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k-HYPONORMALITY AND n-CONTRACTIVITY FOR AGLER-TYPE SHIFTS

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ABSTRACT. We consider k-hyponormality and n-contractivity ($k, n = 1, 2, \ldots$) as "weak subnormalities" for a Hilbert space operator. It is known that k-hyponormality implies $2k$-contractivity; we produce some classes of weighted shifts including a parameter for which membership in a certain n-contractive class is equivalent to k-hyponormality. We consider as well some extensions of these results to operators arising as restrictions of these shifts, or from linear combinations of the Berger measures associated with the shifts.

KEYWORDS: *Weighted shift, subnormal operator, n-contractive, k-hyponormal.*

MSC (2010): 47B37, 47B20.

1. INTRODUCTION

There has been considerable recent study of k-hyponormality and n-contractivity (with k and n positive integers) as two examples of "weak subnormalities" for Hilbert space operators. (For k-hyponormality, the better-established study, see the foundational paper [4], and, for a sampling, [5], [6], [7], and [8]; for more recent n-contractivity work see [1], [10], [11], and [13]; further weak subnormalities which we do not consider here are to be found, for example, in [12] and [14].) We are concerned in this paper with the question of what membership in a k-hyponormal class implies about membership in some n-contractive class, and *vice versa*. We begin with some definitions.

Let $\mathcal{H}$ be a separable complex Hilbert space and T a bounded linear operator on $\mathcal{H}$. Recall that T is normal if $T^*T = TT^*$ and T is subnormal if T is the restriction of a normal operator to an invariant subspace (see [3]). As a yet weaker condition, authors as noted above have considered k-hyponormality: T

is k-hyponormal, $k = 1, 2, \ldots$, if

$$\begin{pmatrix} I & T^* & T^{*2} & \ldots & T^{*k} \\ T & T^*T & T^{*2}T & \ldots & T^{*k}T \\ T^2 & T^*T^2 & T^{*2}T^2 & \ldots & T^{*k}T^2 \\ \vdots & & \vdots & & \vdots \\ T^k & T^*T^k & T^{*2}T^k & \ldots & T^{*k}T^k \end{pmatrix} \geqslant 0.$$

When $k = 1$, we call the operator simply hyponormal, and the condition reduces to the familiar $T^*T \geqslant TT^*$. The study of these classes is motivated in part by the following Bram–Halmos characterization of subnormality (see [2] and [9]).

THEOREM 1.1. *T is subnormal if and only if it is k-hyponormal for all k.*

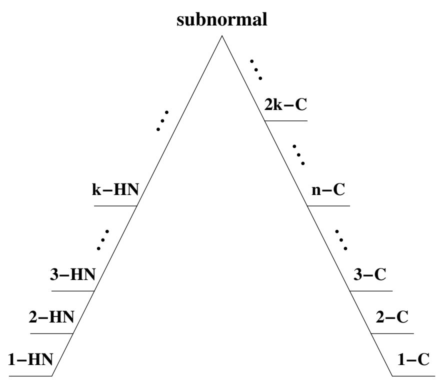
There is an alternative characterization of subnormality (under the mild restriction that the operator is a contraction: $\|T\| \leqslant 1$) which motives another collection of classes. We say T is n-contractive, $n = 1, 2, \ldots$ if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geqslant 0.$$

Observe that "1-contractive" is simply "contractive." The Agler–Embry characterization is essentially to be found in [1] (using the notion of hypercontractivity, which for this theorem is equivalent).

THEOREM 1.2. *A contraction T is subnormal if and only if it is n-contractive for all n.*

Recent study has concerned these classes for various finite k, and, in particular, efforts to compare and contrast the k-hyponormal and n-contractive classes (e.g., [13]). The situation may be captured in a "step-ladder," with the natural questions what achieving height k on one side says about where you are on the other.



One "left-to-right" implication is not difficult: it is known that if T (contractive) is k-hyponormal then it is $2k$-contractive ([13]). One cannot expect too much in the reverse direction, because it is possible to perturb a recursively generated weighted shift so that 1-HN $\Leftarrow$ 2-HN $\Leftrightarrow$ subnormal but so that all the n-C classes are distinct in n.

A standard testing ground for these questions has been the class of weighed shifts, hearkening back to [19]. Consider ℓ^2 with its standard basis $\{e_j\}_{j=0}^{\infty}$ (note that we begin indexing at zero). Given a weight sequence $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \ldots$, we define the weighted shift W_α on ℓ^2 by $W_\alpha e_j = \sqrt{\alpha_j} e_{j+1}$, and extend by linearity. The <u>moments</u> are $\gamma_0 = 1$ and $\gamma_j = \prod_{i=0}^{j-1} \alpha_i$, $j \geqslant 1$. Some particular shifts we consider are the Agler shifts A_j, $j = 2, 3, \ldots$ having weight sequence $\alpha^j : \sqrt{\frac{1}{j}}, \sqrt{\frac{2}{j+1}}, \ldots$, and which were used by Agler as model operators for n-contractive operators ([1]). Observe that A_2 is the Bergman shift.

A common device has been to take a weighted shift known to be subnormal and to "perturb" it in some way. For example, one can introduce a parameter into the m-th weight and consider what classes of interest result for various values of the parameter; alternatively, one may form a "backstep extension" of a known shift by prefixing one or more (parameter) weights to the weight sequence to form a new one. We will consider perturbations in the zeroth weight, yielding a weight sequence $\alpha^j(x) : \sqrt{\frac{x}{j}}, \sqrt{\frac{2}{j+1}}, \ldots$, and write $A_j(x)$ for the shift.

It is well-known that the tests for k-hyponormality or n-contractivity simplify considerably for weighted shifts. A weighted shift is k-hyponormal if and only if certain Hankel moment matrices are positive for $m = 1, 2, \ldots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geqslant 0.$$

A weighted shift is n-contractive if and only if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \gamma_{m+j} \geqslant 0, \quad m = 0, 1, \ldots.$$

It is easy to check that for perturbations in the zeroth weight (that is, $\sqrt{\alpha_0}$) it is the $m = 0$ versions of these that are the only ones in question.

Recall finally that a weighted shift has an associated Berger measure; in particular, there is a probability measure μ^j associated with each Agler shift A_j so that

$$\gamma_n = \int_0^1 t^n \mathrm{d}\mu^j(t), \quad n = 0, 1, \ldots.$$

In fact, $\mu^j(t) = (j-1)(1-t)^{(j-2)}$.

2. k-HYPONORMALITY AND n-CONTRACTIVITY FOR THE PERTURBED AGLER SHIFTS

We turn now to consideration of membership in these k-hyponormality and n-contractivity classes for the various Agler A_j as perturbed in the zeroth weight to $A_j(x)$. Using the Berger measure, n-contractivity for some $A_j(x)$ is easy: with the γ_i those for A_j, one needs

$$
\begin{aligned}
0 &\leqslant 1+\sum_{i=1}^{n}(-1)^i x\binom{n}{i}\gamma_i = 1-x+x\cdot\sum_{i=0}^{n}(-1)^i\binom{n}{i}\gamma_i = \\
&= 1-x+x\cdot\int_0^1\sum_{i=0}^{n}(-1)^i\binom{n}{i}t^i \mathrm{d}\mu^j(t) = 1-x+x\cdot\int_0^1(1-t)^n\mathrm{d}\mu^j(t).
\end{aligned}
$$

Note for future use that this may be expressed as

$$
x \leqslant \frac{1}{1-\int_0^1(1-t)^n\mathrm{d}\mu^j(t)}. \tag{2.1}
$$

In any event, what results is that $A_j(x)$ is n-contractive if and only if $x \leqslant \frac{n+j-1}{n}$.

The matter of k-hyponormality of $A_j(x)$ is harder, as might be expected from the fact that it is intrinsically a matrix condition. For k-hyponormality we need (with the γ_i those for A_j)

$$
\begin{pmatrix}
1 & x\gamma_1 & x\gamma_2 & \cdots & x\gamma_k \\
x\gamma_1 & x\gamma_2 & & \cdots & x\gamma_{k+1} \\
x\gamma_2 & \cdots & & \cdots & x\gamma_{k+2} \\
\vdots & & \vdots & & \vdots \\
x\gamma_k & x\gamma_{k+1} & & \cdots & x\gamma_{2k}
\end{pmatrix} \geqslant 0.
$$

It is known (see [5]) that the set of x for which such a perturbation is k-hyponormal is an interval of the form $[0, x_0]$, and (using the nested determinant test) that x_0 is the (unique) positive value for which the determinant of this matrix is zero. It is clearly equivalent to find that positive value so that

$$
\det\begin{pmatrix}
\frac{1}{x} & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\
\gamma_1 & \gamma_2 & & \cdots & \gamma_{k+1} \\
\gamma_2 & \cdots & & \cdots & \gamma_{k+2} \\
\vdots & & \vdots & & \vdots \\
\gamma_k & \gamma_{k+1} & & \cdots & \gamma_{2k}
\end{pmatrix} = 0.
$$

(There is an opportunity for an "off-by-one" error here: the matrix relevant to k-hyponormality is of size $k+1$ by $k+1$ and not size k by k.) So consider the matrix

$D_{k+1}(y)$ defined by

$$D_{k+1}(y) := \begin{pmatrix} y & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & & \cdots & \gamma_{k+1} \\ \gamma_2 & \cdots & & \cdots & \gamma_{k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_k & \gamma_{k+1} & & \cdots & \gamma_{2k} \end{pmatrix}.$$

As well, set $H_{k+1} = D_{k+1}(1)$ and let $\widehat{H}_{k+1}$ be the lower right k by k submatrix of H_{k+1}. (Note that these are just, respectively, a moment matrix of the original unperturbed shift and one of its submatrices). Set $d_{k+1}(y) = \det(D_{k+1}(y))$, $h_{k+1} = \det(H_{k+1})$ and $\widehat{h}_{k+1} = \det(\widehat{H}_{k+1})$.

LEMMA 2.1. *One has* $d_k(y) = (y-1)\widehat{h}_k + h_k$, $k = 2, 3, \ldots$.

Proof. Observe that

$$d_k(y) = y \cdot \widehat{h}_k + \text{terms not involving } y =: y \cdot \widehat{h}_k + f_k.$$

But $h_k = d_k(1) = \widehat{h}_k + f_k$, so $f_k = h_k - \widehat{h}_k$ and the result follows. ∎

It follows easily that the x we are interested in, for k-hyponormality, are

$$x \leqslant \frac{1}{1 - \frac{h_{k+1}}{\widehat{h}_{k+1}}}. \tag{2.2}$$

Matters therefore come down to finding determinants of the moment matrices, and certain special submatrices, for the original weighted shifts A_j. In the case of the Bergman shift, $j = 2$, what results are the Hilbert matrix and a submatrix. These are Cauchy matrices, and the determinants are available from that theory. For general j, we need further tools.

We rely here on material from [17], in which one may find original references. A sequence of polynomials $(p_n)_{n\geqslant 0}$ is said to be (formally) orthogonal if each p_n has degree n and if there exists a linear functional L and a sequence $(c_n)_{n\geqslant 0}$ of non-zero numbers such that $L(p_n, p_m) = c_n\delta_{mn}$ where δ_{mn} is the Kronecker delta. According to Favard's theorem, there is an intrinsic test to see if a family of polynomials is formally orthogonal.

THEOREM 2.2 (Theorem 12, [17]). *Let $(p_n)_{n\geqslant 0}$ be a sequence of monic polynomials, with each p_n having degree n. The sequence is (formally) orthogonal if and only if there exist sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ with $b_n \neq 0$ for all $n \geqslant 1$ such that*

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x), \quad n \geqslant 1,$$

with $p_0(x) = 1$ and $p_1(x) = x + a_0$.

We also have the following from [17].

THEOREM 2.3 (Theorem 11, [17]). *Let $(\omega_n)_{n\geqslant 0}$ be a sequence of numbers with generating function $\sum\limits_{k=0}^{\infty} \omega_k x^k$ written in the form*

$$\sum_{k=0}^{\infty} \omega_k x^k = \cfrac{\omega_0}{1 + a_0 x - \cfrac{b_1 x}{1+a_1 x - \cfrac{b_2 x^2}{1+a_2 x - \cdots}}}.$$

Then the Hankel determinant $\det(\omega_{i+j})_{0\leqslant i,j\leqslant n-1}$ *equals* $\omega_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-2}^2 b_{n-1}$.

The connection between the theorems is that if the $(\omega_n)_{n\geqslant 0}$ are the moments $L(x^n)$ of the linear functional L, then the a_n and b_n of the two theorems coincide, and we have the following from Theorem 13 of [17].

THEOREM 2.4. *Let $(p_n)_{n\geqslant 0}$ be a sequence of monic polynomials as in Theorem* 2.2 *with associated linear functional L, with recurrence*

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x), \quad n \geqslant 1, \tag{2.3}$$

and initial conditions $p_0(x) = 1$ and $p_1(x) = x + a_0$. Let $(\omega_n)_{n\geqslant 0}$ be the sequence of moments $L(x^n)$ of L. Then

$$\det(\omega_{i+j})_{0\leqslant i,j\leqslant n-1} = \omega_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-2}^2 b_{n-1}. \tag{2.4}$$

In our situation for k-hyponormality (and some $A_j(x)$ with j fixed) we need the determinants of the $k+1$ by $k+1$ moment matrix H_{k+1} and its k by k submatrix $\widehat{H}_{k+1}$. For the first of these, the linear functional L will be integration against $\mu^j(t) = (j-1)(1-t)^{(j-2)}$. For the $\widehat{H}$, it is clear that the appropriate linear functional is integration against $(j-1)t^2 \cdot (1-t)^{(j-2)}$, since we want to produce a matrix with γ_2 as the upper left entry. A glance at the determinant formula (2.4) makes it clear that, since we know $\widehat{\omega}_0 = \gamma_2 = \frac{2}{j(j+1)}$, we need not compute the normalization explicitly. What is needed are the sequences of orthogonal polynomials for these two cases, or, more precisely for our needs, the appropriate sequences of coefficients b_n appearing in the recurrences. Luckily, we need not produce these by hand.

The polynomials we need are (versions of) the Jacobi polynomials, which are available at [15]. There one finds the Jacobi polynomials, orthogonal with respect to integration

$$\int_{-1}^{1} (\cdot)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x, \quad \alpha, \beta > -1.$$

Also to be found there is a normalized recurrence relation

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} p_n(x) \\ &\quad + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} p_{n-1}(x). \end{aligned} \tag{2.5}$$

Upon the substitution $x \to 2t-1$, the polynomials $p_n(2t-1)$ are orthogonal with respect to

$$\int_0^1 (\cdot)(1-t)^{\alpha} t^{\beta}\, \mathrm{d}t. \tag{2.6}$$

To achieve monotonicity we must take $\frac{p_n(2t-1)}{2^n}$; using these polynomials, the recurrence relation (2.5) yields that the b_n of Theorems 2.2, 2.3, and 2.4 are

$$b_n = \frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}. \tag{2.7}$$

From now on we view the j of A_j (some $j \geqslant 2$) as fixed, and suppress it from the notation wherever possible. For $h_{n+1} = \det(H_{n+1})$ we need coefficients relevant to integration against $\mu^j(t) = (j-1)(1-t)^{(j-2)}$; observe that this is (a normalized version of) the kernel in (2.6) with $\alpha = j-2$ and $\beta = 0$. From now on we will reserve "b_n" for the coefficients as in Theorems 2.2, 2.3, and 2.4 for this kernel. For $\widehat{h}_{n+1} = \det(\widehat{H}_{n+1})$ we need coefficients relevant to integration against $\widehat{\mu}^j(t) = (j-1)(1-t)^{(j-2)}t^2$; this is (a multiple of) the kernel in (2.6) with $\alpha = j-2$ and $\beta = 2$. From now on we will use "$\widehat{b}_n$" for the coefficients relevant to this kernel.

We need a computational lemma.

LEMMA 2.5. *For fixed j, and $n = 2, 3, \ldots$,*

$$\widehat{h}_{n+1} = \frac{(n+j-1)(n+1)}{j-1} h_{n+1}. \tag{2.8}$$

Proof. The proof will be by weak induction; recall that we know that $\gamma_0 = 1$ (this is ω_0, the upper left hand entry in the matrix H), and also that $\widehat{\omega}_0 = \gamma_2 = \frac{2}{j(j+1)}$ is the upper left hand entry in the matrix $\widehat{H}$. One may verify directly that $\widehat{h}_2 = \widehat{\omega}_0^1 = \frac{2}{j(j+1)}$ and that $h_2 = \omega_0^2 b_1 = \frac{(j-1)^2}{(j-1)j^2(j+1)}$, and this yields (2.8) in the case $n = 2$. Similarly, using $\widehat{h}_3 = \widehat{\omega}_2 \widehat{b}_1$ and $h_3 = \omega_3 b_1^2 b_2$ one obtains (2.8) for $n = 3$.

For the induction step it is convenient to note (using the determinant formula in (2.4)) that

$$h_{n+1} = h_n \omega_0 b_1 \cdots b_n = h_n \omega_0 b_1 \cdots b_{n-1} b_n = \frac{h_n^2 b_{n-1}}{h_{n-1}}.$$

(Of course there is a similar result for $\widehat{h}_{n+1}$.)

Then $h_{n+2} = \frac{h_{n+1}^2 b_{n+1}}{h_n}$, and, using weak induction,

$$\widehat{h}_{n+2} = \frac{\widehat{h}_{n+1}^2 \widehat{b}_n}{\widehat{h}_n} = \frac{\left[\frac{(n+j-1)(n+1)}{j-1}\right]^2 h_{n+1}^2 \widehat{b}_n}{\left[\frac{(n+j-2)(n)}{j-1}\right] h_n}.$$

Then to verify (2.8) it suffices to show that

$$\frac{(n+j-1)^2(n+1)^2}{n(n+j-2)(j-1)}\widehat{b}_n = \frac{(n+2)(n+2)}{(j-1)}b_{n+1}, \tag{2.9}$$

and this is a straightforward computation using (2.7) keeping in mind that on the right hand side of (2.9) $\alpha = j-2$ and $\beta = 0$ and on the left hand side $\widehat{\alpha} = j-2$ and $\widehat{\beta} = 2$. ∎

Assembling what has gone before, we have the following.

THEOREM 2.6. *The operator $A_j(x)$, for some $j \geqslant 2$, is*

(i) *n-contractive if and only if $x \leqslant \frac{n+j-1}{n}$, and*

(ii) *n-hyponormal if and only if $x \leqslant \frac{n(n+j)+j-1}{n(n+j)}$.*

It follows that $A_j(x)$ is n-hyponormal if and only if it is $n(n+j)$-contractive.

Raúl Curto has well described one of the surprises of the two characterizations of subnormality (Bram–Halmos and Agler–Embry) in that the Agler–Embry conditions have all the T^*'s "on the left," and an operator satisfying all the conditions is subnormal, hence hyponormal, which is an inequality with a T^* "on the right". The theorem above is the only one of which we are aware of which a finite number of Agler–Embry conditions succeed in producing a condition with a T^* on the right; put differently, it is the only of which we are aware that provides an implication from right to left on the "step-ladder" diagram.

3. SOME GENERALIZATIONS

We may generalize the theorem above somewhat in two ways. First, instead of considering a perturbation in the zeroth weight of A_j, we could first compress A_j to the canonical invariant subspace of codimension m ($m = 1, 2, \ldots$). This amounts to forming a new weighted shift A_j^m by discarding the first m weights of the weight sequence for A_j, so, for example, with $j = 3$ and $m = 2$ we consider the shift with weight sequence $\sqrt{\frac{3}{5}}, \sqrt{\frac{4}{6}}, \ldots$. We may then perturb in the (new) zeroth weight, yielding $A_j^2(x)$ with weight sequence $\sqrt{\frac{x\cdot 3}{5}}, \sqrt{\frac{4}{6}}, \ldots$, and consider n-hyponormality and n-contractivity as before. It turns out that the result is not too different from what is above.

THEOREM 3.1. *Consider $A_j^m(x)$, the perturbation in the zeroth weight of the restriction of A_j ($j = 2, 3, \ldots$) to the canonical invariant subspace of codimension m ($m = 1, 2, \ldots$). Then $A_j^m(x)$ is $n(n+j)$-contractive implies it is n-hyponormal.*

Proof. (sketch). The condition on x for n-contractivity is obtained, as before, using the Berger measure for A_j^m, which is the normalization to a probability measure of $t^m(1-t)^{j-2}\mathrm{d}t$, namely $\mu_j^m(t) = \frac{(m+j-1)!}{(j-2)!m!}t^m(1-t)^{j-2}\mathrm{d}t$. The result is $\int_0^1 (1-t)^n \mathrm{d}\mu_m^j(t) = \frac{(m+j-1)!(n^2+(n+1)j-2)!}{(j-2)!(n^2+(n+1)j+m-1)!}$. For n-hyponormality things come down again to determinants of matrices $H_{n+1} = H_{n+1}(j,m)$ and $\widehat{H}_{n+1} = \widehat{H}_{n+1}(j,m)$. These determinants arise from the coefficients in the orthogonal polynomials for integration against $\frac{(m+j-1)!}{(j-2)!m!}t^m(1-t)^{j-2}\mathrm{d}t$ and $\frac{(m+j-1)!}{(j-2)!m!}t^{m+2}(1-t)^{j-2}\mathrm{d}t$ respectively. The analog of Lemma 2.5 is that

$$\frac{h_n}{\widehat{h}_n} = \frac{(m+1)!(j+m+1)!(n-1)!(j+n-3)!}{(j-2)!(m+n)!(j+m+n-2)!},$$

which may be proved using the appropriate versions of (2.4) and induction on n. A comparison of (2.1) and (2.2), and a little algebra, shows that it suffices to show that

$$\frac{(m+j-1)!(n^2+(n+1)j-2)!}{(j-2)!(n^2+(n+1)j+m-1)!} \leqslant \frac{(m+1)!(j+m+1)!(n-1)!(j+n-3)!}{(j-2)!(m+n)!(j+m+n-2)!},$$

which is just a computation. ∎

In an asymptotic sense, one can do better, as follows.

THEOREM 3.2. *For any j, $j \geqslant 2$, and any n, $n \geqslant 1$, there exists M so that for all $m \geqslant M$, $A_j^m(x)$ is (n^2+1)-contractive implies $A_j^m(x)$ is n-hyponormal.*

Proof. This follows as in the proof of the previous theorem by comparing $\int_0^1 (1-t)^{n^2+1}\mathrm{d}\mu_m^j(t)$ and $\frac{h_n}{\widehat{h}_n}$ for m large. ∎

We turn next to the consideration of (perturbations in the first weight of) shifts that are some "combination" of the Bergman shift and another shift. The approach will be through Berger measures; we will consider some shift whose Berger measure is a linear (perhaps convex) combination of $1 \cdot \mathrm{d}t$ (for the Bergman shift) and some other measure. We might take as the other measure $2(1-t)\mathrm{d}t$ (the measure for A_3); we might take $2t\mathrm{d}t$ (the measure for A_2^1, the restriction of the Bergman shift to its canonical invariant subspace of codimension one). In fact we will consider shifts with Berger measure

$$\mu_\varepsilon := (1+(-1+\varepsilon)t) * \Big(\frac{2}{1+\varepsilon}\Big)\mathrm{d}t.$$

It is an easy computation to see that the range $\varepsilon \in [0,1]$ yields the range $\lambda \in [0,1]$ for convex combinations of the form $(\lambda + (1-\lambda) * 2(1-t))\mathrm{d}t$ corresponding to combinations of $B = A_2$ and A_3; the range $\varepsilon \in [1,\infty]$ yields the range $\mu \in [0,1]$ for $(\mu + (1-\mu)2t)\mathrm{d}t$ corresponding to combinations of $B = A_2$ and A_2^1. Note also that $\varepsilon = 0$ is the least value for which the measure of this form produces a probability measure, and $\varepsilon \geqslant 1$ yields the (normalized) measures of the form $(a+bt)\mathrm{d}t$ for which both a and b are non-negative. Denote by A^{ε} the corresponding weighted shift and by $A^{\varepsilon}(x)$ the perturbation of A^{ε} in which the zeroth weight is multiplied by $\sqrt{x}$.

As in the proof of Theorem 3.1, or by a comparison of (2.1) and (2.2) and a little algebra, what we need to compute for n-contractivity is the expression

$$\int_0^1 (1-t)^n \mathrm{d}\mu_{\varepsilon}(t)$$

and what is needed for k-hyponormality of $A^{\varepsilon}(x)$ is the expression

$$\frac{h_{k+1}}{\widehat{h}_{k+1}}.$$

As usual, the term relevant to n-contractivity is easy:

$$\int_0^1 (1-t)^n \mathrm{d}\mu_{\varepsilon}(t) = \int_0^1 (1-t)^n (1+(-1+\varepsilon)t) * \Big(\frac{2}{1+\varepsilon}\Big) \mathrm{d}t = \frac{2(1+n+\varepsilon)}{(1+n)(2+n)(1+\varepsilon)}. \tag{3.1}$$

The computations for what is needed for n-hyponormality of $A^{\varepsilon}(x)$ are considerably more complicated, and we are greatly indebted to Christian Krattenthaler ([18]) for showing us the argument that follows. A computation of moments against the measure μ_{ε} shows that to evaluate $h_{n+1} = \det(H_{n+1})$ we must compute

$$\Big(\frac{2}{1+\varepsilon}\Big)^{n+1} \det(a_{i,j} + b_{i,j})_{0 \leqslant i,j \leqslant n} \tag{3.2}$$

with

$$a_{i,j} = \frac{1}{i+j+1} \quad \text{and} \quad b_{i,j} = \frac{\varepsilon - 1}{i+j+2}.$$

Considering

$$\det(a_{i,j} + b_{i,j})_{0 \leqslant i,j \leqslant n}$$

we may use column multilinearity to see that it equals

$$\sum_{S \subseteq \{0,1,\ldots,n\}} \det(c_{i,j}^{(S)})_{0 \leqslant i,j \leqslant n} \tag{3.3}$$

where $c_{i,j}^{(S)} = a_{i,j}$ if $j \in S$ and $c_{i,j}^{(S)} = b_{i,j}$ otherwise.

Consider now some one of the matrices $(c_{i,j}^{(S)})$. We claim that if there is any column of this matrix consisting of $b_{i,j}$'s to the left of a column consisting of $a_{i,j}$'s then the determinant is zero. In such a case, we may clearly find a column consisting of $b_{i,j}$'s so that the next column consists of $a_{i,j}$'s. But it is easy to see that the j-th column of the matrix $(b_{i,j})$ is $(\varepsilon - 1)$ times the $(j+1)$-st column of the matrix $(a_{i,j})$, for $j = 0, 1, \ldots, n-1$. Therefore the determinant is zero as claimed, and for the sum in (3.3) the only non-zero determinants that arise are from matrices with some number of initial columns all $a_{i,j}$'s filled out with subsequent columns all $b_{i,j}$'s. It follows that the sum in (3.3) is equal to

$$\sum_{k=0}^{n+1}(\varepsilon-1)^{(n+1-k)}\det\Big(\frac{1}{i+j+\eta_{j,k}+1}\Big)_{0\leqslant i,j\leqslant n} \tag{3.4}$$

where $\eta_{j,k}=1$ if $j\geqslant k$ and is zero otherwise.

Fix k for the moment and consider the determinant of $\Big(\frac{1}{i+j+\eta_{j,k}+1}\Big)_{0\leqslant i,j\leqslant n}$ appearing in the previous sum. This matrix is a Cauchy matrix $\Big(\frac{1}{x_i+y_j}\Big)_{0\leqslant i,j\leqslant n}$ with $x_i=i+\frac12$ and $y_j=j+\frac12$ for $j<k$ and $y_j=j+\frac12+1$ if $j\geqslant k$. The Cauchy determinant formula says that

$$\det\Big(\frac{1}{x_i+y_j}\Big)_{0\leqslant i,j\leqslant n}=\frac{\prod_{0\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{0\leqslant i,j\leqslant n}(x_i+y_j)}.$$

Upon insertion of our x_i and y_j, and after considerable simplification, one obtains

$$\det\Big(\frac{1}{i+j+\eta_{j,k}+1}\Big)_{0\leqslant i,j\leqslant n}=\binom{n+1}{k}\binom{n+k+1}{n+1}\frac{(n+1)!\prod_{i=0}^{n}i!^3}{\prod_{i=n+2}^{2n+2}i!}.$$

It follows that the sum in (3.4) is equal to

$$\sum_{k=0}^{n+1}(\varepsilon-1)^{(n+1-k)}\binom{n+1}{k}\binom{n+k+1}{n+1}\frac{(n+1)!\prod_{i=0}^{n}i!^3}{\prod_{i=n+2}^{2n+2}i!}. \tag{3.5}$$

Using the binomial theorem to expand $(\varepsilon-1)^{(n+1-k)}$ we obtain

$$\frac{(n+1)!\prod_{i=0}^{n}i!^3}{\prod_{i=n+2}^{2n+2}i!}\sum_{k=0}^{n+1}\sum_{\ell=0}^{n+1-k}\varepsilon^{\ell}(-1)^{n+1-k-\ell}\binom{n+1-k}{\ell}\binom{n+1}{k}\binom{n+1+k}{n}.$$

Interchanging the order of summation yields

$$\frac{(n+1)!\prod_{i=0}^{n}i!^3}{\prod_{i=n+2}^{2n+2}i!}\sum_{\ell=0}^{n+1}\varepsilon^{\ell}(-1)^{n+1-\ell}\sum_{k=0}^{n+1-\ell}(-1)^{-k}\binom{n+1-\ell}{k}\binom{n+1+k}{k}. \tag{3.6}$$

It turns out that the inner sum is a hypergeometric function: with $(a)_k$ denoting the Pochhammer rising factorial $(a)_k=a(a+1)\cdots(a+k-1)$, a computation shows that

$$\begin{aligned}\sum_{k=0}^{n+1-\ell}(-1)^{-k}\binom{n+1-\ell}{k}\binom{n+1+k}{k}&={}_2F_1(n+2,\ell-n-1;1;1)\\&=\sum_{k=0}^{\infty}\frac{(n+2)_k(\ell-n-1)_k\cdot 1^k}{(1)_k k!}.\end{aligned} \tag{3.7}$$

Upon using Chu–Vandermonde summation (see, for example, [16]), one obtains

$$\sum_{k=0}^{n+1-\ell}(-1)^{-k}\binom{n+1-\ell}{k}\binom{n+1+k}{k}=(-1)^{n+1-\ell}\binom{n+1}{\ell},$$

and therefore finally that (using the previous equation, (3.2), (3.5), and (3.6))

$$h_{n+1} = \det(H_{n+1}) = \Big(\frac{2}{1+\varepsilon}\Big)^{n+1} \frac{(n+1)!\prod_{i=0}^{n} i!^3}{\prod_{i=n+2}^{2n+2} i!} \sum_{\ell=0}^{n+1} \varepsilon^{\ell} \binom{n+1}{\ell}^2. \tag{3.8}$$

We need as well $\widehat{h}_{n+1} = \det(\widehat{H}_{n+1})$, where $\widehat{H}_{n+1}$ is the n by n lower right submatrix of H_{n+1} as in the discussion before Lemma 2.1. The computation is similar to what is above, and we merely give a few points along the way. The determinant becomes

$$\det(\widehat{H}_{n+1}) = \Big(\frac{2}{1+\varepsilon}\Big)^{n} \sum_{k=0}^{n} (\varepsilon-1)^{(n-k)} \det\Big(\frac{1}{i+j+\eta_{j,k}+3}\Big)_{0\leqslant i,j\leqslant n-1}$$

with $\eta_{j,k}$ as above. With $x_i = i + \frac{3}{2}$ and $y_j = j + \frac{3}{2}$ for $j < k$ and $y_j = j + \frac{3}{2} + 1$ if $j \geqslant k$, one again has a Cauchy matrix for each of the matrices whose determinants appear in the sum. One computes that

$$\det\Big(\frac{1}{i+j+\eta_{j,k}+3}\Big)_{0\leqslant i,j\leqslant n-1} = (n+1)\binom{n+k+2}{n}\binom{n}{k}\frac{\prod_{i=0}^{n} i!^3}{\prod_{i=n+3}^{2n+2} i!}.$$

Inserting, reordering the sum, and simplifying using the appropriate hypergeometric function, it follows that

$$\widehat{h}_{n+1} = \det(\widehat{H}_{n+1}) = \Big(\frac{2}{1+\varepsilon}\Big)^{n} \frac{(n+1)\prod_{i=0}^{n} i!^3}{\prod_{i=n+3}^{2n+2} i!} \sum_{\ell=0}^{n} \varepsilon^{\ell} \binom{n}{\ell}\binom{n+2}{\ell}. \tag{3.9}$$

We may finally compare k-hyponormality and n-contractivity. Using the analogs of (2.1) and (2.2) in our situation, we may show that the $n(n+3)$ contractivity cutoff is less than or equal to the n-hyponormality cutoff for any $\varepsilon > 0$, and therefore deduce that $n(n+3)$-contractivity implies n-hyppnormality for our combination, by checking

$$\int_0^1 (1-t)^{n(n+3)}(1+(-1+\varepsilon)t) * \Big(\frac{2}{1+\varepsilon}\Big)\,\mathrm{d}t \leqslant \frac{h_{n+1}}{\widehat{h}_{n+1}}.$$

This is to check that

$$\widehat{h}_{n+1} \cdot \frac{2(\varepsilon + n(n+3)+1)}{(1+n(n+3))(2+n(n+3))(1+\varepsilon)} \leqslant h_{n+1}.$$

Each side of the proposed inequality is a sum of terms with power ε^j with $0 \leqslant j \leqslant n+1$. It turns out, conveniently, that the needed inequality holds term by term in the ε^j, as shown by a modest computation. We have therefore obtained the following.

THEOREM 3.3. *For $\varepsilon \geqslant 0$, let A^{ε} be the weighted shift corresponding to the Berger (probability) measure $\mu_{\varepsilon} := (1+(-1+\varepsilon)t) * \Big(\frac{2}{1+\varepsilon}\Big)\mathrm{d}t$, and let $A^{\varepsilon}(x)$ denote the perturbation of A^{ε} in which the zeroth weight is multiplied by $\sqrt{x}$, with x a parameter. Then $A^{\varepsilon}(x)$ is $n(n+3)$-contractive implies $A_{\varepsilon}(x)$ is n-hyponormal.*

Note finally that with $\varepsilon = 0$ the combination reduces to A_3 for which $n(n+3)$-contractivity is equivalent to n-hyponormality. Thus the theorem cannot be improved to "$n(n+2)$-contractivity" in its current form.

One might also ask about the other direction: what level of contractivity does n-hyponormality imply (perhaps for some ranges of ε)? This is to check for what p

$$\frac{h_{n+1}}{\widehat{h}_{n+1}} \leqslant \int_0^1 (1-t)^p(1+(-1+\varepsilon)t) * \Big(\frac{2}{1+\varepsilon}\Big)\,\mathrm{d}t,$$

which is

$$h_{n+1} \leqslant \widehat{h}_{n+1} \cdot \frac{2(\varepsilon+p+1)}{(1+p)(2+p)(1+\varepsilon)}. \tag{3.10}$$

As $\varepsilon = 1$ yields the operator $A_2(x)$, for which n-hyponormality is equivalent to $n(n+2)$-contractivity, one could hope for no better, but since for both operators $A_2(x)$ and $A_3(x)$ in the conbination n-hyponormality yields $n(n+2)$-contractivity, it seems reasonable to hope for that. It is perhaps surprising to find that n-hyponormality does not imply $n(n+2)$-contractivity in the simplest way. If we compute both sides of the proposed inequality (3.10), it turns out that the inequality does not hold in the simplest "term by term in powers of ε" even for ε in the range $0 < \varepsilon \leqslant 1$, as was true in the (analogous) proof leading up to Theorem 3.3. (This may be shown by a computation with $n = 11$ and the coefficients of ε^{10}.) It turns out, however, that the result is true nonetheless.

PROPOSITION 3.4. *Let $A^{\varepsilon}(x)$ be as in the hypothesis of Theorem* 3.3. *Suppose $0 \leqslant \varepsilon \leqslant 1$. Then $A^{\varepsilon}(x)$ is n-hyponormal implies $A^{\varepsilon}(x)$ is $n(n+2)$-contractive.*

Proof. The result is trivial for ε zero or one. From computations in the proof leading up to Theorem 3.3 it is clear that we require positivity of

$$\widehat{h}_{n+1} \cdot \frac{2(\varepsilon+n(n+2)+1)}{(1+n(n+2))(2+n(n+2))(1+\varepsilon)} - h_{n+1} \tag{3.11}$$

which is a polynomial of degree $n+1$ in ε. Denote this polynomial $\sum\limits_{\ell=0}^{n+1} a_\ell \varepsilon^\ell$, where we have suppressed the dependence of the coefficients on n. A modest computation shows that

$$\begin{aligned} a_\ell = K_{n,\ell} \cdot [(n+2)^2(n+1)^2(n+3-\ell)(n+1-\ell) + \ell^2(n+2)^2 \\ - (n+1)^2(1+(n+1)^2)(n+3-\ell)(n+2-\ell)], \quad 1 \leqslant \ell \leqslant n, \end{aligned}$$

where $K_{n,\ell}$ is positive. This reduces the sign of a_ℓ to that of the second term in the product above, which is a quadratic in ℓ. One computes a_1 is positive and a_n is negative, and it follows readily that there is M so $a_\ell \geqslant 0$ for $0 \leqslant \ell < M$ and $a_\ell \leqslant 0$ for $M \leqslant \ell \leqslant n+1$. (One checks separately that $a_0 \geqslant 0$ and $a_{n+1} \leqslant 0$.)

Since ε=1 yields $A_2(x)$ for which n-hyponormality and $n(n+2)$-contractivity are equivalent, we have the expression in (3.11) is zero at $\varepsilon = 1$. It follows then that

$$\sum_{\ell=0}^{n+1} a_\ell = 0.$$

But since, in $\sum_{\ell=0}^{n+1} a_\ell \varepsilon^\ell$, the positive coefficients occur for lower powers of ℓ than the negative coefficients, we have the positivity required. ∎

A little thought makes it clear that the argument just given, trivially modified, shows that for $\varepsilon \geqslant 1$, if $A^\varepsilon(x)$ is $n(n+2)$-contractive it is n-hyponormal. In fact there is again a certain "term by term" version of the needed inequality, and we turn to proving this slightly stronger result. Recall that ε in the range $[1, \infty]$ corresponds to operators arising from (perturbations in the first weight of) convex combinations of A_2 and A_2^1 (where the latter is the restriction of A_2 to the canonical coinvariant subspace of codimension one). These in turn are equivalent to operators arising from (the normalization of) measures of the form $(1 + c * t)\mathrm{d}t$, with $c \geqslant 0$. It is term by term in this c that the needed inequality holds.

PROPOSITION 3.5. *Let A_c be the weighted shift with the Berger probability measure $\left(\frac{2}{2+c}\right)(1 + c * t)\mathrm{d}t$, $c \geqslant 0$, and let $A_c(x)$ be the shift obtained by multiplying the zeroth weight of A_c by $\sqrt{x}$. Then if $A_c(x)$ is $n(n+2)$-contractive it is n-hyponormal.*

Proof. The argument is very like the argument leading up to Theorem 3.3, and we merely give a sketch. Computations show that the relevant matrix whose determinant is needed for h_{n+1} is almost as in (3.2) but with $\frac{2}{1+\varepsilon}$ replaced by the normalizing factor $\frac{2}{2+c}$ and with c replacing $\varepsilon - 1$ in the $b_{i,j}$. The computations are similar but easier (with, for example, no need to use the binomial theorem); further computations are then tedious but familiar; in establishing a final positivity concerning the coefficients of c^{n+1-k}, it is useful to employ the substitution $n \to k + j$. ∎

Since $c = 0$ corresponds to A_2, it is easy to see that $n(n+1)$-contractivity is not sufficient for the result above.

The question of what n-hyponormality implies in the way of m-contractivity in the $\varepsilon > 1$ realm (equivalently, the $c > 0$ realm) is less tractable than for $0 \leqslant \varepsilon \leqslant 1$. The hope that n-hyponormality would imply, for all $\varepsilon > 1$, $n(n+p)$-contractivity for some p turns out to be false, failing at large n. It is known that n-hyponormality implies $2n$-contractivity in general ([13]), and the result holds in fact term by term in ε. Implications of the general form n-hyponormality implies $(2n + p)$-contractivity do not hold for $p > 0$, although the set of n for which they fail (for fixed p) is a finite set of small n. We leave further such investigations to the interested reader.

REFERENCES

[1] J. AGLER, Hypercontractions and subnormality, *J. Operator Theory* **13**(1985), 203–217.

[2] J. BRAM, Subnormal operators, *Duke Math. J.* **22**(1965), 75–94.

[3] J. CONWAY, *The Theory of Subnormal Operators*, Math. Surveys Monographs, vol. 36, Amer. Math. Soc., Providence, RI 1980.

[4] R. CURTO, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Sym. Math.* **51**(1990), 69–91.

[5] R. CURTO, L. FIALKOW, Recursively generated weighted shifts and the subnormal completion problem. I, *Integral Equations Operator Theory* **17**(1993), 202–246.

[6] R. CURTO, I. JUNG, W. LEE, Extension and extremality of recursively generated weighted shifts, *Proc. Amer. Math. Soc.* **130**(2001), 565–576.

[7] R. CURTO, W. LEE, Towards a model theory for 2-hyponormal operators, *Integral Equations Operator Theory* **44**(2002), 290–315.

[8] R. CURTO, W. LEE, k-hyponormality of finite rank perturbations of unilateral weighted shifts, *Trans. Amer. Math. Soc.* **133**(2005), 1805–1816.

[9] M. EMBRY, A generalizaton of the Halmos–Bram criterion for subnormality, *Acta Sci. Math. (Szeged)* **35**(1973), 61–64.

[10] G. EXNER, On n-contractive and n-hypercontractive operators, *Integral Equations Operator Theory* **56**(2006), 451–468.

[11] G. EXNER, Aluthge transforms and n-contractivity of weighted shifts, *J. Operator Theory* **61**(2009), 419–438.

[12] G. EXNER, I. JUNG, S. PARK, Weakly n-hyponormal weighted shifts and their examples, *Integral Equations Operator Theory* **54**(2006), 215–233.

[13] G. EXNER, I. JUNG, S. PARK, On n-contractive and n-hypercontractive operators. II, *Integral Equations Operator Theory* **60**(2008), 451–467.

[14] I. JUNG, E. KO, C. PEARCY, Sub-n-normal operators, *Integral Equations Operator Theory* **55**(2006), 83–91.

[15] R. KOEKOEK, R. SWARTTOUW, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology Technical Mathematics and Informatics Report no. 98–17, 1998.

[16] W. KOEPF, *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities*, Braunschweig, Vieweg, Germany 1998.

[17] C. KRATTENTHALER, Advanced Determinant Calculus, *Séminaire Lotharingien Combin.* **42**(1999), Article B42q, 67 pp.

[18] C. KRATTENTHALER, private communication.

[19] J. STAMPFLI, Which weighted shifts are subnormal, *Pacific J. Math.* **17**(1966), 367–379.

[20] WOLFRAM RESEARCH, INC., Mathematica, Version 5.0, Wolfram Research Inc., Champaign, IL 1996.

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