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ON MULTI-HYPERCYCLIC ABELIAN SEMIGROUPS OF MATRICES ON $\mathbb{R}^n$

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ABSTRACT. Let G be an abelian semigroup of matrices on $\mathbb{R}^n$ ($n \geqslant 1$). We show that G is multi-hypercyclic if and only if it has a somewhere dense orbit. We also give a necessary and sufficient condition for a multi-hypercyclic semigroup G to be hypercyclic, in terms of the index of G corresponding to negative eigenvalues of elements of G. On the other hand, we prove that the closure $\overline{G(u)}$ of a somewhere dense orbit $G(u)$, $u \in \mathbb{R}^n$, is invariant under multiplication by positive scalars; this answer a question raised by Feldman. We also prove that G^k is multi-hypercyclic for every $k \in \mathbb{N}^p$, ($p \in \mathbb{N}$) whenever G is multi-hypercyclic.

KEYWORDS: *Hypercyclic, matrices, multi-hypercyclic, dense orbit, semigroup, abelian.*

MSC (2010): 47A16.

1. INTRODUCTION

Let $M_n(\mathbb{R})$ be the set of all square matrices over $\mathbb{R}$ of order $n \geqslant 1$ and $\mathrm{GL}(n,\mathbb{R})$ the group of invertible matrices of $M_n(\mathbb{R})$. Let G be an abelian subsemigroup of $M_n(\mathbb{R})$. For a vector $v \in \mathbb{R}^n$, we consider the orbit of v through G: $G(v) = \{Av : A \in G\} \subset \mathbb{R}^n$. A subset $E \subset \mathbb{R}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $G(v) \subset \mathbb{R}^n$ is *dense* (respectively *somewhere dense*) in $\mathbb{R}^n$ if $\overline{G(v)} = \mathbb{R}^n$ (respectively $\mathring{\overline{G(v)}} \neq \emptyset$), where $\overline{E}$ (respectively $\mathring{E}$) denotes the closure of a subset $E \subset \mathbb{R}^n$ (respectively the interior of a subset E). The semigroup G is called *hypercyclic* if there exists a vector $v \in \mathbb{R}^n$ such that $G(v)$ is dense in $\mathbb{R}^n$. We say that G is *multi-hypercyclic* if there exist vectors $v_1, \dots, v_p \in \mathbb{R}^n$ such that the union $G(v_1) \cup \cdots \cup G(v_p)$ is dense in $\mathbb{R}^n$. We refer the reader to the recent papers ([1], [3], [4], [8], [9], [12], [14]), [7], [11], [13] and books ([6], [10]) for a thorough account on hypercyclicity and multi-hypercyclicity. Herrero [11] conjectured that every multi-hypercyclic operator on a Hilbert space is in fact hypercyclic. This conjecture was verified by Costakis [7] and later independently by Peris [13]. The same conjecture is extended to finitely generated abelian sub-semigroups of $M_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) ($n \geqslant 1$) by Feldman [9] and Javaheri [12]. In this direction, Feldman ([9], Corollary 5.8) proved that every multi-hypercyclic finitely generated abelian sub-semigroup of $M_n(\mathbb{C})$ $n \geqslant 1$ is hypercyclic.

In the real case, the situation is different. In this article we settle this conjecture for abelian sub-semigroups of $M_n(\mathbb{R})$. We give a complete characterization of such questions for the abelian case. On the other hand we give further results on hypercyclicity, in particular, we answer a question of Feldman in [9], question (7).

To state our main results, we need to introduce the following notations and definitions for the sequel. Write $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$. Let $n \in \mathbb{N}_0$ be fixed. For each $m = 1, 2, \dots, n$, denote by:

(i) $\mathbb{T}_m(\mathbb{R})$ the set of matrices over $\mathbb{R}$ of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix}; \tag{1.1}$$

$\mathbb{T}_n^+(\mathbb{R})$ the group of matrices over $\mathbb{R}$ of the form (1.1) with $\mu > 0$.

(ii) $\mathbb{S}$ the set of matrices of $M_2(\mathbb{R})$ of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}.$$

For each $1 \leqslant m \leqslant n/2$, denote by

(iii) $\mathbb{B}_m(\mathbb{R})$ the set of matrices of $M_{2m}(\mathbb{R})$ of the form

$$\begin{bmatrix} C & & & 0 \\ C_{2,1} & C & & \\ \vdots & \ddots & \ddots & \\ C_{m,1} & \dots & C_{m,m-1} & C \end{bmatrix} : C, C_{i,j} \in \mathbb{S}, 2 \leqslant i \leqslant m, 1 \leqslant j \leqslant m-1. \tag{1.2}$$

(iv) $\mathbb{B}_m^*(\mathbb{R}) := \mathbb{B}_m(\mathbb{R}) \cap \mathrm{GL}(2m, \mathbb{R})$ the group of matrices over $\mathbb{R}$ of the form (1.2) with C invertible.

Let $r, s \in \mathbb{N}$ and

$$\eta = \begin{cases} (n_1, \dots, n_r; m_1, \dots, m_s) & \text{if } rs \neq 0, \\ (m_1, \dots, m_s) & \text{if } r = 0, \\ (n_1, \dots, n_r) & \text{if } s = 0, \end{cases}$$

be a sequence of positive integers such that

$$(n_1 + \dots + n_r) + 2(m_1 + \dots + m_s) = n. \tag{1.3}$$

In particular, we have $r + 2s \leqslant n$. Denote by

(v) $\mathcal{K}_{\eta,r,s}(\mathbb{R}) := \mathbb{T}_{n_1}(\mathbb{R}) \oplus \cdots \oplus \mathbb{T}_{n_r}(\mathbb{R}) \oplus \mathbb{B}_{m_1}(\mathbb{R}) \oplus \cdots \oplus \mathbb{B}_{m_s}(\mathbb{R})$.

In particular:

(a) If $r = 1, s = 0$, then $\mathcal{K}_{\eta,1,0}(\mathbb{R}) = \mathbb{T}_n(\mathbb{R})$ and $\eta = (n)$.

(b) If $r = 0, s = 1$, then $\mathcal{K}_{\eta,0,1}(\mathbb{R}) = \mathbb{B}_m(\mathbb{R})$ and $\eta = (m)$, $n = 2m$.

(c) If $r = 0, s > 1$, then $\mathcal{K}_{\eta,0,s}(\mathbb{R}) = \mathbb{B}_{m_1}(\mathbb{R}) \oplus \cdots \oplus \mathbb{B}_{m_s}(\mathbb{R})$ and $\eta = (m_1, \dots, m_s)$.

(vi) $\mathcal{K}^*_{\eta,r,s}(\mathbb{R}) := \mathcal{K}_{\eta,r,s}(\mathbb{R}) \cap \mathrm{GL}(n, \mathbb{R})$.

(vii) $\mathcal{K}^+_{\eta,r,s}(\mathbb{R}) := \mathbb{T}^+_{n_1}(\mathbb{R}) \oplus \cdots \oplus \mathbb{T}^+_{n_r}(\mathbb{R}) \oplus \mathbb{B}^*_{m_1}(\mathbb{R}) \oplus \cdots \oplus \mathbb{B}^*_{m_s}(\mathbb{R})$.

PROPOSITION 1.1 ([5]). *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. Then there exists a $P \in GL(n, \mathbb{R})$ such that $P^{-1}GP$ is an abelian sub-semigroup of $\mathcal{K}_{\eta,r,s}(\mathbb{R})$, where $\eta = (n_1, \dots, n_r;\ m_1, \dots, m_s) \in \mathbb{N}_0^{r+s}$ and $r, s \in \mathbb{N}$.*

Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ and denote by $G^* = G \cap \mathrm{GL}(n, \mathbb{R})$, it is a sub-semigroup of $\mathrm{GL}(n, \mathbb{R})$. We call $P^{-1}GP$ the *normal form* of G. For such a choice of matrix P, we let:

For every $M \in G^*$, one can write $\widetilde{M} := P^{-1}MP = \mathrm{diag}(A_1, \dots, A_r; B_1, \dots, B_s) \in \mathcal{K}^*_{\eta,r,s}(\mathbb{R})$. Set $\widetilde{G}^* = P^{-1}G^*P$. Let μ_k be the eigenvalue of A_k, $k = 1, \dots, r$, and define the *index* of $\widetilde{G}^*$ to be

$$\mathrm{ind}(\widetilde{G}^*) := \begin{cases} 0 & \text{if } r = 0, \\ \begin{cases} 1 & \text{if exists } \widetilde{M} \in \widetilde{G}^* \text{ with } \mu_1 < 0, \\ 0 & \text{otherwise,} \end{cases} & \text{if } r = 1, \\ \mathrm{card}\{k \in \{1, \dots, r\} : \exists \widetilde{M} \in \widetilde{G}^* \text{ with } \mu_k < 0, \mu_i > 0, \forall i \neq k\} & \text{if } r \notin \{0, 1\}. \end{cases}$$

where $\mathrm{card}(E)$ denotes the number of elements of a subset E of $\mathbb{N}$. In particular,

(i) if $\widetilde{G}^* \subset \mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ with $r \neq 0$ then $\mathrm{ind}(\widetilde{G}) = 0$;

(ii) if $\widetilde{G}^* \subset \mathbb{B}^*_m(\mathbb{R})$, then $\mathrm{ind}(\widetilde{G}) = 0$ (since $r = 0$).

We define the *index* of G to be $\mathrm{ind}(G) := \mathrm{ind}(\widetilde{G}^*)$. It is plain that this definition does not depend on P.

Our principal results can now be stated as follows:

THEOREM 1.2. *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, $n \in \mathbb{N}_0$. Then G is multi-hypercyclic if and only if it has a somewhere dense orbit.*

COROLLARY 1.3. *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, $n \in \mathbb{N}_0$ and $P \in \mathrm{GL}(n, \mathbb{R})$ such that $P^{-1}GP \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$. Assume that G is multi-hypercyclic. Then G is hypercyclic if and only if* $\mathrm{ind}(G) = r$.

COROLLARY 1.4. *Let G be an abelian sub-semigroup of $\mathbb{B}_n(\mathbb{R})$ $(n \in \mathbb{N}_0)$. Then G is multi-hypercyclic if and only if it is hypercyclic.*

THEOREM 1.5. *For every $r, s \in \mathbb{N}_0$ and $1 \leqslant q \leqslant r$, there exists an abelian sub-semigroup of $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ generated by $(n-s+1)$ matrices, which is 2^q-hypercyclic but not hypercyclic.*

COROLLARY 1.6. *For every $n \in \mathbb{N}_0$, there exists an abelian sub-semigroup of $GL(n,\mathbb{R})$ generated by $(n+1)$ diagonal matrices, which is 2^n-hypercyclic but not hypercyclic.*

Note that Feldman [9] showed that there exists a semigroup generated by $2n$ matrices of $\mathbb{R}^n$ which is 2^n-hypercyclic but not hypercyclic.

COROLLARY 1.7. *$\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$, $r \geqslant 1$, is 2^r-hypercyclic but not hypercyclic.*

On the other hand, in [2], Ansari proved that if a linear operator T on a locally convex space is hypercyclic then T^k is also hypercyclic for every $k \geqslant 1$. It is there natural to ask if a similar result holds for a semigroup G. Recall that for $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$, we denote by G^k the semigroup defined by

$$G^k = \{A_1^{k_1} \dots . A_p^{k_p} : A_1, \dots, A_p \in G\}.$$

Feldman showed ([9], Corollary 5.8) that if an abelian finitely generated semigroup G of matrices over $\mathbb{C}$ is hypercyclic then for any $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$, G^k is also hypercyclic. It is not always the case in the real case. Here for an abelian semigroup G of matrices over $\mathbb{R}$, we prove the following results:

THEOREM 1.8. *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, $n \in \mathbb{N}_0$.*

(i) *If G is multi-hypercyclic, then G^k is also multi-hypercyclic for any $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$.*

(ii) *If for some $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$, G^k is multi-hypercyclic, then G is also multi-hypercyclic.*

COROLLARY 1.9. *If G has a somewhere dense orbit then G^k has also a somewhere dense orbit for any $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$.*

COROLLARY 1.10. *Let $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$. Assume that G is hypercyclic. Then G^k is hypercyclic if and only if* $\mathrm{ind}(G^k) = \mathrm{ind}(G)$.

COROLLARY 1.11. *If G is hypercyclic, then G^k is hypercyclic for any p-tuple of odd integers $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$.*

This paper is organized as follows: In Section 2 we recall some results on hypercyclicity. Section 3 is devoted to the proof of Theorem 1.2, Corollaries 1.3 and 1.4. In Section 4 we prove Theorem 1.5, Corollaries 1.6 and 1.7. In Section 5, we prove Theorem 1.8, Corollaries 1.9, 1.10 and 1.11. In Section 6 we give some others results of independent interest, in particular we answer the question (7) of Feldman [9] for the space $\mathbb{R}^n$.

2. SOME RESULTS

Throughout the paper, we denote by $\mathcal{B}_0 = (e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$. Denote by:

(i) v^{T} the transpose of a vector $v \in \mathbb{R}^n$.

(ii) I_n the identity matrix on $\mathbb{R}^n$.

(iii) $u_0 = [e_{1,1}, \dots, e_{r,1}; f_{1,1}, \dots, f_{s,1}]^{\mathrm{T}} \in \mathbb{R}^n$ where $e_{k,1} = [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^{n_k}$, $f_{l,1} = [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^{2m_l}$, $k = 1, \dots, r; l = 1, \dots, s$.

Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ and $P \in \mathrm{GL}(n, \mathbb{R})$ such that $P^{-1}GP \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$. Denote by:

(iv) $v_0 = Pu_0$.

(v) $\mathrm{g} := \exp^{-1}(G) \cap [P\mathcal{K}_{\eta,r,s}(\mathbb{R})P^{-1}]$.

(vi) $\mathrm{g}_u := \{Bu : B \in \mathrm{g}\}, u \in \mathbb{R}^n$.

In particular when $G \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$, $\mathrm{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r,s}(\mathbb{R})$.

Recall the following results that have been proved.

THEOREM 2.1 ([5]). *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. The following properties are equivalent:*

(i) *G has a somewhere dense orbit.*

(ii) *$G(v_0)$ is somewhere dense in $\mathbb{R}^n$.*

(iii) *g_{v_0} is an additive sub-semigroup, dense in $\mathbb{R}^n$.*

THEOREM 2.2 ([5], Corollary 1.2). *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ and $P \in \mathrm{GL}(n, \mathbb{R})$ such that $P^{-1}GP \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$ for some $0 \leqslant r, s \leqslant n$. The following properties are equivalent:*

(i) *G is hypercyclic.*

(ii) *$G(v_0)$ is dense in $\mathbb{R}^n$.*

(iii) *g_{v_0} is an additive sub-semigroup dense in $\mathbb{R}^n$ and* $\mathrm{ind}(G) = r$.

3. PROOF OF THEOREM 1.2, COROLLARIES 1.3 AND 1.4

We let

$$U := \prod_{k=1}^{r} (\mathbb{R}^* \times \mathbb{R}^{n_k - 1}) \times \prod_{l=1}^{s} ((\mathbb{R}^2 \backslash \{(0,0)\}) \times \mathbb{R}^{2m_l - 2}),$$

$$C_{u_0} = \prod_{k=1}^{r} (\mathbb{R}^*_+ \times \mathbb{R}^{n_k - 1}) \times \prod_{l=1}^{s} ((\mathbb{R}^2 \backslash \{(0,0)\}) \times \mathbb{R}^{2m_l - 2}),$$

where u_0 is defined in (iii) in the beging of the section.

It is plain that U is open and dense in $\mathbb{R}^n$ and that C_{u_0} is the connected component of U through u_0.

Denote by:

(i) Γ the subgroup of $\mathcal{K}^*_{\eta,r,s}(\mathbb{R})$ generated by $(S_k)_{1\leqslant k\leqslant r}$ given by

$$S_k := \mathrm{diag}(\varepsilon_{1,k}I_{n_1},\dots,\varepsilon_{r,k}I_{n_r}; I_{2m_1},\dots,I_{2m_s}) \in \mathcal{K}^*_{\eta,r,s}(\mathbb{R})$$

where

$$\varepsilon_{i,k} := \begin{cases} -1 & \text{if } i = k, \\ 1 & \text{if } i \neq k, 1 \leqslant i,k \leqslant r. \end{cases}$$

It is plain that $\mathrm{card}(\Gamma) = 2^r$. The following lemma is easy to check.

LEMMA 3.1. *Under the notation above, we have:*
(i) $S_kM = MS_k$, *for every* $M \in \mathcal{K}_{\eta,r,s}(\mathbb{R})$, $k = 1,\dots,r$.
(ii) $U = \bigcup_{S\in\Gamma} S(\mathcal{C}_{u_0})$ *and the* $S(\mathcal{C}_{u_0})$, $S \in \Gamma$ *are pairwise disjoint.*
(iii) $S(\mathcal{C}_{u_0})$, $S \in \Gamma$ *are the connected components of* U.

LEMMA 3.2. *Let* G *be an abelian sub-semigroup of* $\mathcal{K}_{\eta,r,s}(\mathbb{R})$, $n \in \mathbb{N}_0$. *Then* $\mathrm{ind}(G) = r$ *if and only if* $G(u_0)$ *meets all connected components of* U.

Proof. If $\mathrm{ind}(G) = r$ then for every $k = 1,\dots,r$ there exists $M^{(k)} \in G^*$ such that $\mu_{k,k} < 0$ and $\mu_{k,j} > 0$ if $j \neq k$, where $\mu_{k,j}$ is the eigenvalue of the j^{th} bloc of $M^{(k)}$. It follows that $S_kM^{(k)} \in \mathcal{K}^+_{\eta,r,s}(\mathbb{R})$. Let $S \in \Gamma$. As $S_k^{-1} = S_k$, $k = 1,\dots,r$, one can write $S = (S_1)^{p_1}\cdots(S_r)^{p_r} \in \Gamma$ with $p_1,\dots,p_r \in \mathbb{N}$. Set $M = (M^{(1)})^{p_1}\cdots(M^{(r)})^{p_r}$, then $M \in G^*$ and by Lemma 3.1(i), $SM = (S_1M^{(1)})^{p_1}\cdots(S_rM^{(r)})^{p_r} \in \mathcal{K}^+_{\eta,r,s}(\mathbb{R})$, so $SMu_0 \in \mathcal{C}_{u_0}$. As $S^{-1} = S$, thus $Mu_0 \in S(\mathcal{C}_{u_0})$. By Lemma 3.1(iii), it follows that every connected component of U meets $G(u_0)$. Conversely, assume that for every $k = 1,\dots,r$, the orbit $G(u_0)$ meets $S_k(\mathcal{C}_{u_0})$, so there is $M^{(k)} \in G$ such that $M^{(k)}u_0 \in S_k(\mathcal{C}_{u_0})$. Then $S_kM^{(k)}u_0 \in \mathcal{C}_{u_0}$, so $S_kM^{(k)} \in \mathcal{K}^+_{\eta,r,s}(\mathbb{R})$. It follows that for every $k = 1,\dots,r$, $M^{(k)} \in G^*$ with $\mu_{k,k} < 0$ and $\mu_{k,j} > 0$ if $j \neq k$, where $\mu_{k,j}$ is the eigenvalue of the j^{th} bloc of $M^{(k)}$. Therefore $\mathrm{ind}(G) = r$. ∎

LEMMA 3.3 ([5], Lemma 3.8). *Let* G *be an abelian sub-semigroup of* $\mathcal{K}_{\eta,r,s}(\mathbb{R})$, $n \in \mathbb{N}_0$. *If* $\overset{\circ}{\overline{G(u_0)}} \neq \emptyset$ *then* $\overline{G(u_0)} \cap \mathcal{C}_{u_0} = \mathcal{C}_{u_0}$.

Proof of Theorem 1.2. If G is multi-hypercyclic so there exist vectors $v_1,\dots,v_p \in \mathbb{R}^n$ such that the union $\bigcup_{1\leqslant i\leqslant p} G(v_i)$ is dense in $\mathbb{R}^n$. Hence there is some v_i such that $\overset{\circ}{\overline{G(v_i)}} \neq \emptyset$, that is $G(v_i)$ is somewhere dense. Conversely, suppose that G has a somewhere dense orbit $G(u)$ for some $u \in \mathbb{R}^n$. One can assume by Proposition 1.1 that $G \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$. By Theorem 2.1, $G(u_0)$ is somewhere dense. By Lemmas 3.1 and 3.3, it follows that:

$$U = \bigcup_{S\in\Gamma} S(\mathcal{C}_{u_0}) \subset \bigcup_{S\in\Gamma} S(\overline{G(u_0)}) \subset \bigcup_{S\in\Gamma} \overline{G(Su_0)}.$$

Since $\overline{U} = \mathbb{R}^n$ and Γ is finite, $\bigcup_{S\in\Gamma} \overline{G(Su_0)} = \mathbb{R}^n$ and so G is multi-hypercyclic. In fact, G is 2^r-hypercyclic since $\mathrm{card}(\Gamma) = 2^r$. ∎

Proof of Corollary 1.3. If G is multi-hypercyclic and $\mathrm{ind}(G) = r$ then by Theorem 1.2, G has a somewhere dense orbit and so G is hypercyclic by Theorems 2.1 and 2.2. Conversely, if G is hypercyclic then $\mathrm{ind}(G) = r$ by Theorem 2.2. ∎

Proof of Corollary 1.4. This follows from Corollary 1.3 since in this case $r = 0$ and $\mathrm{ind}(G) = 0$. ∎

4. ABELIAN SEMIGROUPS THAT ARE MULTI-HYPERCYCLIC BUT NOT HYPERCYCLIC

We need the following lemma:

LEMMA 4.1 ([5], Lemma 5.3). *Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r,s}(\mathbb{R})$. Then $\overset{\circ}{\overline{G(u_0)}} \neq \emptyset$ if and only if $\overset{\circ}{\overline{G^2(u_0)}} \neq \emptyset$ where $G^2 = \{A^2 : A \in G\}$.*

Proof of Theorem 1.5. In Theorem 1.7 of [5] we constructed for every $n \in \mathbb{N}_0$ and $1 \leqslant r, s \leqslant n$, a hypercyclic abelian sub-semigroup G_0 of $\mathcal{K}^*_{\eta,r,s}(\mathbb{R})$ generated by $p = n - s + 1$ matrices. Hence by Theorem 2.1, $\overline{G_0(u_0)} = \mathbb{R}^n$ and by Lemma 4.1, $\overset{\circ}{\overline{G_0^2(u_0)}} \neq \emptyset$. Set $G = G_0^2$. Then G is a sub-semigroup of $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ having a somewhere dense orbit $G(u_0)$. Let $A_1, \dots, A_p$ generate G. Write $A_k = \mathrm{diag}(A_{k,1}, \dots, A_{k,r}, \widetilde{A}_{k,1}, \dots, \widetilde{A}_{k,s})$ $(1 \leqslant k \leqslant p)$ where $A_{k,i} \in \mathbb{T}^+_{n_i}(\mathbb{R})$ and $\widetilde{A}_{k,j} \in \mathbb{B}^*_{m_j}(\mathbb{R})$. For $1 \leqslant q \leqslant r$, denote by

$$B_k = \begin{cases} A_k & \text{if } k \in \{1, \dots, q\} \cup \{r+1, \dots, p\}, \\ S_k A_k & \text{if } q+1 \leqslant k \leqslant r, \end{cases}$$

and consider G_q the abelian semigroup generated by $B_1, \dots, B_p$. Since $S_k^2 = I_n$, one has $G_q^2 = G^2$. By Lemma 4.1, $\overset{\circ}{\overline{G_q^2(u_0)}} \neq \emptyset$ and so $\overset{\circ}{\overline{G_q(u_0)}} \neq \emptyset$. It follows by Lemma 3.3 that

$$\overline{G_q(u_0)} \cap \mathcal{C}_{u_0} = \mathcal{C}_{u_0}. \tag{4.1}$$

For $1 \leqslant k \leqslant p$, write

$$B_k = \mathrm{diag}(B_{k,1}, \dots, B_{k,r}, \widetilde{B}_{k,1}, \dots, \widetilde{B}_{k,s}) \quad \text{and}$$
$$B_k^{(1)} = \mathrm{diag}(B_{k,q+1}, \dots, B_{k,r}, \widetilde{B}_{k,1}, \dots, \widetilde{B}_{k,s}).$$

Denote by $G_q^{(1)}$ the semigroup generated by

$$B_k^{(1)} = \mathrm{diag}(\varepsilon_{k,q+1} A_{k,q+1}, \dots, \varepsilon_{k,r} A_{k,r}, \widetilde{A}_{k,1}, \dots, \widetilde{A}_{k,s}), 1 \leqslant k \leqslant p.$$

Then $G_q^{(1)}$ is an abelian sub-semigroup of $\mathcal{K}^*_{\eta',r-q,s}(\mathbb{R})$ where $\eta' = (n_{q+1}, \dots, n_r; m_1, \dots, m_s)$. Set $m = 2m_1 + \cdots + 2m_s$ and $n' = m + n_{q+1} + \cdots + n_r$. Denote by π_2 the projection on the second factor $\pi_2 : \mathbb{R}^{n-n'} \times \mathbb{R}^{n'} \longrightarrow \mathbb{R}^{n'}; x = (x_1, x_2) \longmapsto x_2$. Set $u_0^{(1)} = \pi_2(u_0)$. One has $G_q^{(1)}(u_0^{(1)}) = \pi_2(G_q(u_0))$. Since π_2 is an open map and $\overset{\circ}{\overline{G_q(u_0)}} \neq \emptyset$, it follows that $\overset{\circ}{\overline{G_q^{(1)}(u_0^{(1)})}} \neq \emptyset$. Moreover $\mathrm{ind}(G_q^{(1)}) = r - q$, so by Corollary 1.3, $G_q^{(1)}$ is hypercyclic and by Theorems 2.1 and 2.2, $\overline{G_q^{(1)}(u_0^{(1)})} = \mathbb{R}^{n'}$. Therefore by (4.1) one has:

$$\overline{G_q(u_0)} \cap \mathcal{C}'_{u_0} = \mathcal{C}'_{u_0} \tag{4.2}$$

where $\mathcal{C}'_{u_0} = \prod_{k=1}^{q} (\mathbb{R}^*_+ \times \mathbb{R}^{n_k - 1}) \times \mathbb{R}^{n'}$. Denote by Γ_q be the group generated by $S_1 \dots, S_q$, so $\mathrm{card}(\Gamma_q) = 2^q$. By Lemma 3.1, $G_q(S(u_0)) = S(G_q(u_0))$ and we have

$$U = \bigcup_{S \in \Gamma} S(\mathcal{C}_{u_0}) \subset \bigcup_{S \in \Gamma_q} S(\mathcal{C}'_{u_0}).$$

Then by (4.2) one has:

$$U \subset \bigcup_{S \in \Gamma_q} S(\mathcal{C}'_{u_0}) = \bigcup_{S \in \Gamma_q} S(\overline{G_q(u_0)} \cap \mathcal{C}'_{u_0}) \subset \bigcup_{S \in \Gamma_q} \overline{G_q(S(u_0))}.$$

Since $\overline{U} = \mathbb{R}^n$, $\bigcup_{S \in \Gamma_q} \overline{G_q(S(u_0))} = \mathbb{R}^n$. Hence G_q is 2^q-hypercyclic. As $A_{k,j} \in \mathbb{T}^+_{n_j}(\mathbb{R})$ for every $1 \leqslant j \leqslant q$ then $G_q(u_0) \subset \mathcal{C}'_{u_0}$ and therefore G_q is not hypercyclic. This completes the proof. ∎

Proof of Corollary 1.6. By taking $q = r = n$ in Theorem 1.5, so $s = 0$ and the corollary follows. ∎

Proof of Corollary 1.7. It is plain that the group $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ is not hypercyclic since any connected component of U is invariant by $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$. On the other hand, by Theorem 1.5 there exists an abelian sub-semigroup of $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ which is 2^r-hypercyclic, therefore $\mathcal{K}^+_{\eta,r,s}(\mathbb{R})$ is also 2^r-hypercyclic. ∎

5. PROOF OF THEOREM 1.8, COROLLARIES 1.9, 1.10 AND 1.11

Proof of Theorem 1.8. Let $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$. One can assume that G is an abelian sub-semigroup of $\mathcal{K}_{\eta,r,s}(\mathbb{R})$.

(i) By Theorem 1.2, G has a somewhere dense orbit, then by Theorem 2.1, $\overset{\circ}{\overline{G(u_0)}} \neq \emptyset$ and by Lemma 3.3 $\overline{G(u_0)} \cap \mathcal{C}_{u_0} = \mathcal{C}_{u_0}$. For any multi-index $\ell =$

$(\ell_1, \dots, \ell_p) \in \mathbb{N}^p$ there exists (by applying the division algorithm) $q_i \in \mathbb{N}$ and $0 \leqslant r_i < k_i$ such that $\ell_i = q_i k_i + r_i$, $i = 1, \dots, p$. As G is abelian, thus

$$A_1^{\ell_1} \cdots A_p^{\ell_p} u_0 = (A_1^{k_1})^{q_1} \cdots (A_p^{k_p})^{q_p} A_1^{r_1} \cdots A_p^{r_p} u_0.$$

Hence

$$G(u_0) = \bigcup_{0 \leqslant r_i < k_i} G^k(A_1^{r_1} \cdots A_p^{r_p} u_0).$$

Therefore

$$\bigcup_{0 \leqslant r_i < k_i} \overline{G^k(A_1^{r_1} \cdots A_p^{r_p} u_0)} \cap \mathcal{C}_{u_0} = \mathcal{C}_{u_0}.$$

Since $\mathcal{C}_{u_0}$ is open, we see that $\overset{\circ}{\overline{G^k(A_1^{r_1} \cdots A_p^{r_p} u_0)}} \neq \emptyset$ for some $0 \leqslant r_i < k_i$ where the interior is taken in $\mathcal{C}_{u_0}$, hence in $\mathbb{R}^n$ since $\mathcal{C}_{u_0}$ is open. We conclude that G^k has a somewhere dense orbit.

(ii) If G^k is multi-hypercyclic for some $k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$ then G^k has a somewhere dense orbit $G^k(u)$, $u \in \mathbb{R}^n$. As $G^k(u) \subset G(u)$ then $G(u)$ is somewhere dense. This proves the theorem. ∎

Proof of Corollary 1.9. The proof results from Theorems 1.2 and 1.8. ∎

Proof of Corollary 1.10. One can assume by Proposition 1.1 that G is an abelian sub-semigroup of $\mathcal{K}_{\eta,r,s}(\mathbb{R})$. Then $G^k \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$. If G is hypercyclic then $\operatorname{ind}(G) = r$ (Theorem 2.2) and G^k is multi-hypercyclic by Theorem 1.8. Thus by Corollary 1.3, G^k is hypercyclic if and only if $\operatorname{ind}(G^k) = r$. ∎

Proof of Corollary 1.11. Since $k_1, \dots, k_p$ are odd, $\operatorname{ind}(G^k) = \operatorname{ind}(G)$ and so the fact that G^k is hypercyclic follows from Corollary 1.10. ∎

6. FURTHER RESULTS ON HYPERCYCLICITY AND A QUESTION OF FELDMAN

THEOREM 6.1. *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ and $u \in \mathbb{R}^n$, then $\overline{G(u)} = \mathbb{R}^n$ if and only if $0 \in \overset{\circ}{\overline{G(u)}}$.*

We need the following lemmas.

LEMMA 6.2. *Let G be an abelian sub-semigroup of $\mathcal{K}_{\eta,r,s}(\mathbb{R})$. Then $\mathbb{R}^n \backslash U$ is a union of $r+s$, G-invariant vector subspaces of $\mathbb{R}^n$.*

Proof. One has $\mathbb{R}^n \backslash U = \bigcup\limits_{k=1}^{r} H_k \cup \bigcup\limits_{l=1}^{s} \widetilde{H}_l$ where

$$H_k = \{u = (x_1, \dots, x_r, y_1, \dots, y_s),\ x_k \in \{0\} \times \mathbb{R}^{n_k - 1}\} \quad \text{and}$$
$$\widetilde{H}_l = \{u = (x_1, \dots, x_r, y_1, \dots, y_s),\ y_l \in \{(0,0)\} \times \mathbb{R}^{2m_l - 2}\}.$$

Each vector space H_k (respectively $\widetilde{H}_l$) is G-invariant: indeed, if $u = (x_1, \dots, x_r, y_1, \dots, y_s) \in H_k$ (respectively $u \in \widetilde{H}_l$), so $x_k \in \{0\} \times \mathbb{R}^{n_k - 1}$ (respectively $y_l \in \{(0,0)\} \times \mathbb{R}^{2m_l - 2}$) and hence since $G \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$, it is plain that $G(u) \subset H_k$ (respectively $G(u) \subset \widetilde{H}_l$). ∎

LEMMA 6.3 ([5], Proposition 4.1). *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ and $u \in \mathbb{R}^n$. Then $G(u)$ is a somewhere dense orbit if and only if so is $G^*(u)$.*

Denote by $\mathrm{vect}(G)$ the vector subspace of $M_n(\mathbb{R})$ generated by G.

LEMMA 6.4 ([5], Proposition 3.7). *If G is an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r,s}(\mathbb{R})$ having a somewhere dense orbit $G(u)$ then for every $v \in U$, there exists $B \in \mathrm{vect}(G) \cap \mathrm{GL}(n, \mathbb{R})$ such that $Bu = v$.*

Proof of Theorem 6.1. One can suppose that $G \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$ by Proposition 1.1. Assume that $0 \in \overset{\circ}{\overline{G(u)}}$. By Lemma 6.3, $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$. By Lemma 6.4, there exists $B \in \mathrm{vect}(G^*) \cap \mathrm{GL}(n, \mathbb{R})$ such that $Bu_0 = u$. Therefore $\overline{G(u)} = B\big(\overline{G(u_0)}\big)$ and hence $0 \in \overset{\circ}{\overline{G(u_0)}}$. So there is an open ball $B_{(0,\varepsilon)}$ of radius $\varepsilon > 0$ centered at 0 such that $B_{(0,\varepsilon)} \subset \overset{\circ}{\overline{G(u_0)}}$. Hence $G(u_0)$ meets all connected components of U. So by Lemma 3.2, $\mathrm{ind}(G) = r$ and hence by Theorem 2.2, $\overline{G(u_0)} = \mathbb{R}^n$. It follows that $\overline{G(u)} = \mathbb{R}^n$, this completes the proof. ∎

In [9], Feldman raised open questions, most of them answered (see Shkarin [14]). We are interested here in the seventh problem.

Question (7). If an orbit of a tuple T is somewhere dense, but not dense in a real locally convex space X, then is the closure of the orbit invariant under multiplication by positive scalars?

We answer positively this question that can be dealt with semigroups on $\mathbb{R}^n$.

PROPOSITION 6.5. *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$ having a somewhere dense orbit $G(u)$, $u \in \mathbb{R}^n$. Then for any real $\lambda > 0$, we have $\lambda\overline{G(u)} \subset \overline{G(u)}$, this means that $\overline{G(u)}$ is invariant under multiplication by positive scalars.*

We need the following lemma.

LEMMA 6.6. *If G is an abelian sub-semigroup of $\mathcal{K}_{\eta,r,s}(\mathbb{R})$ having a somewhere dense orbit $G(u)$ ($u \in \mathbb{R}^n$), then $G(u)$ is dense in every connected component of U meeting it.*

Proof. By Lemma 6.3, $G^*(u)$ is somewhere dense and by Theorem 2.1, $G(u_0)$ is somewhere dense. So by Lemma 3.3, $G(u_0)$ is dense in $\mathcal{C}_{u_0}$. Let V be a connected component of U meeting $G(u)$, so there is $v \in V \cap G(u)$. By Lemma 6.4, there exists $B \in \mathrm{vect}(G^*) \cap \mathrm{GL}(n, \mathbb{R})$ such that $Bv = u_0$. So $G(v) = B^{-1}(G(u_0))$. This implies that $G(v)$ is dense in $B^{-1}(\mathcal{C}_{u_0}) = V$. ∎

Proof of Proposition 6.5. The proof is done by induction on $n \geqslant 1$. For $n = 1$, G is a multiplicative semigroup of $\mathbb{R}$. Let $u \in \mathbb{R}$ so that $G(u)$ is somewhere dense in $\mathbb{R}$. Here $U = \mathbb{R}^*$. By Lemma 6.6, $G(u)$ is dense in each connected component of $\mathbb{R}^*$ meeting it; that is $\overline{G(u)} \cap \mathcal{C} = \mathcal{C}$ where $\mathcal{C} = \mathbb{R}^*_+$ or $\mathcal{C} = \mathbb{R}^*_-$. Therefore $\overline{G(u)} \cap \mathbb{R}^* = \mathbb{R}^*$. Let $\lambda > 0$ be real. Since $\lambda\mathcal{C} \subset \mathcal{C}$ we see that $\lambda(\overline{G(u)} \cap \mathbb{R}^*) \subset \overline{G(u)} \cap \mathbb{R}^*$. Hence if $0 \in \overline{G(u)}$ then $\lambda\overline{G(u)} = \lambda(\overline{G(u)} \cap \mathbb{R}^*) \cup \{0\} \subset \overline{G(u)} \cup \{0\} = \overline{G(u)}$. If $0 \notin \overline{G(u)}$ then $\lambda\overline{G(u)} = \lambda(\overline{G(u)} \cap \mathbb{R}^*) \subset \overline{G(u)}$. In either case, $\lambda(\overline{G(u)}) \subset \overline{G(u)}$. Suppose the proposition is true until $n-1$, $(n \geqslant 2)$ and let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. By Proposition 1.1, one can assume that $G \subset \mathcal{K}_{\eta,r,s}(\mathbb{R})$. By Lemma 6.6, $G(u)$ is dense in each connected component of U meeting it. Hence $\overline{G(u)} \cap U = \bigcup_{j=1}^{p} \mathcal{C}_j$ where $\mathcal{C}_j$, $j = 1, \dots, p$ are the connected components of U meeting $G(u)$. Let $\lambda > 0$ be real. Since $\lambda\mathcal{C}_j \subset \mathcal{C}_j$ for $1 \leqslant j \leqslant p$, we see that $\lambda(\overline{G(u)} \cap U) \subset \bigcup_{j=1}^{p} \mathcal{C}_j = \overline{G(u)} \cap U$. By Lemma 6.2, $\mathbb{R}^n \backslash U$ is a union of $r+s$ G-invariant vector subspaces H_k and $\widetilde{H}_l$ of $\mathbb{R}^n$. By applying the induction hypothesis to the restriction of G on each vector space H_k (respectively $\widetilde{H}_l$) of dimension $n-1$ (respectively $n-2$), we get $\lambda(\overline{G(u)} \cap H_k) \subset \overline{G(u)} \cap H_k$ and $\lambda(\overline{G(u)} \cap \widetilde{H}_l) \subset \overline{G(u)} \cap \widetilde{H}_l$: indeed, if $x \in \overline{G(u)} \cap H_k$ then $G(x) \subset H_k$ and $\lambda\overline{G(x)} \subset \overline{G(x)}$, in particular, $\lambda x \in \overline{G(x)} \subset \overline{G(u)} \cap H_k$. We conclude that $\lambda\overline{G(u)} \subset \overline{G(u)}$. The proof is complete. ∎

REMARK 6.7. The Proposition 6.5 fails if G has nowhere dense orbit, as can be shown by taking any semigroup of $\mathbb{S}$ composed of rotations.

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