MULTIPLIERS OF DIRICHLET SUBSPACES OF THE BLOCH SPACE

CHRISTOS CHATZIFOUNTAS, DANIEL GIRELA and JOSÉ ÁNGEL PELÁEZ

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ABSTRACT. For $0 < p < \infty$ we let $\mathcal{D}^p_{p-1}$ be the space of all functions f which are analytic in the unit disc $\mathbb{D}$ and satisfy $\int_{\mathbb{D}} (1 - |z|)^{p-1} |f'(z)|^p \mathrm{d}A(z) < \infty$.

It is known that, whenever $p \neq q$, the only multiplier from $\mathcal{D}^p_{p-1}$ to $\mathcal{D}^q_{q-1}$ is the trivial one. However, if X is a subspace of the Bloch space and $0 < p \leqslant q < \infty$, then $X \cap \mathcal{D}^p_{p-1} \subset X \cap \mathcal{D}^q_{q-1}$, a fact which implies that the space of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ is non-trivial.

In this paper we study the spaces of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ $(0 < p, q < \infty)$ for distinct classical subspaces X of the Bloch space. Specifically, we shall take X to be H^∞, BMOA and the Bloch space $\mathcal{B}$.

KEYWORDS: *Bloch space, BMOA, Hardy spaces, spaces of Dirichlet type, multipliers, lacunary power series, random power series.*

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1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \mathrm{d}t \right)^{1/p} \quad 0 < p < \infty, \quad \text{and} \quad M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

Whenever $0 < p \leqslant \infty$ the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $\|f\|_{H^p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$ (see [10] for the theory of H^p-spaces).

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p} \stackrel{\text{def}}{=} \left((\alpha + 1) \int_{\mathbb{D}} (1 - |z|)^\alpha |f(z)|^p \mathrm{d}A(z) \right)^{1/p} < \infty.$$

The unweighted Bergman space A_0^p is simply denoted by A^p. Here, $\mathrm{d}A(z) = (1/\pi)\mathrm{d}x\mathrm{d}y$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [11], [23] and [40] for the theory of these spaces.

The space $\mathcal{D}_\alpha^p$ ($0 < p < \infty$, $\alpha > -1$) consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A_\alpha^p$. Hence, if f is analytic in $\mathbb{D}$, then $f \in \mathcal{D}_\alpha^p$ if and only if

$$\|f\|_{\mathcal{D}_\alpha^p}^p \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|_{A_\alpha^p}^p < \infty.$$

If $p < \alpha + 1$ then it is well known that $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$ (see, e.g. Theorem 6 of [12]). On the other hand, if $p > \alpha + 2$ then $\mathcal{D}_\alpha^p \subset H^\infty$. Therefore $\mathcal{D}_\alpha^p$ becomes a "proper Dirichlet space" when $\alpha + 1 \leqslant p \leqslant \alpha + 2$. The spaces $\mathcal{D}_{p-1}^p$ are closely related with Hardy spaces. Indeed, it is well known that $D_1^2 = H^2$. We have also ([26])

$$(1.1) \qquad\qquad H^p \subsetneq \mathcal{D}_{p-1}^p, \quad \text{for } 2 \leqslant p < \infty,$$

and ([12], [36])

$$(1.2) \qquad\qquad \mathcal{D}_{p-1}^p \subsetneq H^p, \quad \text{for } 0 < p \leqslant 2.$$

We remark that for $p \neq q$ there is no relation of inclusion between $\mathcal{D}_{p-1}^p$ and $\mathcal{D}_{q-1}^q$ (see, e.g., [5] and [21]).

We recall that the Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

We refer to [2] for the theory of Bloch functions.

Next, we consider multiplication operators. For $g \in \mathcal{H}ol(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}), z \in \mathbb{D}.$$

If X and Y are two normed (or Fréchet) spaces of analytic functions in $\mathbb{D}$ which are continuously contained in $\mathcal{H}ol(\mathbb{D})$, $M(X,Y)$ will denote the space of multipliers from X to Y,

$$M(X,Y) = \{g \in \mathcal{H}ol(\mathbb{D}) : fg \in Y, \text{ for all } f \in X\},$$

and $\|M_g\|_{(X \to Y)}$ will denote the norm of the operator M_g. If $X = Y$ we simply write $M(X)$. These operators have been studied on the Dirichlet type spaces $\mathcal{D}_\alpha^p$ in [15], [19], [20], where among other results it is proved that

$$(1.3) \qquad\qquad M(\mathcal{D}_{p-1}^p, \mathcal{D}_{q-1}^q) = \{0\}, \quad 0 < p, q < \infty, p \neq q.$$

The following simple observation plays an important role in the motivation of this work.

LEMMA 1.1. *Suppose that $0 < p < q < \infty$ and $f \in \mathcal{D}^p_{p-1} \cap \mathcal{B}$. Then $f \in \mathcal{D}^q_{q-1}$.*

Proof. Since $f \in \mathcal{B}$ we have that $\sup\limits_{z \in \mathbb{D}}(1 - |z|)|f'(z)| = M < \infty$. Using this we obtain

$$\int\limits_{\mathbb{D}} (1 - |z|)^{q-1}|f'(z)|^q \mathrm{d}A(z) = \int\limits_{\mathbb{D}} [(1 - |z|)|f'(z)|]^{q-p}(1 - |z|)^{p-1}|f'(z)|^p \mathrm{d}A(z)$$

$$\leqslant M^{q-p} \int\limits_{\mathbb{D}} (1 - |z|)^{p-1}|f'(z)|^p \mathrm{d}A(z) < \infty.$$

Hence, $f \in \mathcal{D}^q_{q-1}$. ■

Consequently, we have:
If X is a subspace of the Bloch space then

$$(1.4) \qquad X \cap \mathcal{D}^p_{p-1} \subset X \cap \mathcal{D}^q_{q-1}, \quad if\ 0 < p \leqslant q < \infty,$$

a fact which, contrary to (1.3), implies that whenever $0 < p \leqslant q < \infty$, the space of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ is non-trivial.

If $X \subset \mathcal{B}$, the space $X \cap \mathcal{D}^p_{p-1}$ is equipped with the norm

$$\|f\|_{X \cap \mathcal{D}^p_{p-1}} \overset{\mathrm{def}}{=} \|f\|_X + \|f\|_{\mathcal{D}^p_{p-1}}.$$

Our aim in this paper is to obtain a characterization of the spaces of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ $(0 < p, q < \infty)$ for some important subspaces X of the Bloch space.

Let us start with $X = \mathcal{B}$. For $\alpha > 0$, the α-logarithmic-Bloch space $\mathcal{B}_{\log,\alpha}$ consists of those $g \in \mathcal{H}ol(\mathbb{D})$ such that

$$\rho_\alpha(f) \overset{\mathrm{def}}{=} \sup\limits_{z \in \mathbb{D}}(1 - |z|^2)|g'(z)|(\log(\mathrm{e}/(1 - |z|^2)))^\alpha < \infty.$$

It is clear that

$$(1.5) \qquad \mathcal{B}_{\log,\alpha} \subset \mathcal{B}_{\log,\beta}, \quad \alpha \geqslant \beta.$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by $\mathcal{B}_{\log}$.

The multipliers of the Bloch space into itself were characterized independently by several authors (see [3], [6], [38]). Namely, we have the following result:

$$(1.6) \qquad M(\mathcal{B}) = \mathcal{B}_{\log} \cap H^\infty.$$

Let us turn our attention to the spaces $M(\mathcal{D}^p_{p-1} \cap \mathcal{B}, \mathcal{D}^q_{q-1} \cap \mathcal{B})$. Among other results, we shall prove that, for $p > 1$, the space $M(\mathcal{B} \cap \mathcal{D}^p_{p-1})$ coincides $M(\mathcal{B})$. This is part of the following result.

THEOREM 1.2. *Let $0 < p, q < \infty$.*
(i) *If $1 < q$ and $0 < p \leqslant q < \infty$, then,*

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = M(\mathcal{B}).$$

(ii) *If $0 < q < p < \infty$, then*

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = \{0\}.$$

The question of obtaining a complete characterization of the space of multipliers $M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$ in the case $0 < p \leqslant q \leqslant 1$ remains open. However, we remark that the inclusion

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subset M(\mathcal{B}),$$

is true for any p, q (see the proof of Theorem 1.2 in Section 3). Using this, the fact that $M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subset \mathcal{B} \cap \mathcal{D}^q_{q-1}$, and the following result we see that part (i) of Theorem 1.2 does not remain true for $0 < q \leqslant 1$.

THEOREM 1.3. *If $0 < q \leqslant 1$, then $M(\mathcal{B}) \setminus \mathcal{D}^q_{q-1} \neq \{0\}$.*

Let us now consider the spaces $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$. It is easy to prove the following result for the case $p \leqslant q$.

THEOREM 1.4. *If $0 < p \leqslant q < \infty$, then*

$$M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty \cap \mathcal{D}^q_{q-1}.$$

Regarding the case $0 < q < p$, let us notice that if $2 \leqslant q < p$ then we have that $H^\infty \cap \mathcal{D}^p_{p-1} = H^\infty \cap \mathcal{D}^q_{q-1} = H^\infty$. Then it follows that

(1.7) $$M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty, \quad 2 \leqslant q < p.$$

When $0 < q < p$ and $0 < q < 2$ the question is more complicated. It is well known (see Theorem 1 of [17] and [36]) that, whenever $0 < q < 2$, there exists a function $f \in H^\infty \setminus \mathcal{D}^q_{q-1}$. We improve this result in our next theorem.

THEOREM 1.5. *If $0 < q < \min\{p, 2\}$, then there exists a function f such that $f \in (H^\infty \cap \mathcal{D}^p_{p-1}) \setminus (H^\infty \cap \mathcal{D}^q_{q-1})$.*

The functions constructed in Theorem 1.5 are used in a basic way in the proof of part (i) of our following result.

THEOREM 1.6. (i) *If $0 < q < 1$ and $0 < q < p < \infty$ then $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.*
(ii) *If $1 \leqslant q < 2 \leqslant p$ then $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.*

In order to prove part (ii), we use strongly Theorem 1 of [17] which asserts that, whenever $0 < q < 2$, there exists a function $f \in H^\infty$ such that

(1.8) $$\int_0^1 (1-r)^{q-1}|f'(re^{i\theta})|^q \mathrm{d}r = \infty, \quad \text{for almost every } \theta \in \mathbb{R}.$$

The case $1 \leqslant q < p < 2$ of Theorem 1.6 remains open. However, if the answer to the following open question were affirmative then it would follow that the space $M(H^\infty \cap \mathcal{D}_{p-1}^p, H^\infty \cap \mathcal{D}_{q-1}^q)$ would be trivial also for this range of parameters. (See the proof of Theorem 1.6(ii)).

QUESTION 1.7. Suppose that $0 < q < p < 2$. Does there exist a function $f \in H^\infty \cap \mathcal{D}_{p-1}^p$ satisfying (1.8)?

We end up taking $X =$ BMOA, the space of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by John and Nirenberg [24]. A lot of information about the space BMOA can be found in [4], [16], [18]. Let us recall here that

$$H^\infty \subsetneq \mathrm{BMOA} \subsetneq \mathcal{B}, \quad \text{and} \quad H^\infty \subsetneq \mathrm{BMOA} \subsetneq \bigcap_{0<p<\infty} H^p.$$

We emphasize also that BMOA can be characterized in terms of Carleson measures. If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson box* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, 1 - |I|/2\pi \leqslant r < 1\}$. If μ is a positive Borel measure in $\mathbb{D}$, we shall say that μ is a Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leqslant C|I|, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

We have (see, e.g. Theorem 6.5 of [18]):

A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to BMOA *if and only if the Borel measure μ_f in $\mathbb{D}$ defined by $\mathrm{d}\mu_f(z) = (1-|z|^2)|f'(z)|^2 \mathrm{d}A(z)$ is a Carleson measure.*

The multipliers of the space BMOA have been characterized in [28] (see also [34] and [39]). Indeed, we have

(1.9) $$M(\mathrm{BMOA}) = H^\infty \cap \mathrm{BMOA}_{\log}.$$

Here, $\mathrm{BMOA}_{\log}$ is the space of those functions $g \in H^1$ for which there exists a positive constant C such that

$$\int_{S(I)} (1-|z|^2)|g'(z)|^2 \mathrm{d}A(z) \leqslant C|I|(\log(2/|I|))^{-2}, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

Let us mention that $\mathrm{BMOA}_{\log}$ is called LMOA in [34]. Following the terminology of [39], we have:

$\mathrm{BMOA}_{\log}$ *is the space of those functions $g \in H^1$ for which the Borel measure μ_g in $\mathbb{D}$ defined by $\mathrm{d}\mu_g(z) = (1-|z|^2)|g'(z)|^2 \mathrm{d}A(z)$ is a 2-logarithmic Carleson measure.*

In order to make a proper study of the spaces of multipliers $M(\mathcal{D}^p_{p-1} \cap$ BMOA, $\mathcal{D}^q_{q-1} \cap$ BMOA), we shall present in Sections 5 and 6 a series of results concerning the space BMOA$_{\log}$, some of which are of independent interest.

In Section 5 we shall prove directly that BMOA$_{\log} \subsetneqq \mathcal{B}_{\log} \subsetneqq$ BMOA and we shall also find some simple conditions on a function $f \in \mathcal{H}ol(\mathbb{D})$ which implies its membership to BMOA$_{\log}$. As a corollary we shall prove the following result about lacunary power series in BMOA$_{\log}$.

PROPOSITION 1.8. *Let $f \in \mathcal{H}ol(\mathbb{D})$ be given by a lacunary power series, i.e., f is of the form*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} (z \in \mathbb{D}) \quad \text{with } n_{k+1} \geqslant \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1.$$

If $\sum\limits_{k=0}^{\infty} |a_k|^2 (\log n_k)^3 < \infty$, then $f \in$ BMOA$_{\log} \cap H^\infty$.

Section 6 deals with random power series of the form

$$f_t(z) = \sum_{n=0}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D}, \ 0 \leqslant t \leqslant 1,$$

where $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $\{r_n\}_{n=0}^{\infty}$ is the sequence of Rademacher functions (see Section 2). Among other results, we establish a sharp condition on the Taylor coefficients a_n of f which implies the almost sure membership of f_t in BMOA$_{\log}$.

THEOREM 1.9. (i) *If $\sum\limits_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty$ then for almost every $t \in [0,1]$, the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

belongs to BMOA$_{\log} \cap H^\infty$.

(ii) *Furthermore, (i) is sharp in a very strong sense: Given a decreasing sequence of positive numbers $\{\delta_n\}_{n=1}^{\infty}$ with $\delta_n \to 0$, as $n \to \infty$, there exists a sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ with $\sum\limits_{n=1}^{\infty} a_n^2 \delta_n (\log n)^3 < \infty$ such that, for almost every t the function f_t defined by $f_t(z) = \sum\limits_{n=1}^{\infty} r_n(t) a_n z^n$ ($z \in \mathbb{D}$) does not belong to $\mathcal{B}_{\log}$.*

Now we pass properly to study the multipliers from $\mathcal{D}^p_{p-1} \cap$ BMOA to $\mathcal{D}^q_{q-1} \cap$ BMOA ($0 < p, q < \infty$).

If $\lambda \geqslant 2$ then $\mathrm{BMOA} \subset \mathcal{D}^\lambda_{\lambda-1}$. Hence, trivially, we have

$$M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^q_{q-1} \cap \mathrm{BMOA}) = M(\mathrm{BMOA})$$

(1.10)
$$= \mathrm{BMOA}_{\log} \cap H^\infty, \quad 2 \leqslant p, q < \infty.$$

This remains true for other values of p and q.

THEOREM 1.10. *If* $1 < q < \infty$ *and* $0 < p \leqslant q < \infty$, *then*

$$M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^q_{q-1} \cap \mathrm{BMOA}) = M(\mathrm{BMOA}) = \mathrm{BMOA}_{\log} \cap H^\infty.$$

When $q < p$ then 0 is the only multiplier from $\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}$ to $\mathcal{D}^q_{q-1} \cap \mathrm{BMOA}$, except in the cases covered by (1.10).

THEOREM 1.11. *If* $0 < q < p < \infty$ *and* $q < 2$, *then*

$$M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^q_{q-1} \cap \mathrm{BMOA}) = \{0\}.$$

To deal with the remaining case, $0 < p \leqslant q \leqslant 1$, we shall use the above mentioned results about lacunary power series and random power series. Our main results concerning random power series and multipliers are contained in the following theorem.

THEOREM 1.12. *Let* $\{a_n\}^\infty_{n=0}$ *be a sequence of complex numbers satisfying*

(1.11)
$$\sum_{n=1}^\infty |a_n|^2 (\log n)^3 < \infty.$$

For $t \in [0,1]$ *we set*

(1.12)
$$f_t(z) = \sum_{n=0}^\infty r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

where the r_n's *are the Rademacher functions. Then, for almost every* $t \in [0,1]$, *the function* f_t *satisfies the following conditions:*

(i) $\int_0^1 (1-r)(\log(1/(1-r)))^2 [M_\infty(r, f_t')]^2 \mathrm{d}r < \infty$.

(ii) $f_t \in \mathrm{BMOA}_{\log} \cap H^\infty$.

(iii) $f_t \in M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^q_{q-1} \cap \mathrm{BMOA})$ *whenever* $0 < p \leqslant q$ *and* $q > 1/2$.

Furthermore, if $0 < q < 1/2$ *then there exists a sequence* $\{a_n\}$ *which satisfies* (1.11) *and such that* $f_t \notin \mathcal{D}^q_{q-1}$, *for almost every* t. *Thus, for this sequence* $\{a_n\}$ *and for almost every* t *we have:*

(a) $f_t \in M(\mathrm{BMOA})$.

(b) *If* $0 < p \leqslant \lambda$ *and* $\lambda > 1/2$ *then* $f_t \in M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^\lambda_{\lambda-1} \cap \mathrm{BMOA})$.

(c) $f_t \notin M(\mathcal{D}^p_{p-1} \cap \mathrm{BMOA}, \mathcal{D}^q_{q-1} \cap \mathrm{BMOA})$ *whenever* $0 < p \leqslant q$.

We remark that Theorem 1.12 shows that Theorem 1.10 does not remain true for $q < 1/2$.

Finally, let us consider multipliers in $M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$ given by power series with Hadamard gaps. We will show that whenever $0 < p \leqslant q \leqslant 1$ the power series with Hadamard gaps in $M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$ coincide with those in $\mathcal{D}_{q-1}^q \cap \mathrm{BMOA}_{\log}$ and will obtain also the analogue of Theorem 1.12 for lacunary power series in Theorem 7.4. This will give another proof of the impossibility of extending Theorem 1.10 to $q < 1/2$.

2. PRELIMINARY RESULTS

As usual, a sequence of positive integers $\{n_k\}_{k=0}^{\infty}$ is said to be lacunar if there exists $\lambda > 1$ such that $n_{k+1} \geqslant \lambda n_k$, for all k. Also, by a *lacunary power series* (also called *power series with Hadamard gaps*) we mean a power series of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} (z \in \mathbb{D}) \quad \text{with } n_{k+1} \geqslant \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1.$$

For simplicity, we shall let $\mathcal{L}$ denote the class of all function $f \in \mathcal{H}ol(\mathbb{D})$ which are given by a lacunary power series. Several known results on power series with Hadamard gaps will be repeatedly used along the paper, we collect them in the following statement, (see [2], [7], [41]).

PROPOSITION 2.1. *Suppose that* $0 < p < \infty$, $\alpha > -1$ *and* f *is an analytic function in* $\mathbb{D}$ *which is given by a power series with Hadamard gaps,*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} (z \in \mathbb{D}) \quad \text{with } n_{k+1} \geqslant \lambda n_k \text{ for all } k \ (\lambda > 1).$$

Then:

(i) $f \in \mathcal{D}_{\alpha}^p \Leftrightarrow \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p < \infty$, *and*

$$\|f - f(0)\|_{\mathcal{D}_{\alpha}^p}^p \asymp \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p.$$

(ii) $f \in H^{\infty}$ *if and only if* $\sum_{k=0}^{\infty} |a_k| < \infty$, *and*

$$\|f\|_{H^{\infty}} \asymp \sum_{k=0}^{\infty} |a_k|.$$

(iii) $f \in \mathcal{B} \Leftrightarrow \sup_n |a_n| < \infty$, *and*

$$\|f\|_{\mathcal{B}} \asymp \sup_n |a_n|.$$

It is also well known that $\mathcal{L} \cap H^p = \mathcal{L} \cap H^2$ for any $p \in (0, \infty)$ but

$$H^{\infty} \cap \mathcal{L} \subsetneq H^2 \cap \mathcal{L}.$$

In spite of this, for any given lacunary sequence of positive integers $\{n_k\}_{k=1}^\infty$ and any sequence of complex numbers $\{u_k\}_{k=1}^\infty \in \ell^2$, Fournier constructed in [13] a function $f(z) = \sum_{n=0}^\infty a_n z^n \in H^\infty$ with $a_{n_k} = u_k$, for all k. Some properties of the bounded function f which were not stated in [13] will play an important role in the proof of some of our results. Due to this fact and for the sake of the completeness we present a complete proof of Fournier's construction pointing out some extra properties of the constructed function (for simplicity we shall restrict to sequences $\{n_k\}$ satisfying $n_{k+1} \geqslant 2n_k$).

Let start fixing some notation. The unit circle $\partial \mathbb{D}$ will be denoted by $\mathbb{T}$. If $g \in L^1(\mathbb{T})$ its Fourier coefficients $\widehat{g}(n)$ are defined by

$$\widehat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}.$$

If $n_1 < n_2$ are integers we shall write $\lfloor n_1, n_2 \rfloor$ for the set of all integers n with $n_1 \leqslant n \leqslant n_2$. Also, for $g(z) = \sum_{k=0}^\infty b_k z^k \in \mathcal{H}ol(\mathbb{D})$ and $n_2 > n_1 \geqslant 0$, we set

$$S_{n_1, n_2} g(z) = \sum_{k=n_1}^{n_2 - 1} b_k z^k.$$

PROPOSITION 2.2. *Assume that $\{u_k\}_{k=0}^\infty \in \ell^2$ and let $\{n_k\}_{k=0}^\infty$ be a sequence of positive integers such that $n_{k+1} > 2n_k$, for all k. Then, there exists a function $\Psi \in \mathcal{H}ol(\mathbb{D})$ of the form*

$$\Psi(z) = \sum_{n=0}^\infty a_n z^n, \quad z \in \mathbb{D},$$

with the following properties:

(i) $\Psi \in H^\infty$.

(ii) $a_{n_k} = u_k$, *for all k.*

(iii) *If we define $\Lambda_0 = \{n_0\}$ and $\Lambda_k = \lfloor n_k - n_{k-1}, n_k \rfloor$ for $k > 0$, we have that the sets Λ_k are pairwise disjoint and satisfy $\Lambda_k \subset \lfloor n_{k-1} + 1, n_k \rfloor$ for all $k \geqslant 1$. Furthermore, $a_n = 0$ if $n \notin \bigcup_{k=0}^\infty \Lambda_k$.*

(iv) *There is an absolute constant C such that*

$$\|S_{n_k+1, n_{k+1}+1} \Psi\|_{H^\infty} \leqslant C |u_k|, \quad \text{for all } k.$$

Proof. The construction depends on the following equality ([13], p. 402):

(2.1) $$|a + vb|^2 + |b - \overline{v}a|^2 = (1 + |v|^2)(|a|^2 + |b|^2), \quad a, b, v \in \mathbb{C}.$$

Let us define inductively the following sequences of functions on $\mathbb{T}$

(2.2) $$\phi_0(\zeta) = u_0 \zeta^{n_0}, \quad h_0(\zeta) = 1, \quad \zeta \in \mathbb{T},$$

and, for $k > 0$,

$$(2.3)\quad \phi_k(\zeta)=\phi_{k-1}(\zeta)+u_k\zeta^{n_k}h_{k-1}(\zeta), \quad h_k(\zeta)=h_{k-1}(\zeta)-\overline{u}_k\zeta^{-n_k}\phi_{k-1}(\zeta), \quad (\zeta\in\mathbb{T}).$$

Since $n_{k+1} > 2n_k$, it is clear that the sets Λ_k, $k = 1, 2, \ldots$, are disjoint and that $\Lambda_k \subset \lfloor n_{k-1} + 1, n_k \rfloor$ for all $k \geqslant 1$.

We claim that the sequences $\{\phi_k\}$ and $\{h_k\}$ satisfy the following properties:

$$(2.4)\qquad \widehat{\phi}_k(n) = 0, \quad \text{whenever } k \geqslant 0 \text{ and } n \notin \bigcup_{j=0}^{k}\Lambda_j;$$

$$(2.5)\qquad \widehat{h}_k(-n) = 0, \quad \text{whenever } k \geqslant 0 \text{ and } n \geqslant 1 \text{ and } n \notin \bigcup_{j=1}^{k}\Lambda_j;$$

$$(2.6)\qquad \widehat{\phi}_k(n) = \widehat{\phi}_j(n), \quad \text{whenever } k \geqslant j \text{ and } n \leqslant n_j;$$

$$(2.7)\qquad \widehat{\phi}_k(n_j) = u_j, \quad \text{whenever } k \geqslant j.$$

It is clear that (2.4) and (2.5) hold for $k = 0, 1$. Arguing by induction, assume that (2.4) and (2.5) are valid for some value of $k \in \mathbb{N}$. Then,

$$(2.8)\qquad \phi_{k+1}(\zeta) = \phi_k(\zeta) + u_{k+1}\zeta^{n_{k+1}}h_k(\zeta) = \sum_{j=0}^{k} \sum_{n\in\Lambda_j} \widehat{\phi}_k(n)\zeta^n + f_k(\zeta),$$

where $f_k(\zeta) = u_{k+1}\zeta^{n_{k+1}}h_k(\zeta)$. By the induction hypotheses $\widehat{f}_k(n) = 0$ if $n \notin \Lambda_{k+1}$, which gives (2.4) for $k + 1$. The proof of (2.5) is analogous. Now, (2.6) follows from (2.3), (2.4) and the fact that the sets Λ_k are disjoint and (2.5). Using again that the sets Λ_k are disjoint, (2.6), (2.3) and (2.2), we deduce (2.7).

We have that

$$|\phi_0(\zeta)|^2 + |h_0(\zeta)|^2 = 1 + |u_0|^2,$$

so if we assume that $|\phi_k(\zeta)|^2 + |h_k(\zeta)|^2 = \prod_{j=0}^{k}(1 + |u_j|^2)$, bearing in mind (2.1) and (2.3), it follows that

$$|\phi_{k+1}(\zeta)|^2 + |h_{k+1}(\zeta)|^2 = (1 + |u_{k+1}|^2)(|\phi_k(\zeta)|^2 + |h_k(\zeta)|^2) = \prod_{j=0}^{k+1}(1 + |u_j|^2),$$

hence we have proved by induction that

$$|\phi_k(\zeta)|^2 + |h_k(\zeta)|^2 = \prod_{j=0}^{k}(1 + |u_j|^2), \quad \zeta \in \mathbb{T}, \ k = 0, 1, 2, \ldots.$$

This and the fact that $\{u_k\}_{k=0}^{\infty} \in \ell^2$ imply that $\{h_k\}_{k=0}^{\infty}$ and $\{\phi_k\}_{k=0}^{\infty}$ are uniformly bounded sequences of functions in $L^\infty(\mathbb{T})$. Then, using the Banach–Alaoglu theorem, (2.4), (2.6) and (2.7), we deduce that a subsequence of $\{\phi_k\}$ converges in the weak star topology of $L^\infty(\mathbb{T})$ to a function $\phi \in L^\infty(\mathbb{T})$ with $\widehat{\phi}(n) = 0$ for all

$n < 0$, and $\widehat{\phi}(n_k) = u_k$ for all k. Then if we set $a_n = \widehat{\phi}(n)$ $(n \geqslant 0)$ it follows that the function Ψ defined by

$$\Psi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

is analytic in $\mathbb{D}$ and satisfies (i), (ii) and (iii).

Finally, we shall prove (iv). Using (2.6) and (2.8), we see that for any $\zeta \in \mathbb{T}$, we have

$$S_{n_k+1,n_{k+1}+1} \Psi(\zeta) = \sum_{n=n_k+1}^{n_{k+1}} \widehat{\Psi}(n)\zeta^n = \sum_{m=n_k+1}^{n_{k+1}} \left(\lim_{m \to \infty} \widehat{\phi}_m(n) \right) \zeta^n$$

$$= \sum_{n=n_k+1}^{n_{k+1}} \widehat{\phi}_{k+1}(n)\zeta^n = f_k(\zeta) = u_{k+1}\zeta^{n_{k+1}} h_k(\zeta),$$

which, bearing in mind that $\sup_k \|h_k\|_\infty = C < \infty$, implies

$$\|S_{n_k+1,n_{k+1}+1} \Psi\|_{H^\infty} = |u_{k+1}| \|h_k\|_{L^\infty(\mathbb{T})} \leqslant C|u_{k+1}|.$$

This finishes the proof. ∎

Our work will also make use of the Rademacher functions $\{r_n(t)\}_{n=0}^{\infty}$ which are defined by

$$r_0(t) = \begin{cases} 1 & \text{if } 0 < t < 1/2, \\ -1 & \text{if } 1/2 < t < 1, \\ 0, & \text{if } t = 0, 1/2, 1; \end{cases} \quad r_n(t) = r_0(2^n t), \quad n = 1, 2, \ldots.$$

See, e.g., Chapter V, Vol. I of [41] or Appendix A of [10] for the properties of these functions. In particular, we shall use Khinchine's inequality which we state as follows.

PROPOSITION 2.3 (Khinchine's inequality). *If $\{c_k\}_{k=1}^{\infty} \in \ell^2$ then the series $\sum_{k=1}^{\infty} c_k r_k(t)$ converges almost everywhere. Furthermore, for $0 < p < \infty$ there exist positive constants A_p, B_p such that for every sequence $\{c_k\}_{k=0}^{\infty} \in \ell^2$ we have*

$$A_p \left(\sum_{k=0}^{\infty} |c_k|^2 \right)^{p/2} \leqslant \int_0^1 \left| \sum_{k=0}^{\infty} c_k r_k(t) \right|^p dt \leqslant B_p \left(\sum_{k=0}^{\infty} |c_k|^2 \right)^{p/2}.$$

3. MULTIPLIERS ON $\mathcal{B} \cap \mathcal{D}_{p-1}^p$

Proof of Theorem 1.2. (i) Assume that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$. From now and throughout the paper we shall denote by φ_a the Möbius transformation which interchanges the origin and a,

$$\varphi_a(z) = \frac{a-z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

A simple calculation shows that

$$\sup_{a \in \mathbb{D}} \|\varphi_a\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p} < \infty.$$

So, for any $a, z \in \mathbb{D}$

$$(1 - |z|^2)|\varphi_a'(z)g(z)| = (1 - |z|^2)|(\varphi_a \cdot g)'(z) - \varphi_a(z)g'(z)|$$
$$\leqslant \|\varphi_a g\|_{\mathcal{B} \cap \mathcal{D}_{q-1}^q} + (1 - |z|^2)|\varphi_a(z)g'(z)|$$
$$\text{(3.1)} \qquad\qquad \lesssim \|M_g\|_{(\mathcal{B} \cap \mathcal{D}_{p-1}^p \to \mathcal{B} \cap \mathcal{D}_{q-1}^q)} + \|g\|_{\mathcal{B}} < \infty.$$

Since $(1 - |a|^2)|\varphi_a'(a)| = 1$, taking $z = a$ in (3.1) we obtain

$$|g(a)| \lesssim \|M_g\|_{(\mathcal{B} \cap \mathcal{D}_{p-1}^p \to \mathcal{B} \cap \mathcal{D}_{q-1}^q)} + \|g\|_{\mathcal{B}} < \infty,$$

for any $a \in \mathbb{D}$. Thus, $g \in H^\infty$.

Next consider the family of test functions, $f_\theta(z) = \log(1/(1 - ze^{-i\theta}))$, $\theta \in [0, 2\pi)$. A calculation shows that $\{f_\theta\}_{\theta \in [0, 2\pi)}$ is uniformly bounded in $\mathcal{B} \cap \mathcal{D}_{p-1}^p$. Therefore,

$$A = \sup_{\theta \in [0, 2\pi)} \|g f_\theta\|_{\mathcal{B}} \leqslant \sup_{\theta \in [0, 2\pi)} \|g f_\theta\|_{\mathcal{B} \cap \mathcal{D}_{q-1}^q}$$
$$\leqslant \|M_g\|_{(\mathcal{B} \cap \mathcal{D}_{p-1}^p \to \mathcal{B} \cap \mathcal{D}_{q-1}^q)} \sup_{\theta \in [0, 2\pi)} \|f_\theta\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p} < \infty,$$

which implies that

$$(1 - |z|^2)|g'(z)f_\theta(z)| = (1 - |z|^2)|g'(z)f_\theta(z) + g(z)f_\theta'(z) - g(z)f_\theta'(z)|$$
$$\leqslant A + (1 - |z|^2)|g(z)f_\theta'(z)| = A + \|g\|_{H^\infty} \sup_{\theta \in [0, 2\pi)} \|f_\theta\|_{\mathcal{B}}$$
$$< \infty, \quad \text{for all } z \in \mathbb{D} \text{ and } \theta \in [0, 2\pi).$$

Finally, given $z \in \mathbb{D}$ choose $e^{i\theta} = z/|z|$ to deduce that

$$\sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|) \log(1/(1 - |z|)) < \infty,$$

which together with the fact that $g \in H^\infty$ gives that $g \in M(\mathcal{B})$.

Suppose now that $g \in M(\mathcal{B})$ and take $f \in \mathcal{B} \cap \mathcal{D}_{p-1}^p$. Then $fg \in \mathcal{B}$. Using Lemma 1.1 and the closed graph theorem, we obtain

$$\int_{\mathbb{D}} |(fg)'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$\lesssim \int_{\mathbb{D}} |f'(z)g(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z) + \int_{\mathbb{D}} |g'(z)f(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$(3.2) \quad \lesssim \|g\|_{H^\infty}^q \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q + \int_{\mathbb{D}} |f(z)g'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z).$$

We shall distinguish two cases to deal with the last integral which appears in (3.2). First, if $1 < q \leqslant 2$, bearing in mind (1.2) and the fact that $g \in \mathcal{B}_{\log}$, we see that

$$\int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \lesssim \int_0^1 \frac{1}{(1-r)^q \log^q(e/(1-r))} M_q^q(r,f)(1-r)^{q-1} \mathrm{d}r$$

$$(3.3) \qquad\qquad \lesssim \|f\|_{\mathcal{D}_{q-1}^q}^q \int_0^1 \frac{1}{(1-r)\log^q(e/(1-r))} \mathrm{d}r \lesssim \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q.$$

On the other hand, if $2 < q < \infty$, then using that $g \in \mathcal{B}_{\log}$ and the well known fact that

$$M_q(r,f) \leqslant C\|f\|_{\mathcal{B}} (\log(1/(1-r)))^{1/2}, \quad 0 < r < 1,$$

(see, e.g., [8]) we get

$$\int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \lesssim \int_0^1 \frac{1}{(1-r)^q \log^q(e/(1-r))} M_q^q(r,f)(1-r)^{q-1} \mathrm{d}r$$

$$(3.4) \qquad\qquad \lesssim \|f\|_{\mathcal{B}}^q \int_0^1 \frac{1}{(1-r) \log^{q/2}(e/(1-r))} \mathrm{d}r < \infty.$$

Joining (3.2) (3.3) and (3.4), we see that in any case we have $fg \in \mathcal{D}_{q-1}^q$ and, hence, $fg \in \mathcal{B} \cap \mathcal{D}_{q-1}^q$. Thus, we have proved that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$ finishing the proof.

(ii) We borrow ideas from Theorem 12 of [15]. We shall distinguish three cases.

Case 1. $2 < q < \infty$.

Assume that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$ and $g \not\equiv 0$. By the proof of Theorem K in [15] (see also the proofs of Theorems 1.6 and 1.7 in [21]), it follows that there exists a function $f \in \mathcal{D}_{p-1}^p$, given by a lacunary power series, with $f(0) \neq 0$,

and such that its sequence of ordered zeros $\{z_n\}$ (that is, the $z_n's$ are ordered so that $|z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots$) satisfies

$$\prod_{n=1}^{N} \frac{1}{|z_n|} \neq o((\log N)^{1/2-1/q}).$$

Since f is given by a lacunary power series, by Proposition 2.1, the sequence of its Taylor coefficients is in ℓ^p. This implies that $f \in \mathcal{B} \cap \mathcal{D}_{p-1}^p$. If $\{w_n\}$ is the sequence of non-zero zeros of gf arranged so that $|w_1| \leqslant |w_2| \leqslant |w_3| \leqslant \cdots$, we have that $|w_n| \leqslant |z_n|$, for all n, which gives that

$$\prod_{n=1}^{N} \frac{1}{|w_n|} \geqslant \prod_{n=1}^{N} \frac{1}{|z_n|},$$

hence

$$\prod_{n=1}^{N} \frac{1}{|w_n|} \neq o((\log N)^{1/2-1/q}).$$

This together with Theorem 1.6 of [21] implies that $fg \notin \mathcal{D}_{q-1}^q$. This is a contradiction. Thus, $g \equiv 0$.

Case 2. $0 < q \leqslant 2 < p$.

The proof is similar to that of the case 1. Suppose that $g \not\equiv 0$ and $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$. Take $\gamma \in (0, \frac{1}{2} - \frac{1}{p})$. Then, by the proof of Theorem K of [15] and Proposition 2.1, there is a function $f \in \mathcal{B} \cap \mathcal{D}_{p-1}^p$, represented by a lacunary series, with $f(0) \neq 0$ whose sequence of ordered zeros $\{z_n\}$ satisfies

$$(3.5) \qquad \prod_{n=1}^{N} \frac{1}{|z_n|} \neq o((\log N)^{\gamma}).$$

Let $\{w_n\}_{n=1}^{\infty}$ be the sequence of ordered non-zero zeros of fg. Since $fg \in \mathcal{D}_{q-1}^q$ and $q \leqslant 2$, it follows that $fg \in H^q$ and, hence, $\{w_n\}_{n=1}^{\infty}$ satisfies the Blaschke condition which is equivalent to saying that

$$\prod_{k=1}^{N} \frac{1}{|w_k|} = O(1), \quad \text{as } N \to \infty.$$

This is in contradiction with (3.5), because any zero of f is also a zero of fg. Consequently, $g \equiv 0$.

Case 3. $0 < p \leqslant 2$.

Suppose that $g \not\equiv 0$ and $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$. Take $a_n = 1/n^{1/p+\varepsilon}$ with $0 < \varepsilon < 1/q - 1/p$ and $f(z) = \sum_{n=1}^{\infty} a_n z^{2^n}$. Since $\sum_{n=1}^{\infty} a_n^p < \infty$ and $\sum_{n=1}^{\infty} a_n^q = \infty$, then by Proposition 2.1, $f \in \mathcal{B} \cap \mathcal{D}_{p-1}^p \setminus \mathcal{D}_{q-1}^q$.

Let $\{r_k(t)\}$ be the Rademacher functions and let $f_t(z) = \sum\limits_{k=1}^{\infty} r_k(t) a_k z^{2^k}$. By Proposition 2.1(iii)

$$\|f\|_{\mathcal{B}} \asymp \sup_n |a_n| \asymp \|f_t\|_{\mathcal{B}} \quad t \in [0,1], \quad \text{and}$$

$$(3.6) \quad \|f_t\|_{H^2}^{2p} = \Big(\sum_{k=0}^{\infty} |a_k|^2 \Big)^p \leqslant \Big(\sum_{k=0}^{\infty} |a_k|^p \Big)^2 \asymp \|f_t\|_{\mathcal{D}_{p-1}^p}^{2p} \asymp \|f\|_{\mathcal{D}_{p-1}^p}^{2p} \quad t \in [0,1].$$

Then for any $t \in [0,1]$, it follows that

$$(3.7) \quad \int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \lesssim \|f_t\|_{\mathcal{D}_{p-1}^p}^q + \|f_t\|_{\mathcal{B}}^q \asymp \|f\|_{\mathcal{D}_{p-1}^p}^q + \|f\|_{\mathcal{B}}^q < \infty.$$

So, by Fubini's theorem, Khinchine's inequality and the fact that $g \in \mathcal{D}_{q-1}^q$, we obtain

$$\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \mathrm{d}t$$

$$\lesssim \int_0^1 \int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \mathrm{d}t + \int_0^1 \int_{\mathbb{D}} |f_t g'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \mathrm{d}t$$

$$\lesssim \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q + \int_{\mathbb{D}} |g'(z)|^q \int_0^1 |f_t(z)|^q (1-|z|^2)^{q-1} \mathrm{d}t \mathrm{d}A(z)$$

$$\lesssim \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q + \int_{\mathbb{D}} |g'(z)|^q M_2^q(|z|,f)(1-|z|^2)^{q-1} \mathrm{d}A(z)$$

$$(3.8) \quad \lesssim \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q + \|f\|_{\mathcal{D}_{p-1}^p}^q \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \lesssim \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-1}^p}^q.$$

On the other hand, since $g \not\equiv 0$, there exists a positive constant C such that $M_q^q(r,g) \geqslant C$, $1/2 < r < 1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also given by a power series with Hadamard gaps (thus $M_2(r,f') \asymp M_q(r,f')$) we have that

$$\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) \mathrm{d}t = \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} \Big(\int_0^1 |f_t'(z)|^q \mathrm{d}t \Big) \mathrm{d}A(z)$$

$$\asymp \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} M_2^q(|z|,f') \mathrm{d}A(z)$$

$$\geqslant C \int_{1/2}^1 M_q^q(r,g) M_q^q(r,f')(1-r^2)^{q-1} \mathrm{d}r$$

$$\geqslant C \int\limits_{1/2}^{1} M_q^q(r,f')(1-r^2)^{q-1}\mathrm{d}r = +\infty.$$

This is in contradiction with (3.8). It follows that $g \equiv 0$. ∎

We remark that the argument used to prove the inclusion $M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q) \subset M(\mathcal{B})$ in the proof of Theorem 1.1(i) works for any values of p and q, that is we have

$$M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q) \subset M(\mathcal{B}), \quad 0 < p,q < \infty.$$

We do not have a complete characterization of the space $M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$ in the case $0 < p \leqslant q \leqslant 1$, however we find a sharp sufficient condition on a function g to lie in this space of multipliers. We note that Theorem 1.3 is a byproduct of part (ii) of the following stronger result.

PROPOSITION 3.1. *Let $0 < p \leqslant q \leqslant 1$, $\alpha \in (\frac{1}{q}, \infty)$ and $g \in \mathcal{H}ol(\mathbb{D})$. Then,*
(i) *If $g \in \mathcal{B}_{\log,\alpha} \cap H^\infty$, then $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$.*
(ii) *$(\mathcal{B}_{\log,(1/q)} \cap H^\infty) \setminus \mathcal{D}_{q-1}^q \neq \{0\}$.*

Proof. Part (i) can be proved arguing as in (3.2) and (3.3), so we omit a detailed proof.

(ii) Assume first that $0 < q < 1$. Consider the lacunary power series

$$g(z) = \sum_{k=1}^{\infty} \frac{1}{k^{1/q}} z^{2^k}.$$

By Proposition 2.1, $g \in H^\infty \setminus \mathcal{D}_{q-1}^q$. Since $\limsup\limits_{k\to\infty}(1/k^{1/q})(\log 2^k)^{1/q} < \infty$ (see p. 20 of [31]) $g \in \mathcal{B}_{\log,(1/q)}$.

Let us consider now the case $q = 1$. The proof in this case is a little bit more involved. Set

$$u_k = \frac{1}{k+1} \quad \text{and} \quad n_k = 4^k, \ k = 0,1,2,\ldots.$$

Let Ψ be the H^∞-function associated to these sequences via Proposition 2.2. By Lemma 1.6(i) of [36],

$$\|\Psi\|_{\mathcal{D}_0^1} \gtrsim \|\{\widehat{\Psi}(4^k)\}_{k=0}^{\infty}\|_{\ell^1} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty.$$

Finally, we shall see that $\Psi \in \mathcal{B}_{\log}$. Bearing in mind, p. 113 of [30], Lemma 3.1 of [27], and Lemma 2.2(iv), we deduce

$$M_\infty(r,\Psi') \leqslant |\widehat{\Psi}(1)| + \sum_{k=0}^{\infty} M_\infty(r, S_{n_k+1,n_{k+1}}\Psi') \lesssim \|\Psi\|_{H^\infty} + \sum_{k=0}^{\infty} \|S_{n_k+1,n_{k+1}}\Psi'\|_{H^\infty} r^{4^k}$$

$$\lesssim \|\Psi\|_{H^\infty} + \sum_{k=0}^{\infty} 4^k \|S_{n_k+1, n_{k+1}} \Psi\|_{H^\infty} r^{4^k}$$

$$\lesssim \|\Psi\|_{H^\infty} + \sum_{k=0}^{\infty} \frac{4^k}{k+1} r^{4^k}.$$

Since a standard calculation shows that

$$\sum_{k=0}^{\infty} \frac{4^k}{k+1} r^{4^k} \lesssim \frac{1}{(1-r)\log(e/(1-r))}, \quad 0 \leqslant r < 1,$$

this finishes the proof. ∎

Next we provide a sufficient condition, which involves Carleson measures, on a function g to lie in this space of multipliers. It turns out to be also necessary if g is given by a power series with Hadamard gaps.

THEOREM 3.2. *Assume that* $0 < p \leqslant q \leqslant 1$ *and let* g *be an analytic function in* $\mathbb{D}$. *Let* $\mu_{g,q}$ *be the Borel measure in* $\mathbb{D}$ *defined by* $\mathrm{d}\mu_{g,q}(z) = |g'(z)|^q(1 - |z|^2)^{q-1}\mathrm{d}A(z)$.

(i) *If* $g \in H^\infty \cap \mathcal{B}_{\log}$ *and the measure* $\mu_{g,q}$ *is a Carleson measure, then* $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$.

(ii) *If* g *is given by a power series with Hadamard gaps, then* $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$ *if and only if* $g \in H^\infty \cap \mathcal{B}_{\log}$ *and the measure* $\mu_{g,q}$ *is a Carleson measure.*

Proof. Suppose that $g \in H^\infty \cap \mathcal{B}_{\log}$ and the measure $\mu_{g,q}$ is a Carleson measure. Take $f \in \mathcal{B} \cap \mathcal{D}_{p-1}^p$.

Using (1.6), we see that $g \in M(\mathcal{B})$ and, hence, $fg \in \mathcal{B}$.

Using Theorem 2.1 of [36] we deduce that $g \in M(\mathcal{D}_{p-1}^p)$ and, then it follows that $fg \in \mathcal{D}_{p-1}^p$.

Since $\mathcal{D}_{p-1}^p \cap \mathcal{B} \subset \mathcal{D}_{q-1}^q \cap \mathcal{B}$, we have that $fg \in \mathcal{B} \cap \mathcal{D}_{q-1}^q$. Thus, we have proved that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$. This finishes the proof of part (i).

Suppose now that g is given by a power series with Hadamard gaps and $g \in M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q)$. Then $g \in \mathcal{D}_{q-1}^q$. Now, using Theorem 3.2 of [19], we see that this implies that $\mu_{g,q}$ is a Carleson measure. ∎

4. MULTIPLIERS ON $H^\infty \cap \mathcal{D}_{p-1}^p$

Proof of Theorem 1.4. Suppose that $0 < p \leqslant q < \infty$.

If $g \in M(H^\infty \cap \mathcal{D}_{p-1}^p, H^\infty \cap \mathcal{D}_{q-1}^q)$ then, since $H^\infty \cap \mathcal{D}_{p-1}^p$ contains the constant functions, it follows trivially that $g \in H^\infty \cap \mathcal{D}_{q-1}^q$.

On the other hand, if $g \in H^\infty \cap \mathcal{D}^q_{q-1}$ and $f \in H^\infty \cap \mathcal{D}^p_{p-1}$, it is clear that $gf \in H^\infty$. We also have

$$\int_{\mathbb{D}} |(g'f + gf')(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$\lesssim \int_{\mathbb{D}} |(g'f)(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z) + \int_{\mathbb{D}} |g(z)f'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$\lesssim \|f\|^q_{H^\infty} \|g\|^q_{\mathcal{D}^q_{q-1}} + \|g\|^q_{H^\infty} \|f\|^{q-p}_{\mathcal{B}} \|f\|^{q-p}_{\mathcal{D}^p_{p-1}} < \infty.$$

Thus, $gf \in \mathcal{D}^q_{q-1}$ and, hence, $gf \in H^\infty \cap \mathcal{D}^q_{q-1}$. Consequently, we have proved that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$. ∎

Proof of Theorem 1.5. Let $\widetilde{p} = \min\{p, 1\}$ and $p^\star = \min\{p, 2\}$. We shall split the proof in two cases.

Case 1. $0 < q < 1$.

Take a sequence $\{u_k\}^\infty_{k=1} \in \ell^{\widetilde{p}} \setminus \ell^q$ and let f be defined by

$$f(z) = \sum_{k=1}^\infty u_k z^{2^k}, \quad z \in \mathbb{D}.$$

Then, using Proposition 2.1 and the fact that $\widetilde{p} \leqslant 1$, we see that $f \in (\mathcal{D}^p_{p-1} \cap H^\infty) \setminus \mathcal{D}^q_{q-1}$.

Case 2. $1 \leqslant q < 2$.

Let us consider a sequence $\{u_k\}$ such that $\{u_k\}^\infty_{k=1} \in \ell^{p^\star} \setminus \ell^q$ and let choose $n_k = 4^k$. We claim that the function $\Phi \in H^\infty$ associated to $\{u_k\}$ and $\{n_k\}$ via Lemma 2.2 satisfies that $\Phi \in H^\infty \cap \mathcal{D}^p_{p-1} \setminus H^\infty \cap \mathcal{D}^q_{q-1}$.

Arguing as in the proof of Lemma 1.6(i) in [36] and bearing in mind Lemma 2.2(ii), we deduce

$$\|\Phi\|^q_{\mathcal{D}^q_{q-1}} \gtrsim \sum_{k=0}^\infty |\widehat{\Phi(n_k)}|^q = \|\{u_k\}\|^q_{\ell^q} = \infty,$$

that is, $\Phi \notin \mathcal{D}^q_{q-1}$.

By (1.1), if $p \geqslant 2$ we are done. On the other hand, if $0 < p < 2$ by Theorem 1.1(ii) of [21], M. Riesz theorem and Lemma 2.2(iv),

$$\|\Phi\|^q_{\mathcal{D}^p_{p-1}} \leqslant \int_0^1 (1 - r)^{p-1} M_2^p(r, \Phi') \mathrm{d}r \lesssim \sum_{k=0}^\infty (\|S_{2^k, 2^{k+1}} \Phi\|_{H^2})^{p/2}$$

$$\lesssim \sum_{k=0}^\infty (\|S_{4^k+1, 4^{k+1}+1} \Phi\|_{H^2})^{p/2} \lesssim \|\{u_k\}\|^p_{\ell^p} < \infty,$$

which finishes the proof. ∎

Proof of Theorem 1.6(i). Assume that $0 < q < 1$, $0 < q < p$ and that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ and $g \not\equiv 0$. Take

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k^{1/q}}.$$

Then we use the Rademacher functions as in the proof of Case 3 of Theorem 1.2(ii) to get a contradiction. Hence, $g \equiv 0$. ∎

Proof of Theorem 1.6(ii). Assume that $1 \leqslant q < 2 \leqslant p$. By Theorem 1 of [17] there is a function $f \in H^\infty$ such that

$$\int_0^1 (1 - r^2)^{q-1} |f'(re^{i\theta})|^q dr = \infty \quad \text{for every } \theta \in B,$$

where B is a subset of $[0, 2\pi]$ whose Lebesgue measure $|B|$ is 2π.

Suppose that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ and $g \not\equiv 0$. Notice that $g \in H^\infty \cap \mathcal{D}^q_{q-1}$. Since

$$(4.1) \qquad \int_{\mathbb{D}} (1 - |z|^2)^{q-1} |g'(z)f(z)|^q dA(z) \leqslant \|f\|^q_{H^\infty} \|g\|^q_{H^\infty \cap \mathcal{D}^q_{q-1}} < \infty,$$

it follows that

$$(4.2) \qquad \int_{\mathbb{D}} (1 - |z|^2)^{q-1} |g(z)f'(z)|^q dA(z) < \infty$$

Since $g \in H^\infty$ and $g \not\equiv 0$, there is a set $A = A(g) \subset [0, 2\pi]$ with $|A| > 0$ and such that $\lim_{r \to 1^-} g(re^{i\theta}) \neq 0$ if $\theta \in A$. Then, for every $\theta \in A \cap B$ there is $r_0(\theta) \in (0, 1)$ such that $K = \inf_{r_0 < r < 1} |g(re^{i\theta})| > 0$. Then

$$\int_0^1 (1 - r^2)^{q-1} |g(re^{i\theta})|^q |f'(re^{i\theta})|^q dr \geqslant K^q \int_{r_0}^1 (1 - r^2)^{q-1} |f'(re^{i\theta})|^q dr = \infty,$$

since $|A \cap B| > 0$, this is in contradiction with (4.2). Thus g must be identically 0. This finishes the proof. ∎

5. SOME BASIC RESULTS ON THE SPACE BMOA$_{\log}$

We shall start this section by proving some embedding relations between BMOA$_{\log}$, $\mathcal{B}_{\log}$ and BMOA. With this aim, we recall that $g \in$ BMOA$_{\log}$ if and only if

$$\sup_{a \in \mathbb{D}} \frac{\log^2(2/(1 - |a|))}{1 - |a|} \int_{S(a)} |g'(z)|^2 (1 - |z|^2) dA(z) < \infty,$$

where $S(a)$ is the Carleson box associated to the interval

$$I_a = \left\{ e^{it} \in \mathbb{T} : |\arg(ae^{-it})| < \frac{1-|a|}{2} \right\}, \quad a \in \mathbb{D} \setminus \{0\}, \ I_0 = \mathbb{T}.$$

PROPOSITION 5.1. *If $1 > \beta > 1/2$, then $\mathrm{BMOA}_{\log} \subsetneq \mathcal{B}_{\log} \subsetneq \mathcal{B}_{\log,\beta} \subsetneq \mathrm{BMOA}$.*

Proof. First, we prove that $\mathrm{BMOA}_{\log} \subset \mathcal{B}_{\log}$. Take $f \in \mathrm{BMOA}_{\log}$. Let $a \in \mathbb{D}$ and assume without loss of generality that $|a| > 1/2$. Set $a^\star = ((3|a|-1)/2)e^{i \arg a}$ so that the disc $D(a, \frac{1-|a|}{2})$ of center a and radius $(1-|a|)/2$ is contained in the Carleson box $S(a^\star)$. This inclusion together with the subharmonicity of $|f'|^2$ and the fact that $(1-|z|) \asymp (1-|a|)$ for $z \in D(a, \frac{1-|a|}{2})$ gives

$$(\log(2/(1-|a|)))^2 (1-|a|)^2 |f'(a)|^2$$

$$\lesssim (\log(2/(1-|a|)))^2 \int_{D(a,\frac{1-|a|}{2})} |f'(z)|^2 dA(z)$$

$$\asymp \frac{(\log(2/(1-|a|)))^2}{1-|a|} \int_{D(a,\frac{1-|a|}{2})} (1-|z|^2)|f'(z)|^2 dA(z)$$

$$\asymp \frac{(\log(2/(1-|a^\star|)))^2}{1-|a^\star|} \int_{D(a,\frac{1-|a|}{2})} (1-|z|^2)|f'(z)|^2 dA(z)$$

$$\lesssim \frac{(\log(2/(1-|a^\star|)))^2}{1-|a^\star|} \int_{S(a^\star)} (1-|z|^2)|f'(z)|^2 dA(z),$$

so $f \in \mathcal{B}_{\log}$.

Now, let us see that the inclusion is strict. We borrow ideas from Proposition 5.1(D) of [32]. Assume on the contrary to the assertion that $\mathrm{BMOA}_{\log} = \mathcal{B}_{\log}$. By Theorem 1 of [22] (see also [1]) there are $g_1, g_2 \in \mathcal{B}_{\log}$ such that

$$|g_1'(z)| + |g_2'(z)| \gtrsim \frac{1}{(1-|z|)\log(2/(1-|z|))}, \quad z \in \mathbb{D}.$$

Then, for any $a \in \mathbb{D}$

$$\int_{S(a)} \frac{1}{(1-|z|)\log^2(2/(1-|z|))} dA(z)$$

$$\lesssim \int_{S(a)} (|g_1'(z)| + |g_2'(z)|)^2 (1-|z|^2) dA(z)$$

$$\lesssim \int_{S(a)} |g_1'(z)|^2 (1-|z|^2) dA(z) + \int_{S(a)} |g_2'(z)|^2 (1-|z|^2) dA(z) \lesssim \frac{(1-|a|)}{\log^2(2/(1-|a|))},$$

so bearing in mind that

$$\int\limits_{S(a)} \frac{1}{(1-|z|)\log^2(2/(1-|z|))}dA(z) \asymp \frac{(1-|a|)}{\log(2/(1-|a|))},$$

and letting $|a| \to 1^-$, we obtain a contradiction.

Assume now that $\beta \in (\frac{1}{2}, 1)$. Then it is clear that $\mathcal{B}_{\log} \subsetneq \mathcal{B}_{\log,\beta}$. Furthermore, $f(z) = \sum\limits_{k=1}^{\infty} (z^{2^k}/k^\beta) \in \mathcal{B}_{\log,\beta} \setminus \mathcal{B}_{\log}$ (see p. 20 of [31]).

The inclusion $\mathcal{B}_{\log,\beta} \subsetneq \text{BMOA}$, for $\beta > 1/2$, follows easily using the characterization of BMOA in terms of Carleson measures (see p. 669 of [14]). Finally, we observe that $f(z) = \log(1/(1-z)) \in \text{BMOA} \setminus \mathcal{B}_{\log,\beta}$ for any $\beta > 0$. This concludes the proof. ∎

Next we find a simple sufficient condition for the membership of a function $f \in \mathcal{H}ol(\mathbb{D})$ in the space $\text{BMOA}_{\log}$.

PROPOSITION 5.2. *Let f be an analytic function in $\mathbb{D}$. If*

$$(5.1) \qquad \int\limits_0^1 (1-r)(\log(1/(1-r)))^2[M_\infty(r,f')]^2 dr < \infty$$

then $f \in \text{BMOA}_{\log}$.

Proof. Suppose that f satisfies (5.1) Let I be an interval in $\mathbb{T}$ of length h, say $I = \{e^{it} : \theta_0 < t < \theta_0 + h\}$. Then

$$\frac{(\log(2/|I|))^2}{|I|} \int\limits_{S(I)} (1-|z|^2)|f'(z)|^2 dA(z)$$

$$\asymp \frac{(\log(2/h))^2}{h} \int\limits_{1-h}^1 \int\limits_{\theta_0}^{\theta_0+h} (1-r)|f'(re^{it})|^2 dr \lesssim (\log(2/h))^2 \int\limits_{1-h}^1 (1-r)[M_\infty(r,f')]^2 dr$$

$$\leqslant \int\limits_{1-h}^1 (1-r)[M_\infty(r,f')]^2(\log(2/(1-r)))^2 dr$$

$$\leqslant \int\limits_0^1 (1-r)[M_\infty(r,f')]^2(\log(2/(1-r)))^2 dr. \quad ∎$$

Now we turn to the question of finding conditions on the Taylor coefficients of a function $f \in \mathcal{H}ol(\mathbb{D})$ enough to assert that $f \in \text{BMOA}_{\log}$. We shall need two lemmas. The first one estimates an integral which may be viewed as a generalization of the classical beta function (compare with Lemma 2 of [9]) and we omit its proof.

LEMMA 5.3. *Whenever $m = 1, 2, 3, \ldots$ and $\alpha > 0$, we have*

$$(5.2) \qquad \int_0^1 x^n (1-x)^m (\log(1/(1-x)))^\alpha dx \asymp \frac{(\log n)^\alpha}{n^{m+1}}, \quad \text{as } n \to \infty.$$

LEMMA 5.4. *Suppose that $\alpha > 0$ and let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^\infty a_n z^n$ ($z \in \mathbb{D}$). The following two conditions are equivalent:*

(i) $\int_{\mathbb{D}} (1 - |z|^2) |g'(z)|^2 (\log(2/(1-|z|)))^\alpha dA(z) < \infty.$

(ii) $\sum_{n=1}^\infty |a_n|^2 [\log n]^\alpha < \infty.$

Proof. We have

$$\int_{\mathbb{D}} (1 - |z|^2) |g'(z)|^2 (\log(2/(1-|z|)))^\alpha dA(z)$$

$$\asymp \int_0^1 r(1-r) M_2(r, g')^2 (\log(2/(1-r)))^\alpha dr$$

$$= \sum_{n=1}^\infty n^2 |a_n|^2 \int_0^1 (1-r) r^{2n-1} (\log(2/(1-r)))^\alpha dr.$$

Using Lemma 5.3 with $m = 1$ we see that $\int_0^1 (1-r) r^{2n-1} (\log(2/(1-|z|)))^\alpha dr \asymp [\log n]^\alpha / n^2$. Then it follows that

$$\int_{\mathbb{D}} (1 - |z|^2) |g'(z)|^2 (\log(2/(1-|z|)))^2 dA(z) \asymp \sum_{n=1}^\infty |a_n|^2 [\log(n+1)]^\alpha. \qquad \blacksquare$$

We close this section proving Proposition 1.8.

Proof of Proposition 1.8. Suppose that $\sum_{k=0}^\infty |a_k|^2 (\log n_k)^3 < \infty$ and

$$f(z) = \sum_{k=0}^\infty a_k z^{n_k} \ (z \in \mathbb{D}) \quad \text{with } n_{k+1} \geqslant \lambda n_k \text{ for all } k, \text{ and } \lambda > 1.$$

Using the Cauchy–Schwarz inequality and the fact that $\sum_{k=0}^\infty r^{2n_k} \lesssim \log(2/(1-r))$ (because the function h given by $h(z) = \sum z^{2n_k}$ is a Bloch function), we see that

$$[r M_\infty(r, f')]^2 \leqslant \left(\sum_{k=0}^\infty n_k |a_k| r^{n_k} \right)^2 \leqslant \left(\sum_{k=0}^\infty n_k^2 |a_k|^2 r^{2n_k} \right) \left(\sum_{k=0}^\infty r^{2n_k} \right)$$

$$\lesssim (\log(2/(1-r))) \sum_{k=0}^\infty n_k^2 |a_k|^2 r^{2n_k}.$$

Then, using Lemma 5.3 with $m = 1$ and $\alpha = 3$, we obtain

$$\int_0^1 (1-r)(\log(1/(1-r)))^2 [M_\infty(r, f')]^2 dr$$

$$\lesssim \int_0^1 (1-r)(\log(1/(1-r)))^3 \Big(\sum_{k=0}^\infty n_k^2 |a_k|^2 r^{2n_k} \Big) dr$$

$$= \sum_{k=0}^\infty n_k^2 |a_k|^2 \int_0^1 r^{2n_k} (1-r)(\log(1/(1-r)))^3 dr \lesssim \sum_{k=0}^\infty |a_k|^2 (\log n_k)^3 < \infty.$$

Then Proposition 5.2 implies that $f \in \text{BMOA}_{\log}$.

To see that $f \in H^\infty$ observe that $\lambda^k \lesssim n_k$ and $|a_k|^2 \lesssim (\log n_k)^{-3}$. Then it follows that $|a_k| = O(k^{-3/2})$, as $k \to \infty$ and the result follows. ∎

6. RANDOM POWER SERIES

In this section we shall consider random power series analytic in $\mathbb{D}$ of the form

$$\sum_{n=0}^\infty \epsilon_n a_n z^n$$

where the ϵ_n's are random signs. More precisely, if $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ ($z \in \mathbb{D}$), we set

$$f_t(z) = \sum_{n=0}^\infty r_n(t) a_n z^n, \quad 0 \leqslant t < 1, z \in \mathbb{D},$$

where the r_n's are the Rademacher functions. Each function f_t is analytic in $\mathbb{D}$. Littlewood [25] (see also Appendix A of [10]) proved that if $\sum_{n=0}^\infty |a_n|^2 < \infty$ then $f_t \in \bigcap_{0<p<\infty} H^p$ almost surely, that is, for almost every t. On the other hand, the condition $\sum_{n=0}^\infty |a_n|^2 = \infty$ implies that for almost every t, f_t has a radial limit almost nowhere.

Paley and Zygmund [29] gave an example of an f with

$$(6.1) \qquad \sum_{n=1}^\infty |a_n|^2 \log n < \infty$$

such that $f_t \notin H^\infty$ for every t.

Anderson, Clunie and Pommerenke [2] used a result of Salem and Zygmund [33] on the behaviour of the maxima of the partial sums of random trigonometric series to prove that (6.1) implies that $f_t \in \mathcal{B}$ almost surely and that this

condition is the best possible. Later on, Sledd [35] used also the Salem and Zygmund theorem to show that (6.1) actually implies that $f_t \in$ BMOA almost surely.

Duren proved in [9] the following result.

THEOREM 6.1. *If* $0 \leqslant \beta \leqslant 1$ *and* $\sum\limits_{n=1}^{\infty} |a_n|^2 (\log n)^\beta < \infty$, *then for almost every* $t \in [0, 1]$, *the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

satisfies

(6.2) $$\int_0^1 (1 - r)(\log(1/(1 - r)))^{\beta-1} [M_\infty(r, f_t')]^2 dr < \infty.$$

Using this, Duren gave in [9] a new proof of Sledd's theorem. Next we prove an analogue of Duren's theorem for $\beta = 3$. This will allow us to obtain the analogue of Sledd's theorem for BMOA$_{\log}$.

THEOREM 6.2. *If* $\sum\limits_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty$ *then for almost every* $t \in [0, 1]$, *the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

satisfies

(6.3) $$\int_0^1 (1 - r)(\log(1/(1 - r)))^2 [M_\infty(r, f_t')]^2 dr < \infty.$$

Another result of [33] yields that the condition $\sum\limits_{n=1}^{\infty} |a_n|^2 [\log n]^\beta < \infty$ for some $\beta > 1$, implies that for almost every t, f_t has a continuous extension to the closed unit disc. Using this, Proposition 5.2, and Theorem 6.2 we obtain the first part of Theorem 1.9. Part (ii) of this theorem can be proved arguing as in Section 3.4 of [2], and we omit the proof.

The proof of Theorem 6.2 follows the lines of that of Theorem 6.1 in [9]. We shall use the result of Salem and Zygmund already mentioned ([9], Lemma 1), Hilbert's inequality ([9], Lemma 2) and Lemma 5.3 with $m = 3$ and $\alpha = 2$.

Proof of Theorem 6.2. Set

$$B_n^2 = \sum_{k=1}^{\infty} k^2 |a_k|^2, \quad n = 1, 2, \dots,$$

and $\psi(r) = (1 - r) \sum\limits_{n=1}^{\infty} B_n \sqrt{\log n}\, r^n$ $(0 < r < 1)$. Just as in p. 84 of [9], we have

(6.4) $$|f_t'(z)| \leqslant C\psi(r), \quad |z| = r, 0 < r < 1, \text{ almost surely.}$$

Using Lemma 5.3, the simple fact that $\log x / x^{3/2}$ decreases as x increases in $[e^{2/3}, \infty)$, and Hilbert's inequality, we deduce

$$\int_0^1 (1-r)(\log(1/(1-r)))^2 [\psi(r)]^2 dr$$

$$\asymp \int_0^1 (1-r)^3 (\log(1/(1-r)))^2 \Big[\sum_{n=1}^\infty B_n \sqrt{\log n} r^n \Big]^2 dr$$

$$= \sum_{n=1}^\infty \sum_{j=1}^\infty B_n \sqrt{\log n} B_j \sqrt{\log j} \int_0^1 r^{n+j}(1-r)^3 (\log(1/(1-r)))^2 dr$$

$$\lesssim \sum_{n=1}^\infty \sum_{j=1}^\infty \frac{B_n \sqrt{\log n} B_j \sqrt{\log j}}{(n+j)^4} [\log(n+j)]^2$$

$$(6.5) \qquad \leqslant \sum_{n=1}^\infty \sum_{j=1}^\infty \frac{1}{n+j} \frac{B_n [\log n]^{3/2}}{n^{3/2}} \frac{B_j [\log j]^{3/2}}{j^{3/2}} \lesssim \sum_{n=1}^\infty |B_n|^2 \frac{[\log n]^3}{n^3}.$$

Now,

$$\sum_{n=1}^\infty |B_n|^2 \frac{[\log n]^3}{n^3} = \sum_{n=1}^\infty \sum_{k=1}^n k^2 |a_k|^2 \frac{[\log n]^3}{n^3}$$

$$(6.6) \qquad = \sum_{k=1}^\infty k^2 |a_k|^2 \sum_{n=k}^\infty \frac{[\log n]^3}{n^3} \asymp \sum_{k=1}^\infty |a_k|^2 [\log k]^3 < \infty.$$

Then (6.4), (6.5) and (6.6) imply that (6.3) holds for almost every t, finishing the proof. ∎

7. MULTIPLIERS ON $\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}$

In this section we shall prove our results about multipliers from $\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}$ to $\mathcal{D}_{q-1}^q \cap \mathrm{BMOA}$. Let start with the following result.

THEOREM 7.1. *For any p, q with $0 < p, q < \infty$ we have*

$$M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA}) \subset \mathrm{BMOA}_{\log} \cap H^\infty = M(\mathrm{BMOA}).$$

Proof. The proof uses arguments similar to those in that of Theorem 1.2(i) and, hence, we shall omit some details.

Using that the family $\{\varphi_a : a \in \mathbb{D}\}$ is bounded in $\mathcal{D}_{\lambda-1}^\lambda \cap \mathrm{BMOA}$ for all $\lambda > 0$, we deduce that

$$M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA}) \subset H^\infty, \quad 0 < p, q < \infty.$$

Suppose now that $0 < p, q < \infty$ and $g \in M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$. Let us use the test functions f_a ($a \in \mathbb{D}$) defined by

$$f_a(z) = \log(1/(1 - \bar{a}z)), \quad z \in \mathbb{D}.$$

It is easy to see that the family $\{f_a : a \in \mathbb{D}\}$ is also bounded $\mathcal{D}_{\lambda-1}^\lambda \cap \mathrm{BMOA}$ for all $\lambda > 0$. On the other hand, there exists an absolute constant $C > 0$ such that for any arc $I \subset \partial \mathbb{D}$

$$\frac{1}{C} \log(2/|I|) \leqslant |f_a(z)| \leqslant C \log(2/|I|), \quad z \in S(I),$$

where $a = (1 - |I|/2\pi)\xi$ with ξ the center of I.

Then we have

$$\frac{\log^2(2/|I|)}{|I|} \int\limits_{S(I)} (1 - |z|^2)|g'(z)|^2 \mathrm{d}A(z)$$

$$\leqslant \frac{C^2}{|I|} \int\limits_{S(I)} (1 - |z|^2)|f_a(z)|^2 |g'(z)|^2 \mathrm{d}A(z)$$

$$\lesssim \frac{C^2}{|I|} \int\limits_{S(I)} (1 - |z|^2)|(f_a g)'(z)|^2 \mathrm{d}A(z) + \frac{C^2}{|I|} \int\limits_{S(I)} (1 - |z|^2)|f_a'(z)|^2 |g(z)|^2 \mathrm{d}A(z).$$

Since $g \in M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$, the family $\{f_a g : a \in \mathbb{D}\}$ is bounded in BMOA and hence $\sup_I (C^2/|I|) \int_{S(I)} (1 - |z|^2)|(f_a g)'(z)|^2 \mathrm{d}A(z) < \infty$. Also, using that $g \in H^\infty$ and that the family $\{f_a : a \in \mathbb{D}\}$ is bounded in BMOA, we deduce that $\sup_I (C^2/|I|) \int_{S(I)} (1 - |z|^2)|f_a'(z)|^2 |g(z)|^2 \mathrm{d}A(z) < \infty$. Consequently, we have that

$$\sup_I \frac{\log^2(2/|I|)}{|I|} \int\limits_{S(I)} (1 - |z|^2)|g'(z)|^2 \mathrm{d}A(z) < \infty,$$

that is, $g \in \mathrm{BMOA}_{\log}$. ∎

Proof of Theorem 1.10. Suppose that $1 < q < \infty$ and $0 < p \leqslant q < \infty$. In view of Theorem 7.1, we only have to prove that $M(\mathrm{BMOA}) \subset M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$.

Take $g \in M(\text{BMOA})$ and $f \in \text{BMOA} \cap \mathcal{D}_{p-1}^p$. Then, clearly, $fg \in \text{BMOA}$. Using Lemma 1.1 and the closed graph theorem, we obtain

$$\int_{\mathbb{D}} |(fg)'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z)$$

$$\lesssim \int_{\mathbb{D}} |f'(z)g(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z) + \int_{\mathbb{D}} |g'(z)f(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z)$$

$$(7.1) \quad \lesssim \|g\|_{H^\infty}^q \|f\|_{\mathcal{D}_{p-1}^p \cap \text{BMOA}}^q + \int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} \mathrm{d}A(z).$$

Now, Proposition 5.1 implies that $g \in \mathcal{B}_{\log}$. Also, since $\text{BMOA} \subset H^q$, we have that $f \in H^q$. Then we see that the last integral in (7.1) is finite as in the proof of (3.3). Thus, we have proved that $g \in M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA})$ finishing the proof. ∎

Proof of Theorem 1.11. Suppose that $0 < q < p < \infty$, $q < 2$ and $g \in M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA})$ with $g \not\equiv 0$. Take $a_n = 1/n^\lambda$ ($n = 1, 2, \dots$) with

$$\max(\tfrac{1}{2}, \tfrac{1}{p}) < \lambda \leqslant \frac{1}{q}$$

and set $f(z) = \sum_{n=1}^{\infty} a_n z^{2^n}$ ($z \in \mathbb{D}$). We have that $f \in \mathcal{D}_{p-1}^p \cap \text{BMOA} \setminus \mathcal{D}_{q-1}^q$. Then we use the Rademacher functions as in the proof of Case 3 of Theorem 1.2(ii) to get a contradiction. Hence, $g \equiv 0$. ∎

Now we turn to prove Theorem 1.12. Let us notice that (i) follows from Theorem 6.2 and (ii) from Theorem 1.9(i). To prove (iii) we shall use the following lemma.

LEMMA 7.2. *Suppose that $0 < q < 2$ and $\alpha > 0$. Let f be an analytic function in $\mathbb{D}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$), with $\sum_{n=0}^{\infty} |a_n|^2 [\log n]^\alpha < \infty$. If $f \in \text{BMOA}_{\log} \cap H^\infty$ then*

$$f \in M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA}), \quad \text{whenever } 0 < p \leqslant q \text{ and } \frac{q\alpha}{2-q} > 1.$$

For $0 < q \leqslant 1$, (iii) of Theorem 1.12 follows using (ii) and the lemma with $\alpha = 3$, while, for $1 < q < \infty$, it follows from Theorem 1.10.

Proof of Lemma 7.2. Suppose that f is in the conditions of the lemma and that $0 < p \leqslant q$ and $q\alpha/(2-q) > 1$.

Take $h \in \mathcal{D}_{p-1}^p \cap \text{BMOA}$. Since $\text{BMOA}_{\log} \cap H^\infty = M(\text{BMOA})$, it follows that $fh \in \text{BMOA}$.

We have also

$$\int_{\mathbb{D}} (1 - |z|)^{q-1} |(fh)'(z)|^q \mathrm{d}A(z)$$

$$\lesssim \int_{\mathbb{D}} (1-|z|)^{q-1} |f(z)|^q |h'(z)|^q \mathrm{d}A(z) + \int_{\mathbb{D}} (1-|z|)^{q-1} |f'(z)|^q |h(z)|^q \mathrm{d}A(z) = I_1 + I_2.$$

The first summand I_1 is finite because $f \in H^\infty$ and $h \in \mathcal{D}_{p-1}^p \cap \mathrm{BMOA} \subset \mathcal{D}_{q-1}^q \cap$ BMOA. Let us estimate the second one I_2. Using Hölder's inequality with the exponents $2/q$ and $2/(2-q)$, we obtain

$$I_2 = \int_{\mathbb{D}} (1 - |z|)^{q-1} |f'(z)|^q |h(z)|^q \mathrm{d}A(z)$$

$$= \int_{\mathbb{D}} |f'(z)|^q (1 - |z|)^{q/2} (\log(e/(1-|z|)))^{q\alpha/2}$$

$$(\log(e/(1-|z|)))^{-q\alpha/2} |h(z)|^q (1-|z|)^{q/2-1} \mathrm{d}A(z)$$

$$\leqslant \left[\int_{\mathbb{D}} |f'(z)|^2 (1-|z|)(\log(e/(1-|z|)))^\alpha \mathrm{d}A(z) \right]^{q/2}$$

$$\left[\int_{\mathbb{D}} \frac{|h(z)|^{2q/(2-q)}}{(\log(e/(1-|z|)))^{q\alpha/(2-q)}(1-|z|)} \mathrm{d}A(z) \right]^{(2-q)/q}.$$

Using Lemma 5.4, it follows that the first integral in the last product is finite. Now, notice that $f \in H^\lambda$ for all $\lambda < \infty$ to deduce

$$\int_{\mathbb{D}} \frac{|h(z)|^{2q/(2-q)}}{(\log(e/(1-|z|)))^{q\alpha/(2-q)}(1-|z|)} \mathrm{d}A(z)$$

$$\leqslant \|f\|_{H^{2q/(2-q)}}^{2q/(2-q)} \int_0^1 \frac{1}{(\log(e/(1-r)))^{q\alpha/(2-q)}(1-r)} \mathrm{d}r$$

and this integral is finite because $q\alpha/(2-q) > 1$. Thus $I_2 < \infty$. Then we have that $fh \in \mathcal{D}_{q-1}^q$ and, hence, $fh \in \mathcal{D}_{q-1}^q \cap \mathrm{BMOA}$. Consequently, we have proved that $f \in M(\mathcal{D}_{p-1}^p \cap \mathrm{BMOA}, \mathcal{D}_{q-1}^q \cap \mathrm{BMOA})$. ∎

To finish the proof of Theorem 1.12 take $q \in (0, \frac{1}{2})$ and let $\{a_n\}$ be defined as follows:

$$a_{2^k} = (k+1)^{-1/q}, \quad k = 0, 1, \ldots$$

and $a_n = 0$, if n is not a power of 2. Set

$$f(z) = \sum_{n=0}^\infty a_n z^n = \sum_{k=0}^\infty (k+1)^{-1/q} z^{2^k}, \quad z \in \mathbb{D}.$$

It is clear that $\{a_n\}$ satisfies (1.11). Furthermore, for almost every t, f_t is given by a lacunary power series, $f_t(z) = \sum\limits_{k=0}^{\infty} r_{2^k}(t) a_{2^k} z^{2^k}$, which does not belong to $\mathcal{D}_{q-1}^q$ because $\sum\limits_{k=0}^{\infty} |a_{2^k}|^q = \infty$.

Now we turn to consider multipliers in $M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA})$ given by power series with Hadamard gaps. First we show that whenever $0 < p \leqslant q \leqslant 1$ the power series with Hadamard gaps in $M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA})$ coincide with those in $\mathcal{D}_{q-1}^q \cap \text{BMOA}_{\log}$.

THEOREM 7.3. *Suppose that $0 < p \leqslant q \leqslant 1$ and let g be an analytic function in $\mathbb{D}$ given by a power series with Hadamard gaps. Then the following conditions are equivalent:*

(i) $g \in M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA})$.

(ii) $g \in \mathcal{D}_{q-1}^q \cap \text{BMOA}_{\log}$.

Proof. Since $\mathcal{D}_{p-1}^p \cap \text{BMOA}$ contains the constants functions, it is clear that the space $M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA}) \subset \mathcal{D}_{q-1}^q$ and the inclusion $M(\mathcal{D}_{p-1}^p \cap \text{BMOA}, \mathcal{D}_{q-1}^q \cap \text{BMOA}) \subset \text{BMOA}_{\log}$ follows from Theorem 7.1. Thus, the implication (i) $\Rightarrow$ (ii) holds.

Let us prove next the other implication. So take $g \in \mathcal{D}_{q-1}^q \cap \text{BMOA}_{\log} \cap \mathcal{L}$,

$$g(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \ (z \in \mathbb{D}) \text{ with } n_{k+1} \geqslant \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1.$$

We have $\sum\limits_{k=0}^{\infty} |a_k|^q < \infty$ which, since $q \leqslant 1$, implies that $\sum\limits_{k=0}^{\infty} |a_k| < \infty$. Thus $g \in H^\infty$. Then $g \in \text{BMOA}_{\log} \cap H^\infty = M(\text{BMOA})$.

Take $f \in \mathcal{D}_{p-1}^p \cap \text{BMOA}$. Since $g \in M(\text{BMOA})$, we have that $gf \in \text{BMOA}$. Now,

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$\lesssim \int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z) + \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-1} \mathrm{d}A(z)$$

$$= I_1 + I_2.$$

The first summand I_1 is finite because $g \in H^\infty$ and $f \in \mathcal{D}_{q-1}^q$.

Let us estimate the second one. Using Theorem 3.2 of [19] we see that the measure $\mu_{g,q}$ in $\mathbb{D}$ defined by $\mathrm{d}\mu_{g,q}(z) = (1 - |z|^2)^{q-1} |g'(z)|^q \mathrm{d}A(z)$ is a Carleson measure and (see, e.g., Theorem 1 of [37] or Theorem 2.1 of [36]) this implies that $\mu_{g,q}$ is a Carleson measure for $\mathcal{D}_{q-1}^q$, that is, $\mathcal{D}_{q-1}^q \subset L^q(\mathrm{d}\mu_{g,q})$. Hence $f \in L^q(\mathrm{d}\mu_{g,q})$ which is equivalent to saying that $I_2 < \infty$. Hence, $gf \in \mathcal{D}_{q-1}^q$.

So, we have proved that $gf \in \mathcal{D}^q_{q-1} \cap \text{BMOA}$ for any $f \in \mathcal{D}^p_{p-1} \cap \text{BMOA}$, that is, $g \in M(\mathcal{D}^p_{p-1} \cap \text{BMOA}, \mathcal{D}^q_{q-1} \cap \text{BMOA})$.   ∎

Finally, we obtain also the analogue of Theorem 1.12 for lacunary power series.

THEOREM 7.4. *Let $f \in \mathcal{H}ol(\mathbb{D})$ be given by a lacunary power series, of the form*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \ (z \in \mathbb{D}) \quad \text{with } n_{k+1} \geqslant \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1,$$

and suppose that the sequence of coefficients $\{a_k\}_{k=0}^{\infty}$ satisfies

(7.2)
$$\sum_{k=1}^{\infty} |a_k|^2 (\log n_k)^3 < \infty.$$

Then the function f satisfies the following conditions:
  (i) $f \in M(\text{BMOA})$.
  (ii) $f \in M(\mathcal{D}^p_{p-1} \cap \text{BMOA}, \mathcal{D}^q_{q-1} \cap \text{BMOA})$ *whenever $0 < p \leqslant q$ and $q > 1/2$.*
    Furthermore, if $0 < q < 1/2$ then there exists a sequence $\{a_k\}$ which satisfies (7.2) and such that $f \notin \mathcal{D}^q_{q-1}$. Thus for this sequence $\{a_k\}$ the function f satisfies:
  (a) $f \in M(\text{BMOA})$.
  (b) *If $0 < p \leqslant \lambda$ and $\lambda > 1/2$ then $f \in M(\mathcal{D}^p_{p-1} \cap \text{BMOA}, \mathcal{D}^\lambda_{\lambda-1})$ whenever $0 < p \leqslant \lambda$.*
  (c) $f \notin M(\mathcal{D}^p_{p-1} \cap \text{BMOA}, \mathcal{D}^q_{q-1} \cap \text{BMOA})$ *whenever $0 < p \leqslant q$.*

Proof. Part (i) follows from Proposition 1.8. Part (ii) follows from Theorem 1.10 for $q > 1$ and from Lemma 7.2 (with $\alpha = 3$) whenever $0 < q \leqslant 1$.

Now, if $0 < q < 1/2$ take

$$a_k = k^{-1/q}, \quad k = 1, 2, \dots$$

and

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \mathbb{D}.$$

Clearly,

$$\sum_{k=1}^{\infty} |a_k|^2 k^3 < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} |a_k|^q = \infty.$$

Then f satisfies conditions (a), (b) and (c) of Theorem 7.4.   ∎

As we mentioned in Section 1, Theorem 7.4 also shows that Theorem 1.10 does not remain true for $q < 1/2$.

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CHRISTOS CHATZIFOUNTAS, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS, 29071 MÁLAGA, SPAIN
E-mail address: christos.ch@uma.es

DANIEL GIRELA, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS, 29071 MÁLAGA, SPAIN
E-mail address: girela@uma.es

JOSÉ ÁNGEL PELÁEZ, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS, 29071 MÁLAGA, SPAIN
E-mail address: japelaez@uma.es