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WEIGHTED SHIFTS AND DISJOINT HYPERCYCLICITY

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ABSTRACT. We give characterizations for finite collections of disjoint hypercyclic weighted shift operators, both in the unilateral and bilateral cases. It follows that some well-known results about the dynamics of an operator fail to hold true in the disjoint setting. For example, finite collections of disjoint hypercyclic shifts never satisfy the disjoint hypercyclicity criterion, even though they satisfy the disjoint blow-up/collapse property; thus they are densely disjoint hypercyclic, but are never hereditarily densely disjoint hypercyclic. Moreover, they fail to be disjoint weakly mixing. Also, any finite collection of bilateral shifts containing an invertible shift fails to be disjoint hypercyclic. Even more, each of these facts is in sharp contrast with what happens to finite collections of shift operators raised to positive, distinct powers.

KEYWORDS: *Hypercyclic vectors, hypercyclic operators, unilateral weighted backward shift, bilateral weighted shift.*

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1. INTRODUCTION

Throughout this paper, X denotes a separable, infinite dimensional Banach space over the real or complex scalar field $\mathbb{K}$, and $B(X)$ denotes the space of bounded, linear operators on X. By $\mathbb{Z}$ and $\mathbb{N}$, we denote the set of integers and the set of nonnegative integers, respectively. An operator T in $B(X)$ is *hypercyclic* if there exists a vector x in X such that the orbit, $\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$, is dense in X. Such a vector x is said to be a *hypercyclic vector* for the operator T.

The first example of a hypercyclic operator on a Banach space was given in 1969 by Rolewicz [21], who showed that if B is the unweighted unilateral backward shift on ℓ^2, then λB is hypercyclic if and only if $|\lambda| > 1$.

In a fundamental paper in the area, Salas [22] completely characterized the hypercyclic unilateral weighted backward shifts on ℓ^p with $1 \leqslant p < \infty$ and the hypercyclic bilateral weighted shifts on $\ell^p(\mathbb{Z})$ with $1 \leqslant p < \infty$ in terms of their weight sequences. His proof actually showed that each hypercyclic shift operator

T satisfies the so-called *blow-up/collapse* property; that is, that for any three non-empty open subsets U, V, W of X with $0 \in W$, there exists a positive integer n so that $W \cap T^{-n}(U) \neq \emptyset$ and $V \cap T^{-n}(W) \neq \emptyset$.

León-Saavedra and Montes-Rodríguez ([19], p. 544) later used Salas' weight characterization to show that either type of weighted shift is hypercyclic precisely when it satisfies the so-called *hypercyclicity criterion*. This criterion plays a key role in the theory of hypercyclic operators. It was obtained in slightly different formulations independently by Kitai [17] and by Gethner and Shapiro [15]. It provides a sufficient condition for a general operator to be hypercyclic. More recently, Bernal-González and Grosse-Erdmann ([5], Theorem 3.4), and independently León-Saavedra [18] showed that an operator satisfies the hypercyclicity criterion if and only if it satisfies the blow-up/collapse property.

Other equivalent dynamical properties to satisfying the hypercyclicity criterion include that of being *weakly mixing,* or of being *hereditarily hypercyclic*; see Chapter 4 of [3] and Chapter 3 of [16]. We should also stress that examples of hypercyclic operators that do not satisfy the hypercyclicity criterion are highly non-trivial and have taken decades to find. In a landmark result, De la Rosa and Read [13] constructed a Banach space that supports a hypercyclic operator which is not weakly mixing, and thus fails to satisfy the hypercyclicity criterion. Bayart and Matheron [2] later showed that operators of this kind may even be constructed on c_0 and the ℓ^p spaces with $1 \leqslant p < \infty$.

The aim of this paper is to extend Salas' characterization of hypercyclic shift operators to the setting of disjointness in hypercyclicity introduced independently by Bernal-González [4] and by Bès and Peris [8], as well as to consider the role of the hypercyclicity criterion and its equivalent properties in this new setting. For any integer $N \geqslant 2$, we say that hypercyclic operators $T_1, T_2, \dots, T_N$ acting on the same topological vector space X are *disjoint*, or *d-hypercyclic*, provided there is some vector x in X for which the vector $(x, \dots, x)$ in X^N is a hypercyclic vector for the direct sum operator $\bigoplus_{m=1}^{N} T_m$ acting on the product space X^N, endowed with the product topology. Such a vector x is called a *d-hyperyclic vector* for the operators $T_1, T_2, \dots, T_N$.

Among several examples, Bernal-González [4] provided for each integer $N \geqslant 2$ a unilateral weighted backward shift B on $X = c_0$ and ℓ^p with $1 \leqslant p < \infty$ such that the iterates $B, B^2, \dots, B^N$ have a dense set of d-hypercyclic vectors. Bès and Peris [8] introduced a natural extension of the hypercyclicity criterion to this new setting, called the *d-hypercyclicity criterion*, again a sufficient condition for d-hypercyclicity and equivalent to the property of being *hereditarily densely d-hypercyclic*; see Subsection 1.1. They characterized the d-hypercyclic powers of weighted shifts $B_1^{r_1}, \dots, B_N^{r_N}$ with $N \geqslant 2$ for the case when the powers $r_1, \dots, r_N$ are positive and distinct, showing that in all such cases these N operators must satisfy the d-hypercyclicity criterion.

Finally, they constructed for each integer $N \geqslant 2$ an example of N unilateral weighted backward shifts that are d-hypercyclic, by means of the *disjoint blow-up/collapse* property; see Definition 1.4.

In this paper, we provide a characterization of unilateral weighted backward shifts and bilateral weighted shifts that are d-hypercyclic in terms of their weight sequences; see Theorem 2.2 and Theorem 2.1. In particular, we show that each collection of $N \geqslant 2$ d-hypercyclic shifts satisfies the disjoint blow-up/collapse property, and thus that their set of d-hypercyclic vectors form a dense G_δ set. Using this characterization, we generate d-hypercyclic weighted shifts with specific norms; see Proposition 2.4.

As it turns out, some well known results about a single hypercyclic operator T fail to hold true for d-hypercyclic operators $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$. For example, by considering only invertible weighted backward shifts, Feldman in [14] simplified Salas' necessarily complicated characterization for hypercyclic bilateral weighted shifts. There is no such simplification for the case of d-hypercyclicity. In fact, if we have a finite collection of bilateral weighted shifts and at least one of them is invertible, then they fail to be d-hypercyclic; see Corollary 3.1.

Another surprising consequence is that in the setting of d-hypercyclicity the weakly mixing property and the blow-up/collapse property are no longer equivalent. Indeed, any d-hypercyclic weighted shift operators $B_1, \ldots, B_N$ satisfy the disjoint blow-up/collapse property, but are never *d-weakly mixing*; see Definition 1.9. This implies that the shift operators $B_1, \ldots, B_N$ never satisfy the d-hypercyclicity criterion; see Proposition 3.2. In particular, the weighted shifts $B_1, \ldots, B_N$ are densely d-hypercyclic, but they are never *hereditarily* densely hypercyclic.

The paper is organized in the following manner: In Section 2, we provide the characterizations of d-hypercyclic shifts, both in the unilateral and bilateral cases. In Section 3, we derive some consequences of the characterizations, and in Section 4 we characterize hereditary d-hypercyclicity of powers of shift operators with respect to a given strictly increasing sequence of positive integers. For the sake of convenience, all the results in these sections are stated for weighted shifts on the Hilbert spaces ℓ^2 or $\ell^2(\mathbb{Z})$. All of the results do in fact generalize to weighted shifts on ℓ^p or $\ell^p(\mathbb{Z})$ with $1 \leqslant p < \infty$.

We conclude this introduction with a subsection containing definitions and basic results on disjoint hypercyclicity. For a thorough account on linear dynamics, we refer the reader to the excellent books by Bayart and Matheron [3] and by Grosse-Erdmann and Peris [16]. We also note of recent progress in the study of disjoint hypercyclicity, including on the existence of d-hypercyclic operators on arbitrary separable, infinite-dimensional topological vector spaces and on d-hypercyclic composition operators on spaces of holomorphic functions, appearing in the works of Peris, Salas, Shkarin, and the first two authors, see [9], [10], [11], [23], and [26].

1.1. PRELIMINARIES ON DISJOINT HYPERCYCLICITY.

DEFINITION 1.1. We say that $N \geqslant 2$ sequences of operators $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ in $B(X)$ are *d-topologically transitive* provided for any non-empty open subsets $V_0, V_1, \dots, V_N$ of X, there exists a positive integer m so that

$$\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap \dots \cap T_{N,m}^{-1}(V_N). \tag{1.1}$$

If in addition, (1.1) holds for each integer $j \geqslant m$, then the N sequences of operators are called *d-mixing*. The operators $T_1, \dots, T_N$ themselves are called *d-topologically transitive* if the sequences $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ are d-topologically transitive. Following the same lines, the operators $T_1, \dots, T_N$ themselves are *d-mixing* if the sequences $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ are d-mixing.

DEFINITION 1.2. We say that $N \geqslant 2$ sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ in $B(X)$ are *d-universal* if the set

$$\{(T_{1,j}x, T_{2,j}x, \dots, T_{N,j}x) : j \in \mathbb{N}\}$$

is dense in X^N for some vector $x \in X$. We call such vector x a *d-universal vector* for the sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$. The sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ are *hereditarily d-universal* provided for each increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ the sequences of operators $(T_{1,n_k})_{k=1}^{\infty}, \dots, (T_{N,n_k})_{k=1}^{\infty}$ are d-universal. The operators $T_1, T_2, \dots, T_N$ themselves in $B(X)$ are *hereditarily* d-*hypercyclic* with respect to the sequence $(n_k)_{k=1}^{\infty}$ if the sequences of operators $(T_1^{n_k})_{k=1}^{\infty}, \dots, (T_N^{n_k})_{k=1}^{\infty}$ are hereditarily d-universal. To all the concepts in the above definition, we add the word *densely* if the corresponding set of d-universal vectors is dense in X.

PROPOSITION 1.3 ([8], Proposition 2.3). *Let $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ be sequences of operators in $B(X)$. Then the following are equivalent:*

(i) $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ *are* d-*topologically transitive.*
(ii) *The set of* d-*universal vectors for* $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ *is dense.*
(iii) *The set of* d-*universal vectors for* $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ *is a dense* G_δ.

DEFINITION 1.4. We say that $N \geqslant 2$ sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ in $B(X)$ satisfy the *disjoint blow-up/collapse* property provided for any non-empty open sets $W, V_0, V_1, \dots, V_N$ of X with $0 \in W$, there exists a positive integer n so that

$$W \cap T_{1,n}^{-1}(V_1) \cap \dots \cap T_{N,n}^{-1}(V_N) \neq \emptyset$$
$$V_0 \cap T_{1,n}^{-1}(W) \cap \dots \cap T_{N,n}^{-1}(W) \neq \emptyset.$$

We say that the operators $T_1, \dots, T_N$ themselves in $B(X)$ satisfy the disjoint blow-up/collapse property if their corresponding sequences of iterates $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ do.

PROPOSITION 1.5 ([8], Proposition 2.4). *If $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ are $N \geqslant 2$ sequences of operators acting on a Fréchet space X satisfying the disjoint blow-up/collapse property, then they are* d*-topologically transitive.*

DEFINITION 1.6. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers. We say the operators $T_1, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ satisfy the *d-hypercyclicity criterion with respect to the sequence $(n_k)_{k=1}^{\infty}$* provided there exist dense subsets $X_0, X_1, \dots, X_N$ of X and mappings $S_{m,k} : X_m \to X$ with $1 \leqslant m \leqslant N, k \in \mathbb{N}$ satisfying

$$\begin{aligned} T_m^{n_k} &\underset{k\to\infty}{\to} 0 && \text{pointwise on } X_0, \\ S_{m,k} &\underset{k\to\infty}{\to} 0 && \text{pointwise on } X_m, \text{ and} \\ (T_m^{n_k} S_{i,k} - \delta_{i,m} \operatorname{Id}_{X_m}) &\underset{k\to\infty}{\to} 0 && \text{pointwise on } X_m \ (1 \leqslant i \leqslant N). \end{aligned} \tag{1.2}$$

In general, we say the operators $T_1, \dots, T_N$ satisfy the d-hypercyclicity criterion if there exists some sequence $(n_k)_{k=1}^{\infty}$ for which (1.2) is satisfied.

PROPOSITION 1.7 ([8], Proposition 2.6). *If $T_1, T_2, \dots, T_N$ are $N \geqslant 2$ operators on X that satisfy the* d*-hypercyclicity criterion with respect to a sequence $(n_k)_{k=1}^{\infty}$, then the sequences $(T_1^{n_k})_{k=1}^{\infty}, \dots, (T_N^{n_k})_{k=1}^{\infty}$ are* d*-mixing. In particular, the operators $T_1, \dots,$ T_N are* d*-hypercyclic.*

THEOREM 1.8 ([8], Theorem 2.7). *Let $T_1, \dots, T_N$ be operators in $B(X)$ with $N \geqslant 2$. The following are equivalent:*

(i) *The operators $T_1, \dots, T_N$ satisfy the* d*-hypercyclicity criterion.*

(ii) *The operators $T_1, \dots, T_N$ are hereditarily densely* d*-hypercyclic.*

(iii) *For each $r \in \mathbb{N}$, the direct sum operators $\overbrace{T_1 \oplus \cdots \oplus T_1}^{r}, \dots, \overbrace{T_N \oplus \cdots \oplus T_N}^{r}$ are* d*-topologically transitive on X^r.*

Next, recall that an operator T on X is said to be weakly mixing provided the direct sum $T \oplus T$ is topologically transitive on $X \times X$. Motivated by this, we introduce the following defintion.

DEFINITION 1.9. We say that the operators $T_1, \dots, T_N$ on X with $N \geqslant 2$ are *d-weakly mixing* provided the direct sum operators $T_1 \oplus T_1, \dots, T_N \oplus T_N$ on $X \times X$ are d-topologically transitive.

REMARK 1.10. We note that Proposition 1.3 and Theorem 1.8 ensure that when the operators $T_1, \dots, T_N$ with $N \geqslant 2$ are d-mixing, then they must also satisfy the d-hypercyclicity criterion. Also, Theorem 1.8 ensures that the operators $T_1, \dots, T_N$ with $N \geqslant 2$ must be d-weakly mixing whenever they satisfy the d-hypercyclicity criterion. Hence, d-mixing implies d-weakly mixing.

We conclude this section by proving that d-weakly mixing implies the disjoint blow-up/collapse property.

PROPOSITION 1.11. *If the operators $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ on a topological vector space X are* d*-weakly mixing, then they satisfy the disjoint blow-up/collapse property. That is, for any non-empty open subsets $V_0, \ldots, V_N$ and W of X with $0 \in W$, there exists a positive integer n so that*

$$W \cap T_1^{-n}(V_1) \cap \cdots \cap T_N^{-n}(V_N) \neq \varnothing, \quad V_0 \cap T_1^{-n}(W) \cap \cdots \cap T_N^{-n}(V_N) \quad \neq \varnothing.$$

Proof. Consider the non-empty open subsets $U_0 := W \times V_0$, and $U_j := V_j \times W$ with $1 \leqslant j \leqslant N$ of $X \times X$. Since $T_1 \oplus T_1, \ldots, T_N \oplus T_N$ are d-topologically transitive on $X \times X$, there exists a positive integer n so that

$$\begin{aligned}\varnothing &\neq U_0 \cap (T_1 \oplus T_1)^{-n}(U_1) \cap \cdots \cap (T_N \oplus T_N)^{-n}(U_N) \\ &= (W \cap T_1^{-n}(V_1) \cap \cdots \cap T_N^{-n}(V_N)) \times (V_0 \cap T_1^{-n}(W) \cap \cdots \cap T_N^{-n}(W)),\end{aligned}$$

and the conclusion holds. ∎

2. DISJOINT HYPERCYCLIC WEIGHTED SHIFTS

Let $\{e_i : i \in \mathbb{Z}\}$ (or $\{e_i : i \geqslant 0\}$) be the standard orthonormal basis of the Hilbert space $\ell^2(\mathbb{Z})$ (or ℓ^2) over the real or complex scalar field $\mathbb{K}$. An operator B on $\ell^2(\mathbb{Z})$ (or ℓ^2) is a *bilateral weighted shift* (or a *unilateral weighted backward shift*) if there is a bounded weight sequence $\{w_j : j \in \mathbb{Z}\}$ (or $\{w_j : j \geqslant 1\}$) of nonzero scalars for which $Be_i = w_i e_{i-1}$ for all integers $i \in \mathbb{Z}$ (or $Be_0 = 0$ and $Be_i = w_i e_{i-1}$ for all integers $i \geqslant 1$). We first give the characterization for d-hypercyclic bilateral weighted shifts.

THEOREM 2.1. *Let $N \geqslant 2$ and for each integer m with $1 \leqslant m \leqslant N$, let B_m be a bilateral weighted backward shifts on $\ell^2(\mathbb{Z})$ with the weight sequence $\{w_j^{(m)} : j \in \mathbb{Z}\}$. For integers i, n, and m with $n \geqslant 1$ and $2 \leqslant m \leqslant N$, define*

$$\alpha_{i,n}^{(m)} = \prod_{j=1}^{n} \frac{w_{i+j}^{(m)}}{w_{i+j}^{(1)}}.$$

The following statements are equivalent:

(i) *The shifts $B_1, B_2, \ldots, B_N$ are* d*-hypercyclic.*

(ii) *The shifts $B_1, B_2, \ldots, B_N$ satisfy the disjoint blow up/collapse property.*

(iii) *There exists a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that for each integer i and integer m with $1 \leqslant m \leqslant N$, we have*

$$\Big|\prod_{j=1}^{n_k} w_{i+j}^{(1)}\Big| \to \infty \quad \textit{and} \quad \Big|\prod_{j=0}^{n_k-1} w_{i-j}^{(m)}\Big| \to 0 \quad \textit{as } k \to \infty,$$

and the set

$$\{(\ldots, \alpha_{-1,n_k}^{(2)}, \ldots, \alpha_{-1,n_k}^{(N)}, \alpha_{0,n_k}^{(2)}, \ldots, \alpha_{0,n_k}^{(N)}, \alpha_{1,n_k}^{(2)}, \ldots, \alpha_{1,n_k}^{(N)}, \ldots) : k \geqslant 0\}$$

is dense in $\mathbb{K}^{\mathbb{Z}}$ with respect to the product topology.

Proof. Clearly, (ii) implies (i), by Proposition 1.3 and Proposition 1.5. To show (iii) implies (ii), suppose the weight sequences of the shifts $B_1, \ldots, B_N$ satisfy the conditions in statement (iii), and let $V_0, V_1, \ldots, V_N, W$ be non-empty, open subsets of $\ell^2(\mathbb{Z})$ with $0 \in W$. We want to find a positive integer n for which

$$W \cap B_1^{-n}(V_1) \cap B_2^{-n}(V_2) \cap \cdots \cap B_N^{-n}(V_N) \neq \emptyset, \text{ and}$$
$$V_0 \cap B_1^{-n}(W) \cap B_2^{-n}(W) \cap \cdots \cap B_N^{-n}(W) \neq \emptyset.$$

Since the linear span of the orthonormal basis $\{e_i : i \in \mathbb{Z}\}$ is dense in $\ell^2(\mathbb{Z})$, there exists an integer $r \geqslant 1$ and vectors $h_0, h_1, \ldots, h_N$ in $\operatorname{span}\{e_i : |i| \leqslant r\}$ such that $h_m \in V_m$ for integers $0 \leqslant m \leqslant N$. We may further assume $\langle h_0, e_i \rangle \neq 0$ and $\langle h_1, e_i \rangle \neq 0$ for integers $|i| \leqslant r$. Select an $\varepsilon > 0$ such that $B(0; \varepsilon) \subseteq W$, and $B(h_m; \varepsilon) \subseteq V_m$ for integers $0 \leqslant m \leqslant N$. Define the linear map $S : \operatorname{span}\{e_i : i \in \mathbb{Z}\} \to \operatorname{span}\{e_i : i \in \mathbb{Z}\}$ by $Se_i = [w_{i+1}^{(1)}]^{-1} e_{i+1}$ and extending linearly. For integers i, m, n with $2 \leqslant m \leqslant N$, and $n \geqslant 0$, we have

$$S^n e_i = \Big[\frac{1}{\prod_{j=1}^{n} w_{i+j}^{(1)}} \Big] e_{i+n}, \tag{2.1}$$

$$B_1^n S^n e_i = e_i, \quad \text{and} \tag{2.2}$$

$$B_m^n S^n e_i = \Big[\prod_{j=1}^{n} \frac{w_{i+j}^{(m)}}{w_{i+j}^{(1)}} \Big] e_i = \alpha_{i,n}^{(m)} e_i. \tag{2.3}$$

Using the conditions given in statement (iii), equation (2.1), and the fact that $\langle h_0, e_i \rangle \neq 0$ and $\langle h_1, e_i \rangle \neq 0$ for integers $|i| \leqslant r$, there exists an integer $k \geqslant 0$ such that $n_k > 2r + 1$ and

$$\|S^{n_k} h_1\| < \varepsilon, \tag{2.4}$$

$$\|B_1^{n_k} h_0\| < \varepsilon, \tag{2.5}$$

and so that for integers i, m with $|i| \leqslant r$ and $2 \leqslant m \leqslant N$,

$$\|B_m^{n_k} h_0\| < \varepsilon \text{ and} \tag{2.6}$$

$$|\langle h_1, e_i \rangle \alpha_{i,n_k}^{(m)} - \langle h_m, e_i \rangle| < \frac{\varepsilon}{\sqrt{2r+1}}. \tag{2.7}$$

From inequalities (2.5) and (2.6), it follows that $h_0 \in V_0 \cap B_1^{-n_k}(W) \cap B_2^{-n_k}(W) \cap \cdots \cap B_N^{-n_k}(W)$. From inequality (2.4), we get $S^{n_k} h_1 \in B(0, \varepsilon) \subset W$. By equation (2.2), we have $B_1^{n_k} S^{n_k} h_1 = h_1 \in V_1$, and finally by inequality (2.7) and equation (2.3), for integers $2 \leqslant m \leqslant N$,

$$\|B_m^{n_k} S^{n_k} h_1 - h_m\|^2 = \sum_{i=-r}^{r} |\langle h_1, e_i \rangle \alpha_{i,n_k}^{(m)} - \langle h_m, e_i \rangle|^2 < \sum_{i=-r}^{r} \frac{\varepsilon^2}{2r+1} = \varepsilon^2, \tag{2.8}$$

and so $B_m^{n_k} S^{n_k} h_1 \in B(h_m, \varepsilon) \subset V_m$. Hence, $S^{n_k} h_1 \in W \cap B_1^{-n_k}(V_1) \cap B_2^{-n_k}(V_2) \cap \cdots \cap B_N^{-n_k}(V_N)$, which concludes the proof of (iii) implies (ii).

To establish (i) implies (iii), suppose $g \in \ell^2(\mathbb{Z})$ is a d-hypercyclic vector for the shifts $B_1, B_2, \ldots, B_N$. Let

$$\{(\ldots, \lambda_{-1,k}^{(2)}, \ldots, \lambda_{-1,k}^{(N)}, \lambda_{0,k}^{(2)}, \ldots, \lambda_{0,k}^{(N)}, \lambda_{1,k}^{(2)}, \ldots, \lambda_{1,k}^{(N)}, \ldots) : k \geqslant 0\}$$

be a countable, dense set in $\mathbb{K}^{\mathbb{Z}}$ with respect to the product topology. For each integer $k \geqslant 0$, select a positive real number c_k for which

$$\max\{|\lambda_{i,k}^{(m)}| : 2 \leqslant m \leqslant N \text{ and } |i| \leqslant k\} < c_k. \tag{2.9}$$

Since g is a d-hypercyclic vector for the shifts $B_1, B_2, \ldots, B_N$, there exists a sequence $(r_k)_{k=0}^{\infty}$ of positive integers such that for integers m with $1 \leqslant m \leqslant N$, we have

$$\Big\| B_m^{r_k} g - \sum_{|i| \leqslant k} e_i \Big\| < \frac{1}{2^{k+1}}. \tag{2.10}$$

This inequality implies $\langle B_m^{r_k} g, e_i \rangle \to 1$ as $k \to \infty$ for $|i| \leqslant k$.

For each integer $k \geqslant 0$, let $M_k := \min\Big\{ \frac{1}{2^{k+1}}, \frac{1}{c_k 2^{k+1}} \Big\}$ and $\delta_k = \min(A_k)$, where

$$A_k = \{M_k\} \cup \Big\{ M_k \Big| \prod_{j=0}^{r_k - 1} w_{i-j}^{(m)} \Big| : |i| \leqslant k, 1 \leqslant m \leqslant N \Big\}.$$

Choose a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of integers such that for each integer $k \geqslant 0$, we have $n_k > r_k + 2k + 1$ and

$$\Big\| B_1^{n_k + r_k} g - \sum_{|i| \leqslant k} \Big(\prod_{j=0}^{r_k - 1} w_{i-j}^{(1)} \Big) e_{i - r_k} \Big\| < \delta_k, \tag{2.11}$$

and so that for integers m with $2 \leqslant m \leqslant N$,

$$\Big\| B_m^{n_k + r_k} g - \sum_{|i| \leqslant k} \lambda_{i,k}^{(m)} \Big(\prod_{j=0}^{r_k - 1} w_{i-j}^{(m)} \Big) e_{i - r_k} \Big\| < \delta_k. \tag{2.12}$$

From inequality (2.11), it follows that for integers i with $|i| \leqslant k$,

$$|\langle B_1^{r_k} g, e_i \rangle| \Big| \prod_{j=0}^{n_k - 1} w_{i-j}^{(1)} \Big| = |\langle B_1^{n_k}(B_1^{r_k} g), e_{i - n_k} \rangle| < \delta_k \leqslant \frac{1}{2^{k+1}}.$$

Likewise, for integers i, m with $|i| \leqslant k$ and $2 \leqslant m \leqslant N$, inequality (2.12) gives

$$|\langle B_m^{r_k} g, e_i \rangle| \Big| \prod_{j=0}^{n_k - 1} w_{i-j}^{(m)} \Big| < \frac{1}{2^{k+1}}.$$

Therefore, since $\langle B_m^{r_k} g, e_i \rangle \to 1$ as $k \to \infty$, it follows that for integers $1 \leqslant m \leqslant N$,

$$\prod_{j=0}^{n_k-1} w_{i-j}^{(m)} \to 0 \quad \text{as } k \to \infty.$$

Next, observe that for integers i, m with $|i| \leqslant k$ and $1 \leqslant m \leqslant N$, we have

$$\langle B_m^{n_k+r_k} g, e_{i-r_k} \rangle = \langle g, e_{i+n_k} \rangle \prod_{j=1}^{n_k+r_k} w_{i-r_k+j}^{(m)} = \langle g, e_{i+n_k} \rangle \Big(\prod_{j=0}^{r_k-1} w_{i-j}^{(m)} \Big) \prod_{j=1}^{n_k} w_{i+j}^{(m)}.$$

Therefore, by inequality (2.11), we get

$$\Big| \langle g, e_{i+n_k} \rangle \Big(\prod_{j=0}^{r_k-1} w_{i-j}^{(1)} \Big) \prod_{j=1}^{n_k} w_{i+j}^{(1)} - \prod_{j=0}^{r_k-1} w_{i-j}^{(1)} \Big| < M_k \Big| \prod_{j=0}^{r_k-1} w_{i-j}^{(1)} \Big|,$$

which gives us

$$\Big| \langle g, e_{i+n_k} \rangle \prod_{j=1}^{n_k} w_{i+j}^{(1)} - 1 \Big| < M_k = \min\Big\{ \frac{1}{2^{k+1}}, \frac{1}{c_k 2^{k+1}} \Big\}. \tag{2.13}$$

In particular, note that $\langle g, e_{i+n_k} \rangle \neq 0$ for integers $|i| \leqslant k$. Also from inequality (2.12), it follows that for integers i, m with $|i| \leqslant k$ and $1 \leqslant m \leqslant N$,

$$\Big| \langle g, e_{i+n_k} \rangle \prod_{j=0}^{r_k-1} w_{i-j}^{(m)} \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \prod_{j=0}^{r_k-1} w_{i-j}^{(m)} \Big| < \frac{1}{2^{k+1}} \Big| \prod_{j=0}^{r_k-1} w_{i-j}^{(m)} \Big|,$$

and so

$$\Big| \langle g, e_{i+n_k} \rangle \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \Big| < \frac{1}{2^{k+1}}. \tag{2.14}$$

To establish $\Big| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \to \infty$ as $k \to \infty$, note that inequality (2.13) implies that for integers i, k with $k \geqslant 0$ and $|i| \leqslant k$, we have

$$\Big| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| > \frac{1 - \frac{1}{2^{k+1}}}{|\langle g, e_{i+n_k} \rangle|}. \tag{2.15}$$

Since $|\langle g, e_{i+n_k} \rangle| \to 0$ as $k \to \infty$, we get the desired result.

Finally to establish that the set

$$\{ (\ldots, \alpha_{-1,n_k}^{(2)}, \ldots, \alpha_{-1,n_k}^{(N)}, \alpha_{0,n_k}^{(2)}, \ldots, \alpha_{0,n_k}^{(N)}, \alpha_{1,n_k}^{(2)}, \ldots, \alpha_{1,n_k}^{(N)}, \ldots) : k \geqslant 0 \}$$

is dense in $\mathbb{K}^{\mathbb{Z}}$ with respect to the product topology, it suffices to show that for integers i, m with $i \in \mathbb{Z}$ and $2 \leqslant m \leqslant N$, we have $|\alpha_{i,n_k}^{(m)} - \lambda_{i,k}^{(m)}| \to 0$ as $k \to \infty$.

To this end, note that using inequalities (2.13), (2.14), and (2.15), we have that for any integers i and k with $k > |i|$,

$$
\begin{aligned}
|\alpha_{i,n_k}^{(m)} - \lambda_{i,k}^{(m)}| &= \Big| \prod_{j=1}^{n_k} \frac{w_{i+j}^{(m)}}{w_{i+j}^{(1)}} - \lambda_{i,k}^{(m)} \Big| = \frac{1}{\Big| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big|} \Big| \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \\
&\leqslant \frac{|\langle g, e_{i+n_k}\rangle|}{1-\frac{1}{2^{k+1}}} \Big| \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \\
&= \frac{2^{k+1}}{2^{k+1}-1} \Big| \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \\
&= \frac{2^{k+1}}{2^{k+1}-1} \Big| \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \Big| \\
&\quad + \frac{2^{k+1}}{2^{k+1}-1} \Big| \lambda_{i,k}^{(m)} - \lambda_{i,k}^{(m)} \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \\
&= \frac{2^{k+1}}{2^{k+1}-1} \Big| \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \Big| \\
&\quad + \frac{2^{k+1}}{2^{k+1}-1} |\lambda_{i,k}^{(m)}| \Big| 1 - \langle g, e_{i+n_k}\rangle \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \\
&< \frac{2^{k+1}}{2^{k+1}-1} \frac{1}{2^{k+1}} + \frac{2^{k+1}}{2^{k+1}-1} c_k \frac{1}{c_k 2^{k+1}} = \frac{2}{2^{k+1}-1}. \qquad (2.16)
\end{aligned}
$$

Therefore, $|\alpha_{i,n_k}^{(m)} - \lambda_{i,k}^{(m)}| \to 0$ as $k \to \infty$. ∎

Due to the similar shifting patterns, the unilateral weighted backward shift case of the characterization is similar to the bilateral weighted shift case found in Theorem 2.1. Moreover, the proof of the unilateral characterization is a simpler version of the proof of Theorem 2.1. As a result, we include a basic outline of the proof, and leave some of the details for the reader.

THEOREM 2.2. *Let $N \geqslant 2$, and for each integer m with $1 \leqslant m \leqslant N$, let B_m be a unilateral weighted backward shift on ℓ^2 with the weight sequence $\{w_j^{(m)} : j \geqslant 1\}$. For integers i, n, m with $n \geqslant 1$, $i \geqslant 0$, and $2 \leqslant m \leqslant N$, define*

$$\alpha_{i,n}^{(m)} = \prod_{j=1}^{n} \frac{w_{i+j}^{(m)}}{w_{i+j}^{(1)}}.$$

The following statements are equivalent:

(i) *The shifts $B_1, B_2, \ldots, B_N$ are* d-*hypercyclic.*

(ii) *The shifts $B_1, B_2, \ldots, B_N$ satisfy the disjoint blow up/collapse property.*

(iii) *There exists a strictly increasing sequence* $(n_k)_{k=0}^{\infty}$ *of positive integers such that for each integer* $i \geqslant 0$, *we have*

$$\Big|\prod_{j=1}^{n_k} w_{i+j}^{(1)}\Big| \to \infty \quad \text{as } k \to \infty,$$

and the set

$$\{(\alpha_{0,n_k}^{(2)}, \alpha_{0,n_k}^{(3)}, \ldots, \alpha_{0,n_k}^{(N)}, \alpha_{1,n_k}^{(2)}, \alpha_{1,n_k}^{(3)}, \ldots, \alpha_{1,n_k}^{(N)}, \ldots) : k \geqslant 0\}$$

is dense in $\mathbb{K}^{\mathbb{N}}$ *with respect to the product topology.*

Proof. As in the bilateral case, (ii) implies (i) is clear. For (iii) implies (ii), given non-empty, open subsets $V_0, V_1, \ldots, V_N, W$ of ℓ^2 with $0 \in W$, select an integer $r \geqslant 1$, vectors $h_0, h_1, \ldots, h_N$ in $\operatorname{span}\{e_i : 0 \leqslant i \leqslant r\}$ and an $\varepsilon > 0$ such that $B(0;\varepsilon) \subseteq W$, and $B(h_m;\varepsilon) \subseteq V_m$ for integers $1 \leqslant m \leqslant N$. Further assume $\langle h_1, e_i\rangle \neq 0$ for integers $0 \leqslant i \leqslant r$. Using the conditions in statement (iii), select an integer $k \geqslant 0$ for which $n_k > r+1$, $\|S^{n_k} h_1\| < \varepsilon$, and for integers i, m with $0 \leqslant i \leqslant r$ and $2 \leqslant m \leqslant N$,

$$|\langle h_1, e_i\rangle \alpha_{i,n_k}^{(m)} - \langle h_m, e_i\rangle| < \frac{\varepsilon}{\sqrt{r+1}},$$

where the linear map $S : \operatorname{span}\{e_i : i \geqslant 0\} \to \operatorname{span}\{e_i : i \geqslant 0\}$ is given by $Se_i = [w_{i+1}^{(1)}]^{-1} e_{i+1}$. With computations similar to (2.8) together with the fact that $B_m^{n_k} h_0 = 0$, we get

$$h_0 \in V_0 \cap B_1^{-n_k}(W) \cap B_2^{-n_k}(W) \cap \cdots \cap B_N^{-n_k}(W) \text{ and}$$
$$S^{n_k} h_1 \in W \cap B_1^{-n_k}(V_1) \cap B_2^{-n_k}(V_2) \cap \cdots \cap B_N^{-n_k}(V_N),$$

which concludes the proof of (iii) implies (ii).

The establishment of (i) implies (iii) requires fewer steps than its bilateral counterpart. For this case, let g in ℓ^2 be a d-hypercycic vector for the shifts $B_1, B_2, \ldots, B_N$, and let

$$\{(\lambda_{0,k}^{(2)}, \lambda_{0,k}^{(3)}, \ldots, \lambda_{0,k}^{(N)}, \lambda_{1,k}^{(2)}, \lambda_{1,k}^{(3)}, \ldots, \lambda_{1,k}^{(N)}, \ldots) : k \geqslant 0\}$$

be a countable, dense set in $\mathbb{K}^{\mathbb{N}}$ with respect to the product topology. For each integer $k \geqslant 0$, choose a positive real number c_k for which

$$\max\{|\lambda_{i,k}^{(m)}| : 0 \leqslant i \leqslant k, 2 \leqslant m \leqslant N\} < c_k.$$

Next, select a a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that for each integer $k \geqslant 0$, we have

$$\Big\|B_1^{n_k} g - \sum_{l=0}^{k} e_l\Big\| < \min\Big\{\frac{1}{2^{k+1}}, \frac{1}{c_k 2^{k+1}}\Big\}, \tag{2.17}$$

and for integers m with $2 \leqslant m \leqslant N$,

$$\Big\| B_m^{n_k} g - \sum_{l=0}^{k} \lambda_{l,k}^{(m)} e_l \Big\| < \frac{1}{2^{k+1}}. \tag{2.18}$$

From inequalities (2.17) and (2.18), we get that for integers i, k with $0 \leqslant i \leqslant k$, we have

$$\Big| \langle g, e_{i+n_k} \rangle \prod_{j=1}^{n_k} w_{i+j}^{(1)} - 1 \Big| < \min\Big\{ \frac{1}{2^{k+1}}, \frac{1}{c_k 2^{k+1}} \Big\},$$

and for integers m with $2 \leqslant m \leqslant N$,

$$\Big| \langle g, e_{i+n_k} \rangle \prod_{j=1}^{n_k} w_{i+j}^{(m)} - \lambda_{i,k}^{(m)} \Big| < \frac{1}{2^{k+1}}.$$

The remainder of the proof of (i) implies (iii) follows from computations similar to those found in (2.15) and in (2.16) within the proof of the bilateral case. ∎

REMARK 2.3. (i) Due to Proposition 1.3 and Proposition 1.5, the set of d-hypercyclic vectors for a finite family of bilateral weighted shifts or of unilateral weighted backward shifts is either the empty set or a dense G_δ set.

(ii) It is easy to see that the first assumption in condition (iii) of Theorem 2.2, that $\Big| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \Big| \to \infty$ as $k \to \infty$, may be replaced by the assumption that for integers i, m with $i \geqslant 0$ and $1 \leqslant m \leqslant N$, we have $\Big| \prod_{j=1}^{n_k} w_{i+j}^{(m)} \Big| \to \infty$ as $k \to \infty$. Also as expected, because the order in which we list the unilateral weighted backward shifts $B_1, \ldots, B_N$ does not affect their d-hypercyclic property, the second assumption in condition (iii) should not convey any special role to the first weight sequence $\{w_j^{(1)} : j \geqslant 1\}$. A similar observation holds for the bilateral weighted shifts in Theorem 2.1.

(iii) For unilateral weighted backward shifts $B_1, \ldots, B_N$ to be d-hypercyclic, their weight sequences *cannot* be bounded away from zero. Indeed, notice that for any integers m with $2 \leqslant m \leqslant N$ and any integers $n \geqslant 1$, we have $\alpha_{0,n}^{(m)} = \frac{w_1^{(m)} w_{1+n}^{(1)}}{w_1^{(1)} w_{1+n}^{(m)}} \alpha_{1,n}^{(m)}$. So the density of $\{(\alpha_{0,n}^{(m)}, \alpha_{1,n}^{(m)}) : n \geqslant 1\}$ in $\mathbb{K}^2$ that is ensured by Theorem 2.2 gives that for every $2 \leqslant m \leqslant N$ the weight sequence $\{w_j^{(m)} : j \geqslant 1\}$ is not bounded away from zero. Part (ii) of this remark ensures that $\{w_j^{(1)} : j \geqslant 1\}$ is not bounded away from zero, either. With Corollary 3.1, we point out that the weight sequences for d-hypercyclic bilateral weighted shifts cannot be bounded away from zero as well.

(iv) Part (iii) of this remark is in sharp contrast with what happens with unilateral weighted shifts raised to different powers. Indeed, for the unweighted backward shift B on ℓ^2, the operators $\lambda_1 B$, $\lambda_2 B^2$ are d-mixing whenever $1 < |\lambda_1| < |\lambda_2|$, see Corollary 4.7.

For any integer $N \geqslant 2$, Bès and Peris ([8], Theorem 4.6) constructed an example of N d-hypercyclic unilateral weighted backward shifts on ℓ^2. Using Theorem 2.2 and Theorem 2.1, we create other examples of d-hypercyclic unilateral weighted backward shifts, as well as, examples for bilateral weighted shifts. Moreover, we can maintain control over the norms of the shifts.

PROPOSITION 2.4. *For any integer $N \geqslant 2$ and real scalars β, ε with $\beta > 1$ and $0 < \varepsilon < \beta$, there exists* d*-hypercyclic bilateral weighted (or unilateral weight backward) shifts $B_1, B_2, \dots, B_N$ on $\ell^2(\mathbb{Z})$ (or ℓ^2) over the scalar field $\mathbb{C}$ such that $\|B_1\| = \beta$, and for integers m with $2 \leqslant m \leqslant N$, we have $\|B_1 - B_m\| < \varepsilon$.*

Proof. For simplicity, we show the case $N = 2$. First, we establish the bilateral case. Let $\{(\dots, \lambda_{-1}^{(k)}, \lambda_0^{(k)}, \lambda_1^{(k)}, \dots) : k \geqslant 0\}$ be a countable dense set in $\mathbb{C}^{\mathbb{Z}}$ with respect to the product topology. Without loss of generality, assume that for any integers i, k with $k \geqslant 0$ and $|i| \leqslant k$, we have $\lambda_i^{(k)} \neq 0$. To create the d-hypercyclic bilateral weighted shifts B_1, B_2, from Theorem 2.1, it suffices to construct two bounded weight sequences $\{w_j^{(1)} : j \in \mathbb{Z}\}$,$\{w_j^{(2)} : j \in \mathbb{Z}\}$ and a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that for any integers i, k, m with $k \geqslant 0$, $|i| \leqslant k$, and $1 \leqslant m \leqslant 2$, we have

$$\Big|\prod_{j=1}^{n_k} w_{i+j}^{(1)}\Big| > \beta^k, \quad \Big|\prod_{j=0}^{n_k-1} w_{i-j}^{(m)}\Big| < \frac{1}{\beta^k}, \quad \text{and} \quad \prod_{j=1}^{n_k} \frac{w_{i+j}^{(2)}}{w_{i+j}^{(1)}} = \lambda_i^{(k)}. \tag{2.19}$$

Furthermore, we require

$$\sup\{|w_j^{(1)}| : j \in \mathbb{Z}\} = \beta \quad \text{and} \quad \sup\{|w_j^{(1)} - w_j^{(2)}| : j \in \mathbb{Z}\} < \varepsilon. \tag{2.20}$$

To define the nonpositive indexed weights, let $w_j^{(1)} = w_j^{(2)} = \frac{1}{\beta}$ for integers $j \leqslant 0$. Next, we define the positive indexed weights in blocks. To this end, write $\lambda_0^{(0)} = r_0 \mathrm{e}^{\mathrm{i}\theta_0}$ for some $r_0 > 0$ and $\theta_0 \in \mathbb{R}$. Since $\lim_{n\to\infty} r^{1/n} \mathrm{e}^{\mathrm{i}\theta/n} = 1$ for any $r > 0$ and $\theta \in \mathbb{R}$, we may select an integer $n_0 > 1$ such that $|\beta - \beta r_0^{1/n_0} \mathrm{e}^{\mathrm{i}\theta_0/n_0}| < \frac{\varepsilon}{2}$. Set

$$w_1^{(1)} = w_2^{(1)} = \cdots = w_{n_0}^{(1)} = \beta, \tag{2.21}$$

$$w_1^{(2)} = w_2^{(2)} = \cdots = w_{n_0}^{(2)} = \beta r_0^{1/n_0} \mathrm{e}^{\mathrm{i}\theta_0/n_0}. \tag{2.22}$$

Observe that $|w_j^{(1)}| = \beta$ and $|w_j^{(1)} - w_j^{(2)}| < \frac{\varepsilon}{2}$ for integers $1 \leqslant j \leqslant n_0$. Also,

$$\Big|\prod_{j=1}^{n_0} w_j^{(1)}\Big| = \beta^{n_0} > 1, \quad \Big|\prod_{j=0}^{n_0-1} w_{-j}^{(m)}\Big| = \frac{1}{\beta^{n_0}} < 1, \quad \text{and} \quad \prod_{j=1}^{n_0} \frac{w_j^{(2)}}{w_j^{(1)}} = \lambda_0^{(0)}. \tag{2.23}$$

To select the integer n_1 and weights $w_{1+n_0}^{(1)}, \dots, w_{1+n_1}^{(1)}, w_{1+n_0}^{(2)}, \dots, w_{1+n_1}^{(2)}$, write

$$\frac{\lambda_{-1}^{(1)}}{\lambda_0^{(0)}} = r_1 \mathrm{e}^{\mathrm{i}\theta_1} \tag{2.24}$$

for some $r_1 > 0$ and $\theta_1 \in \mathbb{R}$, and select a δ_1 with $0 < \delta_1 < \beta$ such that

$$\delta_1 \max\left\{\Big|1 - \frac{\lambda_i^{(1)} w_i^{(2)}}{\lambda_{i-1}^{(1)} w_i^{(1)}}\Big| : 0 \leqslant i \leqslant 1\right\} < \frac{\varepsilon}{2}. \tag{2.25}$$

Choose an integer $n_1 > n_0 + 2$ such that

$$|\beta - \beta r_1^{1/(n_1-n_0-1)} \mathrm{e}^{\mathrm{i}\theta_1/(n_1-n_0-1)}| < \frac{\varepsilon}{2}, \quad \text{and}$$

$$\beta^{n_1-n_0-1} \min\left\{\delta_1^{i+1} \Big|\prod_{j=i+1}^{n_0} w_j^{(1)}\Big| : |i| \leqslant 1\right\} > \beta. \tag{2.26}$$

From the definition of the nonpositive indexed weights, we may further assume that for integers i, m with $|i| \leqslant 1$ and $1 \leqslant m \leqslant 2$, we have

$$\Big|\prod_{j=0}^{n_1-1} w_{i-j}^{(m)}\Big| < \frac{1}{\beta}.$$

Set

$$w_{1+n_0}^{(1)} = \cdots = w_{-1+n_1}^{(1)} = \beta, \tag{2.27}$$

$$w_{j+n_1}^{(1)} = \delta_1 \quad \text{for integers } 0 \leqslant j \leqslant 1, \tag{2.28}$$

$$w_{1+n_0}^{(2)} = \cdots = w_{-1+n_1}^{(2)} = \beta r_1^{1/(n_1-n_0-1)} \mathrm{e}^{\mathrm{i}\theta_1/(n_1-n_0-1)}, \tag{2.29}$$

$$w_{j+n_1}^{(2)} = \frac{\lambda_j^{(1)} w_j^{(2)}}{\lambda_{j-1}^{(1)} w_j^{(1)}} \delta_1 \quad \text{for integers } 0 \leqslant j \leqslant 1. \tag{2.30}$$

Note $|w_j^{(1)}| \leqslant \beta$ and $|w_j^{(1)} - w_j^{(2)}| < \frac{\varepsilon}{2}$ for integers $1 + n_0 \leqslant j \leqslant 1 + n_1$. For integers i with $|i| \leqslant 1$, we have

$$\Big|\prod_{j=1}^{n_1} w_{i+j}^{(1)}\Big| = \Big|\prod_{j=i+1}^{n_0} w_j^{(1)} \prod_{j=n_0+1}^{i+n_1} w_j^{(1)}\Big|$$

$$
\begin{aligned}
&= \Big|\prod_{j=i+1}^{n_0} w_j^{(1)}\Big|\beta^{n_1-n_0-1}\delta_1^{i+1}, \quad \text{by (2.27), (2.28)}\\
&> \beta, \quad \text{by (2.26).}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\prod_{j=1}^{n_1}\frac{w_{-1+j}^{(2)}}{w_{-1+j}^{(1)}} &= \frac{w_0^{(2)}}{w_0^{(1)}}\prod_{j=1}^{n_0}\frac{w_j^{(2)}}{w_j^{(1)}}\prod_{j=1+n_0}^{-1+n_1}\frac{w_j^{(2)}}{w_j^{(1)}}\\
&= \prod_{j=1}^{n_0}\frac{w_j^{(2)}}{w_j^{(1)}}r_1\mathrm{e}^{\mathrm{i}\theta_1}, \quad \text{by } w_0^{(1)} = w_0^{(2)} \text{ and (2.28), (2.29)}\\
&= \lambda_{-1}^{(1)}, \quad \text{by (2.23), (2.24),} \qquad (2.31)
\end{aligned}
$$

and for integers i with $0 \leqslant i \leqslant 1$

$$
\begin{aligned}
\prod_{j=1}^{n_1}\frac{w_{i+j}^{(2)}}{w_{i+j}^{(1)}} &= \prod_{j=0}^{i}\frac{w_j^{(1)}}{w_j^{(2)}}\prod_{j=1}^{n_1}\frac{w_{-1+j}^{(2)}}{w_{-1+j}^{(1)}}\prod_{j=0}^{i}\frac{w_{j+n_1}^{(2)}}{w_{j+n_1}^{(1)}}\\
&= \prod_{j=0}^{i}\frac{w_j^{(1)}}{w_j^{(2)}}\lambda_{-1}^{(1)}\prod_{j=0}^{i}\frac{\lambda_j^{(1)}w_j^{(2)}}{\lambda_{j-1}^{(1)}w_j^{(1)}}, \quad \text{by (2.28), (2.30), (2.31)}\\
&= \lambda_i^{(1)}.
\end{aligned}
$$

Inductively, suppose for $k \geqslant 2$, the positive integers $n_1, \ldots, n_{k-1}$ and the weights $w_{1+n_0}^{(1)}, \ldots, w_{k-1+n_{k-1}}^{(1)}$, $w_{1+n_0}^{(2)}, \ldots, w_{k-1+n_{k-1}}^{(2)}$ have been chosen such that $|w_j^{(1)}| \leqslant \beta$ and $|w_j^{(1)} - w_j^{(2)}| < \frac{\varepsilon}{2}$ for $1 + n_0 \leqslant j \leqslant k-1+n_{k-1}$, and for integers i, l, m with $1 \leqslant l \leqslant k-1$, $|i| \leqslant l$, and $1 \leqslant m \leqslant 2$, we have $n_l > 2l + n_{l-1}$ together with

$$
\Big|\prod_{j=1}^{n_l} w_{i+j}^{(1)}\Big| > \beta^l, \quad \Big|\prod_{j=0}^{n_l-1} w_{i-j}^{(m)}\Big| < \frac{1}{\beta^l}, \quad \text{and} \quad \prod_{j=1}^{n_l}\frac{w_{i+j}^{(2)}}{w_{i+j}^{(1)}} = \lambda_i^{(l)}. \qquad (2.32)
$$

To select the integer n_k and weights $w_{k+n_{k-1}}^{(1)}, \ldots, w_{k+n_k}^{(1)}$, $w_{k+n_{k-1}}^{(2)}, \ldots, w_{k+n_k}^{(2)}$, write

$$
\frac{\lambda_{-k}^{(k)}}{\lambda_{k-1}^{(k-1)}}\prod_{j=1}^{k-1}\frac{w_j^{(1)}}{w_j^{(2)}} = r_k\mathrm{e}^{\mathrm{i}\theta_k} \qquad (2.33)
$$

for some $r_k > 0$ and $\theta_k \in \mathbb{R}$. Select an δ_k with $0 < \delta_k < \beta$ such that

$$
\delta_k \max\left\{\Big|1 - \frac{\lambda_i^{(k)}w_i^{(2)}}{\lambda_{i-1}^{(k)}w_i^{(1)}}\Big| : -k+1 \leqslant i \leqslant k\right\} < \frac{\varepsilon}{2}. \qquad (2.34)
$$

Choose an integer $n_k > 2k + n_{k-1}$ such that

$$|\beta - \beta r_k^{1/(n_k-n_{k-1}-2k+1)} e^{\mathrm{i}\theta_k/(n_k-n_{k-1}-2k+1)}| < \frac{\varepsilon}{2},$$

$$\beta^{n_k-n_{k-1}-2k+1} \min\left\{\delta_k^{i+k}\Big|\prod_{j=i+1}^{k-1+n_{k-1}} w_j^{(1)}\Big| : |i| \leqslant k\right\} > \beta^k, \tag{2.35}$$

and for integers i, m with $|i| \leqslant k$ and $1 \leqslant m \leqslant 2$,

$$\Big|\prod_{j=0}^{n_k-1} w_{i-j}^{(m)}\Big| < \frac{1}{\beta^k}.$$

Set

$$w_{k+n_{k-1}}^{(1)} = \cdots = w_{-k+n_k}^{(1)} = \beta, \tag{2.36}$$

$$w_{j+n_k}^{(1)} = \delta_k, \quad \text{for } -k+1 \leqslant j \leqslant k, \tag{2.37}$$

$$w_{k+n_{k-1}}^{(2)} = \cdots = w_{-k+n_k}^{(2)} = \beta r_k^{1/(n_k-n_{k-1}-2k+1)} e^{\mathrm{i}\theta_k/(n_k-n_{k-1}-2k+1)}, \tag{2.38}$$

$$w_{j+n_k}^{(2)} = \frac{\lambda_j^{(k)} w_j^{(2)}}{\lambda_{j-1}^{(k)} w_j^{(1)}}\delta_k, \quad \text{for } -k+1 \leqslant j \leqslant k. \tag{2.39}$$

Observe that $|w_j^{(1)}| \leqslant \beta$ and $|w_j^{(1)} - w_j^{(2)}| < \frac{\varepsilon}{2}$ for integers $k + n_{k-1} \leqslant j \leqslant k + n_k$. For integers i with $|i| \leqslant k$, we have

$$\begin{aligned}
\Big|\prod_{j=1}^{n_k} w_{i+j}^{(1)}\Big| &= \Big|\prod_{j=i+1}^{k-1+n_{k-1}} w_j^{(1)} \prod_{j=k+n_{k-1}}^{i+n_k} w_j^{(1)}\Big| \\
&= \Big|\prod_{j=i+1}^{k-1+n_{k-1}} w_j^{(1)}\Big|\beta^{n_k-n_{k-1}-2k+1}\delta_k^{i+k}, \quad \text{by (2.36), (2.37)} \\
&> \beta^k, \quad \text{by (2.35).}
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\prod_{j=1}^{n_k} \frac{w_{-k+j}^{(2)}}{w_{-k+j}^{(1)}} &= \prod_{j=1}^{-k+n_k} \frac{w_j^{(2)}}{w_j^{(1)}}, \quad \text{since } w_j^{(1)} = w_j^{(2)} \text{ for } j \leqslant 0, \\
&= \prod_{j=1}^{k-1} \frac{w_j^{(2)}}{w_j^{(1)}} \prod_{j=1}^{n_{k-1}} \frac{w_{k-1+j}^{(2)}}{w_{k-1+j}^{(1)}} \prod_{j=k+n_{k-1}}^{-k+n_k} \frac{w_j^{(2)}}{w_j^{(1)}} \\
&= \prod_{j=1}^{k-1} \frac{w_j^{(2)}}{w_j^{(1)}} \lambda_{k-1}^{(k-1)} r_k e^{\mathrm{i}\theta_k}, \quad \text{by (2.32), (2.36), (2.38),} \\
&= \lambda_{-k}^{(k)}, \quad \text{by (2.33),}
\end{aligned} \tag{2.40}$$

and for integers i with $-k+1 \leqslant i \leqslant k$,

$$\begin{aligned}\prod_{j=1}^{n_k} \frac{w_{i+j}^{(2)}}{w_{i+j}^{(1)}} &= \prod_{j=-k+1}^{i} \frac{w_j^{(1)}}{w_j^{(2)}} \prod_{j=1}^{n_k} \frac{w_{-k+j}^{(2)}}{w_{-k+j}^{(1)}} \prod_{j=-k+1}^{i} \frac{w_{j+n_k}^{(2)}}{w_{j+n_k}^{(1)}} \\ &= \prod_{j=-k+1}^{i} \frac{w_j^{(1)}}{w_j^{(2)}} \lambda_{-k}^{(k)} \prod_{j=-k+1}^{i} \frac{\lambda_j^{(k)} w_j^{(2)}}{\lambda_{j-1}^{(k)} w_j^{(1)}}, \quad \text{by (2.37), (2.39), (2.40)} \\ &= \lambda_i^{(k)}.\end{aligned}$$

The weight sequences $\{w_j^{(1)} : j \in \mathbb{Z}\}$, $\{w_j^{(2)} : j \in \mathbb{Z}\}$ and the strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers constructed according to this procedure satisfy the inequalities in (2.21), and so the corresponding bilateral weighted shifts B_1, B_2 are d-hypercyclic. Moreover, the weight sequences satisfy $\sup\{|w_j^{(1)} - w_j^{(2)}| : j \in \mathbb{Z}\} \leqslant \frac{\varepsilon}{2} < \varepsilon$, and $\sup\{|w_j^{(1)}| : j \in \mathbb{Z}\} = \beta$ giving us the desired shifts. For the unilateral case, use only the positive index weights $\{w_j^{(1)} : j \geqslant 1\}$, $\{w_j^{(2)} : j \geqslant 1\}$ from the weight sequences given above. Due to the similar shifting patterns, it follows that the corresponding unilateral weighted backward B_1, B_2 shifts are d-hypercyclic satisfying $\|B_1\| = \beta$ and $\|B_1 - B_2\| < \varepsilon$. ∎

3. CONSEQUENCES OF THEOREM 2.2 AND THEOREM 2.1

As in the unilateral case, whenever bilateral weighted shifts $B_1, B_2, \ldots, B_N$ are d-hypercyclic, the weight sequence $\{w_j^{(m)} : j \in \mathbb{Z}\}$ of each B_m with $1 \leqslant m \leqslant N$ must not be bounded away from zero. We also stress that this is in sharp contrast to what happens with bilateral weighted shifts raised to distinct powers. Indeed, any mixing (and hence any invertible and mixing) bilateral weighted shift B satisfies that $B, B^2, \ldots, B^N$ are d-mixing on $\ell^2(\mathbb{Z})$ for each integer $N \geqslant 2$; see Theorem 3.4 of [11].

COROLLARY 3.1. *Let $N \geqslant 2$, and suppose $B_1, B_2, \ldots, B_N$ are* d-*hypercyclic bilateral weighted shifts on $\ell^2(\mathbb{Z})$. Then none of the shifts $B_1, B_2, \ldots, B_N$ is invertible.*

Proof. By way of contradiction, say the shift B_1 is invertible. For integers $m = 1, 2$, let $\{w_j^{(m)} : j \in \mathbb{Z}\}$ be the weight sequence for B_m, and set $\alpha = \inf\{|w_j^{(1)}| : j \in \mathbb{Z}\}$ and $\beta = \sup\{|w_j^{(2)}| : j \in \mathbb{Z}\}$. Note $\alpha > 0$. By Theorem 2.1, there exists a strictly increasing sequence $(m_k)_{k=0}^{\infty}$ of positive integers such that

$$\prod_{j=1}^{m_k} \frac{w_j^{(2)}}{w_j^{(1)}} \to w_1^{(2)} \alpha \quad \text{and} \quad \prod_{j=1}^{m_k} \frac{w_{1+j}^{(2)}}{w_{1+j}^{(1)}} \to 2 w_1^{(1)} \beta \quad \text{as } k \to \infty.$$

Thus,

$$\left|\frac{w_1^{(2)} w_{m_k+1}^{(1)}}{w_1^{(1)} w_{m_k+1}^{(2)}}\right| = \frac{\left|\prod_{j=1}^{m_k} \frac{w_j^{(2)}}{w_j^{(1)}}\right|}{\left|\prod_{j=1}^{m_k} \frac{w_{j+1}^{(2)}}{w_{j+1}^{(1)}}\right|} \to \frac{1}{2}\frac{|w_1^{(2)}|\alpha}{|w_1^{(1)}|\beta} \quad \text{as } k \to \infty. \tag{3.1}$$

On the other hand, for any integer $k \geqslant 0$, we have

$$\left|\frac{w_1^{(2)} w_{m_k+1}^{(1)}}{w_1^{(1)} w_{m_k+1}^{(2)}}\right| \geqslant \frac{|w_1^{(2)}|\alpha}{|w_1^{(1)}|\beta},$$

a contradiction with (3.1). ∎

As mentioned in the Introduction, hypercyclic weighted shift operators satisfy the hypercyclicity criterion, and examples of hypercyclic operators *not* satisfying the hypercyclicity criterion are highly non-trivial. In sharp contrast, in the setting of d-hypercyclicity we have the following.

PROPOSITION 3.2. *If B_1, B_2 are unilateral weighted backward (or bilateral weighted) shifts on ℓ^2 (or $\ell^2(\mathbb{Z})$), then they are not* d*-weakly mixing. In particular, B_1, B_2 fail to satisfy the* d*-hypercyclicity criterion.*

Proof. We prove the unilateral weighted backward shift case. A similar argument will work for the bilateral weighted shift case. By way of contradiction, suppose the operators $B_1 \oplus B_1$, $B_2 \oplus B_2$ have a d-hypercyclic vector $g \oplus f$. Select an integer $n \geqslant 1$ for which

$$\|(B_1 \oplus B_1)^n (g \oplus f) - (-e_0 \oplus e_0)\| < \frac{1}{4} \tag{3.2}$$

$$\|(B_2 \oplus B_2)^n (g \oplus f) - (e_0 \oplus e_0)\| < \frac{1}{4}. \tag{3.3}$$

Now, for integers $m = 1, 2$, let $\{w_j^{(m)} : j \geqslant 1\}$ be the weight sequence for the unilateral weighted shift B_m. From inequalities (3.2) and (3.3), we get

$$\langle g, e_n\rangle \prod_{j=1}^{n} w_j^{(1)} = -1 + \varepsilon_1 \text{ and } \langle f, e_n\rangle \prod_{j=1}^{n} w_j^{(1)} = 1 + \varepsilon_2, \tag{3.4}$$

$$\langle g, e_n\rangle \prod_{j=1}^{n} w_j^{(2)} = 1 + \varepsilon_3 \text{ and } \langle f, e_n\rangle \prod_{j=1}^{n} w_j^{(2)} = 1 + \varepsilon_4. \tag{3.5}$$

where $|\varepsilon_k| < \frac{1}{4}$ for integers $1 \leqslant k \leqslant 4$.

Multiplying the first equality in (3.4) by the second equality in (3.5) yields

$$\langle g, e_n\rangle \prod_{j=1}^{n} w_j^{(1)} \langle f, e_n\rangle \prod_{j=1}^{n} w_j^{(2)} = -1 - \varepsilon_4 + \varepsilon_1 + \varepsilon_1\varepsilon_4. \tag{3.6}$$

Likewise, multiplying the second equality in (3.4) by the first equation in (3.5) yields

$$\langle g, e_n \rangle \prod_{j=1}^{n} w_j^{(1)} \langle f, e_n \rangle \prod_{j=1}^{n} w_j^{(2)} = 1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_2\varepsilon_3. \tag{3.7}$$

Subtracting equation (3.7) from equation (3.6) gives us

$$2 = \varepsilon_1 - \varepsilon_4 + \varepsilon_1\varepsilon_4 - \varepsilon_2 - \varepsilon_3 - \varepsilon_2\varepsilon_3.$$

However, since $|\varepsilon_k| < \frac{1}{4}$ for $1 \leqslant k \leqslant 4$, this above equation implies

$$2 \leqslant \sum_{k=1}^{4} |\varepsilon_k| + |\varepsilon_1\varepsilon_4| + |\varepsilon_2\varepsilon_3| < 4\frac{1}{4} + 2\frac{1}{16} = \frac{9}{8}, \tag{3.8}$$

which gives us a contradiction. ∎

Bernal-González and Grosse-Erdmann in Theorem 3.4 of [5], and independently León-Saavedra [18] established that a single operator satisfies the blow-up/collapse property if and only if it satisfies the hypercyclicity criterion. In contrast, for the setting of disjoint hypercyclicity Theorem 1.8, Theorem 2.2, Theorem 2.1, and Proposition 3.2 give the following.

COROLLARY 3.3. *Let H be the Hilbert space ℓ^2 or $\ell^2(\mathbb{Z})$, and let the integer $N \geqslant 2$. Then there exist operators $T_1, T_2, \ldots, T_N$ on H which satisfy the disjoint blow-up/collapse property yet fail to be* d*-weakly mixing, and thus fail to satisfy the* d*-hypercyclicity criterion.*

A striking result by Bourdon and Feldman [12] states that an orbit of a continuous, linear operator T on a topological vector space X must be either dense or nowhere dense in X. Thus it follows from Theorem 1.8 that any given operators $T_1, \ldots, T_N$ satisfy the d-hypercyclicity criterion if and only if for each (respectively, for some) positive integer r their powers $T_1^r, \ldots, T_N^r$ satisfy the d-hypercyclicity criterion. Proposition 3.2 now gives the following (complementing Theorem 4.1 and Theorem 4.7 of [8]).

COROLLARY 3.4. *Suppose $B_1^{r_1}, \ldots, B_N^{r_N}$ are $N \geqslant 2$ positive powers of unilateral weighted shift operators on ℓ^2 (respectively, of bilateral weighted shift operators on $\ell^2(\mathbb{Z})$) satisfying the* d*-hypercyclicity criterion. Then the integers $r_1, \ldots, r_N$ are distinct.*

We conclude this section with some open problems related to the set of d-hypercyclic vectors. From Remark 2.3, part (i), any d-hypercyclic family of weighted shift operators is densely d-hypercyclic (equivalently, d-topologically transitive). It is then natural to ask whether this holds for d-hypercyclic families of arbitrary operators.

PROBLEM 3.5. Let $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ be d-hypercyclic operators in $B(X)$. Must they be densely d-hypercyclic?

While there are many examples of densely d-hypercyclic operators in the literature, the answer to Problem 3.5 is not even known for powers of bilateral weighted shifts $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ where the powers $r_1, r_2, \ldots, r_N$ are distinct.

Note that for the case of d-hypercyclic unilateral weighted shift operators $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ raised to distinct, positive powers, they always support a dense d-*hypercyclic manifold*; that is, a dense, linear manifold for which every non zero vector is a d-hypercyclic vector. In fact, this holds for any finite family of operators satisfying the d-hypercyclicity criterion, see Proposition 2.10 of [7]. An arbitrary family of d-hypercyclic operators always supports a d-hypercyclic manifold because a nonzero scalar multiple of a d-hypercyclic vector is still a d-hypercyclic vector. This together with the above observation leads us to the following problem.

PROBLEM 3.6. Let $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ be densely d-hypercyclic operators in $B(X)$. Must they support a dense d-hypercyclic manifold?

The answer to Problem 3.6 does not seem to be known even for the case of d-hypercyclic weighted shifts $B_1, B_2, \ldots, B_N$.

Next, we point out that the important notion of hypercyclic subspaces, initiated by Bernal-González and Montes-Rodríguez [6] motivates the following one in the setting of disjointness. A d-*hypercyclic subspace* for given operators $T_1, T_2, \ldots, T_N$ in $B(X)$ is a closed, infinitely dimensional subspace of X for which ever non-zero vector is d-hypercyclic.

We note in Corollary 3.8 below that every Banach space whose dual is also separable supports d-hypercyclic subspaces. We first observe the following.

PROPOSITION 3.7. *Suppose X has separable dual. Let $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ be operators in $B(X)$ which satisfy the* d-*hypercyclicity criterion with respect to some sequence $(n_k)_{k=1}^{\infty}$, and so that for some closed, infinite dimensional subspace Y of X, we have*

$$T_j^{n_k} \to 0$$

pointwise on Y for integers $1 \leqslant j \leqslant N$. Then the operators $T_1, T_2, \ldots, T_N$ support a d-*hypercyclic subspace.*

Proof. For each T in $B(X)$, let $L_T : A(X) \to A(X)$, $V \mapsto TV$, where $A(X)$ denotes the norm closure of the finite rank operators on X. A slight modification of Lemma 3.3 in [1] gives that $L_{T_1}, L_{T_2}, \ldots, L_{T_N}$ satisfy the d-hypercyclicity criterion with respect to $(n_k)_{k=1}^{\infty}$. Let V be an operator in $A(X)$ with norm strictly less than one and which is a d-hypercyclic vector for $L_{T_1}, L_{T_2}, \ldots, L_{T_N}$. It follows that the operator $I + V$ is bounded below, and it is simple to see that Vx is d-hypercyclic vector for the operators $T_1, T_2, \ldots, T_N$ for any $0 \neq x \in X$. It follows that $(I + V)(Y)$ is a d-hypercyclic subspace for the operators $T_1, T_2, \ldots, T_N$. ∎

COROLLARY 3.8. *Suppose X has separable dual. Then for each integer $N \geqslant 2$, there exist* d-*mixing operators $T_1, T_2, \ldots, T_N$ supporting a* d-*hypercyclic subspace.*

Proof. The space X supports operators of the form $T_j = I + K_j$ with K_j compact for integers $1 \leqslant j \leqslant N$ where the operators $T_1, T_2, \ldots, T_N$ are d-mixing, and hence satisfying the d-hypercyclicity criterion with respect to some sequence $(n_k)_{k=1}^{\infty}$; see Theorem 3.1 of [11]. As in the proof of Theorem 4.1 in [1], we may obtain a subsequence $(m_k)_{k=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$ and a closed, infinite dimensional linear subspace Y of X so that $T_j^{m_k} y \to 0$ as $k \to \infty$ for each integer $j = 1, \ldots, N$ and for each $y \in Y$. The conclusion now follows from Proposition 3.7. ∎

Finally, motivated by Example 2.1 of [1] and by our results in Section 2 and Section 4 we pose the following problems.

PROBLEM 3.9. Let T_1, T_2 in $B(X)$ be d-hypercyclic and having a common hypercyclic subspace. Must they have a d-hypercyclic subspace?

PROBLEM 3.10. Let $B_1, B_2, \ldots, B_N$ with $N \geqslant 2$ be densely d-hypercyclic unilateral weighted backward shifts on ℓ_2 (respectively, bilateral weighted backward shifts on $\ell_2(\mathbb{Z})$) which support a common hypercyclic subspace.

(a) When do the shifts $B_1, B_2, \ldots, B_N$ support a d-hypercyclic subspace?

(b) Given integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, when do the operators $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ support a d-hypercyclic subspace?

4. HEREDITARY d-HYPERCYCLICITY FOR POWERS OF SHIFT OPERATORS

We have shown that $N \geqslant 2$ backward shifts $B_1, B_2, \ldots, B_N$ may be densely d-hypercyclic but can never be *hereditarily* densely d-hypercyclic, as they cannot even be d-weakly mixing. This is in sharp contrast with the case of powers of shift operators. We provide in this section a characterization of when a family of powers of shift operators is hereditarily d-hypercyclic with respect to a given strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers.

THEOREM 4.1. *Let $N \geqslant 2$, and for each integer m with $1 \leqslant m \leqslant N$, let B_m be a bilateral weighted backward shift on $\ell^2(\mathbb{Z})$ with the weight sequence $\{w_j^{(m)} : j \in \mathbb{Z}\}$, and let $(n_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive integers. For any integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, the following are equivalent:*

(i) *The shifts $B_1^{r_1}, \ldots, B_N^{r_N}$ are hereditarily densely* d*-hypercyclic with respect to $(n_k)_{k=0}^{\infty}$. That is, the sequences $(B_1^{r_1 n_k})_{k=0}^{\infty}, \ldots, (B_N^{r_N n_k})_{k=0}^{\infty}$ are* d*-mixing.*

(ii) *For each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists a positive integer k_0 so that for any integers $|i| \leqslant q$ and $k \geqslant k_0$, we satisfy the following:*

$$\text{if } 1 \leqslant m \leqslant N, \text{ we have } \begin{cases} \left| \prod_{j=1}^{r_m n_k} w_{i+j}^{(m)} \right| > \frac{1}{\varepsilon}, \\ \left| \prod_{j=0}^{r_m n_k - 1} w_{i-j}^{(m)} \right| < \varepsilon; \end{cases} \quad \text{and} \tag{4.1}$$

$$\text{(4.2)} \qquad \textit{if } 1 \leqslant s < m \leqslant N, \textit{ we have } \begin{cases} \Big| \prod_{j=1}^{r_m n_k} w_{i+j}^{(m)} \Big| > \frac{1}{\varepsilon} \Big| \prod_{j=0}^{r_s n_k - 1} w_{i-j+r_m n_k}^{(s)} \Big|, \\ \Big| \prod_{j=0}^{r_m n_k - 1} w_{i-j+r_s n_k}^{(m)} \Big| < \varepsilon \Big| \prod_{j=1}^{r_s n_k} w_{i+j}^{(s)} \Big|. \end{cases}$$

(iii) *The shifts* $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ *satisfy the* d*-hypercyclicity criterion with respect to* $(n_k)_{k=0}^{\infty}$.

In particular, the shifts $B_1^{r_1}, \ldots, B_N^{r_N}$ *are* d*-mixing if and only if they satisfy the* d*-hypercyclicity criterion with respect to the full sequence* $(k)_{k=0}^{\infty}$ *of positive integers.*

To establish Theorem 4.1, we need Proposition 4.2 below, which is similar to Theorem 4.7 of [8]. Theorem 4.7 in [8] provides necessary and sufficient conditions for powers of bilateral weighted shifts with distinct powers to satisfy the d-hypercyclicity criterion. Our Proposition 4.2 provides necessary and sufficient conditions for such operators to satisfy the d-hypercyclicity criterion with respect to a specific increasing sequence $(n_k)_{k=0}^{\infty}$ of integers, and so the proof of Proposition 4.2 mimics the proof of Theorem 4.7 in [8]. The details are left to the reader.

PROPOSITION 4.2. *Let* $N \geqslant 2$*, and for each integer* m *with* $1 \leqslant m \leqslant N$*, let* B_m *be a bilateral weighted backward shift on* $\ell^2(\mathbb{Z})$ *with weight sequence* $\{w_j^{(m)} : j \in \mathbb{Z}\}$*, and let* $(n_k)_{k=0}^{\infty}$ *be a strictly increasing sequence of positive integers. For any integers* $1 \leqslant r_1 < r_2 < \cdots < r_N$*, the following are equivalent:*

(i) *The sequences* $(B_1^{r_1 n_k})_{k=0}^{\infty}, \ldots, (B_N^{r_N n_k})_{k=0}^{\infty}$ *are densely* d*-universal.*

(ii) *For each* $\varepsilon > 0$ *and* $q \in \mathbb{N}$*, there exists an arbitrarily large integer* $n \in \{n_k\}_{k \in \mathbb{N}}$ *so that for integers* i *with* $|i| \leqslant q$ *the following holds:*

$$\text{(4.3)} \qquad \textit{if } 1 \leqslant m \leqslant N, \textit{ we have } \begin{cases} \Big| \prod_{j=1}^{r_m n} w_{i+j}^{(m)} \Big| > \frac{1}{\varepsilon}, \\ \Big| \prod_{j=0}^{r_m n - 1} w_{i-j}^{(m)} \Big| < \varepsilon; \end{cases} \qquad \textit{and}$$

$$\text{(4.4)} \qquad \textit{if } 1 \leqslant l < m \leqslant N, \textit{ we have } \begin{cases} \Big| \prod_{j=1}^{r_m n} w_{i+j}^{(m)} \Big| > \frac{1}{\varepsilon} \Big| \prod_{j=0}^{r_l n - 1} w_{i-j+r_m n}^{(l)} \Big|, \\ \Big| \prod_{j=0}^{r_m n - 1} w_{i-j+r_l n}^{(m)} \Big| < \varepsilon \Big| \prod_{j=1}^{r_l n} w_{i+j}^{(l)} \Big|. \end{cases}$$

(iii) *The shifts* $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ *satisfy the* d*-hypercyclicity criterion with respect to some subsequence of* $(n_k)_{k=0}^{\infty}$.

Proof of Theorem 4.1. The implication (iii) implies (i) holds by Proposition 1.7. The proof of (ii) implies (iii) is identical to the corresponding implication in Proposition 4.2, by considering $(n_{k_q})_{q=0}^{\infty} = (n_q)_{q=0}^{\infty}$. Lastly, to establish (i) implies (ii) by means of contradiction, suppose that (ii) fails to hold. Then there exist $\varepsilon > 0$, $q \in \mathbb{N}$, and a subsequence $(n_{k_q})_{q=0}^{\infty}$ of $(n_k)_{k=0}^{\infty}$ so that one of (4.1) or (4.2) fails for each $q \in \mathbb{N}$ and for some integer j with $|j| \leqslant q$. Thus by Proposition 4.2 the sequences $(B_1^{n_{k_q} r_1})_{q=1}^{\infty}, \ldots, (B_N^{n_{k_q} r_N})_{q=1}^{\infty}$ fail to be densely d-universal, contradicting condition (i). ∎

We note that for $N \geqslant 2$ and $X = \ell^2(\mathbb{Z})$, a direct sum of powers of bilateral backward shifts $B_1^{r_1} \oplus \cdots \oplus B_N^{r_N}$ is hereditarily hypercyclic on X^N with respect to a given sequence $(n_k)_{k=0}^{\infty}$ if and only if it satisfies the hypercyclicity criterion with respect to $(n_k)_{k=0}^{\infty}$. Using this fact and Theorem 4.1, we have for the special case for when $B_1 = \cdots = B_N$.

COROLLARY 4.3. *Let B be a bilateral weighted backward shift on $X = \ell^2(\mathbb{Z})$, let $(n_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive integers, and let $r_0 = 0 < r_1 < \cdots < r_N$ and $N \geqslant 2$ be integers. Then the powers $B^{r_1}, \ldots, B^{r_N}$ are hereditarily densely* d*-hypercyclic on X if and only if they satisfy the* d*-hypercyclicity criterion with respect to $(n_k)_{k=0}^{\infty}$, and if and only if the direct sum operator $\bigoplus_{0 \leqslant s < \ell \leqslant N} B^{(r_\ell - r_s)}$ on $X^{N(N+1)/2}$ is hereditarily hypercyclic with respect to $(n_k)_{k=0}^{\infty}$.*

We conclude the paper by stating a unilateral version of Theorem 4.1, which follows similarly, and some of its consequences.

THEOREM 4.4. *Let $N \geqslant 2$, and for each integer m with $1 \leqslant m \leqslant N$, let B_m be a unilateral weighted backward shift on ℓ^2 with weight sequence $\{w_j^{(m)} : j \geqslant 1\}$, and let $(n_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive integers. For any integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, the following are equivalent:*

(i) *The shifts $B_1^{r_1}, \ldots, B_N^{r_N}$ are hereditarily densely* d*-hypercyclic with respect to the sequence $(n_k)_{k=0}^{\infty}$.*

(ii) *For each $\varepsilon > 0$ and $q \in \mathbb{N}$ there exists a positive integer k_0 so that for integers with $0 \leqslant i \leqslant q$ and $k > k_0$, we satisfy the following:*

$$\text{if } 1 \leqslant m \leqslant N, \text{ we have } \Big| \prod_{j=1}^{r_m n_k} w_{i+j}^{(m)} \Big| > \frac{1}{\varepsilon}, \quad \text{and}$$

$$\text{if } 1 \leqslant l < m \leqslant N, \text{ we have } \Big| \prod_{j=1}^{r_m n_k} w_{i+j}^{(m)} \Big| > \frac{1}{\varepsilon} \Big| \prod_{j=0}^{r_l n_k - 1} w_{i-j+r_m n_k}^{(l)} \Big|.$$

(iii) *The shifts $B_1^{r_1}, \ldots, B_N^{r_N}$ satisfy the* d*-hypercyclicity criterion with respect to $(n_k)_{k=0}^{\infty}$.*

In particular, the shifts $B_1^{r_1}, \ldots, B_N^{r_N}$ are d*-mixing on ℓ^2 if and only if they satisfy the* d*-hypercyclicity criterion with respect to the full sequence $(k)_{k=0}^{\infty}$.*

When all the shifts coincide, Theorem 4.4 gives the following.

COROLLARY 4.5. *Let B be a unilateral weighted backward shift with weight sequence $\{w_j : j \geqslant 1\}$ on ℓ^2, and let $(n_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive integers. For any integers $r_0 = 0 < r_1 < r_2 < \cdots < r_N$, the following are equivalent:*

(i) *The shifts* $B^{r_1}, B^{r_2}, \dots, B^{r_N}$ *are hereditarily* d-*hypercyclic with respect to the sequence* $(n_k)_{k=0}^{\infty}$.

(ii) *The shifts* $B^{r_1}, B^{r_2}, \dots, B^{r_N}$ *satisfy the* d-*hypercyclicity criterion with respect to* $(n_k)_{k=0}^{\infty}$.

(iii) $\lim_{k\to\infty} \min \left\{ \prod_{i=1}^{n_k(r_l - r_s)} w_i : 0 \leqslant s < l \leqslant N \right\} = \infty.$

(iv) *The operator* $\bigoplus_{0\leqslant s<l\leqslant N} B^{(r_l - r_s)}$ *on* $X^{N(N+1)/2}$ *is hereditarily hypercyclic with respect to* $(n_k)_{k=0}^{\infty}$.

REMARK 4.6. An operator T in $B(X)$ for which there exists a dense subset X_0 of X and a map $S : X_0 \to X_0$ so that $TSy = y$, $T^n y \to 0$ and $S^n y \to 0$ for each $y \in X_0$ satisfies that for each $N \geqslant 2$ the powers $T, T^2, \dots, T^N$ are d-mixing on X for each $N \geqslant 2$ ([11], Theorem 3.4).

Every mixing unilateral weighted backward shift B on ℓ^2 has the former property, by taking $X_0 = \operatorname{span}\{e_i : i \in \mathbb{N}\}$ and the linear mapping S on X_0 determined by $Se_i = w_{i+1}^{-1} e_{i+1}$ for every $i \in \mathbb{N}$, as its weight sequence $\{w_j : j \geqslant 1\}$ satisfies $\left| \prod_{j=1}^{n} w_j \right| \to \infty$; see p. 97 of [16]. It follows that a unilateral weighted backward shift B is mixing on ℓ^2 if and only if the operators $B, B^2, \dots, B^N$ are d-mixing on X for each $N \geqslant 2$.

The bilateral case of this fact is also true and was observed in Remark 3.5 of [11].

We also note that for $N \geqslant 2$ scalar multiples of powers of the unilateral unweighted backward shift, being d-hypercyclic coincides with being d-mixing.

COROLLARY 4.7. *Let* B *be the unweighted unilateral backward shift on* ℓ^2. *Let* $r_1, \dots, r_N$ *be integers with* $1 \leqslant r_1 \leqslant \cdots \leqslant r_N$, *and let* $\lambda_1, \dots, \lambda_N$ *be scalars. The following are equivalent:*

(i) *The shifts* $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ *are* d-*hypercyclic on* X.

(ii) *The shifts* $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ *are* d-*mixing on* X.

(iii) *We have* $1 \leqslant r_1 < r_2 < \cdots < r_N$ *and* $1 < |\lambda_1| < \cdots < |\lambda_N|$.

Proof. The fact that (i) and (iii) are equivalent was established in Corollary 4.2 of [8], and statement (i) follows immediately from (ii).

Now, if (iii) holds, then $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ satisfy the d-hypercyclicity criterion with respect to the full sequence $(k)_{k=0}^{\infty}$, by considering $X_0 = X_1 = \cdots = X_N = \operatorname{span}\{e_i : i \in \mathbb{N}\}$ and $S_{\ell,n} : X_\ell \to X$ given by $S_{\ell,n} = \lambda_\ell^{-nr_l} S^{nr_l}$, where S denotes the unilateral unweighted forward shift. Statement (ii) then follows by Proposition 1.7. ∎

We conclude the paper with the following observation. While for a single operator the properties of being weakly mixing, satisfying the hypercyclicity criterion, satisfying the blow-up/collapse condition, and being hereditarily densely hypercyclic are all equivalent, this is no longer true within the setting of disjointness. Indeed, we have within this setting the following implications/non-implications:

$$\begin{array}{ccc} \text{hereditary dense d-hypercyclicity} & \Leftrightarrow & \text{d-hypercyclicity criterion} \\ \Downarrow & & \Downarrow \quad \not\Uparrow \\ \text{d-weakly mixing} & \begin{array}{c} \not\Leftarrow \\ \Rightarrow \end{array} & \text{disjoint blow-up/collapse} \end{array}$$

This raises the question whether the missing implication in the diagram is true.

PROBLEM 4.8. Let $T_1, \ldots, T_N$ be d-weakly mixing operators with $N \geqslant 2$. Must they satisfy the d-hypercyclicity criterion?

REMARK 4.9. Recently, Sanders and Shkarin [25] have answered Problem 3.5 and Problem 4.8 in the negative. Salas [24] has also recently provided a partial answer to Problem 3.6 by means of a stronger disjoint blow-up/collapse property.

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