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LINEAR ORTHOGONALITY PRESERVERS OF HILBERT C^*-MODULES

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ABSTRACT. We show in this paper that the module structure and the orthogonality structure of a Hilbert C^*-module determine its inner product structure.

Let A be a C^*-algebra, and E and F be Hilbert A-modules. Assume $\Phi : E \to F$ is an A-module map satisfying $\langle \Phi(x), \Phi(y) \rangle_A = 0$ whenever $\langle x, y \rangle_A = 0$. Then Φ is automatically bounded. In case Φ is bijective, E is isomorphic to F.

More precisely, let J_E be the closed two-sided ideal of A generated by the set $\{\langle x, y \rangle_A : x, y \in E\}$. We show that there exists a unique central positive multiplier $u \in M(J_E)_+$ such that $\langle \Phi(x), \Phi(y) \rangle_A = u \langle x, y \rangle_A$ $(x, y \in E)$. As a consequence, the induced map $\Phi_0 : E \to \overline{\Phi(E)}$ is adjointable, and $\overline{Eu^{1/2}}$ is isomorphic to $\overline{\Phi(E)}$ as Hilbert A-modules.

KEYWORDS: *Orthogonality preservers, Hilbert C^*-modules, Uhlhorn theorem, auto continuity.*

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1. INTRODUCTION

It is well known that the norm and the inner product of a (complex) Hilbert space H determine each other, through a polarization formula. By the Uhlhorn theorem (which generalized the famous Wigner theorem; see, e.g., [18]), the orthogonality structure of the projective space of H also determines its Hilbert space structure up to unitary or anti-unitary if $\dim H \geqslant 3$ (see, e.g., Corollary 2.2.2 in [17]). In the case when the (complex) linear structure of the Hilbert space is also considered, one can relax the two-way orthogonality preserving assumption in the Uhlhorn theorem and obtain the following result:

If θ is a bijective $\mathbb{C}$-linear map between Hilbert spaces satisfying

$$\langle \theta(x), \theta(x) \rangle = 0 \quad \textit{whenever } \langle x, y \rangle = 0,$$

then θ is a scalar multiple of a unitary.

It is interesting to ask whether it is possible to generalize the above to the case of Hilbert C^*-modules. Recall that a (right) Hilbert A-module E (where A is a complex C^*-algebra) is a right A-module equipped with an "A-valued inner product" $\langle\cdot,\cdot\rangle_A$ such that E is complete under the norm defined by $\|x\| = \sqrt{\|\langle x,x\rangle_A\|}$ (see e.g. [11] for the precise definition). It is well-known that every surjective A-linear isometry $T : E \to F$ between Hilbert A-modules is a unitary (see e.g. [11]), i.e.,

$$\langle T(x), T(y)\rangle_A = \langle x,y\rangle_A \quad (x,y \in E).$$

In this paper, we will verify that every bijective A-linear orthogonality preserver is an "A-scalar" multiple of a unitary. More precisely, let E and F be Hilbert A-modules. Suppose that J_E is the closed two-sided ideal of A generated by $\{\langle x,y\rangle_A : x,y \in E\}$ and $M(J_E)$ is its multiplier algebra. Our main results in Section 3 can be formulated as follows:

Suppose that $\Phi : E \to F$ is an A-module map, which is not assumed to be bounded. The following are all equivalent:

(i) *Φ is orthogonality preserving, in the sense that*

$$\langle\Phi(x),\Phi(y)\rangle_A = 0 \quad \textit{whenever } \langle x,y\rangle_A = 0 \quad (x,y \in E). \tag{1.1}$$

(ii) *There exists a (unique) positive central element $u \in M(J_E)$ such that*

$$\langle\Phi(x),\Phi(y)\rangle_A = u\langle x,y\rangle_A \quad (x,y \in E). \tag{1.2}$$

(iii) *There exist a (unique) positive central element $w \in M(J_E)$ and a Hilbert A-module isomorphism $\Theta : \overline{Ew} \to \overline{\Phi(E)}$ such that*

$$\Phi(x) = \Theta(xw) \quad (x \in E).$$

In particular, every orthogonality preserving module map Φ between Hilbert A-modules is automatically continuous. In the case when Φ is bijective, $w = u^{1/2}$ is invertible and $x \mapsto \Phi(x)w^{-1}$ is a Hilbert A-module isomorphism from E onto F.

The last statement implies that the A-module structure and the orthogonality structure of E determine the Hilbert A-module E up to a Hilbert A-module automorphism.

The above can be considered as a generalization of the Uhlhorn theorem to Hilbert A-modules, where only one-way orthogonality preserving property is assumed but the A-linear structure is also considered. We would like to emphasize that, to line up with Uhlhorn and Wigner theorems, it is better not to assume any boundedness condition on the map Φ (but expect the boundedness in the conclusion). On the other hand, we are almost forced to take into account the A-module structure. Indeed, Example 1.1 below tells us that the above results will not be true if Φ is only a $\mathbb{C}$-linear map instead of an A-linear map.

EXAMPLE 1.1. The conjugate Hilbert space $\overline{H}$ of a complex Hilbert space H can be regarded as a Hilbert $\mathcal{K}(H)$-module (where $\mathcal{K}(H)$ is the C^*-algebra of all compact operators on H), and for any $\overline{x}, \overline{y} \in \overline{H}$, one has $\langle \overline{x}, \overline{y} \rangle_{\mathcal{K}(H)} = 0$ if and only if either $\overline{x} = 0$ or $\overline{y} = 0$ (recall that $\langle \overline{x}, \overline{y} \rangle_{\mathcal{K}(H)}(z) = y\langle x, z\rangle$ for any $z \in H$).

Orthogonality preservers of C^*-modules have been studied by many authors, e.g., [1], [3], [6], [10], [22]. In the case when A is a standard C^*-algebra, the equivalence of (1.1) and (1.2) was established by D. Ilišević and A. Turnšek [8]. When A is commutative and E is full (i.e. $J_E = A$), this equivalence was established by the authors of this paper in [12]. In [13], we proved this result in the case when A has real rank zero and E is full. Moreover, with the extra assumption on the boundedness of Φ, this was obtained by M. Frank, A.S. Mishchenko and A.A. Pavlov in [7]. Note that the first version of this paper (namely, [14]) was finished and circulated at almost the same time as [7] and the two works are independent. Note also that this version is merely the same as in [14] except that the materials concerning the linking algebras in Section 4 there is removed (see Remark 3.6). It happens that the ideas of the proofs in these papers are very different, and none of them seem to be suitable for the general case.

We also note that J. Schweizer also studied bounded orthogonality preserving map in Theorem 9.6 of [20]. However, there seems to be no overlap between his work and the current paper. For instance, even in the very simple case when $X = \overline{H}$ as in Example 1.1, the result in Theorem 9.6 in [20] gives us merely the trivial conclusion that a bounded orthogonality preserving $\mathbb{C}$-linear map $T : X \to X$ is $\mathbb{C}$-linear. Our main theorem, however, implies that any orthogonality preserving $\mathcal{K}(H)$-module map $T : X \to X$ is a scalar multiple of an isometry. Therefore, Schweizer's result does not seem to shed too much light on the proof of the main theorems in this paper.

2. NOTATIONS AND PRELIMINARIES

Let us first set some notations. Throughout this article, A is a C^*-algebra and A^{**} is the bidual of A (which is a von Neumann algebra). Write A_{sa} and A_+ for the self-adjoint and positive parts of A, and $Z(A)$ and $M(A)$ for the center and the space of multipliers of A, respectively. Moreover, $\mathrm{Proj}_1(A)$ is the collection of all non-zero projections in A.

If $a \in A_+$, we consider $C^*(a)$ to be the C^*-subalgebra generated by a, and let $\mathbf{c}(a)$ be the central cover of a in A^{**} (see e.g. Section 2.6.2 in [19]). If $\alpha, \beta \in \mathbb{R}_+$, we set $e_a(\alpha, \beta)$ and $e_a(\alpha, \beta]$ to be the spectral projections (in A^{**}) of a corresponding respectively, to the sets $(\alpha, \beta) \cap \sigma(a)$ and $(\alpha, \beta] \cap \sigma(a)$. When $(a_\lambda)_{\lambda\in\Lambda}$ is an increasing net (respectively, a decreasing net) in A^{**}_{sa}, the notation $a_\lambda \uparrow a$ (respectively, $a_\gamma \downarrow a$) means that $a_\lambda \to a$ in the weak*-topology. Note that $p \in \mathrm{Proj}_1(A^{**})$ is an open projection exactly when there is an increasing net $(a_\lambda)_{\lambda\in\Lambda}$ from A_+

such that $a_\lambda \uparrow p$. In this case, the C^*-subalgebra $A \cap pA^{**}p$ is weak*-dense in $pA^{**}p$ (see e.g. [2] or Proposition 3.11.9 in [19]).

On the other hand, throughout this article, E and F are non-zero Hilbert A-modules. It is well-known that E is an essential right A-module. Thus, E is unital whenever A is. If A is not unital and A_1 denotes the C^*-algebra obtained by adjoining an identity 1 to A then E becomes a unital Hilbert A_1-module if (and only if) we define $x1 = x$ (cf. page 5 in [11]). On the other hand, for any C^*-subalgebra $B \subseteq A$, we put $EB := \{xb : x \in E; b \in B\}$. By the Cohen factorisation theorem, EB coincides with its norm closed linear span.

For simplicity, we write $\langle x, y\rangle$ instead of $\langle x, y\rangle_A$ when both x and y are in E (or F). Recall that E is said to be *full* if $J_E = A$, where J_E is the closed two-sided ideal of A generated by all the A-valued inner products of elements in E.

Unless specified otherwise, $\Phi : E \to F$ is an orthogonality preserving A-module map (i.e. satisfying (1.1)), but Φ is *not* assumed to be bounded. When A is unital, Φ is an orthogonality preserving A-module map between unital essential Hilbert A-modules, otherwise Φ can be regarded as an orthogonality preserving A_1-module map between unital essential Hilbert A_1-modules.

We now recall the following elementary result (see e.g. [13]).

LEMMA 2.1. *Suppose that $p \in \mathrm{Proj}_1(A^{**})$. If $b \in Z(pA^{**}p)_+$, then $\|\mathbf{c}(b)\| = \|b\|$, $\mathbf{c}(b)p = b$ and $\mathbf{c}(b)\mathbf{c}(p) = \mathbf{c}(b)$.*

In the following lemma, we collect some simple useful facts concerning Hilbert C^*-modules (which are probably known). Recall that E^{**} is a Hilbert A^{**}-module with the module action and the inner product extending the ones in E.

LEMMA 2.2. *Let $p \in \mathrm{Proj}_1(A^{**})$, $\delta \in [0,1)$ and $x \in E \setminus \{0\}$. Set $a := \langle x, x\rangle / \|x\|^2$, $q_\delta := e_a(\delta, 1]$, $q_x := e_a(0,1]$ and $F_\Phi := \overline{\Phi(E)}$.*

(i) *If p is open and $y \in E$ satisfying $\langle x, y\rangle p = 0$, then $\langle \Phi(x), \Phi(y)\rangle p = 0$.*

(ii) *If $v \in A^{**}$ such that $\langle x, x\rangle v \in A$, then $xv \in E$.*

(iii) *If $u, v \in A^{**}$ with $au = av$, then $q_\delta u = q_\delta v$. Thus, $ap = a$ will imply that $q_x \leqslant p$.*

(iv) *$xp = x$ if and only if $a \in pAp$, which is also equivalent to $x \in E \cdot (A \cap pA^{**}p)$.*

(v) *$xq_x = x$ and $\Phi(x)q_x = \Phi(x)$.*

(vi) *$F_\Phi \cdot J_E = F_\Phi$ and $J_{F_\Phi} \subseteq J_E$.*

Proof. In the following, let $(e_n)_{n\in\mathbb{N}}$ be an approximate unit in $C^*(a)$. Notice that $\|xe_n - x\| \to 0$ since $\|x - xe_n\|^2 = \|x\|^2\|a - e_n a - ae_n + e_n a e_n\|$.

(i) Pick any increasing net $(a_\lambda)_{\lambda\in\Lambda}$ in $A_+ \cap pA^{**}p$ with $a_\lambda \uparrow p$ (note that p is open). As $a_\lambda = pa_\lambda$, one has $\langle x, ya_\lambda\rangle = 0$ (for any λ). Thus, $\langle \Phi(x), \Phi(y)\rangle a_\lambda = 0$ (for any λ), and hence $\langle \Phi(x), \Phi(y)\rangle p = 0$.

(ii) As $e_n v \in A$ (by the hypothesis) and $\|xv - xe_n v\|^2_{E^{**}} = \|x\|^2\|v^*(1 - e_n)a(1 - e_n)v\|$, we see that $xv \in E$.

(iii) Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $C^*(a)_+$ such that $b_n \uparrow q_\delta$. As $b_n(u-v)=0$ ($n \in \mathbb{N}$), we see that $q_\delta u = q_\delta v$. By taking $\delta = 0$, we obtain also the second statement.

(iv) If $xp = x$, then $a = pap$. If $a \in pAp$, then $e_n \in pAp$ and $x \in E\cdot(A \cap pA^{**}p)$ (as $\|xe_n - x\| \to 0$). Finally, if $x \in E\cdot(A\cap pA^{**}p)$, then clearly $xp = x$.

(v) As $xe_n = xe_nq_x \to xq_x$ in norm, one has $x = xq_x$. Now, part (iv) implies that $x = zb$ for some $z \in E$ and $b \in A\cap q_xA^{**}q_x$. Thus, $\Phi(x) = \Phi(z)b \in F\cdot(A\cap q_xA^{**}q_x)$, which gives $\Phi(x)q_x = \Phi(x)$.

(vi) As E is a Hilbert J_E-module, any $z \in E$ is of the form $z = yb$ for some $y \in E$ and $b \in J_E$. Thus, $\Phi(E) \subseteq F_\Phi \cdot J_E$. The second statement follows from the first one (as J_E is a closed two-sided ideal of A). ∎

3. THE MAIN RESULTS

We may now start proving our main theorem. We use open projections in our proof. Notice that this proof is not a translation of the one for the real rank zero case in [13] (because most of the techniques used there cannot be carried over to the general case) and it is much more difficult and technical. On the other hand, none of the approaches in [7], [8], [12] seems to work in the general case neither.

LEMMA 3.1. *Suppose that $x \in E\setminus\{0\}$. If $a := \langle x,x\rangle/\|x\|^2$ and $q_x := e_a(0,1]$, there is $u_x \in Z(q_xA^{**}q_x)_+$ such that*

$$\langle\Phi(y),\Phi(x)\rangle = \langle y,x\rangle u_x \quad (y\in E).$$

Proof. Without loss of generality we assume that $\|x\| = 1$ and A is unital. If $\varepsilon \in (0,1)$ and $q_\varepsilon := e_a(\varepsilon,1]$, pick any $b \in C^*(a)_+$ satisfying $q_\varepsilon \leqslant ab \leqslant 1$ and set $x_\varepsilon := xb^{1/2} \in E$. Then we have $\langle x_\varepsilon q_\varepsilon, x_\varepsilon\rangle_{A^{**}} = \langle x_\varepsilon, x_\varepsilon q_\varepsilon\rangle_{A^{**}} = \langle x_\varepsilon q_\varepsilon, x_\varepsilon q_\varepsilon\rangle_{A^{**}} = q_\varepsilon$. Moreover,

$$b^{1/2}q_\varepsilon\Big(b^{1/2}+\frac{q_\varepsilon}{n}\Big)^{-1} \uparrow q_\varepsilon \quad \text{when } n\to\infty. \tag{3.1}$$

Put $u_\varepsilon := \langle\Phi(x_\varepsilon),\Phi(x_\varepsilon)\rangle q_\varepsilon \in Aq_\varepsilon$. Consider $c \in q_\varepsilon A^{**}q_\varepsilon \cap A_+$ to be a norm one element, and set $p := e_c(\alpha,\beta) \in q_\varepsilon A^{**}q_\varepsilon$ for some $\alpha < \beta$ in $\mathbb{R}_+$. Let $b_n \in C^*(c) \subseteq A\cap q_\varepsilon A^{**}q_\varepsilon$ such that $0 \leqslant b_n \uparrow p$ and $b_nb_{n+1} = b_n$ ($n\in\mathbb{N}$). Set $c_n := 1-b_n$, and observe that $1 \geqslant c_n \downarrow (1-p)$, $b_nc_{n+k} = 0$, $b_np = b_n$, and $c_{n+k}(1-p) = 1-p$ ($n,k\in\mathbb{N}$). Since

$$\langle x_\varepsilon b_n, x_\varepsilon c_{n+k}\rangle = b_nq_\varepsilon\langle x_\varepsilon,x_\varepsilon\rangle c_{n+k} = b_nq_\varepsilon c_{n+k} = b_nc_{n+k} = 0,$$

we have $b_nu_\varepsilon c_{n+k} = \langle\Phi(x_\varepsilon b_n),\Phi(x_\varepsilon c_{n+k})\rangle q_\varepsilon = 0$ (by Lemma 2.2(i)). By letting $k\to\infty$ and then $n\to\infty$, we see that $pu_\varepsilon(1-p) = 0$, i.e., $pu_\varepsilon = pu_\varepsilon p$. Similarly, we have $pu_\varepsilon p = u_\varepsilon p$ and so, $pu_\varepsilon = u_\varepsilon p$. As c can be approximated in norm by linear combinations of projections of the form $e_c(\alpha,\beta)$, one concludes that u_ε commutes with an arbitrary element in $A \cap q_\varepsilon A^{**} q_\varepsilon$. Thus, u_ε commutes with elements in $q_\varepsilon A^{**} q_\varepsilon$ (as q_ε is open). In particular, $u_\varepsilon = u_\varepsilon q_\varepsilon = q_\varepsilon u_\varepsilon q_\varepsilon = q_\varepsilon \langle \Phi(x_\varepsilon), \Phi(x_\varepsilon)\rangle q_\varepsilon \in q_\varepsilon A q_\varepsilon$, which means that $u_\varepsilon \in Z(q_\varepsilon A^{**} q_\varepsilon)_+$.

For any $y \in E$, the element $y - x_\varepsilon \langle x_\varepsilon, y\rangle \in E$ is orthogonal to $x_\varepsilon q_\varepsilon \in E^{**}$. By Lemma 2.2(i), we have

$$\langle \Phi(y), \Phi(x_\varepsilon)\rangle q_\varepsilon = \langle y, x_\varepsilon\rangle \langle \Phi(x_\varepsilon), \Phi(x_\varepsilon)\rangle q_\varepsilon = \langle y, x_\varepsilon\rangle u_\varepsilon,$$

which implies that

$$\langle \Phi(y), \Phi(x)\rangle b^{1/2} q_\varepsilon = \langle y, x\rangle u_\varepsilon b^{1/2} q_\varepsilon$$

(because $b^{1/2} q_\varepsilon = q_\varepsilon b^{1/2} q_\varepsilon \in q_\varepsilon A^{**} q_\varepsilon$). Now relation (3.1) tells us that

$$\langle \Phi(y), \Phi(x)\rangle q_\varepsilon = \langle y, x\rangle u_\varepsilon \quad (y \in E). \tag{3.2}$$

If $0 < \delta \leqslant \varepsilon < 1$, we have $q_\varepsilon \leqslant q_\delta$ and $q_\varepsilon A^{**} q_\varepsilon \subseteq q_\delta A^{**} q_\delta$. Hence,

$$a u_\delta q_\varepsilon = \langle x, x\rangle u_\delta q_\varepsilon = \langle \Phi(x), \Phi(x)\rangle q_\delta q_\varepsilon = \langle \Phi(x), \Phi(x)\rangle q_\varepsilon = a u_\varepsilon,$$

and Lemma 2.2(iii) tells us that $u_\delta q_\varepsilon = q_\delta u_\delta q_\varepsilon = q_\delta u_\varepsilon = q_\delta q_\varepsilon u_\varepsilon = u_\varepsilon$. By taking adjoint, we see that u_δ commutes with q_ε, which gives

$$0 \leqslant u_\varepsilon = u_\delta^{1/2} q_\varepsilon u_\delta^{1/2} \leqslant u_\delta \quad (0 < \delta \leqslant \varepsilon < 1). \tag{3.3}$$

Next, we show that $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is a bounded set. Suppose on the contrary that there is a strictly decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\|u_{\varepsilon_n}\| > \|u_{\varepsilon_{n-1}}\| + n^5$ for every $n \in \mathbb{N}$ (see relation (3.3)). Let $b_n, d_n \in C^*(a)_+$ such that $e_a(\varepsilon_{4n-1}, \varepsilon_{4n-2}] \leqslant b_n \leqslant e_a(\varepsilon_{4n}, \varepsilon_{4n-3}]$ ($\leqslant q_{\varepsilon_{4n}}$) and $q_{\varepsilon_{4n}} \leqslant a d_n \leqslant 1$. As $b_n, q_{\varepsilon_{4n-1}}, q_{\varepsilon_{4n-2}} \in q_{\varepsilon_{4n}} A^{**} q_{\varepsilon_{4n}}$ and $u_{\varepsilon_{4n}} \in Z(q_{\varepsilon_{4n}} A^{**} q_{\varepsilon_{4n}})_+$, we see that

$$\|u_{\varepsilon_{4n}} b_n\| \;\geqslant\; \|u_{\varepsilon_{4n}}(q_{\varepsilon_{4n-1}} - q_{\varepsilon_{4n-2}})\| \;=\; \|u_{\varepsilon_{4n-1}} - u_{\varepsilon_{4n-2}}\| \geqslant (4n-1)^5.$$

If $x_n := x b_n^{1/2} d_n^{1/2}$, then $\langle x_n, x_n\rangle = b_n q_{\varepsilon_{4n}} a d_n = b_n$. Moreover, if $m \neq n$, then

$$\langle x_n, x_m\rangle = d_n^{1/2} b_n^{1/2} e_a(\varepsilon_{4n}, \varepsilon_{4n-3}] a e_a(\varepsilon_{4m}, \varepsilon_{4m-3}] b_m^{1/2} d_m^{1/2} = 0$$

(as $(\varepsilon_{4n}, \varepsilon_{4n-3}] \cap (\varepsilon_{4m}, \varepsilon_{4m-3}] = \emptyset$). Let $y := \sum_{n=1}^{\infty} x_n / n^2 \in E$ (note that $\|x_n\|^2 = \|b_n\| \leqslant 1$). For any $m \in \mathbb{N}$, we have $\langle \Phi(y), \Phi(y)\rangle \geqslant \langle \Phi(x_m), \Phi(x_m)\rangle / m^4$ (as Φ preserves orthogonality), and by relation (3.2),

$$\begin{aligned} m^4 \langle \Phi(y), \Phi(y)\rangle \geqslant \langle \Phi(x_m), \Phi(x_m)\rangle &= \langle \Phi(x_m), \Phi(x)\rangle q_{\varepsilon_{4m}} b_m^{1/2} d_m^{1/2} \\ &= \langle x_m, x\rangle u_{\varepsilon_{4m}} b_m^{1/2} d_m^{1/2} = b_m u_{\varepsilon_{4m}} \end{aligned} \tag{3.4}$$

(since $b_m^{1/2} d_m^{1/2} \in q_{\varepsilon_{4m}} A^{**} q_{\varepsilon_{4m}}$ and $u_{\varepsilon_{4m}} \in Z(q_{\varepsilon_{4m}} A^{**} q_{\varepsilon_{4m}})_+$). Consequently,

$$\|\Phi(y)\|^2 \geqslant \frac{(4m-1)^5}{m^4}, \quad \text{for all } m \in \mathbb{N},$$

which is a contradiction.

Now, the bounded sequence $(u_{1/n})_{n\in\mathbb{N}}$ in $(q_x A^{**} q_x)_+$ has a subnet having a weak*-limit $u_x \in (q_x A^{**} q_x)_+$. As $q_{1/n} \uparrow q_x$, we have $\bigcup_{n\in\mathbb{N}} q_{1/n} A^{**} q_{1/n}$ being weak*-dense in $\bigcup_{n\in\mathbb{N}} q_{1/n}A^{**}q_x$ and hence also weak*-dense in $q_xA^{**}q_x$. Thus, $u_x \in Z(q_xA^{**}q_x)_+$ (as $q_{1/m}u_x = u_xq_{1/m} = u_{1/m} \in Z(q_{1/m}A^{**}q_{1/m})$ for any $m \in \mathbb{N}$). By relation (3.2) and Lemma 2.2(v), we have

$$\langle \Phi(y), \Phi(x)\rangle = \langle \Phi(y), \Phi(x)\rangle q_x = \langle y, x\rangle u_x \quad (y \in E). \quad \blacksquare$$

Recall that $J_E \subseteq A$ is the closed two-sided ideal generated by the inner products of elements in E.

THEOREM 3.2. *Suppose that $\Phi : E \to F$ is a $\mathbb{C}$-linear map (not assumed to be bounded). Then $\Phi : E \to F$ is an orthogonality preserving A-module map if and only if there exists $u \in Z(M(J_E))_+$ such that*

$$\langle \Phi(x), \Phi(y)\rangle = u\langle x, y\rangle \quad (x, y \in E).$$

In this case, u is unique and Φ is automatically bounded.

Proof. As E is a full Hilbert J_E-module, it is easy to see that u is unique if it exists, and in this case, $\|\Phi\|^2 \leqslant \|u\|$.

The sufficiency is obvious, and we will establish the necessity in the following. Since $J_{F_\Phi} \subseteq J_E$ (see Lemma 2.2(vi)), by replacing Φ with the induced map $\Phi_0 : E \to F_\Phi := \overline{\Phi(E)}$, we may assume that $J_E = A$.

Let M be a maximal family of orthogonal norm-one elements in E (whose existence is ensured by applying Zorn's lemma), and $\mathcal{F}$ be the collection of all non-empty finite subsets of M. If $\{y, z\} \in \mathcal{F}$, then by Lemma 3.1,

$$\langle y, y\rangle u_y = \langle \Phi(y), \Phi(y)\rangle = \langle \Phi(y), \Phi(y+z)\rangle = \langle y, y\rangle u_{y+z},$$

which implies that $\|y(u_{y+z} - u_y)\|^2_{E^{**}} \leqslant \|u_{y+z} - u_y\|\|\langle y, y\rangle(u_{y+z} - u_y)\| = 0$, and so,

$$yu_y = yu_{y+z}. \tag{3.5}$$

On the other hand, $\langle y, y\rangle q_{y+z} = \langle y, y+z\rangle q_{y+z} = \langle y, y\rangle$ (by Lemma 2.2(v)) and thus $q_y \leqslant q_{y+z}$ (by Lemma 2.2(iii)). If $p \in \mathrm{Proj}_1(A^{**})$ such that $q_y \leqslant p$ and $q_z \leqslant p$, then $\langle y+z, y+z\rangle p = \langle y, y\rangle q_y p + \langle z, z\rangle q_z p = \langle y+z, y+z\rangle$, which tells us that $q_{y+z} \leqslant p$ (again by Lemma 2.2(iii)). Thus, $q_{y+z} = q_y \vee q_z$ in $\mathrm{Proj}_1(A^{**})$. Inductively, if $S \in \mathcal{F}$ and

$$x_S := \sum_{x\in S} x,$$

then by Lemma 3.1 and relation (3.5) we have

$$\langle \Phi(y), \Phi(x)\rangle = \langle y, x\rangle u_x = \langle y, x\rangle u_{x_S} \quad (y \in E; x \in S), \tag{3.6}$$

$$q_{x_S} = \bigvee_{x\in S} q_x \quad (\text{as elements in } \mathrm{Proj}_1(A^{**})). \tag{3.7}$$

If $S' \in \mathcal{F}$ with $S \subseteq S'$, then

$$\langle x_S, x_S\rangle u_{x_{S'}} = \langle \Phi(x_S), \Phi(x_S)\rangle = \langle x_S, x_S\rangle u_{x_S}$$

(by relation (3.6)). Thus, Lemma 2.2(iii) tells us that

$$u_{x_S} = q_{x_S} u_{x_S} = q_{x_S} u_{x_{S'}}. \tag{3.8}$$

By taking adjoint, we see that q_{x_S} commutes with $u_{x_{S'}}$, and relation (3.8) implies that $(u_{x_S})_{S\in\mathcal{F}}$ is an increasing net in A^{**}_+.

We now show that $(u_{x_S})_{S\in\mathcal{F}}$ is a bounded net. Suppose on the contrary that there is an increasing sequence $\emptyset \subsetneq S(0) \subsetneq S(1) \subsetneq \cdots$ in $\mathcal{F}$ with

$$\|u_{x_{S(n)}}\| \geqslant \|u_{x_{S(n-1)}}\| + n^5 \quad (n \in \mathbb{N})$$

(notice that $\|u_{x_S}\| \leqslant \|u_{x_{S'}}\|$ if $S \subseteq S'$). Denote by

$$y_n := \sum_{x\in S(n)\setminus S(n-1)} x = x_{S(n)} - x_{S(n-1)} \quad (n \in \mathbb{N}).$$

By Proposition V.1.6 in [21], one has a partial isometry $w \in A^{**}$ such that

$$q_{x_{S(n)}} - q_{x_{S(n-1)}} = q_{x_{S(n-1)}} \vee q_{y_n} - q_{x_{S(n-1)}} = w(q_{y_n} - q_{x_{S(n-1)}} \wedge q_{y_n})w^*,$$

which implies

$$\begin{aligned} u_{x_{S(n)}} = u^{1/2}_{x_{S(n)}} q_{x_{S(n)}} u^{1/2}_{x_{S(n)}} &\leqslant u^{1/2}_{x_{S(n)}} (q_{x_{S(n-1)}} + w q_{y_n} w^*) u^{1/2}_{x_{S(n)}} \\ &= u_{x_{S(n-1)}} + u^{1/2}_{x_{S(n)}} w q_{y_n} w^* u^{1/2}_{x_{S(n)}} \end{aligned}$$

(see also (3.8)). On the other hand, by (3.7) and Lemma 2.1,

$$\begin{aligned} &u^{1/2}_{x_{S(n)}} w q_{y_n} w^* u^{1/2}_{x_{S(n)}} \\ &\quad = \mathbf{c}(u^{1/2}_{x_{S(n)}}) q_{x_{S(n)}} w q_{y_n} w^* q_{x_{S(n)}} \mathbf{c}(u^{1/2}_{x_{S(n)}}) = q_{x_{S(n)}} w q_{y_n} \mathbf{c}(u^{1/2}_{x_{S(n)}}) \mathbf{c}(u^{1/2}_{x_{S(n)}}) w^* q_{x_{S(n)}} \\ &\quad = q_{x_{S(n)}} w q_{y_n} q_{x_{S(n)}} \mathbf{c}(u^{1/2}_{x_{S(n)}}) \mathbf{c}(u^{1/2}_{x_{S(n)}}) w^* q_{x_{S(n)}} = q_{x_{S(n)}} w q_{y_n} u_{x_{S(n)}} w^* q_{x_{S(n)}}. \end{aligned}$$

Consequently,

$$u_{x_{S(n)}} - u_{x_{S(n-1)}} \leqslant q_{x_{S(n)}} w q_{y_n} u_{x_{S(n)}} w^* q_{x_{S(n)}},$$

which gives

$$\|q_{y_n} u_{x_{S(n)}}\| > n^5.$$

Let $a_n := \langle y_n, y_n\rangle / \|y_n\|^2$. Since $\{a_n b : b \in C^*(a_n)\}$ is a norm-dense ideal of $C^*(a_n)$, there is $b_n \in C^*(a_n)_+$ such that

$$\|a_n b_n\| \leqslant 1 \quad \text{and} \quad \|a_n b_n u_{x_{S(n)}}\| > n^5.$$

Define $x_n := y_n b_n^{1/2} / \|y_n\|$. Then clearly $(x_n)_{n\in\mathbb{N}}$ is an orthogonal sequence with $\langle x_n, x_n\rangle = a_n b_n$. Let $z := \sum_{n=1}^{\infty} x_n / n^2 \in E$ (notice that $\|x_n\| \leqslant 1$). As in (3.4), since Φ preserves orthogonality, for any $m \in \mathbb{N}$ we have

$$\langle \Phi(z), \Phi(z)\rangle \geqslant \frac{b_m^{1/2} \langle y_m, y_m\rangle u_{x_{S(m)}} b_m^{1/2}}{m^4 \|y_m\|^2} = \frac{a_m b_m u_{x_{S(m)}}}{m^4}$$

(because of relation (3.6) as well as the facts that $b_m^{1/2} \in q_{x_{S(m)}} A^{**} q_{x_{S(m)}}$ and $u_{x_{S(m)}} \in Z(q_{x_{S(n)}} A^{**} q_{x_{S(n)}})_+$). This gives the contradiction that $\|\Phi(z)\|^2 > m$ for all $m \in \mathbb{N}$.

For any $x \in E$, we set $v_x := \mathbf{c}(u_x)$. By Lemmas 3.1, 2.1 and 2.2(v), we have

$$\langle \Phi(y), \Phi(x) \rangle = \langle y, x \rangle q_x v_x = \langle y, x \rangle v_x \quad (y \in E). \tag{3.9}$$

Moreover, by Lemma 2.1, the net $(v_{x_S})_{S \in \mathcal{F}}$ is also bounded. Let $v \in Z(A^{**})_+$ be the weak*-limit of a subnet of $(v_{x_S})_{S \in \mathcal{F}}$. Note that if $S \in \mathcal{F}$ and $x \in S$, then by Lemmas 2.2(v) and 2.1 as well as relations (3.7) and (3.8), we have

$$\langle y, x \rangle v_{x_S} = \langle y, x \rangle q_x q_{x_S} v_{x_S} = \langle y, x \rangle u_x = \langle \Phi(y), \Phi(x) \rangle \quad (y \in E).$$

Therefore,

$$\langle \Phi(y), \Phi(x) \rangle = \langle y, x \rangle v \quad (y \in E, x \in M). \tag{3.10}$$

If I is the closed two-sided ideal of A generated by $\{\langle y, x \rangle : y \in E, x \in M\}$, then $Iv \subseteq A$. For any $z \in E \cdot I \setminus \{0\}$, one has $zv \in E$. On the other hand, as $\langle z, z \rangle v_z \in A$ (see (3.9)), we know that $zv_z \in E$ (by Lemma 2.2(ii)). Furthermore, one has

$$\langle x, z \rangle v_z = \langle \Phi(x), \Phi(z) \rangle = v \langle x, z \rangle = \langle x, z \rangle v \quad (x \in M).$$

This shows that the element $z(v - v_z)$ in E is orthogonal to any $x \in M$. This forces $zv = zv_z$ (by the maximality of M). As a consequence,

$$\langle \Phi(x), \Phi(y) \rangle a = \langle x, ya \rangle v_{ya} = \langle x, y \rangle av \quad (x, y \in E, a \in I).$$

If q is the central open projection in A^{**} with $I = A \cap qA^{**}q$ (see e.g. Section 3.11 of [19]), then q is the weak*-limit of a net in I, and we have

$$\langle \Phi(x), \Phi(y) \rangle q = v \langle x, y \rangle q \quad (x, y \in E). \tag{3.11}$$

We now claim that $\phi : a \mapsto qa$ is an injection from A onto qA (which is a C^*-subalgebra of A^{**} as ϕ is a $*$-homomorphism). Indeed, if $a \in \ker \phi$, then $\langle x, ya \rangle = \langle x, y \rangle qa = 0$ (for every $x \in M$ and $y \in E$), and the maximality of M as well as the fullness of E will imply that $a = 0$. Consequently, ϕ induces a $*$-isomorphism $\widetilde{\phi} : M(A) \to M(qA)$.

By equation (3.11) and the fullness of E, we see that v induces an element $m \in Z(M(qA))_+$ such that

$$q \langle \Phi(x), \Phi(y) \rangle = m(q \langle x, y \rangle) \quad (x, y \in E).$$

If $u := (\widetilde{\phi})^{-1}(m)$, then $u \in Z(M(A))_+$ and the injectivity of ϕ gives the required relation

$$\langle \Phi(x), \Phi(y) \rangle = u \langle x, y \rangle \quad (x, y \in E). \quad \blacksquare$$

Suppose that $v \in M(J_E)$. Since E is a Hilbert J_E-module, it becomes a unital right Banach $M(J_E)$-module in a canonical way. We denote by $R_v : E \to E$ the right multiplication of v, i.e., $R_v(x) = xv$ $(x \in E)$.

COROLLARY 3.3. *Suppose that Φ is an orthogonality preserving A-module map. Denote by u_Φ the unique element in $Z(M(J_E))_+$ associated with Φ as in Theorem 3.2. Set $w_\Phi := u_\Phi^{1/2}$.*

(i) *$J_{F_\Phi} = \overline{u_\Phi J_E}$ and $\ker \Phi = \ker R_{w_\Phi}$. Moreover, there is a Hilbert A-module isomorphism $\Theta : \overline{Ew_\Phi} \to F_\Phi$ such that $\Phi = \Theta \circ R_{w_\Phi}$. Consequently, the induced map $\Phi_0 : E \to F_\Phi$ is adjointable with Φ_0^* being orthogonality preserving.*

(ii) *If Φ is injective, then $\Phi^{-1} : \Phi(E) \to E$ is also orthogonality preserving.*

(iii) *If $J_{F_\Phi} = J_E$, then Ew_Φ is dense in E and Φ is injective.*

Proof. (i) The first equality follows directly from Theorem 3.2. As

$$\|\Phi(x)\| = \|R_{w_\Phi}(x)\| \quad (x \in E),$$

we see that $\ker \Phi = \ker R_{w_\Phi}$. Thus, we can define $\Theta : Ew_\Phi \to F$ by

$$\Theta(R_{w_\Phi}(x)) := \Phi(x).$$

Since Θ preserves the A-valued inner products, it extends to a Hilbert A-module isomorphism from $\overline{Ew_\Phi}$ onto F_Φ that satisfies the required condition. Furthermore, it is easy to see that both $R_{w_\Phi} : E \to \overline{Ew_\Phi}$ and Θ are adjointable, and so is Φ_0. Finally, as $\Phi_0^* = R_{w_\Phi} \circ \Theta^{-1}$, we see that Φ_0^* also preserves orthogonality.

(ii) Suppose that $a \in J_E$ with $au_\Phi = 0$. Then $aw_\Phi = 0$ as $w_\Phi \in C^*(u_\Phi)$ and so, $xa \in \ker \Phi$ for any $x \in E$ (by part (i)). As Φ is injective and E is a full Hilbert J_E-module, we have $a = 0$. Consequently, if $x, y \in E$ satisfying $\langle \Phi(x), \Phi(y) \rangle = 0$, then by Theorem 3.2, $\langle x, y \rangle = 0$.

(iii) Part (i) tells us that $u_\Phi J_E$ is dense in $J_{F_\Phi} = J_E$, and so, $w_\Phi J_E \supseteq w_\Phi(w_\Phi J_E)$ is dense in J_E. Consequently, $Ew_\Phi = (E \cdot J_E)w_\Phi$ is dense in E. By part (i) again, we see that E is isomorphic to F_Φ. Moreover, if $x \in \ker R_{w_\Phi}$, then

$$\langle x, yw_\Phi \rangle = \langle xw_\Phi, y \rangle = 0 \quad \text{for any } y \in E,$$

which implies that $x = 0$. Consequently, part (i) tells us that $\ker \Phi = \{0\}$. ∎

By Corollary 3.3(i), if $\Phi : E \to F$ is an orthogonality preserving A-module map with dense range, then F and Φ can be represented by an element $w_\Phi \in Z(M(J_E))_+$, up to an isomorphism. On the other hand, Φ might not have closed range even if it is injective (see Example 3.5(ii) below), and Corollary 3.3(ii) does not give us any good information about Φ^{-1}. Furthermore, it is not true that all orthogonality preserving A-module maps are adjointable (see Example 3.5(iii) below), and it is only true if we restrict the range of the map.

THEOREM 3.4. *Let $\Phi : E \to F$ be an orthogonality preserving A-module map (not assumed to be bounded), $F_\Phi := \overline{\Phi(E)}$, and J_E be the closed two-sided ideal generated by the inner products of elements in E.*

(i) *If $J_{F_\Phi} = J_E$, there is a Hilbert A-module isomorphism $\Theta : E \to F_\Phi$ such that $\Phi(x) = \Theta(xw_\Phi)$ $(x \in E)$.*

(ii) *If Φ is bijective, then $J_F = J_E$ and there is a unique invertible $w \in Z(M(J_E))_+$ such that $x \mapsto \Phi(x)w^{-1}$ is a Hilbert A-module isomorphism from E onto F.*

Proof. (i) This follows directly from Corollary 3.3.

(ii) By Lemma 2.2(vi), we have $J_F \subseteq J_E$ and we might assume that E is full. Notice that $\Phi^{-1} : F \to E$ is an orthogonality preserving A-module map because of Corollary 3.3(ii). Thus, Theorem 3.2 gives $u_{\Phi^{-1}} \in Z(M(J_F))_+$ such that

$$\langle x, y\rangle = \langle \Phi^{-1}(\Phi(x)), \Phi^{-1}(\Phi(y))\rangle = u_{\Phi^{-1}} u_\Phi \langle x, y\rangle \quad (x, y \in E).$$

As E is full, the above implies that for any $a \in A$, one has $a = u_{\Phi^{-1}} u_\Phi a \in u_{\Phi^{-1}} J_F \subseteq J_F$ (by Corollary 3.3(i)). This shows that $J_F = A$ and u_Φ is invertible (and so is w_Φ). Now, part (ii) follows directly from part (i) (note that the uniqueness of w follows from the uniqueness of u_Φ). ∎

We remark that in the case of complex Hilbert spaces (i.e., $A = \mathbb{C}$), the condition that $J_{\overline{\Phi(E)}} = J_E$ is the same as Φ being nonzero. However, in the general case, one cannot even replace the requirement $J_{\overline{\Phi(E)}} = J_E$ in Theorem 3.4(i) with Φ being either injective or surjective (see Example 3.5(i) and (iv) below; note that a Hilbert A-module isomorphism is isometric). We remark also that even in the situation of Theorem 3.4(i), the submodule $\Phi(E)$ needs not be closed in F and w_Φ needs not be invertible (see Example 3.5(ii) below).

EXAMPLE 3.5. (i) Let

$$A := C[0,1], \quad E := C[0,1], \quad \text{and} \quad F := C_0(0,1].$$

If $a \in A_+$ is given by $a(t) := t$ ($t \in [0,1]$) and $\Phi : E \to F$ is defined by $\Phi(x) := xa$, then Φ is an injective orthogonality preserving A-module map. However, there is no isometric A-module map from E into F. Suppose on the contrary that $\Theta : E \to F$ is such a map. Then $\Theta(b) = \Theta(1)b$ ($b \in A$). Since $f := \Theta(1)$ is in $C_0(0,1]$, one can find $t_0 \in (0,1)$ such that $|f(t)| < 1/2$ for $t \leqslant t_0$. Now, if $b \in A$ such that $\|b\| = 1$ and b vanishes on $[t_0, 1]$, then

$$\|\Theta(b)\| \leqslant \frac{1}{2} < 1 = \|b\|$$

which is a contradiction.

(ii) Let $A := C_0(0,1]$ and $a \in A_+$ be the function defined by

$$a(t) := t \quad (t \in (0,1]).$$

If we set $E := A$ and $F := A$, and define $\Phi : E \to F$ by $\Phi(x) := xa$, then Φ is an orthogonality preserving A-module map with dense range and $J_{F_\Phi} = A = J_E$, but Φ is not surjective, and $a = w_\Phi$ is not invertible in $M(A)$.

(iii) Let

$$A := C_0(0,1), \quad E := \{f \in A : f(1/2) = 0\}, \quad F := A \quad \text{and} \quad \Phi : E \to F$$

be the canonical injection. Then Φ is an orthogonality preserving A-module map with closed range and $J_{F_\Phi} = J_E$, but Φ is not an adjointable map from E into F. Indeed, suppose that Φ is adjointable, and $g \in F$ with $g(1/2) \neq 0$. Then

$$\langle \Phi^*(g), f\rangle_E - \langle g, f\rangle_F = 0 \quad \text{for any } f \in E \subseteq F,$$

which implies that $\Phi^*(g) - g = 0$ (because 0 is the only element in F being orthogonal to E). Thus, we have a contradiction $g = \Phi^*(g) \in E$.

(iv) Let

$$A = \mathbb{C} \oplus \mathbb{C}, \quad E = A \quad \text{and} \quad F = \mathbb{C} \oplus \{0\} \subseteq E.$$

Define

$$\Phi(x) := x(1,0) \quad (\text{for any } x \in E).$$

Then Φ is a surjective orthogonality preserving A-module map, but $E \ncong F$.

REMARK 3.6. Since E and F can be embedded into their respective linking algebras, some readers may consider the possibility of extending the orthogonality preserving map Φ to a disjointness preserver between the linking algebras, and then use the corresponding results for disjointness preservers in the literature (e.g., [3], [4], [5], [9], [15], [16], [23], [24]) to obtain Theorem 3.2. However, in order to extend Φ to a disjointness preserver on the linking algebra, one needs a canonical map from $\mathcal{K}(E)$ into $\mathcal{K}(F)$ which is compatible with Φ. It seems difficult to obtain such a map because Φ is not even assumed to be bounded. Nevertheless, after obtaining Theorem 3.2, we can use it to show that such an extension is possible, but we do not see any easy way to obtain it without our main theorems. Readers are referred to Section 4 in [14] for the details.

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