mathbb{D}) \otimes \mathcal{D}_P$

$$\eta(\mathrm{Ran} M_{\Theta_p} f) = \eta_1 \Theta_P f = \Theta_{P_1}\mu_1 f = M_{\Theta_{P_1}}(\mu_1 f).$$

Therefore,

$$\eta(\mathbb{H}_P) = \mathbb{H}_{P_1}, \quad \text{as } \mathbb{H}_P = \mathrm{Ran}(M_{\Theta_P})^{\perp} \text{ and } \mathbb{H}_{P_1} = \mathrm{Ran}(M_{\Theta_{P_1}})^{\perp}.$$

Now clearly

$$\eta(M_z \otimes I_{\mathcal{D}_{P^*}})^* = (M_z \otimes I_{\mathcal{D}_{P_1^*}})^* \eta,$$

which shows that $\eta(\mathbb{H}_P)$ i.e., $\mathbb{H}_{P_1}$ is co-invariant under $M_z \otimes I_{\mathcal{D}_{P_1^*}}$ and $P_{\mathbb{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})|_{\mathbb{H}_P}$ coincides with $P_{\mathbb{H}_{P_1}}(M_z \otimes I_{\mathcal{D}_{P_1^*}})|_{\mathbb{H}_{P_1}}$, i.e., $P$ defined on $\mathcal{H}$ coincides with $P_1$ defined on $\mathcal{H}_1$.

Again

$$\begin{aligned}\eta(I \otimes F_*{}^* + M_z \otimes F_*)^* &= \eta(I \otimes F_* + M_z^* \otimes F_*{}^*) = I \otimes \eta_1 F_* + M_z^* \otimes \eta_1 F_*^* \\ &= I \otimes F_{1*}\eta_1 + M_z^* \otimes F_{1*}^*\eta_1 = (I \otimes F_{1*} + M_z^* \otimes F_{1*}^*)(I \otimes \eta_1) \\ &= (I \otimes F_{1*}^* + M_z \otimes F_{1*})^*(I \otimes \eta_1),\end{aligned}$$

which shows that $S(\equiv P_{\mathbb{H}_P}(I \otimes F_*^* + M_z \otimes F_*)|_{\mathbb{H}_P})$ and $S_1(\equiv P_{\mathbb{H}_{P_1}}(I \otimes F_{1*}^* + M_z \otimes F_{1*})|_{\mathbb{H}_{P_1}})$ are unitarily equivalent. Hence $(S, P)$ and $(S_1, P_1)$ are also unitarily equivalent and the proof is complete. ∎

Combining the last two propositions we obtain the main result of this section.

THEOREM 4.4. *Let $(S, P)$ and $(S_1, P_1)$ be two pure $\Gamma$-contractions on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ respectively and let $F_*$ and $F_{1*}$ be the fundamental operators of $(S^*, P^*)$ and $(S_1^*, P_1^*)$. Then $(S, P)$ is unitarily equivalent to $(S_1, P_1)$ if and only if the characteristic functions of $P$ and $P_1$ coincide and $F_*$ and $F_{1*}$ are unitarily equivalent by the unitary from $\mathcal{D}_{P^*}$ and $\mathcal{D}_{P_1^*}$ that establishes the coincidence of the characteristic functions of $P$ and $P_1$.*

*Proof.* Since $(S, P)$ and $(S_1, P_1)$ are unitarily equivalent, so are $(S^*, P^*)$ and $(S_1^*, P_1^*)$. Now we apply Proposition 4.2 to the $\Gamma$-contractions $(S^*, P^*)$ and $(S_1^*, P_1^*)$ to have the unitary equivalence of $F_*$ and $F_{1*}$. ∎

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## REFERENCES

[1] J. AGLER, Rational dilation on an annulus, *Ann. of Math.* **121**(1985), 537–563.

[2] J. AGLER, J. HARLAND, B.J. RAPHAEL, Classical function theory, operator dilation theory, and machine computation on multiply-connected domains, *Mem. Amer. Math. Soc.* **191**(2008), no. 892.

[3] J. AGLER, N.J. YOUNG, A commutant lifting theorem for a domain in $\mathbb{C}^2$ and spectral interpolation, *J. Funct. Anal.* **161**(1999), 452–477.

[4] J. AGLER, N.J. YOUNG, Operators having the symmetrized bidisc as a spectral set, *Proc. Edinburgh Math. Soc.* **43**(2000), 195–210.

[5] J. AGLER, N.J. YOUNG, The two-point spectral Nevanlinna–Pick problem, *Integral Equations Operator Theory* **37**(2000), 375–385.

[6] J. AGLER, N.J. YOUNG, A Schwarz lemma for symmetrized bidisc, *Bull. London Math. Soc.* **33**(2001), 175–186.

[7] J. AGLER, N.J. YOUNG, A model theory for $\Gamma$-contractions, *J. Operator Theory* **49**(2003), 45–60.

[8] J. AGLER, N.J. YOUNG, The two-by-two spectral Nevanlinna–Pick problem, *Trans. Amer. Math. Soc.* **356**(2003), 573–585.

[9] J. AGLER, N.J. YOUNG, The hyperbolic geometry of the symmetrized bidisc, *J. Geom. Anal.* **14**(2004), 375–403.

[10] J. AGLER, N.J. YOUNG, The magic functions and automorphisms of a domain, *Complex Anal. Oper. Theory* **2**(2008), 383–404.

[11] W.B. ARVESON, Subalgebras of $C^*$-algebra. II, *Acta Math.* **128**(1972), 271–308.

[12] W.B. ARVESON, Subalgebras of $C^*$-algebras. III, Multivariable operator theory, *Acta Math.* **181**(1998), 159–228.

[13] T. BHATTACHARYYA, J. ESCHMEIER, J. SARKAR, Characteristic function of a pure commuting contractive tuple, *Integral Equations Operator Theory* **53**(2005), 23–32.

[14] T. BHATTACHARYYA, G. MISRA, Contractive and completely contractive homomorphisms of planar algebras, *Illinois J. Math.* **49**(2005), 1181–1201.

[15] T. BHATTACHARYYA, S. PAL, S. SHYAM ROY, Dilations of $\Gamma$-contractions by solving operator equations, *Adv. in Math.* **230**(2012), 577–606.

[16] R.G. DOUGLAS, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17**(1966), 413–415.

[17] R.G. DOUGLAS, G. MISRA, J. SARKAR, Contractive Hilbert modules and their dilations, *Israel J. Math.* **187**(2012), 141–165.

[18] M.A. DRITSCHEL, S. MCCULLOUGH, The failure of rational dilation on a triply connected domain, *J. Amer. Math. Soc.* **18**(2005), 873–918.

[19] B.A. FRANCIS, *A Course in $H^\infty$ Control Theory*, Lecture Notes in Control and Inform. Sci., vol. 88, Springer-Verlag, Berlin 1987.

[20] M. JARNICKI, P. PFLUG, On automorphisms of the symmetrized bidisc, *Arch. Math.* **83**(2004), 264–266.

[21] V. MÜLLER, F.-H. VASILESCU, Standard models for some commuting multioperators, *Proc. Amer. Math. Soc.* **117**(1993), 979–989.

[22] V. PAULSEN, *Completely Bounded Maps and Operator Algebras*, Cambridge Univ. Press, Cambridge 2002.

[23] G. POPESCU, Characteristic functions for infinite sequences of noncommuting operators, *J. Operator Theory* **22**(1989), 51–71.

[24] G. POPESCU, Isometric dilations for infinite sequences of noncommuting operators, *Trans. Amer. Math. Soc.* **316**(1989), 523–536.

[25] S. POTT, Invariant subspaces for tuples with normal dilations, in *Banach Algebras '97 (Blaubeuren)*, de Gruyter, Berlin 1998, pp. 397–413.

[26] B. SZ.-NAGY, C. FOIAS, H. BERCOVICI, L. KERCHY, *Harmonic Analysis of Operators on Hilbert Space*, Universitext, Springer, New York 2010.

[27] J.L. TAYLOR, A general framework for multi-operator functional calculus, *Adv. in Math.* **9**(1972), 183–252.

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