J. OPERATOR THEORY
71:2(2014), 507–515
doi: 10.7900/jot.2012jul03.1986

AMALGAMATED FREE PRODUCTS OF COMMUTATIVE C^*-ALGEBRAS ARE RESIDUALLY FINITE-DIMENSIONAL

ANTON KORCHAGIN

Communicated by Şerban Strătilă

ABSTRACT. We prove that an amalgamated free product of separable commutative C^*-algebras is residually finite-dimensional.

KEYWORDS: *C^*-algebra, amalgamated product, RFD-property.*

MSC (2010): 46L45.

1. PREMIMINARIES

Recall that a C^*-algebra is residually finite-dimensional (RFD) if it separable and admits an embedding in a direct product of finite-dimensional C^*-algebras. In other terms, a C^*-algebra A is RFD if $\|a\| = \sup \|\varphi(a)\|$ for any $a \in A$, where the supremum is taken over all finite-dimensional representations φ of A. In this paper we prove the RFD property for amalgamated products of commutative C^*-algebra.

At the end of the article we demonstrate an application of this theorem to one interesting example.

Recall that if $\phi_A : C \to A$, $\phi_B : C \to B$ are unital $*$-homomorphisms of unital C^*-algebras then their amalgamated free product (or simply amalgam) $A \underset{C}{\star} B$ is a C^*-algebra with the following properties:

(i) There exist $*$-homomorphisms $\varphi_A : A \to A \underset{C}{\star} B$ and $\varphi_B : B \to A \underset{C}{\star} B$ such that the square

$$\begin{tikzcd} C \arrow[r, "\phi_A"] \arrow[d, "\phi_B"'] & A \arrow[d, "\varphi_A"] \\ B \arrow[r, "\varphi_B"] & A \underset{C}{\star} B \end{tikzcd}$$

commutes;

(ii) For any C^*-algebra D and for any commutative square

$$\begin{array}{ccc} C & \xrightarrow{\varphi_A} & A \\ \Big\downarrow{\scriptstyle \varphi_B} & & \Big\downarrow{\scriptstyle \widetilde{\varphi}_A} \\ B & \xrightarrow{\widetilde{\varphi}_B} & D \end{array}$$

there is a unique $*$-homomorphism $\Phi : A \underset{C}{\star} B \to D$ such that $\Phi \circ \varphi_A = \widetilde{\varphi}_A$ and $\Phi \circ \varphi_B = \widetilde{\varphi}_B$.

Such C^*-algebra exists and is unique up to isomorphism (for information see Section 2.3 of [7]).

We give some examples:

(i) $C(\mathbb{T}) \underset{\mathbb{C}}{\star} C(\mathbb{T}) \cong C^*(\mathbb{F}_2)$, where $C(\mathbb{T})$ is the algebra of continuous functions over the circle $\mathbb{T}$ and $C^*(\mathbb{F}_2)$ is the full group C^*-algebra of a free group on two generators.

(ii) $\mathbb{C}^2 \underset{\mathbb{C}}{\star} \mathbb{C}^2 \cong \mathbb{C}^*(p, q)$ is the universal C^*-algebra generated by two selfadjoint projections without any additional relations ([7], Remark 5.6).

(iii) $A \underset{A}{\star} A \cong A$.

(iv) $A \underset{\mathbb{C}}{\star} \mathbb{C} \cong A$.

(v) $C^*(G) \underset{C^*(H)}{\star} C^*(T) \cong C^*(G \underset{H}{\star} T)$, where $G \underset{H}{\star} T$ is the amalgamated free product of groups ([2], Theorem 4).

Unlike the above examples, most of the amalgams have no explicit description and can be described only by their universality property.

If a separable C^*-algebra can be embedded in a direct product (one can make it countable) of matrix algebras,

$$A \hookrightarrow \prod_k M_{n_k}(\mathbb{C})$$

then we say that A has the RFD property or simply that A is an RFD algebra.

Recall that every RFD algebra has a trace, e.g. in the countable case it can be defined by the formula $\tau = \sum\limits_k \tau_k / 2^k$, where τ_k is the normalized matrix trace on $M_{n_k}(\mathbb{C})$. Non-existence of a trace often is a way to disprove the RFD property, but we do not deal with traces in this paper.

Here are some basic examples:

(i) Finite-dimensional C^*-algebras are all RFD.

(ii) If A and B are RFD algebras then $A \underset{\mathbb{C}}{\star} B$ is an RFD algebra [3].

(iii) If A and B are RFD algebras then $A \oplus B$ is RFD.

(iv) For any compact Hausdorff space X, the algebra $C(X)$ of continuous functions over X is RFD.

(v) The C^*-algebra $\mathbb{K}(\mathbb{H})$ of all compact operators over a separable Hilbert space $\mathbb{H}$ is not an RFD algebra as it has no trace.

(vi) $M_2(\mathbb{C}) \underset{\mathbb{C}^2}{\star} M_3(\mathbb{C})$, where the amalgamation is constructed from the embeddings $\mathbb{C}^2 \hookrightarrow M_2(\mathbb{C}) : (x, y) \mapsto \mathrm{diag}(x, y)$ and $\mathbb{C}^2 \hookrightarrow M_3(\mathbb{C}) : (x, y) \mapsto \mathrm{diag}(x, y, 0)$, is not an RFD algebra because it has no trace ([5], Example 2.1).

Here is the main result of the paper:

THEOREM 1.1. *Let $A, B \supseteq C$ be separable commutative unital C^*-algebras. Then $A \underset{C}{\star} B$ is an* RFD *algebra.*

2. MAIN SECTION

The proof of the theorem will be obtained as a consequence of the following lemmas:

LEMMA 2.1 (Abundance of invariant subspaces). *Let*

$$\varphi : \mathcal{M} = C(X) \underset{C(Z)}{\star} C(Y) \to \mathbb{B}(\mathbb{H})$$

be a unital representation on some Hilbert space $\mathbb{H}$. Then, for every Borel set $\mu \subseteq Z$ we can construct an invariant subspace $\mathbb{H}_\mu \subseteq \mathbb{H}$ with the properties:

(i) $\mathbb{H}_\mu \perp \mathbb{H}_\nu$, *whenever* $\mu \cap \nu = \varnothing$;

(ii) *if* $Z = \coprod\limits_{k=1}^{N} \mu_k$ *then* $\mathbb{H} = \bigoplus\limits_{k=1}^{N} \mathbb{H}_{\mu_k}$;

(iii) $\mathbb{H}_\mu \subseteq \mathbb{H}_\nu$, *whenever* $\mu \subseteq \nu$.

Proof. In Sections 7.3–7.4 of [8] for every Hausdorff compact Z and representation φ the commutative triangle is constructed:

$$\begin{tikzcd} C(Z) \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{B}(\mathbb{H}) \\ \mathcal{B}(Z) \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd}$$

where $\mathcal{B}(Z)$ is the algebra of all bounded complex-valued Borel functions, $\widehat{\varphi}$ is continuous with respect to ω_0-WOT-topology, where WOT- is the weak topology on $\mathbb{B}(\mathbb{H})$ and ω_0 is the weak-measure topology, which is defined by the collection of semi-norms $\|f\|_\alpha = \left| \int\limits_Z f \mathrm{d}\alpha \right|$ parametrized by regular Borel measure α with bounded variation.

It is also known that for bounded sequences ω_0-topology is equivalent to the point-wise convergence topology.

By definition, put

$$\mathbb{H}_\mu = \mathrm{Im}\widehat{\varphi}(\chi_\mu),$$

where χ_μ is a characteristic function of $\mu \subseteq Z$. Properties (i)–(iii) easily follow from similar properties of characteristic functions. To prove that $\mathbb{H}_\mu$ is invariant subspace, we first consider a sequence of continuous functions $\{f_k\}$ such that

$$f_k \xrightarrow{\text{point-wise}} \chi_\mu.$$

The convergence

$$\varphi(f_k) \xrightarrow{\text{WOT}} \widehat{\varphi}(\chi_\mu)$$

easy follows from the ω_0-WOT continuity of the operator $\widehat{\varphi}$. Recall that commutativity of the triangle, which defines the operator $\widehat{\varphi}$, allows us to identify φ and the restriction of $\widehat{\varphi}$ to $C(Z)$. Using universal property of the amalgam, we obtain that $f_k \in C(Z)$ lies in the center of the amalgam $\mathcal{M}$, then $\varphi(f_k)$ commute with all $\varphi(\mathcal{M})$. Thus, for arbitrary $\omega \in \varphi(\mathcal{M})$ we have

$$\omega\widehat{\varphi}(\chi_\mu) = \omega \lim\text{-WOT}\varphi(f_k) = \lim\text{-WOT}\varphi(f_k)\omega = \widehat{\varphi}(\chi_\mu)\omega,$$

which proves the invariance of $\mathbb{H}_\mu$. ∎

LEMMA 2.2 (Topological). *Let K be a metric compact space and let $\nu_k \subseteq K$, $k \in \mathbb{N}$ be compact subsets with the property that for every $n \in \mathbb{N}$ one has*

$$\nu_{n+1} \subseteq \nu_n.$$

Set $\Gamma = \bigcap\limits_{n\in\mathbb{N}} \nu_n$. Then for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n > N$

$$\nu_n \subseteq \mathcal{O}_\varepsilon(\Gamma),$$

i.e. the ε- neighbourhood of Γ contains every ν_n with $n > N$.

The proof is an easy exercise.

Recall that $\mathcal{M} = C(X) \underset{C(Z)}{\star} C(Y)$. By Gelfand theory for commutative algebras there are natural continuous maps $p_X : X \to Z$ and $p_Y : Y \to Z$.

For arbitrary $\nu \subseteq Z$ introduce the notation $\widehat{\nu} = p_X^{-1}(\nu)$ and $\check{\nu} = p_Y^{-1}(\nu)$.

Let $Z \supseteq \nu_1 \supseteq \nu_2 \supseteq \cdots$ be compact sets with $\mathrm{diam}(\nu_n) \to 0$. We have $\bigcap\limits_n \nu_n = \bullet$, where $\bullet$ denotes a point. Set

$$\mathcal{M}_n = C(\widehat{\nu}_n) \underset{C(\nu_n)}{\star} C(\check{\nu}_n), \quad \mathcal{M}_\infty = C(\widehat{\bullet}) \underset{C(\bullet)}{\star} C(\check{\bullet}).$$

Then, for nested compact sets, we can construct the chains

$$C(Z) \to C(\nu_1) \to C(\nu_2) \to \cdots \to C(\bullet),$$
$$C(X) \to C(\widehat{\nu}_1) \to C(\widehat{\nu}_2) \to \cdots \to C(\widehat{\bullet}),$$
$$C(Y) \to C(\check{\nu}_1) \to C(\check{\nu}_2) \to \cdots \to C(\check{\bullet}).$$

Gelfand theory can describe subalgebras of commutative algebras

$$C(Z) = \{f \in C(X) : f|_{p_X^{-1}(z)} = \text{const } \forall z \in Z\}.$$

Due to this characterization, we can lift these homomorphisms to amalgams (homomorphisms are admissible on common subalgebras $C(\nu_n)$, i.e. form necessary commutative squares):

$$\mathcal{M} \xrightarrow{\alpha_0} \mathcal{M}_1 \xrightarrow{\alpha_1} \mathcal{M}_2 \xrightarrow{\alpha_2} \cdots \to \mathcal{M}_\infty.$$

LEMMA 2.3 (Main). *One has*

$$\varinjlim \mathcal{M}_n = \mathcal{M}_\infty.$$

Proof. One can check (using nice commutation properties of our chains) that α_n induce a well-defined homomorphism

$$\varinjlim \mathcal{M}_n \xrightarrow{\alpha_\infty} \mathcal{M}_\infty.$$

Let us construct a homomorphism $\gamma_\infty : \mathcal{M}_\infty \to \varinjlim \mathcal{M}_n$. Since $C(\widehat{\bullet})$ and $C(\check{\bullet})$ generate $\mathcal{M}_\infty$ [4], it suffices to define γ_∞ only on the elements of these subalgebra. For $\omega \in C(\widehat{\bullet})$ (similarly for $C(\check{\bullet})$) set

$$\gamma_\infty(\omega) = \{\Omega|_{\nu_1}, \Omega|_{\nu_2}, \ldots\}$$

where Ω is an arbitrary extension of ω from $\widehat{\bullet}$ to X by Titze–Urysohn theorem.

The map γ_∞ is well-defined, as for another extension Σ of ω, we have

$$(\Omega - \Sigma)|_{\widehat{\bullet}} = 0.$$

Since $\Omega - \Sigma$ is uniformly continuous on the compact Z, then by Lemma 2.2 for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\|(\Omega - \Sigma)|_{\widehat{\nu}_n}\| \leqslant \varepsilon.$$

This means that in $\varinjlim \mathcal{M}_n$ one has the equality

$$\{\Omega|_{\nu_1}, \Omega|_{\nu_2}, \ldots\} = \{\Sigma|_{\nu_1}, \Sigma|_{\nu_2}, \ldots\}.$$

It easy to check that γ_∞ is a homomorphism (à la product of admissible sequences is admissible for product...). Since γ_∞ is unital on $C(\widehat{\bullet})$ (and on its twin $C(\check{\bullet})$) and $C(\bullet) = \mathbb{C}$, then we can extend this homomorphism to $\mathcal{M}_\infty$. It is easy to check that $\gamma_\infty \circ \alpha_\infty = \text{id}$. If we combine this with surjectivity of the α_∞, we obtain that α_∞ is an isomorphism. ∎

We remark that the reader can find this lemma in more general terms in Proposition 4.12 of [7].

Let $\{\mu_1, \ldots, \mu_N\}$ be a finite covering of Z by compact sets. By definition, put

$$P_{\mu_n}\mathcal{M} = C(\widehat{\mu}_n) \underset{C(\mu_n)}{\star} C(\check{\mu}_n).$$

Using Gelfand duality, for an arbitrary compact subspace $\mu_n \subseteq Z$ we can construct two homomorphisms: $\widehat{\gamma}_n : x \mapsto x|_{\widehat{\mu}_n}$ from $C(X)$ to $C(\widehat{\mu}_n)$ and $\check{\gamma}_n : x \mapsto$

$x|_{\check{\mu}_n}$ from $C(Y)$ to $C(\check{\mu}_n)$. By the universal property they have a unique extension to amalgam

$$\gamma_n : \mathcal{M} \to P_{\mu_n}\mathcal{M}.$$

LEMMA 2.4 (Decomposition of an amalgam). *The following map is injective:*

$$\gamma = \prod_{m=1}^{N} \gamma_m : \mathcal{M} \to \prod_{m=1}^{N} P_{\mu_m}\mathcal{M}.$$

Proof. To prove injectivity it suffices to check, for arbitrary $\omega \in \mathcal{M}$, the equality $\|\omega\| = \max_m \|\gamma_m(\omega)\|$. By elementary properties of $*$-homomorphisms we have

$$\max_m \|\gamma_m(\omega)\| \leqslant \|\omega\|.$$

By Gelfand–Naimark theorem we can construct a faithful representation

$$\varphi : \mathcal{M} \hookrightarrow \mathbb{B}(\mathcal{H}).$$

By Lemma 2.1 we can restrict representation on $\mathbb{H}_{\mu_m}$. Let $\varphi_m = P_{\mathbb{H}_{\mu_m}}\varphi$ be this restriction

$$\mathcal{M} \xrightarrow{\varphi_m} \mathbb{B}(\mathbb{H}_{\mu_m}).$$

As $Z = \bigcup_{m=1}^{N} \mu_m$ we obtain by Lemma 2.1 that

$$\mathbb{H} \subseteq \mathbb{H}_{\mu_1} + \cdots + \mathbb{H}_{\mu_N}, \quad \|\omega\| \leqslant \max_m \|\varphi_m(\omega)\|.$$

We can easily find disjoint Borel (also F_σ-type is possible) sets $\widetilde{\mu}_m \subseteq \mu_n$ with the property $Z = \coprod_{m=1}^{N} \widetilde{\mu}_m$, which makes $\mathbb{H}_{\widetilde{\mu}_m}$ be orthogonal by Lemma 2.1. Then by properties of block-diagonal operators one has $\|\varphi(\omega)\| = \max_m \|P_{\mathbb{H}_{\widetilde{\mu}_m}}\varphi(\omega)\|$. Using the embedding $\widetilde{\mu}_m \subseteq \mu_m$, we can easily obtain

$$\|\omega\| = \max_m \|P_{\mathbb{H}_{\widetilde{\mu}_m}}\varphi(\omega)\| \leqslant \max_m \|\varphi_m(\omega)\|.$$

Using the equality $P_{\mathbb{H}_{\mu_m}}\varphi(\widehat{x}) = \widehat{\varphi}(\widehat{x}\chi_{\mu_m})$, we can construct well-defined homomorphism $\widehat{\delta} : C(\widehat{\mu}_m) \to \varphi_m(\mathcal{M})$, which is defined by formula $\widehat{\delta} : x \mapsto P_{\mathbb{H}_{\mu_m}}\varphi(\widehat{x})$, where $\widehat{x}$ is an arbitrary extension of x to X by Titze–Urysohn theorem. This homomorphism with its twin $\check{\delta}$ together determine the homomorphism

$$\delta : P_{\mu_m}\mathcal{M} = C(\widehat{\mu}_m) \underset{C(\mu_m)}{\star} C(\check{\mu}_m) \to \varphi_m(\mathcal{M}).$$

Since φ is injective, then $\varphi_m(x) = 0$ implies $x|_{\widehat{\mu}_m} = 0$. It makes possible to construct the homomorphism $\widehat{\Delta} : \varphi_m(x) \mapsto x|_{\widehat{\mu}_m}$ from $\varphi_m(C(X))$ to $C(\widehat{\mu}_m) \subseteq C(\widehat{\mu}_m) \underset{C(\mu_m)}{\star} C(\check{\mu}_m)$. It is easy to check that $\widehat{\Delta}$ together with its twin $\check{\Delta}$ determine the homomorphism $\Delta : \varphi_m(\mathcal{M}) \to P_{\mathbb{H}_{\mu_m}}\mathcal{M}$. We obviously have $\Delta \circ \delta = \mathrm{id}$. If we combine this with surjectivity of δ, we obtain that Δ is an isomorphism between $P_{\mathbb{H}_{\mu_m}}\mathcal{M}$ and $\varphi_M(\mathcal{M})$.

Finally, equality $\Delta \circ \varphi_m(\omega) = \gamma_m(\omega)$ implies $\|\gamma_m(\omega)\| = \|\varphi_m(\omega)\|$ and $\|\omega\| \leqslant \max\limits_m \|\gamma_m(\omega)\|$. ∎

LEMMA 2.5 (RFD norms). *Let A, B be C^*-algebras, $a \in A$. Set*

$$\|a\|_{\mathrm{RFD}} = \sup_{\substack{\theta \text{ finite} \\ \text{dimemsional} \\ \text{representation}}} \|\theta(a)\|.$$

If $\sigma : A \to B$ is a $*$-homomorphism then $\|\sigma(a)\|_{\mathrm{RFD}} \leqslant \|a\|_{\mathrm{RFD}}$.

The proof is obvious.

By Gelfand–Naimark theorem we have that $A = C(X), B = C(Y), C = C(Z)$, where X, Y and Z are some metric compact spaces. Suppose that $\mathcal{M}$ is not an RFD algebra. Then for some $\varepsilon > 0$ and some $0 \neq \omega \in \mathcal{M}$ we have

$$\|\omega\| \geqslant (1+\varepsilon)\|\omega\|_{\mathrm{RFD}}.$$

Let $\{\mu_1, \dots, \mu_N\}$ be a finite covering of Z by compact sets such that $\mathrm{diam}(\mu_n) \leqslant \mathrm{diam}(Z)/2$.

By Lemmas 2.4, 2.5, we have

$$\|\omega\| = \max_n \|\gamma_n(\omega)\|, \quad \|\omega\|_{\mathrm{RFD}} \geqslant \max_n \|\gamma_n(\omega)\|_{\mathrm{RFD}}.$$

Then for some n_0, we receive

$$\|\omega\| = \|\gamma_{n_0}(\omega)\| \geqslant (1+\varepsilon)\|\gamma_{n_0}(\omega)\|_{\mathrm{RFD}}.$$

Let $\nu_1 = \mu_{n_0}$, $\alpha_0 = \gamma_{n_0}$ and $\mathcal{M}_1 = C(\widehat{\nu}_1) \underset{C(\nu_1)}{\star} C(\check{\nu}_1)$. Now, let us apply this decomposition method to ν_1 in place of Z, namely, let us find compacts μ_k (with corresponding homomorphisms) such that $\nu_1 = \bigcup\limits_k \mu_k$ and $\mathrm{diam}(\mu_k) \leqslant \mathrm{diam}(\nu_1)/2$. Now we can find n_1 such that

$$\|\omega\| = \|(\gamma_{n_1} \circ \alpha_0)(\omega)\| \geqslant (1+\varepsilon)\|(\gamma_{n_1} \circ \alpha_0)(\omega)\|_{\mathrm{RFD}}.$$

Let $\nu_2 = \mu_{n_1}$, $\alpha_1 = \gamma_{n_1}$ and $\mathcal{M}_2 = C(\widehat{\nu}_2) \underset{C(\nu_2)}{\star} C(\check{\nu}_2)$. Then let us apply this decomposition method to ν_2, etc.

Inductively we have

$$\mathcal{M} \xrightarrow{\alpha_0} \mathcal{M}_1 \xrightarrow{\alpha_1} \mathcal{M}_2 \xrightarrow{\alpha_2} \cdots .$$

Using Lemma 2.3, we obtain that $\varinjlim \mathcal{M}_n \cong \mathcal{M}_\infty = C(\widehat{\bullet}) \underset{C(\bullet)}{\star} C(\check{\bullet})$ and

$$\begin{aligned} 0 \neq \|\omega\| = \|\Phi_\infty(\omega)\| = \liminf_n \|\Phi_n(\omega)\| &\geqslant (1+\varepsilon) \liminf_n \|\Phi_n(\omega)\|_{\mathrm{RFD}} \\ &\geqslant (1+\varepsilon)\|\Phi_\infty(\omega)\|_{\mathrm{RFD}}, \end{aligned}$$

where $\Phi_\infty(\omega) = \{\Phi_0(\omega), \Phi_1(\omega), \dots\} \in \varinjlim \mathcal{M}_n$ and $\Phi_n = \alpha_n \circ \cdots \circ \alpha_0$.

The last inequality follows from the existence of the canonical $*$-homomorphism $\mathcal{M}_n \to \mathcal{M}_\infty : \Phi_n(\omega) \mapsto \Phi_\infty(\omega)$ and from Lemma 2.5. Since $C(\bullet)$ is one-dimensional, then $\mathcal{M}_\infty$ is RFD (as a free product of RFD algebras [3]). But this contradicts our inequality $\|\Phi_\infty(\omega)\| \geqslant (1+\varepsilon)\|\Phi_\infty(\omega)\|_{\mathrm{RFD}}$, so our supposition was wrong.

3. CONCLUDING REMARKS

AN EXAMPLE (OF APPLICATION OF THE THEOREM). In [6] the authors consider the universal C^*-algebras $\mathcal{A}_\lambda = \mathbb{C}^*(U, V)$, which are generated by two unitaries U and V with the property

$$\|\mathrm{Re}(U+V)\| \leqslant \lambda, \quad \lambda \in [0,2].$$

They show that

$$\mathcal{A}_0 \cong C(\mathbb{T}) \underset{C(\mathbb{I})}{\star} C(\mathbb{T}),$$

where $C(\mathbb{T})$ is the algebra of continuous functions over the unit circle $\mathbb{T}$ and $C(\mathbb{I})$ is the algebra of continuous functions over the segment $[-1,1]$. We consider $\mathbb{T}$ and $\mathbb{I}$ as subsets on the complex plane. The map $\mathbb{T} \to \mathbb{I}$, $z \mapsto \mathrm{Re}(z)$, defines $*$-homomorphisms of the algebras in the natural way. RFD property for $\mathcal{A}_0$ follows from our theorem. For $\mathcal{A}_2 = \mathbb{C}^*(\mathbb{F}_2)$, RFD property was proved by Choi in [1]. For $\mathcal{A}_\lambda$, where $\lambda \in (0,2)$, RFD property is an open question.

REMARKS. The construction of the commutative triangle

$$\begin{tikzcd} C(X) \arrow[r, "\varphi"] \arrow[d] & \mathbb{B}(\mathbb{H}) \\ \mathcal{B}(X) \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd}$$

can be found in [8]. In [4] we can find a proof of the fact that finite sums of finite products of elements of algebras, which define an amalgam, are dense in it. Manuilov was the first to introduce and study the algebras $\mathcal{A}_\lambda$. In fact the work done in [6] was the main motivation for our result. For other interesting results on this subject the reader is encouraged to consult [5].

We also remark that, using this method, one can prove RFD property for an amalgamated product of many commutative algebras, i.e.

$$A_1 \underset{C}{\star} A_2 \underset{C}{\star} \cdots \underset{C}{\star} A_n,$$

where C is a common subalgebra of the commutative algebras $A_1, A_2, \ldots, A_n$: our considerations about projections, embedding, block-diagonal operators and decompositions do not depend on the number of amalgamated commutative algebras.

Acknowledgements. The author is greatful to V.M. Manuilov for grand help, partial support by the RFBR grant No. 13-01-00232 and by the Russian Government grant No. 11.G34.31.0005. and my Muse Mary for inspiration.

REFERENCES

[1] M.-D. CHOI, The full C^*-algebra of free group on two generators, *Pacific J. Math.* **87**(1980), 41–48.

[2] A.P. DONSIG, S.P. HAATAJA, J.C. MEAKIN, Amalgams of inverse semigroups and C^*-algebras, *Indiana Univ. Math. J.* **60**(2011), 1059–1076; arXiv:1007.1192v1, 2010.

[3] R. EXEL, T.A. LORING, Finite-dimensional representations of free product C^*-algebras, *Internat. J. Math.* **l3**(1992), 469–476.

[4] K.K. JENSEN, K. THOMSEN, *Elements of KK-Theory*, Birkhauser, Berlin 1991.

[5] Q. LI, J. SHEN, Unital full amalgamed free products of MF-algebras, *Oper. Matrices* **7**(2013), 333–355. ; arXiv:1006.2608v1, 2010.

[6] V.M. MANUILOV, C. YOU, On C^*-algebra, related to constrained representation of a free group, *Russ. J. Math. Phys.* **17**(2010), 280–287; arXiv:1005.2608v1, 2010.

[7] G.K. PEDERSEN, Pullback and pushout constructions in C^*-algebra theory, *J. Funct. Anal.* **167**(1999), 243–344.

[8] [Pir] A.YU. PIRKOVSKII, *Spectral Theory and Functional Calculi for Linear Operators*, Moscow, MCCME 2010.

ANTON KORCHAGIN, DEPARTMENT OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW, 119991, RUSSIA
E-mail address: MogilevMedved@yandex.ru

Received July 3, 2012; posted on June 1, 2014.

ADDED IN PROOFS. We also remark that the construction in [3] can be modified for our case. Representation of the common subalgebra is the main difficulty in this construction, but the simple nature of representations of commutative algebras (every finite-dimensional representation is a direct sum of one-dimensional representations) and a decomposition of our representation space in subspaces where the common subalgebra acts by scalar multiplication are keys for the possibility of a complete proof.