J. OPERATOR THEORY
75:1(2016), 3–19
doi: 10.7900/jot.2014sep05.2051

GENERALIZED BUNDLE SHIFT WITH APPLICATION TO MULTIPLICATION OPERATOR ON THE BERGMAN SPACE

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Communicated by Kenneth R. Davidson

ABSTRACT. Following upon results of Putinar, Sun, Wang, Zheng and the first author, we provide models for the restrictions of the multiplication by a finite Blaschke product on the Bergman space in the unit disc to its reducing subspaces. The models involve a generalization of the notion of bundle shift on the Hardy space introduced by Abrahamse and the first author to the Bergman space. We develop generalized bundle shifts on more general domains. While the characterization of the bundle shift is rather explicit, we have not been able to obtain all the earlier results appeared; in particular, the facts that the number of the minimal reducing subspaces equals the number of connected components of the Riemann surface $B(z) = B(w)$ and the algebra of commutant of T_B is commutative, are not proved. Moreover, the role of the Riemann surface is also not made clear.

KEYWORDS: *Bergman bundle shift, Bergman space, finite Blaschke product, reducing subspace.*

MSC (2010): 47B32, 47B35.

INTRODUCTION

In the study of bounded linear operators, we can ask about the reducing subspaces of a general bounded operator defined on a separable Hilbert space. But, in general, we can not say much about reducing subspaces of an arbitrary bounded linear operator. If we restrict attention to some special class of operators, then we can get more information about their reducing subspaces. One set of examples consists of multiplication operators by finite Blachke products on the Bergman space on the unit disc $\mathbb{D}$.

Let $\mathcal{O}(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$, $L^2_a(\mathbb{D})$ be the Bergman space of functions in $\mathcal{O}(\mathbb{D})$ satisfying

$$\|f\|^2 = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{D}} |f(z)|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z} < \infty.$$

Let T be a bounded linear operator on $L^2_a(\mathbb{D})$, a subspace $\mathfrak{M}$ of $L^2_a(\mathbb{D})$ is called a *reducing subspace* of T if $T(\mathfrak{M}) \subseteq \mathfrak{M}$ and $T^*(\mathfrak{M}) \subseteq \mathfrak{M}$. A reducing subspace $\mathfrak{M}$ of T is called *minimal* if for every reducing subspace $\widetilde{\mathfrak{M}}$ of T such that $\widetilde{\mathfrak{M}} \subseteq \mathfrak{M}$ then either $\widetilde{\mathfrak{M}} = \mathfrak{M}$ or $\widetilde{\mathfrak{M}} = 0$.

An n-th order Blaschke product B is the analytic function on $\mathbb{D}$ given by

$$B(z) = \mathrm{e}^{\mathrm{i}\theta} \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a}_i z},$$

where θ is a real number and $a_i \in \mathbb{D}$ for $1 \leqslant i \leqslant n$.

Let T_B be the multiplication operator on $L^2_a(\mathbb{D})$ by B. Zhu first studied the reducing subspaces of T_B, and showed that T_B has exactly two distinct minimal reducing subspaces when the Blachke product is of order 2 (cf. [17]). Motivated by this fact, Zhu conjectured that the number of minimal reducing subspaces of T_B equals to the order of B (cf. [17]). Guo, Sun, Zheng and Zhong showed that in general this is not true (cf. [15]), and they found that the number of minimal reducing subspaces of T_B equals to the number of connected components of the Riemann surface of $B(z) = B(w)$ when the order of B is 3, 4, 6. Then they conjectured that the number of minimal reducing subspaces of T_B equals the number of connected components of the Riemann surface of $B(z) = B(w)$ for any finite Blaschke product (the refined Zhu's conjecture, cf. [15]).

Set

$$B(z) := \mathrm{e}^{\mathrm{i}\theta} \prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a}_i z} = \frac{P(z)}{Q(z)} \quad \text{where } P(z) = \mathrm{e}^{\mathrm{i}\theta} \prod_{i=1}^{n}(z-a_i) \text{ and } Q(z) = \prod_{i=1}^{n}(1-\overline{a}_i z),$$

and let $(\mathcal{W}^*(T_B))'$ denote the *double commutant* of T_B, i.e., all T in $\mathcal{L}(L^2_a(\mathbb{D}))$ such that $TT_B = T_B T$ and $TT_B^* = T_B^* T$. It is easy to see $(\mathcal{W}^*(T_B))'$ is a von Neumann algebra. The first author, Sun and Zheng proved the following theorem in [9].

THEOREM 0.1. *If B is a finite Blaschke product, then $(\mathcal{W}^*(T_B))'$ has dimension q, where q is the number of connected components of the Riemann surface S_f, where $f(z,w) = P(w)Q(z) - P(z)Q(w)$.*

Zhu's refined conjecture is then proved once one can prove $(\mathcal{W}^*(T_B))'$ is commutative. In the same paper, the conjecture is proved for the cases $n \leqslant 8$. The general case was proved in [8] using the notion of local inverses of a finite Balschke product introduced by Thompson in [16].

THEOREM 0.2. *$(\mathcal{W}^*(T_B))'$ is commutative for any finite Blaschke product.*

One can show that the restrictions of T_B onto its minimal reducing subspaces are not unitary equivalent to each other. Then the following question arises naturally: what are these operators obtained by restricting T_B onto its minimal reducing subspaces?

Firstly let us consider the special case when $B(z) = z^n$. For $0 \leqslant i < n$, set

$$\mathcal{L}_{n,i} = \bigvee \{z^l : l = i \bmod n\}.$$

$\{\mathcal{L}_{n,i}\}_{i=0}^{n-1}$ are minimal reducing subspaces of T_{z^n} and they are pairwise orthogonal. Set $L_a^{2,i}(\mathbb{D}\setminus\{0\}) = L_a^2(\mathbb{D}\setminus\{0\}, |\frac{1}{z^{(n-i-1)/n}}|^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z})$, $0\leqslant i<n$. Define the map $U_i : L_a^{2,i}(\mathbb{D}\setminus\{0\})\to\mathcal{L}_{n,i}$ as follows;

$$U_i(f)(z) = \sqrt{n}z^i f(z^n), \quad \text{for } z\in\mathbb{D}.$$

Then U_i is a unitary map such that $U_i\circ T_z = T_{z^n}|_{\mathcal{L}_{n,i}}\circ U_i$. Thus restrictions of T_{z^n} to its minimal reducing subspaces are unitarily equivalent to the multiplication operators T_z on weighted Bergman spaces, and there exists a distinguished minimal reducing subspace such that the restriction operator is unitarily equivalent to the Bergman shift. Further, one can calculate the curvatures of these restrictions, and prove that they are not unitarily equivalent, (cf. [7]).

We are interested in describing model spaces for the minimal reducing subspaces of Blaschke products B of order n. We show they are weighted Bergman spaces obtained from the analogue of bundle shifts building by using the Bergman space, and such that the multiplication operators on them are unitarily equivalent to restrictions of T_B on its minimal reducing subspaces. The goal of this paper is trying to provide generalized Bergman bundle shifts as the model spaces viz, complex geometry and Riemann surface, etc.

To describe the progress in this direction, we need to introduce some notations. For the *flat unitary vector bundle E* (see Definition 1.3 below), we denote the multiplication operator on the space of holomorphic L^2 sections of E, $L_a^2(E)$, by T_E. Set $\mathcal{S} = B(\{\beta\in\mathbb{D} : B'(\beta)=0\})$. One of our main results is the following.

THEOREM 0.3. *Let B be a finite Blaschke product of order n. Let T_B be the multiplication operator on $L_a^2(\mathbb{D})$ by B. Let E_B be the flat unitary vector bundle over $\mathbb{D}\setminus\mathcal{S}$ determined by B. Let T_{E_B} be the multiplication operator on $L_a^2(E_B)$ defined by z. Then the operator T_B is unitarily equivalent to the operator T_{E_B}.*

The definition of E_B will be explained in the next section.

1. GENERALIZED BUNDLE SHIFTS

For supplying the model spaces of T_B on its reducing subspaces, in Subsection 1.1 a generalized bundle shift is defined in two ways over domains in $\mathbb{C}^n$. We prove the two definitions are equivalent and then classify generalized bundle shifts under unitary equivalence. In Subsection 1.2, Bergman bundle shifts are defined over domains in $\mathbb{C}$, and we characterize their commutant algebra. The investigation of this section extends the paper [1].

1.1. GENERALIZED BERGMAN BUNDLE SHIFTS.

Let Ω be an open subset of $\mathbb{C}^m$. Let $\mathfrak{K}$ be a Hilbert space.

DEFINITION 1.1. A *continuous vector bundle*, which is a family of Hilbert spaces over Ω, is a topological space E together with:

(i) A continuous map $q : E \to \Omega$.

(ii) A Hilbert space structure on each fibre $E_z = q^{-1}(z)$, $z \in \Omega$, such that the Hilbert space topology on E_z is the same as the topology inherited from E.

(iii) For each $z \in \Omega$, there exists a neighborhood U of z in Ω and a homeomorphism $\phi_U : q^{-1}(U) \to U \times \mathfrak{K}$ such that:

(a) for each $(w,k) \in U \times \mathfrak{K}$, the point $\phi_U^{-1}(w,k) \in E_w$;

(b) for each $w \in U$, the map $(\phi_U^w)^{-1} : \mathfrak{K} \to E_w$ defined by $(\phi_U^w)^{-1}(k) = \phi_U(w,k)$ is a continuous linear isomorphism.

DEFINITION 1.2. (i) A continuous vector bundle E is called a *holomorphic* vector bundle if for all open sets U and V such that $U \cap V \neq \varnothing$ the map

$$\phi_U \circ \phi_V^{-1}|_{(U\cap V)\times\mathfrak{K}} : (U \cap V) \times \mathfrak{K} \to (U \cap V) \times \mathfrak{K}$$

is given by the following expression

$$\phi_U \circ \phi_V^{-1}(w,k) = (w, \phi_{UV}(w)k),$$

where $\phi_{UV} : U \cap V \to \mathcal{GL}(\mathfrak{K})$ is holomorphic and $\mathcal{GL}(\mathfrak{K})$ is the set of invertible bounded linear operators on the Hilbert space $\mathfrak{K}$.

(ii) A holomorphic vector bundle E is called a *Hermitian* holomorphic vector bundle if an inner product is given on E_z which varies smoothly as a function of z, that is, for given any two smooth local sections s, t of E, the function $z \mapsto \langle s(z), t(z)\rangle_z$ is a smooth function.

For a given Hermitian holomorphic vector bundle E, let $\Gamma_a(\Omega)$ be the space of holomorphic sections of E over Ω. The Bergman space $L_a^2(E)$ is defined as

$$L_a^2(E) = \Big\{ f \in \Gamma_a(\Omega) : \|f\|^2_{L_a^2(E)} := \int_{\Omega} \|f(z)\|^2_{E_z}\, \mathrm{d}V(z) < \infty \Big\}.$$

Usually, a reproducing kernel Hilbert space consists of vector-valued functions taking value in $\mathbb{C}^k$, but a more general notion is possible. A point evaluation $ev_z : L_a^2(E) \to E_z$ is a continuous function for all $z \in \Omega$, so $L_a^2(E)$ is a reproducing kernel Hilbert space. In fact, the function $K_E : \Omega \times \Omega \to \mathcal{L}(E^*, E)$ defined by $K_E(z,w) = ev_z \circ ev_w^*$, for all $z, w \in \Omega$, takes values in $\mathcal{L}(E_w^*, E_z)$, and for $\eta \in E_z^*$ and $\xi \in E_w^*$ we have

$$\begin{aligned}\langle \eta, K_E(z,w)\xi\rangle_{E_z} &= \langle \eta, ev_z \circ ev_w^* \xi\rangle_{E_z} = \langle ev_z^*\eta, ev_w^*\xi\rangle_{L_a^2(E)} \\ &= \langle ev_w \circ ev_z^*\eta, \xi\rangle_{E_w} = \langle K_E(w,z)\eta, \xi\rangle_{E_w} = \langle \eta, K_E(w,z)^*\xi\rangle_{E_z},\end{aligned}$$

which implies

$$K_E(z,w) = K_E(w,z)^*.$$

Moreover,

$$\langle f, K_E(\cdot,w)\xi\rangle_{L_a^2(E)} = \langle f, ev_w^*\xi\rangle_{L_a^2(E)} = \langle ev_w(f), \xi\rangle_{E_w} = \langle f(w), \xi\rangle_{E_w}.$$

Hence, K_E is the reproducing kernel of $L_a^2(E)$.

There is another way to view the Bergman space over the flat vector bundle. In particular, one can view it as a space of functions on the universal covering space. Sometime this representation is quite useful.

DEFINITION 1.3. Let E be a vector bundle. A *unitary coordinate covering* for E is a coordinate covering $\{U, \phi_U\}$ such that for each open set U and $z \in U$, the fiber map $\phi_U|_{E_z} : E_z \to \{z\} \times \mathfrak{K}$ is unitary. The unitary coordinate covering $\{U, \phi_U\}$ is said to be *flat* if the matrix valued functions $\phi_{UV} : U \cap V \to \mathcal{U}(\mathfrak{K})$ are constants, where $\phi_U \circ \phi_V^{-1}(z, v) = (z, \phi_{UV}(z)v)$ for $z \in U \cap V$ and $v \in \mathfrak{K}$. A *flat* unitary vector bundle is a vector bundle equipped with a flat unitary coordinate covering.

A flat structure on a vector bundle over Ω gives rise to a representation

$$\alpha : \pi_1(\Omega) \to \mathcal{U}(\mathfrak{K})$$

of the fundamental group of Ω via parallel displacement. Conversely, suppose we have a representation α and let $\widetilde{\Omega}$ be the universal covering space of Ω, we can construct a flat bundle as follows: the equivalence relation $(z_1, v_1) \sim (z_2, v_2)$ on $\widetilde{\Omega} \times \mathfrak{K}$ is defined as if $z_2 = A(z_1)$ and $v_2 = \alpha(A)v_1$ for some $A \in \pi_1(\Omega)$. This equivalence relation gives rise to a flat vector bundle $E_\alpha = \widetilde{\Omega} \times \mathfrak{K}/\sim$ with natural projection π. The vector bundle E_α constructed above is a flat unitary vector bundle. Discussions about flat vector bundles can be found in [1], [2], [14]. The group $\mathcal{U}(\mathfrak{K})$ acts on $\mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$ via conjugation $(V, \alpha) \to V\alpha V^*$. The set of equivalence classes is denoted by $\mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))/\mathcal{U}(\mathfrak{K})$.

PROPOSITION 1.4 ([1]). *There is a one to one correspondence between*

$$\mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))/\mathcal{U}(\mathfrak{K})$$

and the set of equivalence classes of flat unitary vector bundles over Ω.

For a flat unitary vector bundle E, the m-tuple multiplication operator $(T_{z_1}, \ldots, T_{z_m})$ on $L_a^2(E)$ is denoted by T_E.

DEFINITION 1.5. Let X and Y be two topological spaces.

(i) A continuous map $q : Y \to X$ is a *covering map* if there is a neighborhood U for each point $x \in X$ such that:

(a) the set $q^{-1}(U)$ is a disjoint union of open sets V_i,

(b) the restriction map of q to V_i is a homeomorphism of V_i onto U.

(ii) We say that a covering map $q : Y \to X$ is the *universal* covering map if Y is a simply connected topological space.

The group of covering transformations for q is the group of homeomorphisms A of Y such that $q \circ A = q$. Two covering spaces (Y_1, q_1) and (Y_2, q_2) of X are said to be *equivalent* if there is a homeomorphism ϕ from Y_1 onto Y_2 such that $q_2 \circ \phi = q_1$.

Let $q : \widetilde{\Omega} \to \Omega$ be a universal covering map. Let $J_q(z) = ((\frac{\partial q_i}{\partial z_j}(z)))_{i,j=1}^m$ be the Jacobian of $q = (q_1, \ldots, q_m)$. For $\alpha \in \mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$, $L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \mathrm{d}V(z))$ is defined to be the space of functions f in $L_a^2(\widetilde{\Omega}, \mathfrak{K}, |\det J_q(z)|^2 \,\mathrm{d}V(z))$ such that

$$\int_{\widetilde{\Omega}} \|f(z)\|_{\mathfrak{K}}^2 |\det J_q|^2 \,\mathrm{d}V(z) < \infty$$

and $f(A(z)) = \alpha(A) f(z)$ for all $A \in \pi_1(\Omega)$ and $z \in \widetilde{\Omega}$. A bounded holomorphic function ϕ defined on $\widetilde{\Omega}$ is said to be $\pi_1(\Omega)$*-automorphic* if $\phi \circ A = \phi$ for each A in $\pi_1(\Omega)$. Let $H^\infty(\widetilde{\Omega}, \pi_1(\Omega))$ denote the set of all $\pi_1(\Omega)$-automorphic bounded holomorphic functions. The space $L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))$ is invariant for $H^\infty(\widetilde{\Omega}, \pi_1(\Omega))$, that is, if $\phi \in H^\infty(\widetilde{\Omega}, \pi_1(\Omega))$ and $f \in L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))$ then $\phi f \in L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))$. In particular, it is invariant for multiplication by the components $q_i, 1 \leqslant i \leqslant m$, of the covering map q and we define the operator $T_\alpha = (T_{\alpha_1}, \ldots, T_{\alpha_m})$ on $L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))$ by $T_{\alpha_i}(f) = q_i f$, for $1 \leqslant i \leqslant m$.

PROPOSITION 1.6. *If α is in $\mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$ and E_α is the flat unitary bundle over Ω determined by α, then T_α is unitarily equivalent to the bundle shift T_{E_α}.*

Proof. Define a map $F : L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z)) \to L_a^2(E_\alpha)$ as follows

$$F(f)(w) = [(z, f(z))] \quad \text{for } z \in q^{-1}(w),\ w \in \Omega,$$

where $[(z, f(z))]$ denotes the equivalence class of $(z, f(z))$. It is easy to see that

$$\|F(f)(w)\|_{(E_\alpha)_w}^2 = \|f(z)\|_{\mathfrak{K}}^2.$$

Step 1. F is an isometry. We have

$$\begin{aligned} \|F(f)\|_{L_a^2(E)}^2 &= \int_{\Omega} \|F(f)(w)\|_{(E_\alpha)_w}^2 \,\mathrm{d}V(w) \\ &= \int_{\widetilde{\Omega}} \|F(f)(q(z))\|_{(E_\alpha)_w}^2 |\det J_q(z)|^2 \,\mathrm{d}V(z) \\ &= \int_{\widetilde{\Omega}} \|f(z)\|_{\mathfrak{K}}^2 |\det J_q(z)|^2 \,\mathrm{d}V(z) = \|f\|_{L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))}^2. \end{aligned}$$

Step 2. F is onto. Let s be a section of E_α such that $s \in L_a^2(E_\alpha)$. Define the function $f_s : \mathbb{D} \to \mathfrak{K}$ such that $s(q(z)) = [(z, f_s(z))]$ for $z \in \widetilde{\Omega}$. One can show that f_s is holomorphic and $q(z) = q(A(z))$ implies that $f_s(A(z)) = \alpha(A) f_s(z)$ for all $A \in \pi_1(\Omega)$ and $z \in \widetilde{\Omega}$. One can easily see that

$$\|s\|_{L_a^2(E_\alpha)}^2 = \int_{\widetilde{\Omega}} \|f_s(z)\|_{\mathfrak{K}}^2 |\det J_q(z)|^2 \,\mathrm{d}V(z).$$

Hence $f_s \in L_a^2(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 \,\mathrm{d}V(z))$ and $F(f_s) = s$.

Step 3. F intertwines T_α and T_{E_α}. For $f \in L^2_a(\widetilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2\, \mathrm{d}V(z))$, $w \in \Omega$, $z \in q^{-1}(w)$ and $1 \leqslant i \leqslant m$

$$\begin{aligned}(F \circ T_{\alpha_i})(f)(w) &= F(q_i f)(w) = [(z, (q_i f)z)] = [(z, q_i(z) f(z))] \\ &= [(z, w_i f(z))] = w_i[z, f(z)] = w_i F(f)(w) = (T_{z_i} \circ F)(f)(w).\end{aligned}$$

Thus $F \circ T_{\alpha_i} = T_{z_i} \circ F$. ∎

1.2. BERGMAN BUNDLE SHIFTS. Although most of the results in this part can be extended to multivariable cases or weighted Bergman spaces, we restrict ourselves to the one variable case for further use. In this section we consider the case where the domain has a nice boundary and follow the approach from [1]. In Section 1.4, we show how to provide an alternative approach covering the general case.

Let R be a multiply connected domain in $\mathbb{C}$ whose boundary consists of $n+1$ analytic Jordan curves and $\widetilde{q} : \mathbb{D} \to R$ be the covering map. Let G be the group of covering transformations of $\widetilde{q}$. The group G is isomorphic to the fundamental group of R.

A function f on $\mathbb{D}$ is said to be *G-automorphic* if $f \circ A = f$ for each A in G. Let $L^2(\mathbb{D}, \mathbb{C}^n, G)$ be the subspace of G-automorphic functions in $L^2(\mathbb{D}, \mathbb{C}^n)$ and let $L^2_a(\mathbb{D}, \mathbb{C}^n, G)$ be the subspace of G-automorphic functions in $L^2_a(\mathbb{D}, \mathbb{C}^n)$. Let $L^\infty(\mathbb{D}, G)$ be the algebra of G-automorphic functions in $L^\infty(\mathbb{D})$ and let $H^\infty(\mathbb{D}, G)$ be the algebra of G-automorphic functions in $H^\infty(\mathbb{D})$.

PROPOSITION 1.7. (i) *The smallest invariant subspace for $L^\infty(\mathbb{D}, G)$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, G)$ is $L^2(\mathbb{D}, \mathbb{C}^n, G)$, where $L^\infty(\mathbb{D}, G)$ acts on $L^2(\mathbb{D}, \mathbb{C}^n, G)$ pointwisely.*

(ii) *The smallest invariant subspace for $H^\infty(\mathbb{D})$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, G)$ is $L^2_a(\mathbb{D}, \mathbb{C}^n)$, where $H^\infty(\mathbb{D})$ acts on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ pointwisely.*

(iii) *The smallest invariant subspace for $L^\infty(\mathbb{D})$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, G)$ is $L^2(\mathbb{D}, \mathbb{C}^n)$, where $L^\infty(\mathbb{D})$ acts on $L^2(\mathbb{D}, \mathbb{C}^n)$ pointwisely.*

Proof. Since $L^2_a(\mathbb{D}, \mathbb{C}^n, G)$ contains the constant functions, the proof is immediate. ∎

An operator T on $L^2_{\mathbb{C}^n}(\mathbb{D})$ will be called *decomposable* if there is a weakly measurable function $w \mapsto T_w$ from $\mathbb{D}$ to $B(\mathbb{C}^n)$ such that for $f \in L^2_{\mathbb{C}^n}(\mathbb{D})$, $(Tf)(w) = T_w f(w)$, a.e. $\mathrm{d}v$. It is well known that an operator on $L^2_{\mathbb{C}^n}(\mathbb{D})$ is decomposable if and only if it commutes with $L^\infty(\mathbb{D})$. Moreover, the algebraic operations can be performed pointwisely, that is $[(T+S)f](w) = T_w f(w) + S_w f(w)$, $(T^* f)(w) = T^*_w f(w)$, etc., and the norm is $\|T\| = \operatorname{ess\,sup} \|T_w\|$. One can see [5], [6] for more details.

PROPOSITION 1.8. *For every unitary representation α of G on $\mathbb{C}^n$, there is a decomposable operator Φ_α on $L^2_{\mathbb{C}^n}(\mathbb{D})$ such that $\Phi_\alpha(L^2_a(\mathbb{D}, \mathbb{C}^n, G)) = L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. Moreover, Φ_α satisfies $\Phi_{\alpha, g(\lambda)} = \alpha(g)\Phi_{\alpha,\lambda}$ a.e. $\mathrm{d}m$ for every $g \in G$.*

Proof. Note that the covering map $q : \mathbb{D} \to R$ can be extended to a covering map from an open set $\widetilde{\mathbb{D}}$ containing $\mathbb{D}$ to an open set $\widetilde{R}$ containing the closure of R. Given α, let E_α be the flat unitary bundle over $\widetilde{\mathbb{D}}$, and let π denote the projection from $\widetilde{\mathbb{D}} \times \mathbb{C}^n$ onto E_α. E_α is analytically equivalent to the trivial bundle $\widetilde{R} \times \mathbb{C}^n$, so there exists an analytic isomorphism ϕ from E_α to $\widetilde{R} \times \mathbb{C}^n$. It is clear that the composition $\phi \circ \pi$ is a bundle map from $\widetilde{\mathbb{D}} \times \mathbb{C}^n$ onto $\widetilde{R} \times \mathbb{C}^n$ which yields an analytic function Φ from $\widetilde{\mathbb{D}}$ to $\mathcal{GL}(\mathbb{C}^n)$, the set of all invertible operators on $\mathbb{C}^n$, such that $\Phi(g(z)) = \alpha(g)\Phi(z)$ for all $z \in \widetilde{\mathbb{D}}$ and $g \in G$. Let Φ_α be the restriction of Φ to the closed disk, it follows that there is a $K > 0$ such that $|\Phi_\alpha(z)| \leqslant K$ and $|\Phi_\alpha(z)^{-1}| \leqslant K$ for all z in the closed disk. Let $f \in L^2_{\mathbb{C}^n}(\mathbb{D})$, consider

$$\begin{aligned}\int\limits_{\mathbb{D}} |(\Phi_\alpha f)(z)|^2 \mathrm{d}m &= -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{D}} |(\Phi_\alpha f)(z)|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{D}} |\Phi_\alpha(z) f(z)|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &\leqslant -\frac{K}{2\pi \mathrm{i}} \int\limits_{\mathbb{D}} |f(z)|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z} = K\|f\|^2_{L^2_{\mathbb{C}^n}(\mathbb{D})}.\end{aligned}$$

Hence, Φ_α is an invertible decomposable operator on $L^2_{\mathbb{C}^n}(\mathbb{D})$. Since Φ_α is analytic and $\Phi_\alpha(z)$ and $\Phi_\alpha(z)^{-1}$ are uniformly bounded for $z \in \overline{\mathbb{D}}$, we have $\Phi_\alpha(L^2_a(\mathbb{D}, \mathbb{C}^n, G)) = L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. ∎

PROPOSITION 1.9. (i) *The smallest invariant subspace for $L^\infty(\mathbb{D}, G)$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ is $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$.*

(ii) *The smallest invariant subspace for $H^\infty(\mathbb{D})$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ is $L^2_a(\mathbb{D}, \mathbb{C}^n)$.*

(iii) *The smallest invariant subspace for $L^\infty(\mathbb{D})$ containing $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ is $L^2(\mathbb{D}, \mathbb{C}^n)$.*

The proof follows from Proposition 1.7 and Proposition 1.8.

LEMMA 1.10. *An operator on $L^2(\mathbb{D}, \mathbb{C}^n, G)$ commutes with $L^\infty(\mathbb{D}, G)$ if and only if it is a G-automorphic decomposable operator.*

Proof. The comnutant of $L^\infty(\Omega)$ defined on $L^2(\Omega, \mathbb{C}^n)$ consists of decomposable operators on $L^2(\Omega, \mathbb{C}^n)$ ([5], Part II). Since $L^2(\mathbb{D}, \mathbb{C}^n, G)$ is isometrically isomorphic to $L^2(\Omega, \mathbb{C}^n)$, the lemma follows. ∎

PROPOSITION 1.11. *An operator $W : L^2(\mathbb{D}, \mathbb{C}^n, \alpha) \to L^2(\mathbb{D}, \mathbb{C}^n, \beta)$ intertwines $L^\infty(\mathbb{D}, G)$ if and only if W is decomposable and $W_{g(\lambda)} = \beta(g) W_\lambda \alpha(g)^*$ a.e. $\mathrm{d}v$ for all $g \in G$.*

Proof. Given W, the operator $(\Phi_\beta)^{-1} W \Phi_\alpha$ on $L^2(\mathbb{D}, \mathbb{C}^n, G)$ commutes with $L^\infty(\mathbb{D}, G)$. Thus, by Lemma 1.10, there is a G-automorphic decomposable operator V such that $V = (\Phi_\beta)^{-1} W \Phi_\alpha$. Hence $W = \Phi_\beta V (\Phi_\alpha)^{-1}$ and thus W is decomposable. Moreover,

$$\begin{aligned}W_{g(\lambda)} &= (\Phi_{\beta, g(\lambda)}) V_{g(\lambda)} (\Phi_{\alpha, g(\lambda)})^{-1} = \beta(g) \Phi_{\beta,\lambda} V_\lambda (\alpha(g) \Phi_{\alpha,\lambda})^{-1} \\ &= \beta(g) (\Phi_{\beta,\lambda} V_\lambda (\Phi_{\alpha,\lambda})^{-1}) (\alpha(g))^{-1} = \beta(g) W_\lambda \alpha(g)^*.\end{aligned}$$

The sufficiency part is obivious. ∎

We state a series of lemmas, culminating in the characterization of reducing subspaces of Bergman bundle shifts (Theorem 1.19).

LEMMA 1.12. *T is a decomposable operator on $L^2(\mathbb{D}, \mathbb{C}^n)$.*

(i) *If $T(L^2_a(\mathbb{D}, \mathbb{C}^n)) \subseteq L^2_a(\mathbb{D}, \mathbb{C}^n)$ then $F : \mathbb{D} \to \mathbb{C}^n$, defined as $F(\lambda) = T_\lambda u$, is a holomorphic map, for each fixed $u \in \mathbb{C}^n$.*

(ii) *If $T^*(L^2_a(\mathbb{D}, \mathbb{C}^n)) \subseteq L^2_a(\mathbb{D}, \mathbb{C}^n)$ then $\widetilde{F} : \mathbb{D} \to \mathbb{C}^n$, defined as $\widetilde{F}(\lambda) = T^*_\lambda u$, is a holomorphic map, for each fixed $u \in \mathbb{C}^n$.*

Proof. (i) For $u \in \mathbb{C}^n$, consider the constant function $f(\lambda) = u$ for all $\lambda \in \mathbb{D}$. Clearly, $f \in L^2_a(\mathbb{D}, \mathbb{C}^n)$. By the given condition, $T(f) \in L^2_a(\mathbb{D}, \mathbb{C}^n)$. In particular, $T(f)$ is a holomorphic function on $\mathbb{D}$. For any $\lambda_0 \in \mathbb{D}$, $\lim\limits_{\lambda \to \lambda_0} \frac{T(f)(\lambda) - T(f)(\lambda_0)}{\lambda - \lambda_0}$ exists. Now,

$$\frac{T(f)(\lambda) - T(f)(\lambda_0)}{\lambda - \lambda_0} = \frac{T_\lambda(f(\lambda)) - T_{\lambda_0}(f(\lambda_0))}{\lambda - \lambda_0} = \frac{T_\lambda(u) - T_{\lambda_0}(u)}{\lambda - \lambda_0} = \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}.$$

Hence $\lim\limits_{\lambda \to \lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}$ exists. Thus F is a holomorphic function on $\mathbb{D}$.

(ii) The proof is the same as the proof of (i). ∎

LEMMA 1.13. *A decomposable operator T on $L^2(\mathbb{D}, \mathbb{C}^n)$ commutes with the projection onto $L^2_a(\mathbb{D}, \mathbb{C}^n)$ if and only if it is a constant.*

Proof. The "if" part is clear. On the other side, assume that $L^2_a(\mathbb{D}, \mathbb{C}^n)$ reduces T, that is, $T(L^2_a(\mathbb{D}, \mathbb{C}^n)) \subseteq L^2_a(\mathbb{D}, \mathbb{C}^n)$ and $T^*(L^2_a(\mathbb{D}, \mathbb{C}^n)) \subseteq (L^2_a(\mathbb{D}, \mathbb{C}^n))$.

Fix u and v in $\mathbb{C}^n$ and define $h(\lambda) = \langle T_\lambda u, v \rangle$. Since $L^2_a(\mathbb{D}, \mathbb{C}^n)$ is invariant for T, so by Lemma 1.12(i), the function h is holomorphic on $\mathbb{D}$. Similarly, since $L^2_a(\mathbb{D}, \mathbb{C}^n)$ is invariant for T^*, so by Lemma 1.12(ii), the function $\widetilde{h}(\lambda) = \overline{h(\lambda)}$ is holomorphic on $\mathbb{D}$. Hence, h is a constant function on $\mathbb{D}$. Since u and v are arbitrary, so $\lambda \mapsto T_\lambda$ is a constant function from $\mathbb{D}$ to $\mathcal{L}(\mathbb{C}^n)$ also. ∎

THEOREM 1.14. *The operators T_α and T_β are unitarily equivalent if and only if α and β are unitarily equivalent.*

Proof. First suppose that $\alpha, \beta \in \mathrm{Hom}(\pi_1(\Omega), \mathcal{U}(\mathbb{C}^n))$ are unitarily equivalent, i.e., there exists a unitary $U_0 \in \mathcal{U}(\mathbb{C}^n)$ such that $U_0 \alpha(A) = \beta(A) U_0$ for all $A \in \pi_1(\Omega)$. Define a map $\tau : L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha, |q'(z)|^2 \mathrm{d}z) \to L^2_a(\mathbb{D}, \mathbb{C}^n, \beta, |q'(z)|^2 \mathrm{d}z)$ as follows

$$\tau(f)(z) = U_0(f(z)) \quad \text{for } z \in \mathbb{D}.$$

We show that τ is well defined, unitary and intertwines T_α and T_β.

Step 1. τ is a well defined and an isometric map. Letting $A \in \pi_1(\Omega)$ and $f \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha, |q'(z)|^2 \mathrm{d}z)$, we have

$$\begin{aligned}\tau(f)(A(z)) &= U_0(f(A(z))) = U_0(\alpha(A)f(z)) = (U_0 \circ \alpha(A))f(z) \\ &= (\beta(A) \circ U_0)f(z) = \beta(A)U_0(f(z)) = \beta(A)(\tau(f)(z))\end{aligned}$$

and

$$\begin{aligned}\|\tau(f)\|^2 &= \int_{\mathbb{D}} \|\tau(f)(z)\|^2_{\mathbb{C}^n} |q'(z)|^2 \mathrm{d}z = \int_{\mathbb{D}} \|U_0(f(z))\|^2_{\mathbb{C}^n} |q'(z)|^2 \mathrm{d}z \\ &= \int_{\mathbb{D}} \|f(z)\|^2_{\mathbb{C}^n} |q'(z)|^2 \mathrm{d}z = \|f\|^2.\end{aligned}$$

Step 2. τ is an onto map. Let $g \in L^2_a(\mathbb{D}, \mathbb{C}^n, \beta, |q'(z)|\mathrm{d}z)$ and set $\widetilde{g}(z) = U_0^{-1}(g(z))$ for $z \in \mathbb{D}$. It is easy to see that $\widetilde{g} \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha, |q'(z)|\mathrm{d}z)$ and $\tau(\widetilde{g}) = g$.

Step 3. τ intertwines T_α and T_β. For $f \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha, |q'(z)|^2\mathrm{d}z)$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}((\tau \circ T_\alpha)f)(z) &= (\tau(T_\alpha f))(z) = \tau(qf)(z) = U_0(q(z)f(z)) = q(z)U_0(f(z)) \\ &= q(z)\tau(f)(z) = (q\tau(f))(z) = (T_\beta(\tau(f)))(z) = ((T_\beta \circ \tau)f)(z).\end{aligned}$$

Thus

$$\tau \circ T_\alpha = T_\beta \circ \tau.$$

Conversely, assume that U is an isometry from $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ onto $L^2_a(\mathbb{D}, \mathbb{C}^n, \beta)$ with $UT_\alpha = T_\beta U$. Then, by the fact that the minimal normal extension for a subnormal operator is unique, U can be extended to a unitary operator which commutes with their minimal normal extensions. The minimal extension of T_α is the multiplication by q on $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$, and the extension of U is a unitary operator $\widetilde{U}$ from $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ to $L^2(\mathbb{D}, \mathbb{C}^n, \beta)$, and it intertwines $L^\infty(\mathbb{D}, G)$. Thus $\widetilde{U}$ is decomposable and $\widetilde{U}(L^2_a(\mathbb{D}, \mathbb{C}^n)) = L^2_a(\mathbb{D}, \mathbb{C}^n)$. Hence $L^2_a(\mathbb{D}, \mathbb{C}^n)$ reduces $\widetilde{U}$, $\widetilde{U}$ has to be a constant operator with respect to λ. Thus there is a unitary operator V on $\mathbb{C}^n$ such that $\widetilde{U}_\lambda = V$ a.e. $\mathrm{d}v$. But $V = \widetilde{U}_{g(\lambda)} = \beta(g)\widetilde{U}_\lambda \alpha(g)^* = \beta(g)V\alpha(g)^*$ a.e. $\mathrm{d}v$, and thus $\alpha(g) = V^*\beta(g)V$, which shows that α and β are unitarily equivalent. ∎

COROLLARY 1.15. *If E and E' are flat unitary bundles over Ω, then the operators T_E and $T_{E'}$ are unitarily equivalent if and only if E and E' are equivalent as flat unitary bundles.*

Proof. First suppose that the flat unitary vector bundles E and E' are equivalent. Let $\Theta : E \to E'$ be a bundle isomorphism. Then, the map $f(z) \mapsto \Theta(z)f(z)$ for $z \in \Omega$ defines a module isomorphism between $A(\Omega)$ modules $\Gamma_a(E)$ and

$\Gamma_a(E')$. For $z \in \Omega$, $\|\Theta(z)f(z)\|^2_{E'_z} = \|f(z)\|^2_{E_z}$. If $f \in L^2_a(E)$, then

$$\|\Theta f\|^2_{L^2_a(E')} = -\frac{1}{2\pi \mathrm{i}} \int_\Omega \|\Theta(z)f(z)\|^2_{E'_z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$= -\frac{1}{2\pi \mathrm{i}} \int_\Omega \|f(z)\|^2_{E_z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \|f\|^2_{L^2_a(E)}.$$

Hence Θ defines an isometry from $L^2_a(E)$ onto $L^2_a(E')$. Since Θ is a module isomorphism, it intertwines T_E and $T_{E'}$, namely, for $f \in L^2_a(E)$ and $z \in \Omega$

$$((\Theta \circ T_E)f)(z) = \Theta(T_E(f))(z) = \Theta(z)T_E(f)(z) = \Theta(z)(zf(z)) = z(\Theta(z)f(z))$$
$$= z(\Theta f)(z) = (T_{E'}(\Theta f))(z) = ((T_{E'} \circ \Theta)f)(z),$$

thus

$$\Theta \circ T_E = T_{E'} \circ \Theta.$$

Conversely suppose that operators T_E and $T_{E'}$ are unitarily equivalent, so by Proposition 1.6 and Theorem 1.14, the bundles E and E' are unitarily equivalent as flat unitary vector bundles. ∎

If Ω_1 and Ω_2 are two multiply connected domains in the complex plane whose boundaries consist of Jordan curves, and biholomorphic to each other by the map φ, and E is a flat unitary bundle over Ω_2, then the pull back bundle φ^*E is a flat unitary bundle over Ω_1.

COROLLARY 1.16. *The bundle shifts T_E and T_{φ^*E} are unitarily equivalent.*

1.3. THE COMMUTANT OF $W^*(T_\alpha)$. We recall some results about the relation between a subnormal operator and its normal extension (cf. [3]). Let S be a subnormal operator on a Hilbert space $\mathcal{H}$ and let N be its minimal normal extension on the Hilbert space $\mathcal{K}$. If an operator commutes with the operators S and S^*, that is, it is in the commutant of the W^*-algebra $\mathcal{W}^*(S)$ generated by S, then it lifts to an operator that commute with N (cf. [13]).

Let $P : \mathcal{K} \to \mathcal{H}$ be the orthogonal projection. Let $\mathcal{A}$ denote the algebra of operators on $\mathcal{K}$ that commute with N and P, and let $(\mathcal{W}^*(S))'$ denote the commutants of $\mathcal{W}^*(S)$.

THEOREM 1.17. *If E is an n-dimensional flat unitary bundle over Ω, then the algebra of the commutants of T_E is $H^\infty_{L(E)}(\Omega)$.*

Proof. Every analytic vector bundle over Ω is analytically trivial by Grauert theorem [11], and we assume that $V : \Omega \times \mathbb{C}^n \to E$ is an analytic isomorphism. Then V defines the similarity $\widetilde{V} : L^2_a(E) \to L^2_a(\Omega, \mathbb{C}^n)$ between $T_{\mathbb{C}^n}$ and T_E, that is, $T_E = \widetilde{V}^{-1} T_{\mathbb{C}^n} \widetilde{V}$. If $L^2_a(\Omega, \mathbb{C}^n)$ is rewritten as $L^2_a(\Omega) \otimes \mathbb{C}^n$, then $T_{\mathbb{C}^n} = T_z \otimes I$, where I is the identity operator on $\mathbb{C}^n$. If A is an operator on $L^2_a(E)$ commuting with T_E, then

$$A\widetilde{V}^{-1}(T_z \otimes I)\widetilde{V} = AT_E = T_E A = \widetilde{V}^{-1}(T_z \otimes I)\widetilde{V}A.$$

This means $(T_z \otimes I)(\widetilde{V}^{-1}A\widetilde{V}) = (\widetilde{V}^{-1}A\widetilde{V})(T_z \otimes I)$, so $(\widetilde{V}^{-1}A\widetilde{V}) \in (T_z \otimes I)' = H^\infty(\Omega) \otimes M_n(\mathbb{C}^n)$. It follows that $A \in \widetilde{V}(H^\infty(\Omega) \otimes M_n(\mathbb{C}^n))\widetilde{V}^{-1} = H^\infty_{L(E)}(\Omega)$.

Conversely, it is clear that every $\Phi \in H^\infty_{L(E)}(\Omega)$ commutes with T_E, and the proof is completed. ∎

THEOREM 1.18 ([13]). *The map $A \mapsto A|_{\mathcal{H}}$ is a $*$-isometric isomorphism from $\mathcal{A}$ onto $(\mathcal{W}^*(S))'$.*

Let $\mathcal{W}^*(\alpha)$ be the W^*-subalgebra of $\mathcal{L}(\mathbb{C}^n)$ generated by $\alpha(G)$.

THEOREM 1.19. *There is a $*$-isometric isomorphism $(\mathcal{W}^*(\alpha))' \to (\mathcal{W}^*(T_\alpha))'$.*

Proof. Let $\mathcal{U}$ be the algebra of operators on $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ that commute with $L^\infty(\mathbb{D}, G)$ and with the projection onto $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. For B in $(\mathcal{W}^*(\alpha))'$ and f in $L^2(\mathbb{C}^n, \alpha)$, observe that $(Bf) \circ A = B(f \circ A) = B(\alpha(A)f) = \alpha(A)Bf$, for all A in G. Thus, there is a decomposable $\tau_\alpha(B)$ on $L^2(\mathbb{D}, \mathbb{C}^n)$ defined by $\tau_\alpha(B)(f) = Bf$.

Since a constant decomposable operator maps $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ and since $\tau_\alpha(B)$ maps $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ into $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$, it also maps $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ into $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. In other words, it takes $(\mathcal{W}^*(\alpha))'$ into $\mathcal{U}$. Clearly, $\tau_\alpha(B^*)f = B^*f = (\tau_\alpha B)^*f$ and $\|\tau_\alpha B\| = \operatorname*{ess\,sup}_\lambda \|(\tau_\alpha(B))_\lambda\| = \|B\|$. Thus τ_α is a $*$-isometric map. If X is in $\mathcal{U}$, then by Proposition 1.11, the operator X is decomposable $X_{g(\lambda)} = \alpha(g)X_\lambda\alpha(g)^*$ for all $g \in G$. Since X is reduced by $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$, so $X(L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)) \subseteq L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ and $X^*(L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)) \subseteq L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. Thus, both X and X^* map $L^2_a(\mathbb{D}, \mathbb{C}^n)$ to $L^2_a(\mathbb{D}, \mathbb{C}^n)$. Hence by Lemma 1.13, X is a decomposable operator, and so it is a constant, thus there is a B in $\mathcal{L}(\mathbb{C}^n)$ with $X_\lambda = B$ a.e. $\mathrm{d}m$. Now,

$$B = X_{g(\lambda)} = \alpha(g)X_\lambda\alpha(g)^* = \alpha(g)B\alpha(g)^*.$$

Thus the operator B is in $(\mathcal{W}^*(\alpha))'$ and so $X = \tau_\alpha(B)$. This proves that τ_α is onto. ∎

Note that the algebra $W^*(T_\alpha)'$ is finite dimensional for a Bergman bundle shift even though the Bergman space $L^2_a(E)$ is infinite dimensional.

1.4. THE CASE OF BAD BOUNDARY. In Section 3, we need similar results about bundle shifts over multiply connected domains to $D_0 = \mathbb{D} \setminus \mathcal{S}$ obtained by removing a finite subset from $\mathbb{D}$. Although the proceding results do not apply directly in this case, the Bergman bundle shift is a Cowen–Douglas operator on D_0, and we can prove similar results by applying ideas from [4]. In particular, D_0 is a multiply connected domain with fundamental group $\pi_1(D_0)$, and its universal covering is $\mathbb{D}$. $\mathbb{D} \times \mathbb{C}^n$ is the trivial bundle over $\mathbb{D}$, E_α is the corresponding flat unitary vector bundle over D_0 defined by a representation $\alpha : \pi_1(D_0) \to \mathcal{U}_n$. The Bergman space $L^2_a(E, \alpha)$ of holomorphic sections on E_α is defined similarly. It can be characterized as the space of all α-automorphic functions in $L^2_a(\mathbb{D}, \mathbb{C}^n)$, denoted by $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. And we define the Bergman bundle shift T_α also. For

D_0, we have similar arguments and establish results which are similar to Theorem 1.14.

THEOREM 1.20. *There is an onto $*$-isometric isomorphism between $(\mathcal{W}^*(T_\alpha))'$ and $(\mathcal{W}^*(\alpha))'$.*

Proof. If A is an operator commuting with T_α, then there exists a bundle map $\Phi \in H^\infty_{M_n(\mathbb{C})}(\mathbb{D})$ such that $Af(z) = \Phi(z)f(z)$ for every $f \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. (Note that the techniques for the Cowen–Douglas operator carry over to flat bundle case, cf. [4]). And for $f \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$, $\gamma \in \pi_1(D_0)$, we have

$$(\Phi f)(\gamma z) = \Phi(\gamma z)f(\gamma z) = \Phi(\gamma z)\alpha(\gamma)f(z),$$

and

$$(\Phi f)(\gamma z) = \alpha(\gamma)(\Phi f)(z) = \alpha(\gamma)\Phi(z)f(z),$$

since $(\Phi f) \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. Then

$$\Phi(\gamma z) = \alpha(\gamma)\Phi(z)\alpha(\gamma)^*.$$

For every projection $P \in (\mathcal{W}^*(T_\alpha))'$, there a $\Psi \in H^\infty_{M_n(\mathbb{C})}(\mathbb{D})$ such that $Pf(z) = \Psi(z)f(z)$ for every $f \in L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$. $P^2 = P$ and $P^* = P$ implies that $\Psi(z)^* = \Psi(z)$ and $\Psi(z)^2 = \Psi(z)$ for $z \in \mathbb{D}$, and so $\Psi(z)$ equals a constant projection Q since $\Psi(z)$ and $\Psi(z)^*$ are analytic. Moreover,

$$\Psi(\gamma z) = \alpha(\gamma)\Psi(z)\alpha(\gamma)^*, \quad \text{for } \gamma \in \pi_1(D_0),$$

so,

$$Q = \alpha(\gamma)Q\alpha(\gamma)^*,$$

that is, $Q \in (\mathcal{W}^*(\alpha))'$. Conversely, it is clear that $\Psi(z) = Q \in (\mathcal{W}^*(T_\alpha))'$ for every $Q \in (\mathcal{W}^*(\alpha))'$, and the correspondence is one-to-one and $*$-isometric. ∎

2. THE MULTIPLICATION OPERATOR T_B ON THE BERGMAN SPACE AS A BERGMAN SHIFT

Let B be a finite Blaschke product of order n. We are ready now to show that T_B can be represented as a Bergman bundle shift. Recall the set

$$\mathcal{S} = B(\{\beta : B'(\beta) = 0\}).$$

Note that $\mathcal{S}$ is finite and $|\mathcal{S}| \leqslant n-1$ where $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. Let $\mathcal{S} = \{\beta_1, \ldots, \beta_k\}$ and l_i be the line obtained by joining β_i to boundary of $\mathbb{D}$ with the assumption that no two lines intersect each other. Set $\mathbb{D}_B = \mathbb{D} \setminus \bigcup_{i=1}^k l_i$, clearly $\mathbb{D}_B$ is a simply connected domain. For an open set $U \subset \mathbb{D}$, we define an *inverse* of B in U to be a holomorphic function f in U with $f(U) \subset \mathbb{D}$ such that $B(f(z)) = z$ for every z in U.

For each $z \in \mathbb{D} \setminus \mathcal{S}$, $B^{-1}(z) = \{z_1, z_2, \ldots, z_n\}$ with $z_i \neq z_j$ for $i \neq j$ and B is one to one in some open neighborhood U_{z_i} of each point z_i. Let U be an open neighborhood of z in $\mathbb{D}_B$ and σ_i be the biholomorphic map from U to U_{z_i}, for

$1 \leqslant i \leqslant n$, such that $B(\sigma_i(z)) = z$ for $z \in U$. Since $\mathbb{D}_B$ is simply connected, by the monodromy theorem we can extend holomorphically each σ_i to the domain $\mathbb{D}_B$ and we denote the extended holomorphic function by the same symbol σ_i.

Note that here we use inverses of B, not local inverses as in [8], [9].

LEMMA 2.1. *The images of the σ_i's are disjoint, that is, $\sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B) = \emptyset$ for $i \neq j$.*

Proof. Suppose $w_0 \in \sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B)$, so there exists $z_0 \in \mathbb{D}_B$ such that $w_0 = \sigma_i(z_0) = \sigma_j(z_0)$. Set $\sigma = \sigma_j^{-1} \circ \sigma_i : \mathbb{D}_B \to \mathbb{D}_B$; here σ is a biholomorphic map (because σ_i's are biholomorphic). Since $\mathbb{D}_B$ is simply connected, by the Riemann mapping theorem there exists a biholomorphic map ψ from $\mathbb{D}_B$ to $\mathbb{D}$. Set $f = \psi \circ \sigma \circ \psi^{-1}$; clearly f is a biholomorphic map of $\mathbb{D}$ and $f(\psi(z_0)) = \psi(z_0)$. By Schwarz lemma we have

$$f(z) = \frac{\psi(z_0) - e^{i\theta}\phi_{\psi(z_0)}(z)}{1 - e^{i\theta}\overline{\psi(z_0)}\phi_{\psi(z_0)}(z)}$$

where $\phi_{\psi(z_0)}(z) = \frac{\psi(z_0) - z}{1 - \overline{\psi(z_0)}\, z}$ and $\theta \in [0, 2\pi)$. Thus $\sigma = \psi^{-1} \circ f \circ \psi$ which implies that $\sigma_i = \sigma_j \circ \psi^{-1} \circ f \circ \psi$. Since σ_i's are inverses of B, so we have $\psi^{-1} \circ f \circ \psi(z) = z$ for all $z \in \mathbb{D}_B$. Thus $\sigma_i = \sigma_j$ which is a contradiction. Hence $\sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B) = \emptyset$ for $i \neq j$. ∎

LEMMA 2.2. *Let f be a holomorphic function defined on a domain Ω such that $f(z_0) = w_o$, $f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$. The behavior of f near z_0 is similar to that of the function z^m near* 0.

The detailed proof of Lemma 2.2 can be found in p. 236 of [10].

Let $w_0 \in \mathbb{D} \setminus \mathcal{S}$ and U be an open set in $\mathbb{D}_B$ containing w_0. Let $\{\sigma_i\}_{i=1}^n$ be inverses of B defined on U. Let γ_i be a closed curve in $\mathbb{D} \setminus \mathcal{S}$ at w_0 which enclose the point β_i for $1 \leqslant i \leqslant k$. If we move σ_j's along closed curve γ_i by analytic continuation we get a permutation on elements $\{1, \dots, n\}$, say τ_i, which defines a unitary operator V_i on $\mathbb{C}^n$ as follows:

$$V_i(x_1, \dots, x_n) = (x_{\tau_i(1)}, \dots, x_{\tau_i(n)}) \quad 1 \leqslant i \leqslant k.$$

Let $\alpha : \pi_1(\mathbb{D} \setminus \mathcal{S}) \to \mathcal{U}(\mathbb{C}^n)$ be a representation. Since $\pi_1(\mathbb{D} \setminus \mathcal{S})$ is a free group on k generators, the representation is determined by k unitary operators $V_1, \dots, V_k$. The representation α defines a flat unitary vector bundle E_B over $\mathbb{D} \setminus \mathcal{S}$. Informally, we define a topology on $\mathbb{D} \times \mathbb{C}^n$ so that as a point (z, v) moves continuously across the cut l_i the vector v becomes $V_i(v)$. Now we are ready to identify T_B as a Bergman bundle shift which is the main result of the paper.

Proof of Theorem 0.3. Define a map $\Gamma : L_a^2(\mathbb{D}) \to L_a^2(E_\alpha)$ as follows

$$\Gamma(f) = \frac{1}{\sqrt{n}}[((f \circ \sigma_1)\sigma_1', \dots, (f \circ \sigma_n)\sigma_n')^{\mathrm{tr}}]$$

where $[((f\circ\sigma_1)\sigma_1',\dots,(f\circ\sigma_n)\sigma_n')^{\mathrm{tr}}]$ denotes the equivalence class of $((f\circ\sigma_1)\sigma_1'$, ..., $(f\circ\sigma_n)\sigma_n')^{\mathrm{tr}}$. We show that Γ is a unitary map:

Step 1. Γ preserves the inner product. We have:

$$\begin{aligned}\langle\Gamma(f),\Gamma(g)\rangle &= \frac{1}{n}\langle[((f\circ\sigma_1)\sigma_1',\dots,(f\circ\sigma_n)\sigma_n')^{\mathrm{tr}}],[((g\circ\sigma_1)\sigma_1',\dots,(g\circ\sigma_n)\sigma_n')^{\mathrm{tr}}]\rangle\\ &= -\frac{1}{2n\pi\mathrm{i}}\sum_{i=1}^{n}\int\limits_{\sigma_i(\mathbb{D}_B)}(f\circ\sigma_i)(z)\overline{(g\circ\sigma_i)(z)}\;\;|\sigma_i'(z)|^2\mathrm{d}z\wedge\mathrm{d}\overline{z}\\ &= -\frac{1}{2n\pi\mathrm{i}}\sum_{i=1}^{n}\int\limits_{\mathbb{D}_B}f(z)\overline{g(z)}\,\mathrm{d}z\wedge\mathrm{d}\overline{z} = -\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathbb{D}_B}f(z)\overline{g(z)}\,\mathrm{d}z\wedge\mathrm{d}\overline{z}\\ &= -\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathbb{D}}f(z)\overline{g(z)}\,\mathrm{d}z\wedge\mathrm{d}\overline{z} = \langle f,g\rangle.\end{aligned}$$

Step 2. Γ is onto. Let $[(g_1,\dots,g_n)^{\mathrm{tr}}]$ be an element of $L^2_a(E_B)$. Define a function g on $\mathbb{D}$ as follows

$$g(z)=\begin{cases}\sqrt{n}\;g_i\circ B(z)\cdot((\sigma_i)'(B(z)))^{-1} & \text{if } z\in\sigma_i(\mathbb{D}_B),\ 1\leqslant i\leqslant n;\\ \text{defined by the equivalence relation} & \text{if } z\in B^{-1}(\bigcup_{j=1}^{k}l_j).\end{cases}$$

Since the values match up on the boundary by construction, g is a holomorphic function on $\mathbb{D}$ and

$$\begin{aligned}\Gamma(g) &= [((g\circ\sigma_1)\cdot\sigma_1',\dots,(g\circ\sigma_n)\cdot\sigma_n')^{\mathrm{tr}}]\\ &= [(g_1\cdot((\sigma_1)')^{-1}\sigma_1',\dots,g_n\cdot((\sigma_n)')^{-1}\sigma_n')^{\mathrm{tr}}] = [(g_1,\dots,g_n)^{\mathrm{tr}}].\end{aligned}$$

Step 3. Γ intertwines T_B and T_{E_B}. For $f\in L^2_a(\mathbb{D})$

$$\begin{aligned}\Gamma\circ T_B(f) = \Gamma(Bf) &= \frac{1}{\sqrt{n}}[((B\circ\sigma_1)(f\circ\sigma_1)\cdot\sigma_1',\dots,(B\circ\sigma_n)(f\circ\sigma_n)\cdot\sigma_n')^{\mathrm{tr}}]\\ &= \frac{1}{\sqrt{n}}[(z(f\circ\sigma_1)\cdot\sigma_1',\dots,z(f\circ\sigma_n)\cdot\sigma_n')^{\mathrm{tr}}]\\ &= z\frac{1}{\sqrt{n}}[((f\circ\sigma_1)\cdot\sigma_1',\dots,(f\circ\sigma_n)\cdot\sigma_n')^{\mathrm{tr}}] = T_{E_B}\circ\Gamma(f).\end{aligned}$$

Thus

$$\Gamma\circ T_B = T_{E_B}\circ\Gamma. \quad ∎$$

Some arguments in the proof of this result are related to the paper [9], but the proof is new.

COROLLARY 2.3. *$(\mathcal{W}^*(T_B))'$ is isomorphic to the commutant algebra of the unitary matrices which define the bundle.*

COROLLARY 2.4. *The restriction of T_B to a reducing subspace is a Bergman bundle shift also, which corresponds to the reduction of the unitaries on that subbundle determined by that subspace.*

In [7], the model space for the special case $B = z^n$ is discussed explicitly.

As we mentioned earlier in this section, there is no way to put a canonical order on the set $B^{-1}(w)$ for $w \in \mathbb{D} \setminus \mathcal{S}$. Different choices of lines will yield different orders unless $\mathcal{S}$ has only one element. As a consequence, the representation of the covering group by permutation group is not unique.

Representing T_B as a bundle shift allows us to recover most of the results in [8], [9] except for two key ones: the fact that $(\mathcal{W}^*(T_B))'$ is abelian and its linear dimension. A more careful analysis of the covering group associated to the Riemann surface $\{(z_1, z_2) : B(z_1) = B(z_2)\}$ for B will be required for that. Also the latter is a rational function in z_1 and z_2, its polynomial numerator is closely related to the permutation covering group, most likely via Galois theory. We leave such analyses to a later paper.

Acknowledgements. The second and third authors thank Department of Mathematics, Texas A&M University for warm hospitality. The work of Dinesh Kumar Keshari was supported partially in the form of a fellowship by Indo-US The Virtual Institute for Mathematics and Statistical Sciences (VI-MSS) and partially in the form of Stipend by Department of Mathematics, Texas A&M University. The work of Anjian Xu was supported by Chongqing Research Program of Basic Research and Frontier Tehnology (No.cstc2015jcyjA00045), Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No.KJ1500938), China Scholarship Council and NSFC(11271388,11501068).

The authors would like to thank the referee for valuable suggestions.

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Received September 5, 2014.