PARAGRASSMANN ALGEBRAS AS QUANTUM SPACES, PART II: TOEPLITZ OPERATORS

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ABSTRACT. This paper continues the study of paragrassmann algebras begun in part I with the definition and analysis of Toeplitz operators in the associated holomorphic Segal–Bargmann space. These are defined in the usual way as multiplication by a symbol followed by the projection defined by the reproducing kernel. These are non-trivial examples of spaces with Toeplitz operators whose symbols are not functions and which themselves are not spaces of functions.

KEYWORDS: *Toeplitz operators, paragrassmann algebras, quantum spaces, reproducing kernels.*

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1. INTRODUCTION

The paper is organized in three sections. Here in the first we briefly survey some relevant material from part I [9]. The intention is that the notation in this paper be identical to that in [9]. The material in this section will be used repeatedly without explicit reference. See [9] for further details. Then in the second section we present our results in full. And in the last section we comment about possibilities for future research. For background information on this field of research in physics and mathematics, see [1] and references given there. Throughout this paper we let $l \geqslant 2$ be an integer and $q \in \mathbb{C} \setminus \{0\}$.

DEFINITION 1.1. The *paragrassmann algebra* $PG_{l,q}(\theta, \overline{\theta})$ with *paragrassmann variables* θ and $\overline{\theta}$ is defined to be

$$PG_{l,q} = PG_{l,q}(\theta, \overline{\theta}) := \mathbb{C}\{\theta, \overline{\theta}\}/J.$$

Here J is the two sided ideal generated by $\theta^l, \overline{\theta}^l, \theta\overline{\theta} - q\overline{\theta}\theta$ and $\mathbb{C}\{\theta, \overline{\theta}\}$ is the free algebra over the field of complex numbers $\mathbb{C}$ generated by the set $\{\theta, \overline{\theta}\}$ of two elements. We say that θ is a *holomorphic* variable while $\overline{\theta}$ is an *anti-holomorphic* variable.

We let $I_l := \{0, 1, \ldots, l-1\}$ be an index set. If we use a variable, such as i, without specifying its domain, it is understood that $i \in I_l$.

We have $\dim_{\mathbb{C}} PG_{l,q}(\theta, \overline{\theta}) = l^2$, which is left to the reader as an exercise. We will be using the *anti-Wick* basis $AW = \{\theta^i \overline{\theta}^j : i, j \in I_l\}$ of $PG_{l,q}(\theta, \overline{\theta})$.

The *Segal–Bargmann space* (or *holomorphic space*) is defined to be

$$\mathcal{B}_H = \mathcal{B}_H(\theta) := \mathrm{span}_{\mathbb{C}} \{\theta^i : i \in I_l\} \subset PG_{l,q}(\theta, \overline{\theta}).$$

This is a commutative sub-algebra which is not isomorphic as an algebra to an algebra of complex valued functions, since $\theta \neq 0$ is nilpotent. On the other hand, the *anti-Segal–Bargmann space* (or *anti-holomorphic space*) is defined to be

$$\mathcal{B}_{AH} = \mathcal{B}_{AH}(\overline{\theta}) := \mathrm{span}_{\mathbb{C}} \{\overline{\theta}^i : i \in I_l\} \subset PG_{l,q}(\theta, \overline{\theta}).$$

We define a conjugation in $PG_{l,q}(\theta, \overline{\theta})$ by putting $(\theta^i \overline{\theta}^j)^* := \theta^j \overline{\theta}^i$ for the basis elements in AW and then extending *anti-linearly* to $PG_{l,q}(\theta, \overline{\theta})$. This conjugation satisfies the $*$-algebra condition (which says that $(fg)^* = g^* f^*$ for all $f, g \in PG_{l,q}(\theta, \overline{\theta})$) if and only if $q \in \mathbb{R} \setminus \{0\}$. However, the $*$-algebra condition is not used anywhere in this paper, which is why we consider the more general situation $q \in \mathbb{C} \setminus \{0\}$.

We define a *Berezin type integral* that maps $PG_{l,q}$ to $\mathbb{C}$ by setting

$$\iint \mathrm{d}\theta \; \theta^i \overline{\theta}^j \; \mathrm{d}\overline{\theta} := \delta_{i,l-1} \delta_{j,l-1}$$

for basis elements in AW and then extending *linearly* to $PG_{l,q}$. Here $\delta_{n,m}$ is the Kronecker delta for the integers n, m.

Take $f = f(\theta, \overline{\theta})$ and $g = g(\theta, \overline{\theta})$ in $PG_{l,q}$. Let w_n for $n \in I_l$ be a finite sequence of strictly positive real numbers. These can be thought of as "weights" if one likes. We define

$$\langle f, g \rangle_w := \sum_{m \in I_l} w_{l-1-m} \iint \mathrm{d}\theta \; \theta^m |f(\theta, \overline{\theta})^*| \, |g(\theta, \overline{\theta})| \overline{\theta}^m \; \mathrm{d}\overline{\theta},$$

where the *anti-Wick product* $|\cdot| \, |\cdot|$ is defined as the $\mathbb{C}$-bilinear extension to $PG_{l,q} \times PG_{l,q}$ of

$$|\theta^a \overline{\theta}^b| \, |\theta^c \overline{\theta}^d| \equiv \theta^{a+c} \overline{\theta}^{b+d}$$

for any pair of elements of AW. The anti-Wick product maps $PG_{l,q} \times PG_{l,q}$ to $PG_{l,q}$. An important consequence of this definition is

$$\langle \theta^a \overline{\theta}^b, \theta^c \overline{\theta}^d \rangle_w = \delta_{a+d,b+c} \, \chi_l(a+d) \, w_{a+d},$$

where χ_l is the characteristic function of I_l, that is for every integer k we put $\chi_l(k) = 1$ if $k \in I_l$ and $\chi_l(k) = 0$ if $k \notin I_l$. Also we put $w_n := 0$ if $n \geqslant l$. Another consequence of this definition is that there is always an element $f \neq 0$ such that $\langle f, f \rangle_w = 0$ and another element g such that $\langle g, g \rangle_w < 0$. However, the inner product $\langle \cdot, \cdot \rangle_w$ restricted to the Segal–Bargmann space $\mathcal{B}_H(\theta)$ is a positive

definite inner product with $\langle \theta^j, \theta^k \rangle_w = \delta_{j,k} w_j$. So the weight w_j is associated to the monomial θ^j. So it turns out that

$$\phi_j(\theta) := w_j^{-1/2} \theta^j$$

is an orthonormal basis of $\mathcal{B}_H(\theta)$, where we take the positive square root of $w_j > 0$. We define $\|f\|_w^2 := \langle f, f \rangle_w$ for any $f \in PG_{l,q}$.

The *reproducing kernel* for the Segal–Bargmann space $\mathcal{B}_H(\theta)$ exists and is unique. It is given by

$$K(\theta, \eta) = \sum_j \frac{1}{w_j} \overline{\theta}^j \otimes \eta^j,$$

where η is another paragrassmann variable. Specifically, for every $f(x) \in \mathbb{C}[x]$, the polynomial ring in x, we have the *reproducing formula*

$$f(\theta) = \langle K(\theta, \eta), f(\eta) \rangle_w,$$

where the inner product is taken with respect to the variables $\eta, \overline{\eta}$. If

$$f(x) = \sum_{j=0}^{N} \alpha_j x^j \in \mathbb{C}[x],$$

where $\alpha_j \in \mathbb{C}$, then $f(\theta) := \sum\limits_{j=0}^{N} \alpha_j \theta^j$ defines a *functional calculus* that plays the role here played by "evaluation at a point" in the classical theory of reproducing kernel Hilbert spaces of functions. Similarly, $f(\eta) := \sum\limits_{j=0}^{N} \alpha_j \eta^j$. Of course, in this case we have that

$$f(\theta) = \sum_{j=0}^{\min(N,l-1)} \alpha_j \theta^j.$$

Also, we have a canonical linear isomorphism $\delta_{\theta \to \eta} : \mathcal{B}_H(\theta) \to \mathcal{B}_H(\eta)$ induced by $\theta^i \mapsto \eta^i$ for all $i \in I_l$.

2. TOEPLITZ OPERATORS

The reproducing kernel on the Segal–Bargmann space allows us to define Toeplitz operators in more or less the usual way. The only subtle point is that multiplication operators can be defined as acting on the left or on the right. The basic idea is to take an arbitrary element $g \in PG_{l,q}(\eta, \overline{\eta})$ and define a linear map $\widetilde{T}_g : \mathcal{B}_H(\eta) \to \mathcal{B}_H(\theta)$ by a formula such as

(2.1) $$\widetilde{T}_g f(\theta) := \langle K(\theta, \eta), f(\eta) g(\eta, \overline{\eta}) \rangle_w,$$

where $f \in \mathcal{B}_H(\eta)$. Or, if we wish, we could define it by

(2.2) $$T_g^{\#} f(\theta) := \langle K(\theta, \eta), g(\eta, \overline{\eta}) f(\eta) \rangle_w,$$

where $f \in \mathcal{B}_H(\eta)$. In fact, we will take (2.1) to be our preliminary definition of a Toeplitz operator $\widetilde{T}_g$ with *symbol* $g(\eta, \overline{\eta})$. The definition (2.2) gives a similar theory (though with some extra factors of q), so we do not develop it here.

Of course, this is a quantization scheme in a generalized sense since it sends "functions" g to operators $\widetilde{T}_g$. We recall here the famous criterion that "quantization is operators instead of functions". And even though this operator does not map a Hilbert space to itself, we can precompose (or postcompose) $\widetilde{T}_g$ with the canonical isomorphism $\delta_{\theta \to \eta}$ to get an operator from either of these two Hilbert spaces to itself. Simply as a convention we take

$$T_g := \widetilde{T}_g \circ \delta_{\theta \to \eta} : \mathcal{B}_H(\theta) \to \mathcal{B}_H(\theta)$$

as our definition of the *Toeplitz operator T_g* with *symbol* $g = g(\eta, \overline{\eta}) \in PG_{l,q}(\eta, \overline{\eta})$. Since we have a functional calculus of $\eta, \overline{\eta} \in PG_{l,q}(\eta, \overline{\eta})$ using the non-commuting polynomials $g \in \mathbb{C}\{\eta, \overline{\eta}\}$, we can also use $\mathbb{C}\{\eta, \overline{\eta}\}$ as the space of symbols here instead of using $PG_{l,q}(\eta, \overline{\eta})$.

We remark that these Toeplitz operators (which form a quantization) should not be confused with the operators A_f introduced in [1] nor with that quantization. Those operators A_f are called the *coherent state quantization* and rely on the resolution of the identity provided by the coherent states. Here we are using the reproducing kernel to project after multiplying by a symbol, and this is the standard definition of a Toeplitz operator in analysis. (See [3] or [5].) These are conceptually quite different constructs. For the coherent state quantization one needs a measure even to be able to write down the integral formula for the resolution of the identity and then the corresponding integral formula for the definition of A_f. Or, as is the case in [1] and in this paper, one needs some reasonable generalization of an integral instead of a measure *per se*. For the Toeplitz quantization one only needs a reproducing kernel, whose basic reproducing property requires an inner product. And that inner product does not necessarily arise from a measure. However, two conceptually distinct constructions could turn out to be equivalent, at least in this particular case. We will return to this question.

Before studying the Toeplitz operators themselves, let us consider the operator associated with the reproducing kernel K. We define this operator $P_K : PG_{l,q}(\theta, \overline{\theta}) \to PG_{l,q}(\theta, \overline{\theta})$ by

$$P_K F(\theta) := \langle K(\theta, \eta), F(\eta, \overline{\eta}) \rangle_w$$

for all $F(\theta, \overline{\theta}) \in PG_{l,q}(\theta, \overline{\theta})$. It is clear that $P_K F(\theta)$ lies in the Segal–Bargmann space $\mathcal{B}_H(\theta)$ and that the range of P_K is $\mathcal{B}_H(\theta)$ since P_K restricted to $\mathcal{B}_H(\theta)$ is the identity map. The classical theory suggests that P_K is an orthogonal projection. And this is so.

THEOREM 2.1. *The linear map P_K is an orthogonal projection, that is to say, it satisfies $P_K = P_K^2 = P_K^*$.*

REMARK 2.2. The adjoint P_K^* is defined with respect to the nondegenerate sesquilinear form $\langle \cdot, \cdot \rangle_w$ on $PG_{l,q}(\theta, \overline{\theta})$. (See Theorem 8.1 in [9].) So adjoints exist and are unique with respect to that form. Even though $P_K^2 = P_K$ follows from the comments above, we prove it explicitly anyway.

Proof. We will use the notation $F_{ab} := \theta^a \overline{\theta}^b$ for the elements in the basis AW. Acting with P_K on this basis we obtain

$$P_K F_{ab}(\theta) = \langle K(\theta, \eta), F_{ab}(\eta, \overline{\eta}) \rangle_w = \langle K(\theta, \eta), \eta^a \overline{\eta}^b \rangle_w = \sum_k \frac{1}{w_k} \langle \eta^k, \eta^a \overline{\eta}^b \rangle_w \, \theta^k$$

$$= \sum_k \frac{1}{w_k} \delta_{k+b,a} \chi_l(a) \chi_l(k) w_a \, \theta^k = \frac{w_a}{w_{a-b}} \chi_l(a - b) \theta^{a-b}.$$

Applying P_K twice we have that

$$P_K^2 F_{ab}(\theta) = P_K \left(\frac{w_a}{w_{a-b}} \chi_l(a - b) \theta^{a-b} \right) = \frac{w_a}{w_{a-b}} \chi_l(a - b) P_K(\theta^{a-b} \overline{\theta}^0)$$

$$= \frac{w_a}{w_{a-b}} \chi_l(a - b) \frac{w_{a-b}}{w_{a-b}} \chi_l(a - b) \theta^{a-b} = \frac{w_a}{w_{a-b}} \chi_l(a - b) \theta^{a-b} = P_K F_{ab}(\theta),$$

which implies that $P_K^2 = P_K$.

Next here are the matrix elements for P_K:

$$\langle F_{ab}, P_K F_{cd} \rangle_w = \left\langle \theta^a \overline{\theta}^b, \frac{w_c}{w_{c-d}} \chi_l(c - d) \theta^{c-d} \right\rangle_w$$

$$= \frac{w_c}{w_{c-d}} \chi_l(c - d) \delta_{a,b+c-d} \chi_l(a) w_a = \frac{w_a w_c}{w_{c-d}} \chi_l(c - d) \delta_{a-b,c-d}.$$

Now we calculate the matrix elements for P_K^* obtaining

$$\langle F_{ab}, P_K^* F_{cd} \rangle_w = \langle P_K F_{ab}, F_{cd} \rangle_w = \left\langle \frac{w_a}{w_{a-b}} \chi_l(a - b) \theta^{a-b}, \theta^c \overline{\theta}^d \right\rangle_w$$

$$= \frac{w_a}{w_{a-b}} \chi_l(a - b) \delta_{a-b+d,c} \chi_l(c) w_c = \frac{w_a w_c}{w_{a-b}} \chi_l(a - b) \delta_{a-b,c-d}.$$

Since these matrix entries for P_K^* are equal to those for P_K, we see that $P_K^* = P_K$. ∎

For any $g \in PG_{l,q}$ we define the linear map $M_g : PG_{l,q}(\theta, \overline{\theta}) \to PG_{l,q}(\theta, \overline{\theta})$ to be multiplication by g on the right, that is

$$M_g F := Fg$$

for all $F \in PG_{l,q}(\theta, \overline{\theta})$. Then we have the following result.

THEOREM 2.3. *Say $f_1, f_2 \in \mathcal{B}_H(\theta)$. Then we have*

$$\langle f_1, T_g f_2 \rangle_w = \langle f_1, M_g f_2 \rangle_w.$$

Proof. We first note that $T_g = P_K M_g$, that is, the Toeplitz operator is right multiplication by g followed by the projection associated to K. Then we calculate

the following as we claimed:

$$\langle f_1, T_g f_2 \rangle_w = \langle f_1, P_K M_g f_2 \rangle_w = \langle P_K^* f_1, M_g f_2 \rangle_w = \langle P_K f_1, M_g f_2 \rangle_w = \langle f_1, M_g f_2 \rangle_w. \quad \blacksquare$$

Now we note that the correspondence $g \mapsto T_g$ gives us a mapping

$$T : PG_{l,q}(\eta, \overline{\eta}) \to \mathcal{L}(\mathcal{B}_H(\theta)),$$

where $\mathcal{L}(\mathcal{B}_H(\theta))$ is the $*$-algebra of all linear endomorphisms of the Hilbert space $\mathcal{B}_H(\theta)$. It is easily verified that T is linear and that $T_1 = I_{\mathcal{B}_H(\theta)}$, the identity. We note that the dimensions of the domain and codomain of T are equal, since $\dim PG_{l,q}(\eta, \overline{\eta}) = l^2$ while $\dim \mathcal{L}(\mathcal{B}_H(\theta)) = (\dim \mathcal{B}_H(\theta))^2 = l^2$. One would like to know whether T is an isomorphism of vector spaces and how it relates to the algebra structure on its domain and codomain. Of course, the non-commutativity of $PG_{l,q}(\eta, \overline{\eta})$ is completely determined by the non-commutativity of the two generators $\eta, \overline{\eta}$, while the non-commutativity of $\mathcal{L}(\mathcal{B}_H(\theta))$ ($\cong l \times l$ complex matrices) is not so simply described in general. It seems that these two algebras in general are not isomorphic as algebras, though (as is well known) they are for $q = -1$ and $l = 2$. But we do have:

THEOREM 2.4. *The linear map* $T : PG_{l,q}(\eta, \overline{\eta}) \to \mathcal{L}(\mathcal{B}_H(\theta))$ *is a vector space isomorphism.*

Proof. It suffices to show that the kernel of $\widetilde{T}$ is zero. So we take $g \in \ker \widetilde{T}$, which means that $\widetilde{T}_g = 0$. In particular, this implies that $\widetilde{T}_g f_d = 0$ for all $d \in I_l$, where $f_d = \theta^d$. Now, writing $g = \sum_{ij} g_{ij} \eta^i \overline{\eta}^j$ with $g_{ij} \in \mathbb{C}$ we calculate

$$\widetilde{T}_g f_d(\theta) = \langle K(\theta, \eta), f_d(\eta) g(\eta, \overline{\eta}) \rangle_w = \sum_c \frac{1}{w_c} \sum_{ij} g_{ij} \langle \overline{\theta}^c \otimes \eta^c, \eta^d \eta^i \overline{\eta}^j \rangle_w$$

$$= \sum_c \frac{1}{w_c} \sum_{ij} g_{ij} \langle \eta^c, \eta^d \eta^i \overline{\eta}^j \rangle_w \theta^c = \sum_c \frac{1}{w_c} \sum_{ij} g_{ij} \langle \eta^c \overline{\eta}^d, \eta^i \overline{\eta}^j \rangle_w \theta^c.$$

So, $\widetilde{T}_g f_d = 0$ for all $d \in I_l$ implies $\sum_{ij} g_{ij} \langle \eta^c \overline{\eta}^d, \eta^i \overline{\eta}^j \rangle_w = 0$ for all $c, d \in I_l$. These are l^2 homogeneous linear equations in the l^2 coefficients g_{ij}. As we have already seen ([9], Theorem 8.1) the $l^2 \times l^2$ matrix $\langle \eta^c \overline{\eta}^d, \eta^i \overline{\eta}^j \rangle_w$ is invertible and so we must have $g_{ij} = 0$ for all $i, j \in I_l$. Hence $g = 0$. $\quad \blacksquare$

So for any non-zero g in $PG_{l,q}(\eta, \overline{\eta})$ the associated Toeplitz operator T_g is non-zero and hence its operator norm is strictly positive, $\|T_g\|_{\mathrm{op}} > 0$. But $\langle g, g \rangle_w$ can be zero or negative. So there is no way to bound the operator norm above by a multiple of $\langle g, g \rangle_w$, nor even by a multiple of $|\langle g, g \rangle_w|$. This contrasts with the situation in the classical case. (See [3].)

Here is another useful general result about these Toeplitz operators.

THEOREM 2.5. *The matrix of $T_{\eta^i \overline{\eta}^j}$ with respect to the ordered, orthogonal basis*

$$\{\theta^a : a \in I_l\}$$

of $\mathcal{B}_H(\theta)$ (where the usual order for the integers in I_l induces the order in the basis) is a matrix whose columns contain either all zeros or exactly one non-zero entry. Column a of this matrix has a non-zero entry in row $a + i - j$ if and only if $a + i \in I_l$ as well as $a + i - j \in I_l$. In this case the non-zero entry is w_{i+a}/w_{i+a-j}. Otherwise, column a contains only zeros.

Proof. We will examine first the image under $\widetilde{T}$ of $\eta^i \overline{\eta}^j \in PG_{l,q}(\eta, \overline{\eta})$. So, taking $f_a(\eta) = \eta^a \in \mathcal{B}_H(\eta)$ with $a \in I_l$, we obtain

$$(\widetilde{T}_{\eta^i \overline{\eta}^j} f_a)(\theta) = \langle K(\theta, \eta), f_a(\eta) \, \eta^i \overline{\eta}^j \rangle_w = \Big\langle \sum_k \frac{1}{w_k} \overline{\theta}^k \otimes \eta^k, \, \eta^a \eta^i \overline{\eta}^j \Big\rangle_w$$

$$= \sum_k \frac{1}{w_k} \theta^k \otimes \langle \eta^k, \eta^{i+a} \overline{\eta}^j \rangle_w = \sum_k \frac{1}{w_k} \delta_{j+k, i+a} \chi_l(j+k) \chi_l(k) w_{j+k} \, \theta^k$$

(2.3) $$= \frac{w_{i+a}}{w_{i+a-j}} \chi_l(i+a) \chi_l(i+a-j) \, \theta^{i+a-j}.$$

Strictly speaking, we must define θ^k and w_k for $k \notin I_l$. But we can give these expressions arbitrary definitions, since the χ_l factors give zero in this case. Of course, we already have $\theta^k = 0$ for $k \geq l$. We also define $\theta^k = 0$ for $k < 0$ for convenience.

So $T_{\eta^i \overline{\eta}^j}$ sends the basis element θ^a to a multiple of the element θ^{i+a-j}, which is a basis element exactly when $i + a - j \in I_l$. (Otherwise, $\theta^{i+a-j} = 0$.) This multiple is non-zero exactly when we also have $a + i \in I_l$. And by linear algebra the coefficients of the expansion of the image of θ^a in the basis θ^b, $b \in I_l$, are the entries in column a of the matrix associated to $T_{\eta^i \overline{\eta}^j}$. ∎

REMARK 2.6. If instead we had used the orthonormal basis $\phi_a(\theta)$, $a \in I_l$, then the associated matrix would have non-zero entries in exactly the same entries, only now the non-zero entry in column a would become

$$\frac{w_{a+i}}{(w_a w_{a+i-j})^{1/2}}$$

provided that both $a + i \in I_l$ and $a + i - j \in I_l$ hold.

We now derive some consequences of (2.3). In particular, when $i = j = 0$ we have that $T_{\eta^i \overline{\eta}^j} = T_1$ sends θ^a to itself for all $a \in I_l$. And this agrees with the fact, noted earlier, that $T_1 = I$, the identity. For the more general case $i = j$ we have that $T_{\eta^i \overline{\eta}^i}$ sends θ^a to $(w_{i+a}/w_a)\theta^a$ for $a = 0, 1, \ldots, l - 1 - i$, while it sends θ^a to zero for $a \geq l - i$. So the matrix of $T_{\eta^i \overline{\eta}^i}$ in the above basis is diagonal with non-negative eigenvalues and rank equal to $l - i$.

By taking $j = 0$ in (2.3), we have that

$$T_{\eta^i} : \theta^a \mapsto \chi_l(a + i)\theta^{a+i} = \theta^{a+i},$$

which we can write as $T_{\eta^i} = M_{\theta^i}$, the operation of multiplying by θ^i. (Since $\mathcal{B}_H(\theta)$ is commutative, we do not have to distinguish between right and left multiplication.) In particular, $T_{\eta^i} = (T_\eta)^i$ for all $i \in I_l$. Of course, this is as it must be since T_η is really multiplication by θ (which maps $\mathcal{B}_H(\theta)$ to itself) followed by the projection induced by the reproducing kernel, which acts as the identity on $\mathcal{B}_H(\theta)$. Clearly, T_η is not invertible, and its kernel is given by $\ker T_\eta = \mathbb{C}\,\theta^{l-1}$. It easily follows that $(T_\eta)^i \neq 0$ for $0 \leqslant i < l$ and $(T_\eta)^l = 0$. So T_η is nilpotent of order l. Clearly, the family of operators T_{η^i} for $i \in I_l$ is commutative, since each one is a power of a fixed operator, namely T_η.

Coming back to the problem of estimating the operator norm of a Toeplitz operator, we note that the above shows that $T_\eta(\phi_a) = (w_{a+1}/w_a)^{1/2}\phi_{a+1}$. This implies $\|T_\eta\|_{\mathrm{op}}^2 \geqslant \max_a(w_{a+1}/w_a)$, while $\|\eta\|_w^2 = w_1$. (Here $w_l = 0$.) So even for a holomorphic symbol, estimating the operator norm depends on information about all the weights w_n which are l independent parameters.

The case $i = 0$ in (2.3) is a bit different. (Note that we have already discussed the case $i = j = 0$.) In this case we have

$$T_{\bar{\eta}^j} : \theta^a \mapsto \frac{w_a}{w_{a-j}}\theta^{a-j}.$$

In particular

$$T_{\bar{\eta}} : \theta^a \mapsto \frac{w_a}{w_{a-1}}\theta^{a-1},$$

which can be viewed as a weighted shift operator or as a generalized derivative operator. Taking $a = 0$, we have that $T_{\bar{\eta}} : 1 \mapsto 0$, and so $T_{\bar{\eta}}$ is not invertible. We have that $\ker T_{\bar{\eta}} = \mathbb{C}\,1$. We note that $T_{\bar{\eta}^j} = (T_{\bar{\eta}})^j$ follows immediately. As above, we easily see that $T_{\bar{\eta}}$ is nilpotent of order l. We will use the notation $\partial_w := T_{\bar{\eta}}$ to indicate that this is a type of derivative. Note that the parameters w_a arise from the definition of the sesquilinear form which are, in general, independent from the other parameter q. We also note that the family of operators $T_{\bar{\eta}^i}$ for $i \in I_l$ is commutative, since again each one is a power of a fixed operator, namely $T_{\bar{\eta}}$.

We now come back to the question whether this Toeplitz quantization is distinct from the coherent state quantization in [1]. The definition in equation (24) in [1] of the coherent state quantization of the "function" $f \in PG_{l,q}$ amounts to the following with our conventions:

$$A_f := \sum_m w_{l-1-m} \iint \mathrm{d}\theta \, |\theta\rangle \, \theta^m f(\theta, \bar{\theta}) \, \bar{\theta}^m \, \langle\bar{\theta}| \, \mathrm{d}\bar{\theta}.$$

Here we are using these definitions from [1] for the coherent states:

$$|\theta\rangle := \sum_r \frac{1}{w_r^{1/2}} \theta^r \otimes |e_r\rangle \in \mathcal{B}_H(\theta) \otimes \mathcal{H},$$

$$\langle\bar\theta| := \sum_s \frac{1}{w_s^{1/2}} \bar\theta^s \otimes \langle e_s| \in \mathcal{B}_{AH}(\bar\theta) \otimes \mathcal{H}'.$$

In these formulas $\mathcal{H}$ is an auxiliary Hilbert space of finite dimension l. We let $\{|e_n\rangle\}$ be an orthonormal basis of $\mathcal{H}$, and $\{\langle e_n|\}$ be its dual orthonormal basis in the dual space $\mathcal{H}'$. So A_f acts in $\mathcal{H}$, which is to say that $A : PG_{l,q} \to \mathcal{L}(\mathcal{H})$. We now calculate A_f explicitly for $f = \theta^i\bar\theta^j$ since this was not written out in [1], though certain special cases were shown there. Suppressing the tensor product notation as in [1] we obtain

$$A_{\theta^i\bar\theta^j} = \sum_m w_{l-1-m} \int\int \mathrm{d}\theta\, |\theta\rangle\, \theta^m \theta^i \bar\theta^j \bar\theta^m\, \langle\bar\theta|\, \mathrm{d}\bar\theta$$

$$= \sum_m w_{l-1-m} \int\int \mathrm{d}\theta \Big(\sum_r \frac{1}{w_r^{1/2}} \theta^r |e_r\rangle\Big) \theta^{m+i}\bar\theta^{m+j} \Big(\sum_s \frac{1}{w_s^{1/2}} \bar\theta^s \langle e_s|\Big) \mathrm{d}\bar\theta$$

$$= \sum_m w_{l-1-m} \sum_{rs} \frac{1}{w_r^{1/2} w_s^{1/2}} \int\int \mathrm{d}\theta\, \theta^{m+i+r} \bar\theta^{m+j+s} \mathrm{d}\bar\theta\, |e_r\rangle\langle e_s|$$

$$= \sum_m w_{l-1-m} \sum_{rs} \frac{1}{w_r^{1/2} w_s^{1/2}} \delta_{l-1,m+i+r}\delta_{l-1,m+j+s} |e_r\rangle\langle e_s|$$

$$= \sum_{rs} \frac{1}{w_r^{1/2} w_s^{1/2}} \Big(\sum_m w_{l-1-m}\chi_l(m)\delta_{l-1,m+i+r}\delta_{l-1,m+j+s}\Big) |e_r\rangle\langle e_s|$$

$$= \sum_{rs} \frac{1}{w_r^{1/2} w_s^{1/2}} w_{i+r}\chi_l(r)\chi_l(i+r)\delta_{i+r,j+s} |e_r\rangle\langle e_s|$$

$$= \sum_{s} \frac{1}{w_{j-i+s}^{1/2} w_s^{1/2}} w_{j+s}\chi_l(j-i+s)\chi_l(j+s) |e_{j-i+s}\rangle\langle e_s|.$$

In an equivalent notation (with $e_a = |e_a\rangle$) this reads as

(2.4) $$A_{\theta^i\bar\theta^j} : e_a \mapsto \frac{w_{j+a}}{w_{j-i+a}^{1/2} w_a^{1/2}} \chi_l(j-i+a)\chi_l(j+a) e_{j-i+a}$$

for all $a \in I_l$. Note that the left and right versions of the coherent state quantization in [1], denoted by $A_{\theta^i\bar\theta^j}^L$ and $A_{\theta^i\bar\theta^j}^R$ respectively, give the same expression modulo some factors of q.

We now compare the formula (2.4) with (2.3) which we write equivalently as

$$T_{\theta^i\bar\theta^j} : \phi_a \mapsto \frac{w_{i+a}}{w_{i-j+a}^{1/2} w_a^{1/2}} \chi_l(i-j+a)\chi_l(i+a) \phi_{i-j+a}$$

using the orthonormal basis ϕ_a of $\mathcal{B}_H(\theta)$. This assumes the form of equation (2.4) provided that we interchange i and j, which corresponds to replacing $\theta^i\bar\theta^j$ with its conjugate $(\theta^i\bar\theta^j)^* = \theta^j\bar\theta^i$. This motivates the definition of a *linear* map

$Z : PG_{l,q} \to PG_{l,q}$ defined by $f \mapsto f^*$ for the elements f in the basis AW. Please note that Z, being the linear extension, is *not* the conjugation. Also we define a unitary map $U : \mathcal{B}_H(\theta) \to \mathcal{H}$ by $\phi_a \mapsto e_a$ and its induced C^*-algebra isomorphism $\widetilde{U} : \mathcal{L}(\mathcal{B}_H(\theta)) \to \mathcal{L}(\mathcal{H})$ given by $\widetilde{U} : T \mapsto UTU^*$ for all $T \in \mathcal{L}(\mathcal{B}_H(\theta))$. So we have proved the following.

THEOREM 2.7. *The coherent state quantization A and the Toeplitz quantization T are related by $AZ = \widetilde{U}T$.*

COROLLARY 2.8 (Proved in Section 7 of [1]). *The coherent state quantization $A : PG_{l,q} \to \mathcal{L}(\mathcal{H})$ is a vector space isomorphism, where we are using the notation established above.*

Proof. The linear map Z is invertible, so we can write $A = \widetilde{U}TZ^{-1}$. This exhibits A as the composition of three vector space isomorphisms. ∎

Whether Theorem 2.7 says that these two quantizations are equivalent will depend on one's definition of equivalence of quantizations. That in turn will depend on one's definition of quantization. I would rather not give a definition of quantization in general, let alone in the context of this paper. However, we made some rather arbitrary choices in our definition of a Toeplitz operator. Another reasonable definition is $T_g^\flat := P_{\overline{K}} M_g^L$, where M_g^L is multiplication on the left by $g \in PG_{l,q}$ and

$$P_{\overline{K}} : PG_{l,q}(\theta, \overline{\theta}) \to \mathcal{B}_{AH}(\overline{\theta})$$

is the projection associated to the reproducing kernel $\overline{K}$ of $\mathcal{B}_{AH}(\overline{\theta})$. (See [9].) Then the Toeplitz quantization $T^\flat : PG_{l,q} \to \mathcal{L}(\mathcal{B}_{AH}(\overline{\theta}))$ has properties corresponding to those of T, though some details undergo minor changes. In particular, as the reader can verify, instead of (2.3) we obtain

$$T_{\theta^i \overline{\theta}^j}^\flat : \phi_a^*(\overline{\theta}) \mapsto \frac{w_{j+a}}{w_{j-i+a}^{1/2} w_a^{1/2}} \chi_l(j-i+a) \chi_l(j+a) \phi_{j-i+a}^*(\overline{\theta}).$$

So in *any* reasonable definition of equivalence of quantizations we have:

THEOREM 2.9. *The quantizations A and $T^\flat$ are equivalent.*

In the next result we consider the adjoint operator $(T_g)^*$ of some Toeplitz operator $T_g \in \mathcal{L}(\mathcal{B}_H(\theta))$.

THEOREM 2.10. *For all $g \in PG_{l,q}$ we have $(T_g)^* = T_{g^*}$ in $\mathcal{L}(\mathcal{B}_H(\theta))$.*

REMARK 2.11. While there probably is no such thing as the "right" quantization nor is any definition of the conjugation in $PG_{l,q}$ the "correct" one, it does turn out that our Toeplitz quantization and our conjugation are compatible. One recognizes that this means that T is a $*$-linear map.

Proof. We first consider the case $g = \eta^i \overline{\eta}^j$, an arbitrary element in the basis AW. We use the elements $h_a(\theta) = \theta^a$, which form an orthogonal basis of $\mathcal{B}_H(\theta)$.

Using equation (2.3) we calculate matrix elements of $(T_g)^*$ as follows:

$$\langle h_a, (T_{\eta^i\overline{\eta}^j})^* h_b\rangle_w = \langle T_{\eta^i\overline{\eta}^j} h_a, h_b\rangle_w$$

$$= \left\langle \frac{w_{i+a}}{w_{i+a-j}}\chi_l(i+a)\chi_l(i+a-j)\theta^{i+a-j}, \theta^b\right\rangle_w$$

$$= \frac{w_{i+a}}{w_{i+a-j}}\chi_l(i+a)\chi_l(i+a-j)\langle \theta^{i+a-j}, \theta^b\rangle_w$$

$$= \frac{w_{i+a}}{w_{i+a-j}}\chi_l(i+a)\chi_l(i+a-j)\, w_{i+a-j}\, \delta_{i+a-j,b}$$

$$= w_{i+a}\, \chi_l(i+a)\chi_l(i+a-j)\, \delta_{i+a-j,b}.$$

Next we calculate matrix entries of T_{g^*}, again using equation (2.3):

$$\langle h_a, T_{\eta^j\overline{\eta}^i} h_b\rangle_w = \left\langle \theta^a, \frac{w_{j+b}}{w_{j+b-i}}\chi_l(j+b)\chi_l(j+b-i)\theta^{j+b-i}\right\rangle_w$$

$$= \frac{w_{j+b}}{w_{j+b-i}}\chi_l(j+b)\chi_l(j+b-i)\langle \theta^a, \theta^{j+b-i}\rangle_w$$

$$= \frac{w_{j+b}}{w_{j+b-i}}\chi_l(j+b)\chi_l(j+b-i)\delta_{a,j+b-i}w_{j+b-i}$$

$$= w_{j+b}\chi_l(j+b)\chi_l(j+b-i)\delta_{a,j+b-i}.$$

Because of the Kronecker deltas, the only non-zero values for these two matrix elements occur for $i + a = j + b$. But in that case we have $\chi_l(i+a-j) = \chi_l(b) = 1$ and $\chi_l(j+b-i) = \chi_l(a) = 1$. And the other factors in these two expressions for the matrix elements are also equal. So we conclude that

$$(T_{\eta^i\overline{\eta}^j})^* = T_{\eta^j\overline{\eta}^i} = T_{(\eta^i\overline{\eta}^j)^*}.$$

This proves the theorem for the special case when $g = \eta^i\overline{\eta}^j$. The proof for a general element g follows immediately by expanding g in the basis AW. ∎

For the corollary of this result we first recall a standard definition.

DEFINITION 2.12. An element r in a vector space with a $*$-operation is *real* or *self-adjoint* if $r^* = r$.

COROLLARY 2.13. *A Toeplitz operator T_g is self-adjoint if and only if the element g is self-adjoint.*

Proof. If $g^* = g$, then by Theorem 2.10 we have $(T_g)^* = T_{g^*} = T_g$. And conversely, T_g self-adjoint implies $T_g = (T_g)^* = T_{g^*}$, again by Theorem 2.10. Since T is injective we get $g = g^*$ as desired. ∎

While T is not an algebra morphism, it does have some nice properties with respect to the products. The following is a typical property of Toeplitz operators. The proof is essentially the same as in the context of reproducing kernel Hilbert spaces of functions.

THEOREM 2.14. *Suppose that $g_1, g_2 \in \mathcal{B}_H(\eta)$. Then $T_{g_1} T_{g_2} = T_{g_1 g_2} = T_{g_2} T_{g_1}$. For $h_1, h_2 \in \mathcal{B}_{AH}(\overline{\eta})$ one has $T_{h_1} T_{h_2} = T_{h_1 h_2} = T_{h_2} T_{h_1}$. So T restricted to either $\mathcal{B}_H(\eta)$ or $\mathcal{B}_{AH}(\overline{\eta})$ is an algebra homomorphism.*

Proof. Using $T_g = P_K M_g$ and $M_{g_1} M_{g_2} = M_{g_2 g_1}$ we have that

$$T_{g_1} T_{g_2} = P_K M_{g_1} P_K M_{g_2} = P_K M_{g_1} M_{g_2} = P_K M_{g_2 g_1} = T_{g_2 g_1}.$$

Interchanging g_1 and g_2 gives

$$T_{g_2} T_{g_1} = T_{g_1 g_2}.$$

But $g_2 g_1 = g_1 g_2$ giving $T_{g_1} T_{g_2} = T_{g_1 g_2} = T_{g_2} T_{g_1}$ The adjoint of this is

$$T_{g_2^*} T_{g_1^*} = T_{(g_1 g_2)^*} = T_{g_1^*} T_{g_2^*}.$$

Now $(g_1 g_2)^* = g_1^* g_2^*$ (even though $PG_{l,q}$ need not be a $*$-algebra) as the reader can show. But we can write any $h \in \mathcal{B}_{AH}(\overline{\eta})$ as $h = g^*$ with $g \in \mathcal{B}_H(\eta)$ and so $T_{h_1} T_{h_2} = T_{h_1 h_2} = T_{h_2} T_{h_1}$ then follows. ∎

We now consider the commutation relation between M_θ and ∂_w. First

(2.5) $$(M_\theta \partial_w) \theta^a = M_\theta \left(\frac{w_a}{w_{a-1}} \chi_l(a-1) \right) \theta^{a-1} = \frac{w_a}{w_{a-1}} \chi_l(a-1) \, \theta^a$$

holds for $a \in I_l$. In the other order we get

$$(\partial_w M_\theta) \theta^a = \partial_w (\chi_l(a+1) \, \theta^{a+1}) = \frac{w_{a+1}}{w_a} \chi_l(a+1) \, \theta^a.$$

In general, this does not lead to a simple formula for the commutator unless we suppose that the weights w_n satisfy some other relations. Nonetheless, by (2.3) we do have

$$T_{\eta \overline{\eta}} \theta^a = \frac{w_{a+1}}{w_a} \, \chi_l(a+1) \theta^a.$$

And so $T_{\eta \overline{\eta}} = \partial_w M_\theta = T_{\overline{\eta}} T_\eta$. We can also write this, strangely enough, as

$$q T_{\overline{\eta} \eta} = T_{q \overline{\eta} \eta} = T_{\eta \overline{\eta}} = T_{\overline{\eta}} T_\eta.$$

And again, in general, this has no simple relation with $T_\eta T_{\overline{\eta}} = M_\theta \partial_w$ as we noted above.

We now note that by Theorem 2.10 these two operators $\partial_w = T_{\overline{\eta}} = T_{\eta^*}$ and $M_\theta = T_\eta$ are adjoints in $\mathcal{L}(\mathcal{B}_H(\theta))$ with respect to the sesquilinear form. The operators M_θ and $M_{\overline{\theta}}$ (both defined as multiplication on the right acting on $PG_{l,q}$) satisfy this *q-commutation relation* in $PG_{l,q}(\theta, \overline{\theta})$:

$$M_{\overline{\theta}} M_\theta - q M_\theta M_{\overline{\theta}} = 0.$$

However, the projection P_K from $PG_{l,q}$ to $\mathcal{B}_H(\theta)$ defined by the reproducing kernel is not an algebra homomorphism and so this particular relation is not necessarily preserved. Of course, this is to be expected since the parameters w_a in the reproducing kernel in general have nothing whatsoever to do with the parameter q. (But in [1] these parameters are related. This can be considered an advantage of the approach of those authors.)

Operators satisfying the condition $TT^* = qT^*T$ were introduced in [6]. They are called *q-normal operators* and have been studied in [4] and [7]. Now M_θ and $M_{\overline{\theta}}$ are not necessarily adjoint operators in $PG_{l,q}$. But $T_g = \iota^* M_g \iota$, where the inclusion $\iota : \mathcal{B}_H \to PG_{l,q}$ is isometric and $\iota^* = P_K$, where we are considering

$$P_K : PG_{l,q} \to \mathcal{B}_H.$$

One says that $T_g = \iota^* M_g \iota$ is the *compression* of M_g to $\mathcal{B}_H$. We feel that the following definitions have been given some inspiration by this discussion and the material in [2].

DEFINITION 2.15. Let $q \in \mathbb{C}$. Let $A, B \in \mathcal{L}(X)$ for some Hilbert space X. We say that A and B *q-commute up to compression* if there exists a complex inner product space X' containing X and $A', B' \in \mathcal{L}(X')$ whose compressions to X are A and B, respectively, and such that A', B' are *q-commuting operators* in X', that is $A'B' = qB'A'$. Here the compression of A' to X is defined to be $\iota^* A' \iota$, where the inclusion $\iota : X \to X'$ is isometric. We say that A is a *sub-q-normal operator* if A and A^* *q-commute up to compression*.

Note that neither the space X' nor the pair of operators A', B' will be unique. Given this terminology we have that the pair of operators ∂_w, M_θ (in that order!) acting on $\mathcal{B}_H(\theta)$ q-commute up to compression. The point here is that these types of operators possibly could be new and therefore of interest to researchers in operator theory. (For example, see [2]). However, we do not elaborate on this for now.

Taking the point of view of physics, we also define a *creation operator* by $A_w^\dagger := M_\theta = T_\eta$, which we now know is the adjoint of the *annihilation operator* $A_w := \partial_w = T_{\overline{\eta}}$. Though this terminology comes from physics, the mathematical fingerprint of an annihilation operator is that it lowers the degree (of homogeneous elements) by 1, while a creation operator raises the degree by 1. For this reason they are often called the *lowering* and *raising operators*, respectively. Thus in this Toeplitz quantization, it is unambiguous that η corresponds to the creation operator and that $\overline{\eta}$ corresponds to the annihilation operator. So the quantization scheme breaks the symmetry in the roles of η and $\overline{\eta}$ in the pre-quantization space $PG_{l,q}(\eta, \overline{\eta})$. But there is another, quite natural Toeplitz quantization scheme available in this setting. This is the linear map

$$PG_{l,q}(\eta, \overline{\eta}) \to \mathcal{L}(\mathcal{B}_{AH}(\overline{\theta}))$$

defined by multiplying elements in the anti-Segal–Bargmann space $\mathcal{B}_{AH}(\overline{\theta})$ on the right by a fixed element in the symbol space $PG_{l,q}(\eta, \overline{\eta})$ and then projecting back into the anti-Segal–Bargmann space by using its reproducing kernel "function". This gives a theory that is isomorphic (or anti-isomorphic, depending on your definitions) to what we have presented. Next, by applying the above "fingerprint" test we see that this new Toeplitz quantization breaks the symmetry in the roles of η and $\overline{\eta}$ in $PG_{l,q}(\eta, \overline{\eta})$ by sending η to an annihilation operator while

sending $\overline{\eta}$ to a creation operator. And this is the reverse of what happens in our Toeplitz quantization! Actually, naming the two subspaces $\mathcal{B}_H(\theta)$ and $\mathcal{B}_{AH}(\overline{\theta})$ as holomorphic and anti-holomorphic, respectively, is totally arbitrary and is most likely due to a desire to maximize creature comfort more than anything else. In other words, $\overline{\theta}$ is just as good a complex variable as θ and so the names of these two subspaces can be interchanged with no damage to mathematical ideas.

Toeplitz quantization in the original context of Segal–Bargmann analysis is related to the anti-Wick quantization. (See [3].) To see what is happening in the current setting, we take elements $\phi \in \mathcal{B}_H(\theta)$ and $\psi \in \mathcal{B}_{AH}(\overline{\theta})$. Using $T_g = P_K M_g$ and $M_{g_1 g_2} = M_{g_2} M_{g_1}$, we get another multiplicative relation:

$$T_{\phi\psi} = P_K M_{\phi\psi} = P_K M_\psi M_\phi = P_K M_\psi P_K M_\phi = T_\psi T_\phi.$$

The second P_K in the fourth expression acts as the identity since the last factor M_ϕ leaves $\mathcal{B}_H(\theta)$ invariant. In particular, on a basis element in AW we get

$$(2.6) \qquad\qquad T_{\eta^i \overline{\eta}^j} = T_{\overline{\eta}^j} T_{\eta^i} = (T_{\overline{\eta}})^j (T_\eta)^i = (A_w)^j (A_w^\dagger)^i,$$

which is in anti-Wick order. So for an arbitrary symbol g we use linearity to write T_g as a linear combination of terms in anti-Wick order. We can change this Toeplitz quantization by using the left multiplication operator instead of the right. Also we can use the anti-Segal–Bargmann space as noted above, and in this case we have two choices for the multiplication operator as well. In all of these other Toeplitz type quantizations, we get an anti-Wick expression of the form in (2.6). Moreover, we have this interesting consequence.

THEOREM 2.16. *The set* $\{(A_w)^j (A_w^\dagger)^i : i, j \in I_l\}$ *of anti-Wick ordered elements is a basis of* $\mathcal{L}(\mathcal{B}_H(\theta))$.

Proof. By equation (2.6) this set of linear maps is the image under the vector space isomorphism T of the basis AW. ∎

REMARK 2.17. Since we lack simple commutation relations for the creation and annihilation operators, we do not make any statement for now about whether the set $\{(A_w^\dagger)^i (A_w)^j : i, j \in I_l\}$ of Wick ordered elements is also a basis. This is a most curious situation which merits further consideration.

The *number operator* N_w in this context is defined to be

$$N_w := A_w^\dagger A_w = M_\theta \partial_w = T_\eta T_{\overline{\eta}} = T_\eta T_\eta^*.$$

Clearly, $N_w \geqslant 0$. Of course, $A_w^\dagger, A_w, N_w$ are operators in the Hilbert space $\mathcal{B}_H(\theta)$. We have shown in (2.5) that the basis θ^a with $a \in I_l$ diagonalizes the operator N_w and that its spectrum is $\{w_a / w_{a-1} : a \in I_l\}$, where $w_0 / w_{-1} \equiv 0$. N_w serves as a Hamiltonian operator in this theory. One can think of N_w as a deformed harmonic oscillator Hamiltonian, possibly up to an additive constant. It defines a *Dirichlet form* $\langle f, N_w f \rangle_w = \|A_w f\|_w^2$ for all $f \in \mathcal{B}_H$. We can define $[a]_w := w_a / w_{a-1}$, the *w-deformation* of the integer $a \in I_l$. A *w*-deformation of the factorial function can

also be introduced. Note that in this formulation, commutation relations for the operators $A_w^\dagger, A_w, N_w$ do not necessarily play a role. The material in this paragraph gives some notational concordance with other papers.

3. CONCLUDING REMARKS

We consider the theory of Toeplitz operators developed in this paper to be just the beginning of a longer story that should eventually include results for infinite dimensional quantum spaces with a reproducing kernel. This will allow one to consider other properties that are not relevant in finite dimensions, such as whether the Toeplitz operator for a given symbol is bounded, compact, in a Schatten class and so forth.

The "functions" in this paper are not functions, but elements in an algebra. We do think of these elements as arising from a functional calculus, of course. Here the algebras of "functions" is $PG_{q,l}$, which is not commutative (except in the case $q = 1$). So the Toeplitz quantization $g \mapsto T_g$ for $g \in PG_{q,l}$ is analogous to *second quantization* in physics, where one quantizes a theory that is itself a quantum (that is, non-commutative) theory to begin with. We also expect that other second quantizations of interest in mathematics and physics can be analyzed along the lines indicated in this paper. In this regard, see the more recent work of the author in [10].

We have only considered in this paper the case of one pair of complex paragrassmann variables. In [1] the generalization to a finite family of such pairs is given. It is plausible to conjecture that their holomorphic subspaces have reproducing kernels and associated Toeplitz operators in that case, too. Also we have a reproducing kernel for $PG_{q,l}(\theta, \bar{\theta})$ (see [9]), and this space can be embedded into any space with a finite family of complex paragrassmann variables. So there could very well be a theory of Toeplitz operators acting on $PG_{q,l}(\theta, \bar{\theta})$ for such embeddings.

The nilpotency conditions gave us a finite dimensional space. But it seems reasonable that Toeplitz operators could also be defined on the *quantum plane* defined by $\mathbb{C}\{\theta, \bar{\theta}\} / \langle \theta\bar{\theta} - q\bar{\theta}\theta \rangle$. This is a well known and studied object in non-commutative geometry, but this would be way of viewing it from the perspective of analysis. After this paper was completed, the author has done some research in this direction in [8]. The motivation for this paper as well as for part I (see [9]) is to introduce ideas from analysis into the study of non-commutative spaces. We expect that the ideas of reproducing kernels and their associated Toeplitz operators, as well as other aspects of analysis, will find more applications in non-commutative geometry.

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