J. OPERATOR THEORY
75:1(20016), 209–223
doi: 10.7900/jot.2015jan23.2065

RELATIVE COMMUTANT OF AN UNBOUNDED OPERATOR AFFILIATED WITH A FINITE VON NEUMANN ALGEBRA

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Communicated by Kenneth R. Davidson

ABSTRACT. This paper is concerned with the commutant of unbounded operators affiliated with finite von Neumann algebras. We prove an unbounded Fuglede–Putnam type theorem and present examples of closed operators affiliated with some II_1 factor with trivial relative commutant in the factor.

KEYWORDS: *Fuglede–Putnam theorem,* II_1 *factors, unbounded operators, relative commutant, transitive lattice.*

MSC (2010): Primary 47C15; Secondary 47A05.

1. INTRODUCTION

The celebrated Fuglede theorem ([4], Theorem I) states that if a bounded operator acting on a Hilbert space commutes with a normal (maybe unbounded) operator, then it also commutes with any function of the normal operator, e.g. the adjoint of the normal operator. Putnam generalized this fact in 1951 ([13], Lemma). The proof of the Fuglede–Putnam theorem cited in many textbooks is given by Rosenblum [14]. Since then, there have been some attempts to extend the Fuglede–Putnam theorem. We refer the interested reader to the survey [10] by M.H. Mortad. One purpose of this paper is to prove a version of Fuglede–Putnam theorem for unbounded operators affiliated with finite von Neumann algebras.

Given a finite von Neumann algebra $\mathfrak{A}$, we will use $\mathfrak{A}'$ to denote its commutant. A densely defined closed operator T is affiliated with $\mathfrak{A}$, denoted by $T\eta\mathfrak{A}$, if $TU = UT$ for any unitary U in $\mathfrak{A}'$. Murray and von Neumann showed that all densely defined closed operators affiliated with a II_1 factor $\mathfrak{A}$ form a $*$-algebra under the operations of addition $\widehat{+}$ and multiplication $\widehat{\cdot}$ (see Section 2). We will use $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ to denote this algebra and prove that if $T \mathbin{\widehat{\cdot}} N = M \mathbin{\widehat{\cdot}} T$, then $T \mathbin{\widehat{\cdot}} N^* = M^* \mathbin{\widehat{\cdot}} T$ where $T, M, N \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ and M, N are normal. As a consequence, we deduce that there exists a $T \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ for any separable II_1 factor $\mathfrak{A}$ such that T commutes with no non-scalar normal operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$.

Since $\{T\}' \cap \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})(\supseteq \{T\})$ is never trivial, it would be interesting to investigate whether the relative commutant of $T \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ in $\mathfrak{A}$ could be trivial, i.e., $\{T\}' \cap \mathfrak{A} = \mathbb{C}I$. By using the group-measure space construction, we can provide some examples of closed operators with trivial relative commutant. As a corollary, we show the existence of relative transitive subspace lattices consisting of four nontrivial projections in some II_1 factors. This answers one of the problems listed in [1] on the number of nontrivial projections in relative transitive lattices in II_1 factors.

This paper is organized as follows. We first recall some definitions and basic properties of the group-measure space construction and the algebra $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ in Section 2. The unbounded version of Fuglede–Putnam theorem for the elements in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ is proved in Section 3. The examples of closed operators with trivial relative commutant and transitive subspace lattices of projections consisting of four nontrivial elements in some II_1 factors are given in Section 4.

2. PREMIMINARIES

If not explicitly stated otherwise, we will use $\mathcal{H}$ to denote a separable Hilbert space throughout this paper. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$. A von Neumann algebra $\mathfrak{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology and contains the identity operator I. If the center $\mathfrak{A} \cap \mathfrak{A}'$ of $\mathfrak{A}$ is trivial, then $\mathfrak{A}$ is called a *factor*. A von Neumann algebra $\mathfrak{A}$ is called *finite* if there is a faithful normal tracial state on it. An infinite dimensional finite factor is called a II_1 *factor*.

The left regular representation of icc (infinite conjugacy classes) groups provide us ample examples of II_1 factors. Given a discrete group G, let $l^2(G) = \Big\{\xi : G \to \mathbb{C} \Big| \sum_{g\in G} |\xi(g)|^2 < +\infty\Big\}$ be the Hilbert space with inner product defined by

$$\langle \xi, \beta \rangle = \sum_{g\in G} \xi(g)\overline{\beta(g)}.$$

For any $h \in G$, we define a unitary operator l_h by $(l_h\xi)(g) = \xi(h^{-1}g)$. The group von Neumann algebra $\mathcal{L}_G$ is the von Neumann algebra generated by the unitary operators l_h, $h \in G$. If G is an icc group, then $\mathcal{L}_G$ is a factor of type II_1.

The crossed product (or group-measure space construction) is a generalization of the above construction. Suppose $(X, \mathcal{B}, \mu)$ is a non-atomic probability space and G is a countable group. Let $L^2(X)$ be the Hilbert space of all square integrable complex functions on X. $L^\infty(X)$, the space of all essentially bounded measurable complex functions on X, is a maximal abelian subalgebra of $\mathcal{B}(L^2(X))$. For notational simplicity, we will use f to denote the multiplication operator on $L^2(X)$, i.e., $(f\xi)(x) = f(x)\xi(x)$, $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Let G act as a group of transformations on X preserving measurability, i.e., for any $g \in G$, $g(S) \in \mathcal{B}$ if and only if $S \in \mathcal{B}$. For the sake of simplicity, we only consider the action that also keep the measure invariant, i.e., $\mu \circ g(S) = \mu(S)$ for any $S \in \mathcal{B}$. For the general case, we refer the reader to the Section 8.6 of [8]. It is clear that the mapping $g \to \alpha_g$, where α_g is defined as $\alpha_g(f)(x) = f(g^{-1}x)$ for every measurable function f, induces a homomorphism from G into the group of $*$-automorphisms of the von Neumann algebra $L^\infty(X)$. Furthermore this representation is unitarily implemented. Indeed, let U_g be the unitary defined by $U_g\xi(x) = \xi(g^{-1}x)$ where $\xi \in L^2(X)$. Then $\alpha_g(f) = U_g f U_g^*$.

The crossed product of the von Neumann algebra $L^\infty(X)$ by the action α of G is the von Neumann algebra $L^\infty(X) \rtimes_\alpha G$, acting on the Hilbert space $L^2(X) \otimes l^2(G)$, generated by the operators

$$\Psi(f) = \sum_{g \in G} \alpha_g^{-1}(f) \otimes E_g, \quad L_g = I \otimes l_g \quad \forall f \in L^\infty(X), g \in G,$$

where E_g is the orthogonal projection from $l^2(G)$ onto the one-dimensional subspace spanned by the vector $e_g \in l^2(G)$, i.e., $e_g(h) = \delta_{g,h}$, $g, h \in G$. We say that G acts ergodically if $S \in \mathcal{B}$ and $\mu(gS \setminus S) = 0$ for each $g \in G$ implies $\mu(S) = 0$ or $\mu(X \setminus S) = 0$. If we further assume that G acts freely, i.e., $\{x \in X : g(x) = x\}$ is a null set for each $g \in G \setminus \{e\}$, then $L^\infty(X) \rtimes_\alpha G$ is a factor of type II_1 (see Proposition 8.6.10 of [8]).

Recall that a densely defined operator T acting on $\mathcal{H}$ is closed if its graph $\mathscr{G}(T) = \{(\xi, T\xi) : \xi \in \mathscr{D}(T)\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$, where $\mathscr{D}(T)$ is the domain of T. A densely defined operator T is called *closable* if the closure of $\mathscr{G}(T)$ is the graph of an operator. Murray and von Neumann proved the following maximality result for the closed operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$, the set of operators affiliated with a finite von Neumann algebra $\mathfrak{A}$.

PROPOSITION 2.1 ([11], Theorem 16.4.2). *Let $T_1, T_2 \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If $T_1 \subseteq T_2$, i.e., $\mathscr{D}(T_1) \subseteq \mathscr{D}(T_2)$ and $T_1\xi = T_2\xi$ for any $\xi \in \mathscr{D}(T_1)$, then $T_1 = T_2$.*

Furthermore if two elements T_1, T_2 of $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ agree on a dense subspace of $\mathcal{H}$, then $T_1 = T_2$ (see Lemma 3.3 of [15]). By Lemma 16.4.3 of [11], for any two elements T_1 and T_2 in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$, $T_1 + T_2$ and T_1T_2 are densely defined and closable. And the closed extensions of $T_1 + T_2$ and T_1T_2 are in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. We will use $T_1 \hat{+} T_2$ and $T_1 \hat{\cdot} T_2$ to denote these closures. Then $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$, provided with the operations $\hat{+}$ and $\hat{\cdot}$, is a $*$-algebra. For the simplicity of notations and without special statement, we still use $T_1 + T_2$ and T_1T_2 to denote the sum $T_1 \hat{+} T_2$ and the product $T_1 \hat{\cdot} T_2$.

3. AN UNBOUNDED VERSION FUGLEDE–PUTNAM THEOREM

The celebrated Fuglede–Putnam theorem in its classical form is as follows:

THEOREM 3.1 (Fuglede–Putnam theorem [4], [13]). *If T is a bounded operator acting on a Hilbert space and M and N are (maybe unbounded) normal operators, then*

$$TN \subseteq MT \Rightarrow TN^* \subseteq M^*T.$$

Throughout this section, we will assume that $\mathfrak{A}$ is a finite von Neumann algebra and τ is a faithful normal tracial state on $\mathfrak{A}$. We will prove the following Fuglede–Putnam type theorem for elements in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$.

THEOREM 3.2. *Let $T \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If N, M are normal operators in $\mathfrak{A}$ and $TN = MT$, then $TN^* = M^*T$.*

The following fact is well-known. We include the proof for the sake of completeness.

LEMMA 3.3. *Let N be a normal operator in $\mathfrak{A}$. If P is a projection in $\mathfrak{A}$ such that $(I-P)NP = 0$, then $PN = NP$.*

Proof. Let $N_1 = PNP$, $N_2 = PN(I-P)$ and $N_3 = (I-P)N(I-P)$. We need to show that $N_2 = 0$. Since $N = N_1 + N_2 + N_3$ and

$$\begin{aligned} N_1N_1^* + N_2N_2^* &= P(N_1 + N_2 + N_3)(N_1^* + N_2^* + N_3^*)P \\ &= PNN^*P = PN^*NP = N_1^*N_1, \end{aligned}$$

we have

$$\tau(N_1N_1^* + N_2N_2^*) = \tau(N_1^*N_1 + N_2N_2^*) = \tau(N_1^*N_1).$$

Thus $\tau(N_2N_2^*) = 0$ and $N_2 = 0$. ∎

To prove Theorem 3.2, we first show the following technical lemma.

LEMMA 3.4. *Let H be a positive element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If M and N are normal operators in $\mathfrak{A}$ and $NH = HM$, then $N^*H = HM^*$, $NH = HN$ and $MH = HM$. Furthermore, if $\operatorname{Ker}(H) = \{0\}$, then $M = N$.*

Proof. Let E_0 be the orthogonal projection from $\mathcal{H}$ onto $\operatorname{Ker}(H)$. If $E_0 \neq \{0\}$, then it is not hard to check that $(I - E_0)ME_0 = 0$. By Lemma 3.3, $E_0M = ME_0$. Note that $HN^* = M^*H$. The same argument shows that $E_0N^* = N^*E_0$. Thus $E_0N = NE_0$.

By considering $(I - E_0)H$, $(I - E_0)N$ and $(I - E_0)M$, we could assume $\operatorname{Ker}(H) = \{0\}$. Let $\{E_\lambda\}$ be the resolution of the identity for H in $\mathfrak{A}$ such that

$$H = \int_0^\infty \lambda \mathrm{d}E_\lambda.$$

For fixed $\lambda > 0$, let $P_1 = E_\lambda$, $P_2 = I - P_1$ and

$$H = H_1 + H_2 := P_1H + P_2H \quad \text{and} \quad H^{-1} = H_1^{-1} + H_2^{-1} = P_1H^{-1} + P_2H^{-1}.$$

Note that H_1 and H_2^{-1} are bounded and $H_iH_i^{-1} = Pi$, $i = 1,2$. Let $N = \sum_{i=j=1}^{2} N_{ij} := \sum_{i=j=1}^{2} P_iNP_j$ and $M = \sum_{i=j=1}^{2} M_{ij} := \sum_{i=j=1}^{2} P_iMP_j$.

Since $H^{-1}NH = M$, we have

$$M_{11} = H_1^{-1}N_{11}H_1, \quad M_{12} = H_1^{-1}N_{12}H_2,$$
$$M_{21} = H_2^{-1}N_{21}H_1, \quad M_{22} = H_2^{-1}N_{22}H_2.$$

Since M is normal, we have $\tau(P_1M^*MP_1) = \tau(P_1MM^*P_1)$ and

$$\tau(H_1N_{11}^*H_1^{-2}N_{11}H_1 + H_1N_{21}^*H_2^{-2}N_{21}H_1) = \tau(H_1^{-1}N_{11}H_1^2N_{11}^*H_1^{-1} + H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1}).$$

Note that

$$\tau(H_1N_{11}^*H_1^{-2}N_{11}H_1) = \tau(M_{11}^*M_{11}) = \tau(M_{11}M_{11}^*) = \tau(H_1^{-1}N_{11}H_1^2N_{11}^*H_1^{-1}).$$

We have

$$\tau(H_1N_{21}^*H_2^{-2}N_{21}H_1) = \tau(H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1}).$$

Since $\|H_1\| \leqslant \lambda$ and $\|H_2^{-1}\| \leqslant 1/\lambda$, we have

$$\tau(H_1N_{21}^*H_2^{-2}N_{21}H_1) \leqslant \frac{1}{\lambda^2}\tau(H_1N_{21}^*N_{21}H_1) = \frac{1}{\lambda^2}\tau(N_{21}H_1^2N_{21}^*) \leqslant \tau(N_{21}N_{21}^*) = \tau(N_{21}^*N_{21}).$$

Let $Q = E_\beta - E_\lambda$ where $\beta > \lambda$. Then

$$\begin{aligned}\tau(H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1}) &\geqslant \beta^2\tau(H_1^{-1}N_{12}(I-Q)N_{12}^*H_1^{-1}) + \lambda^2\tau(H_1^{-1}N_{12}QN_{12}^*H_1^{-1})\\ &= \beta^2\tau((I-Q)N_{12}^*H_1^{-2}N_{12}(I-Q)) + \lambda^2\tau(QN_{12}^*H_1^{-2}N_{12}Q)\\ &\geqslant \frac{\beta^2}{\lambda^2}\tau((I-Q)N_{12}^*N_{12}(I-Q)) + \tau(QN_{12}^*N_{12}Q)\\ &= \frac{\beta^2}{\lambda^2}\tau(N_{12}(I-Q)N_{12}^*) + \tau(N_{12}QN_{12}^*).\end{aligned}$$

By $N^*N = NN^*$, it is not hard to check that $\tau(N_{12}N_{12}^*) = \tau(N_{21}^*N_{21})$. Therefore

$$\frac{\beta^2}{\lambda^2}\tau(N_{12}(I-Q)N_{12}^*) + \tau(N_{12}QN_{12}^*) \leqslant \tau(N_{12}N_{12}^*).$$

It is clear that

$$\frac{\beta^2}{\lambda^2}\tau(N_{12}(I-Q)N_{12}^*) \leqslant \tau(N_{12}(I-Q)N_{12}^*)$$

implies $N_{12}(I-Q)N_{12}^* = 0$. Since $E_\lambda = \bigwedge_{\alpha>\lambda} E_\alpha$, we have $N_{12}N_{12}^* = 0$. By Lemma 3.3, we have $E_\lambda N = NE_\lambda$. Since this is true for any $\lambda > 0$, N commutes with any element in the abelian von Neumann algebra generated by the projections $\{E_\lambda\}$. Specially, $N(I+H)^{-1} = (I+H)^{-1}N$. It is now clear that $NH = HN$. Recall that $\mathrm{Ker}(H) = \{0\}$. Thus $H(N-M) = 0$ implies $N = M$. ∎

With the help of the preceding lemma we can now prove Theorem 3.2.

Proof of Theorem 3.2. Let $T = UH$ be the polar decomposition of T (since $\mathfrak{A}$ is a finite von Neumann algebra, we can assume that U is unitary). $TN = MT$ is equivalent to $HN = (U^*MU)H$. By Lemma 3.4, $HN^* = (U^*M^*U)H$. Thus $TN^* = M^*T$. ∎

We can extend Theorem 3.2 to make it work for normal elements in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$.

COROLLARY 3.5. *Let T and N be closed operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If N is normal and $NT = TN$, then $N^*T = TN^*$.*

Proof. For each positive integer n, let E_n be the spectral projection for N corresponding to the set $\{z : |z| \leqslant n\}$. It is clear that $\{E_n\}$ is an increasing sequence of projections which converges to I in the strong operator topology. Since $NT = TN$, we have

$$(E_nNE_n)(E_nTE_n) = (E_nTE_n)(E_nNE_n).$$

Note that E_nNE_n is bounded. By Theorem 3.2,

$$E_nN^*TE_n = (E_nN^*E_n)(E_nTE_n) = (E_nTE_n)(E_nN^*E_n) = E_nTN^*E_n.$$

Note that $E_n \leqslant E_m$ if $n \leqslant m$. Multiplying both sides of the equation $E_mN^*TE_m = E_mTN^*E_m$ from right by $E_n (n \leqslant m)$, we have

$$E_mN^*TE_n = E_mTN^*E_n.$$

Let m tend to infinity, we get $N^*TE_n = TN^*E_n$. Thus, $E_nT^*N = E_nNT^*$. Now, let n tend to infinity, we have $T^*N = NT^*$ and $N^*T = TN^*$. ∎

COROLLARY 3.6. *Let T, N and M be closed operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If N and M are normal and $MT = TN$, then $M^*T = TN^*$.*

Proof. Note that

$$\mathfrak{A} \otimes M_2(\mathbb{C}) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : A_{ij} \in \mathfrak{A} \right\}$$

is also a finite von Neumann algebra. Consider the matrices of operators

$$N_1 = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

N_1 and T_1 are in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A}) \otimes M_2(\mathbb{C})$. It is well-known that $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A}) \otimes M_2(C) \cong \mathscr{A}_{\mathfrak{F}}(\mathfrak{A} \otimes M_2(\mathbb{C}))$. The operator N_1 is normal and $N_1T_1 = T_1N_1$. By Corollary 3.5, we have $N_1^*T_1 = T_1N_1^*$. Comparing the $(2,1)$-entry then gives $M^*T = TN^*$. ∎

Recall that the numerical range of a closed operator T, denoted by $W(T)$, is defined as

$$W(T) = \{\langle T\xi, \xi\rangle : \xi \in \mathscr{D}(T), \|\xi\|_2 = 1\}.$$

In [3], Embry proved the following theorem.

THEOREM 3.7 ([3], Theorem 1). *Let N and M be two commuting bounded normal operators and T a bounded operator such that $0 \notin W(T)$. If $MT = TN$, then $N = M$.*

We will obtain similar result for unbounded operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. For the proof of this fact we give first the following few useful facts.

LEMMA 3.8. *Let T be an element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If $A \in \mathfrak{A}$ such that $A \hat{\cdot} T = T \hat{\cdot} A$, then $\mathscr{D}(T) \subseteq \mathscr{D}(TA) = \{\beta : A\beta \in \mathscr{D}(T)\}$. Thus $AT \subseteq TA$.*

Proof. Let $\xi \in \mathscr{D}(T) \subseteq \mathscr{D}(A \hat{\cdot} T) = \mathscr{D}(T \hat{\cdot} A)$. By the definition of $T \hat{\cdot} A$, for any $1 \leqslant n \in \mathbb{N}$, there is $\xi_n \in \mathscr{D}(TA)$ such that

$$\|\xi_n - \xi\| \leqslant \frac{1}{n}, \quad \|T(A\xi_n) - T \hat{\cdot} A\xi\| \leqslant \frac{1}{n}.$$

Since $A\xi_n \to A\xi$ and T is closed, we have $T(A\xi) = (T \hat{\cdot} A)\xi$. Thus $A\xi \in \mathscr{D}(T)$. ∎

COROLLARY 3.9. *Let T be an element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If N is a normal operator in $\mathfrak{A}$ such that $NT = TN$, then $AT = TA$ for each A affiliated with the abelian von Neumann algebra $\mathfrak{A}$ generated by N.*

Proof. By Theorem 3.2, Lemma 3.8 and Lemma 5.6.13 of [8], we have $TA = AT$ for any $A \in \mathfrak{A}$. Let $B = UH \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ where U is a unitary in $\mathfrak{A}$ and H is a positive element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. Since $(I + H)^{-1}T = T(I + H)^{-1}$, we have $HT = TH$. Therefore $BT = TB$. ∎

The next corollary follows easily from Corollary 3.9 and the argument of Corollary 3.6, and we leave it to the reader to supply the reasonably easy proof.

COROLLARY 3.10. *Let T be an element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If N is a normal operator in $\mathfrak{A}$ such that $NT = TN$, then $AT = TA$ for each A affiliated with the abelian von Neumann algebra $\mathcal{A}$ generated by N.*

Proof. By Theorem 3.2, Lemma 3.8 and Lemma 5.6.13 of [], we have $TA = AT$ for any $A \in \mathcal{A}$. Let $B = UH \in \mathscr{A}_{\mathfrak{F}}(\mathcal{A})$ where U is a unitary in $\mathcal{A}$ and H is a positive element in $\mathscr{A}_{\mathfrak{F}}(\mathcal{A})$. Since $(I + H)^{-1}T = T(I + H)^{-1}$, we have $HT = TH$. Therefore $BT = TB$. ∎

By Corollary 3.10 and an argument parallel to that used in [3], we have the following fact.

COROLLARY 3.11. *Let T, N and M be closed operators in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. Suppose that N and M are two commuting normal elements and $MT = TN$. If $0 \notin W(T)$, then $N = M$.*

Proof. Let $N = N_1 + iN_2$ and $M = M_1 + iM_2$ where N_1, N_2, M_1 and M_2 are selfadjoint elements in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. Note that $MT = TN$ implies that $M^*T = TN^*$ by Corollary 3.6. Thus we know $M_iT = TN_i$, $i = 1, 2$. Similarly, as $NM = MN$

and $NM^* = M^*N$, we have $NM_i = M_iN$, $i = 1,2$. Therefore, $N_iM_j = M_jN_i$, $i,j \in \{1,2\}$.

To show $N = M$, we only need to prove that $N_i = M_i$ for $i = 1,2$. By the above argument, we could assume that N and M are two commuting selfadjoint elements. If N and M are selfadjoint, then $(iI+M)^{-1}$ and $(iI+N)^{-1}$ are bounded. Note that $MT = TN$ if and only if $(iI+M)^{-1}T = T(iI+N)^{-1}$. Let χ be the characteristic function for a Borel subset of $\mathbb{C}$. By Corollary 3.10, we know $ET = TF$ where $E = \chi((iI+M)^{-1})$ and $F = \chi((iI+N)^{-1})$. Noting that $EF = FE$, we have

$$[FT^*(I-F)]T[(I-F)TF] = T^*E(I-F)T(I-F)TF = T^*(I-F)TF(I-F)TF = 0,$$

and

$$[(I-F)T^*F]T[FT(I-F)] = T^*(I-E)FTFT(I-F) = T^*FT(I-F)FT(I-F) = 0.$$

Since $0 \notin W(T)$, the above equations imply that $(I-F)TF = 0$ and $FT(I-F) = 0$. Thus $TF = FT$. Consequently, $T(iI+N)^{-1} = (iI+N)^{-1}T$ by Lemma 5.6.13 of [8]). Note that $0 \notin W(T)$ implies that $\text{Ker}(T^*) = \{0\}$. Therefore $(iI+N)^{-1}T = (iI+M)^{-1}T$ implies $(iI+N)^{-1} = (iI+M)^{-1}$ and $N = M$. ∎

The following result is well-known. For the sake of completeness, we give the proof here.

LEMMA 3.12. *Let $\mathfrak{A}$ be a separable* II_1 *factor. There exist two maximal abelian selfadjoint subalgebras $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$.*

Proof. By Corollary 4.1 in [12], there is a hyperfinite subfactor $\mathcal{R}$ such that $\mathcal{R}' \cap \mathfrak{A} = \mathbb{C}I$. Let $\widetilde{\mathfrak{M}}_1$ and $\widetilde{\mathfrak{M}}_2$ be two orthogonal maximal abelian selfadjoint subalgebras which generate $\mathcal{R}$. There exist two maximal abelian selfadjoint subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{A}$ containing $\widetilde{\mathfrak{M}}_1$ and $\widetilde{\mathfrak{M}}_2$ respectively. If $T \in \mathfrak{M}_1 \cap \mathfrak{M}_2$, then T commutes with all elements in $\widetilde{\mathfrak{M}}_1$ and $\widetilde{\mathfrak{M}}_2$. Hence $T \in \mathcal{R}' \cap \mathfrak{A} = \mathbb{C}I$. ∎

COROLLARY 3.13. *If $\mathfrak{A}$ is a separable* II_1 *factor, then there exists a closed operator $T \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ such that $NT \neq TN$ for any normal element $N \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A}) \setminus \mathbb{C}I$.*

Proof. By Lemma 3.12, there exist two maximal abelian selfadjoint subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{A}$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. Let $T = H_1 + iH_2$ where H_1 and H_2 are two positive invertible (the inverse is a bounded positive operator in $\mathfrak{A}$) operators that generate $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Suppose that N is a nontrivial normal operator in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ and $NT = TN$. By Corollary 3.5, $N^*T = TN^*$. Hence $NT^* = T^*N$. This implies that $NH_1 = H_1N$ and $NH_2 = NH_2$. Note that $(I+N^*N)^{-1}$ is in $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. Thus N^*N must be a scalar and $N = cU$ where $c \in \mathbb{C}$ and U is a unitary. If N is a unitary, then N is in $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. ∎

4. UNBOUNDED OPERATOR WITH TRIVIAL RELATIVE COMMUTANT

In this section, we will construct unbounded operators affiliated with some II_1 factors with trivial relative commutant in the factors.

As in Section 2, let $(X, \mathcal{B}, \mu)$ be a non-atomic probability space. Consider the von Neumann algebra $\mathfrak{A} = L^\infty(X) \rtimes_\alpha G$ where G is a countable discrete group acting on X and leaving μ invariant. Suppose that G acts ergodically and freely, then $\mathfrak{A}$ is a factor of type II_1. Recall that $\mathfrak{A}$, as a subalgebra of $\mathcal{B}(L^2(X) \otimes l^2(G))$, is generated by the operators

$$\Psi(f) = \sum_{g \in G} \alpha_g^{-1}(f) \otimes E_g, \quad L_g = I \otimes l_g, \quad \forall f \in L^\infty(X), g \in G,$$

where E_g is the orthogonal projection from $l^2(G)$ onto the subspace spanned by the vector $e_g \in l^2(G)$.

Fix $n (\in \mathbb{N})$ different elements $s_1, s_2, \ldots, s_n$ in G. Let $\{h_{s_i}\}_{i=1}^n$ be n measurable functions on X satisfying $\mu(\{x : h_{s_i}(x) = 0 \text{ or } \infty\}) = 0$. It is easy to see that $\Psi(h_{s_i})$ is affiliated with the von Neumann algebra $\{\Psi(f) : f \in L^\infty(X)\}$. Thus

$$T = \sum_{i=1}^n \Psi(h_{s_i}) L_{s_i} = \sum_{i=1}^n L_{s_i} \Psi(\alpha_{s_i}^{-1}(h_{s_i})) \tag{4.1}$$

is an element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. Let $\{\chi_m\}_{m=1}^\infty \in L^\infty(X)$ be a sequence of characteristic functions satisfies the following three conditions:

(I) $\chi_{m_1} \chi_{m_2} = \chi_{m_1}$ for $m_1 \leqslant m_2$;

(II) $\bigcup_m \chi_m L^2(X)$ is dense in $L^2(X)$;

(III) for each m, $h_{s_i} \chi_m$, $\alpha_{s_i}^{-1}(h_{s_i}) \chi_m$ are bounded, $i = 1, \ldots, n$.

Let

$$P_m = \Psi(\chi_m) = \sum_{g \in G} \alpha_g^{-1}(\chi_m) \otimes E_g. \tag{4.2}$$

Then it is not hard to check that TP_m and $T^* P_m$ are both bounded. Thus $P_m \mathcal{H} \subseteq \mathscr{D}(T) \cap \mathscr{D}(T^*)$. By Proposition 2.1, it is easy to see that $\bigcup_m P_m \mathcal{H}$ is a common core for T and T^*. To proceed further, we will need the following technical result.

LEMMA 4.1. *With the above notations, let* $A = \sum_s \Psi(f_s) L_s \in L^\infty(X) \times_\alpha G$. *If* $AT = TA$ *(*T *is the element in* $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ *defined by equation* (4.1)*), then*

$$\sum_{i=1}^n \alpha_g^{-1}(f_{s s_i^{-1}}) \alpha_{s_i s^{-1} g}^{-1}(h_{s_i}) = \sum_{i=1}^n \alpha_g^{-1}(h_{s_i}) \alpha_{s_i^{-1} g}^{-1}(f_{s_i^{-1} s}), \quad \forall g, s \in G. \tag{4.3}$$

Proof. Since $AT = TA$, $\mathscr{D}(TA) = \mathscr{D}(AT) \supseteq \mathscr{D}(T) \cap \mathscr{D}(T^*) \supseteq \bigcup_m P_m \mathcal{H}$. Note that by the definition of P_m (see equation (4.2)), there is a dense linear subspace

$$\mathscr{D}_g = \bigcup_m \alpha_g^{-1}(\chi_m) L^2(X) \subseteq L^2(X)$$

such that $\mathscr{D}_g \otimes e_g = \{\xi \otimes e_g : \xi \in \mathscr{D}_g\} \subseteq \bigcup_m P_m \mathcal{H}$. Since $\{\chi_m\}$, as projections in $\mathcal{B}(L^2(X))$, tend to I in the strong operator topology, it is not hard to see that $\mathscr{D}_g \cap \mathscr{D}_l$ is also a dense subspace for g and l in G. For $\xi, \beta \in \mathscr{D}_g \cap \mathscr{D}_l$, we have

$$\begin{aligned}\langle TA\xi \otimes e_l, \beta \otimes e_g \rangle &= \Big\langle \Big(\sum_s \Psi(f_s) L_s \Big) \xi \otimes e_l, \Big(\sum_{i=1}^n L_{s_i^{-1}} \Psi(\overline{h}_{s_i}) \Big) \beta \otimes e_g \Big\rangle \\ &= \Big\langle \sum_s \alpha_{sl}^{-1}(f_s) \xi \otimes e_{sl}, \sum_{i=1}^n \alpha_g^{-1}(\overline{h}_{s_i}) \beta \otimes e_{s_i^{-1} g} \Big\rangle \\ &= \Big\langle \sum_{i=1}^n \alpha_g^{-1}(h_{s_i}) \alpha_{s_i^{-1} g}^{-1}(f_{s_i^{-1} g l^{-1}}) \xi, \beta \Big\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle AT\xi \otimes e_l, \beta \otimes e_g \rangle &= \Big\langle \Big(\sum_{i=1}^n \Psi(h_{s_i}) L_{s_i} \Big) \xi \otimes e_l, \Big(\sum_s L_{s^{-1}} \Psi(\overline{f}_s) \Big) \beta \otimes e_g \Big\rangle \\ &= \Big\langle \sum_{i=1}^n \alpha_{s_i l}^{-1}(h_{s_i}) \xi \otimes e_{s_i l}, \sum_s \alpha_g^{-1}(\overline{f}_s) \beta \otimes e_{s^{-1} g} \Big\rangle \\ &= \Big\langle \sum_{i=1}^n \alpha_g^{-1}(f_{g l^{-1} s_i^{-1}}) \alpha_{s_i l}^{-1}(h_{s_i}) \xi, \beta \Big\rangle.\end{aligned}$$

Since $\mathscr{D}_g \cap \mathscr{D}_l$ is dense in $L^2(X)$, the above two equations imply

$$\sum_{i=1}^n \alpha_g^{-1}(h_{s_i}) \alpha_{s_i^{-1} g}^{-1}(f_{s_i^{-1} g l^{-1}}) = \sum_{i=1}^n \alpha_g^{-1}(f_{g l^{-1} s_i^{-1}}) \alpha_{s_i l}^{-1}(h_{s_i}).$$

Let $gl^{-1} = s$, we obtain the desired equation (4.3). ∎

With the help of the preceding lemma, we can now give some unbounded operators affiliated with some II_1 factor $\mathfrak{A}$ with trivial relative commutant in $\mathfrak{A}$.

4.1. HYPERFINITE CASE. Let $X = [0, 1]$ be the unit interval endowed with the normalized Lebesgue measure and $G = \mathbb{Z}$. Fix an irrational number r in $[0, 1]$, we consider the action of G on X given by $n(x) = (x - nr)\mathrm{mod}1$ for $n \in \mathbb{Z}$. Clearly the action satisfies $\mu \circ n = \mu$. Let $\alpha_n(f)(x) = f((x + nr)\mathrm{mod}1)$ for any measurable function f on X. It is well-known that the action is free, ergodic and $\mathcal{R} = L^\infty(X) \rtimes_\alpha G$ is the hyperfinite II_1 factor. In the following, we will use α to denote α_1. Note that $\alpha_n = \alpha^n$.

Let $T = \Psi(h_1)L_1 \in \mathscr{A}_{\mathfrak{F}}(\mathcal{R})$ and $A = \sum_{k \in \mathbb{Z}} \Psi(f_k) L_k \in \mathcal{R}$. If $AT = TA$ then let $n = 1, s_1 = 1, g = n, s = m + 1$ in equation (4.3). We have

$$\alpha_{-n}(f_m)\alpha_{m-n}(h_1) = \alpha_{-n}(h_1)\alpha_{1-n}(f_m).$$

Applying α_n to both side of the above equation, we get

$$f_m \alpha_m(h_1) = h_1 \alpha(f_m). \tag{4.4}$$

Recall that h_1 is a measurable function on X such that $\mu(\{x : h_1(x) = 0 \text{ or } \infty\}) = 0$.

Let

$$k_m = \begin{cases} h_1\alpha_1(h_1)\cdots\alpha_{m-1}(h_1) & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ \alpha_{-1}(1/h_1)\alpha_{-2}(1/h_1)\cdots\alpha_m(1/h_1) & \text{if } m < 0. \end{cases}$$

Then equation (4.4) implies that $\alpha(f_m/k_m) = f_m/k_m$.

By Lemma 8.6.6 of [8]), there exist constants $c_n, n \in \mathbb{Z}$, such that $f_m = c_m k_m$ almost everywhere. If we choose h_1 such that k_m is unbounded for each $m \neq 0$ (for example $h_1 = (1-x)/x$ satisfies the condition), then A is bounded if and only if $c_m = 0$ for all $m \neq 0$. Thus the relative commutant of T in $\mathcal{R}$ is trivial.

Recall that a Cartan subalgebra $\mathcal{M}$ in a II_1 factor $\mathfrak{A}$ is a maximal abelian $*$-subalgebra with normalizer $\mathcal{N}_{\mathfrak{A}}(\mathcal{M}) = \{U \in \mathcal{U}(\mathcal{M}) : U^*\mathcal{M}U = \mathcal{M}\}$ generating $\mathfrak{A}$, where $\mathcal{U}(\mathfrak{A})$ is the group of all unitary operators in $\mathfrak{A}$. By the above discussion, we have the following fact.

LEMMA 4.2. *Let $\mathcal{R} = L^\infty(X) \rtimes_\alpha \mathbb{Z}$ be the hyperfinite II_1 factor. There exists a closed operator $T \in \mathscr{A}_{\mathfrak{F}}(\mathcal{R})$ such that $\{T\}' \cap \mathcal{R} = \mathbb{C}I$ and T generates $\mathcal{R}$, i.e., U and $(I+H)^{-1}$ generate $\mathcal{R}$, where $T = HU$ is the polar decomposition of T. Furthermore, $(I+H)^{-1}$ generates a Cartan subalgebra of $\mathcal{R}$.*

Proof. As stated above, let $h_1 = (1-x)/x$, then $T = \Psi(h_1)L_1$ is an element in $\mathscr{A}_{\mathfrak{F}}(\mathcal{R})$ such that $\{T\}' \cap \mathcal{R} = \mathbb{C}I$. Note that $\{\Psi(f) : f \in L^\infty(X)\}$ is a Cartan subalgebra of $\mathcal{R}$ since the action is free (see Theorem 8.6.1 of [8])). It is clear that $(I+h_1)^{-1} = x$ generates $\{\Psi(f) : f \in L^\infty(X)\}$. And $(I+h_1)^{-1}$ and L_1 generate $\mathcal{R}$. ∎

4.2. A II_1 FACTOR WITH ABELIAN CENTRAL SEQUENCE ALGEBRA. We now consider the factor studied in [16]. Let $X = [0,1]$ be the unit interval endowed with the normalized Lebesgue measure and $G = F_2$ be the free group generated by two generators a, b. The action of G on X is determined by $\alpha_a(h)(x) = h(a^{-1}(x)) = h((x+r)\mathrm{mod}1)$ and $\alpha_b(h)(x) = h(x)$ for any $h \in L^\infty(X)$, where $r \in [0,1]$ is a fixed irrational number.

It is proved in Proposition 3.1 of [16] that $\mathfrak{A} = L^\infty(X) \rtimes_\alpha F_2$ is a prime II_1 factor with nontrivial abelian central sequence algebra.

PROPOSITION 4.3. *With the above notations, there exists $T \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ with trivial relative commutant in the factor $\mathfrak{A} = L^\infty(X) \rtimes_\alpha F_2$.*

Proof. Let $h_a(x) = (1-x)/x$ and $T = \Psi(h_a)L_a \in \mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. If $A = \sum_s \Psi(f_s)L_s$ in $\mathfrak{A}$ commutes with T, then by equation (4.3) we have

$$\alpha_g^{-1}(h_a)\alpha_{a^{-1}g}^{-1}(f_{a^{-1}s}) = \alpha_g^{-1}(f_{sa^{-1}})\alpha_{as^{-1}g}^{-1}(h_a), \quad \forall g, s \in F_2. \tag{4.5}$$

For simplicity of notation, we will use α^n to denote α_{a^n} . Let ρ be the group homomorphism from F_2 to $\mathbb{Z}$ such that $\rho(a) = 1, \rho(b) = 0$. Substitute g and s in equation (4.5) with a and sa respectively. If $\rho(s) = m$, then we have

$$\alpha^{-1}(h_a)f_{a^{-1}sa} = \alpha^{-1}(f_s)\alpha^{m-1}(h_a), \quad \forall s \in G \text{ and } \rho(s) = m. \tag{4.6}$$

Let

$$k_m = \begin{cases} h_a\alpha^1(h_a)\alpha^2(h_a)\cdots\alpha^{m-1}(h_a) & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ \alpha^{-1}(1/h_a)\alpha^{-2}(1/h_a)\cdots\alpha^{m}(1/h_1) & \text{if } m < 0. \end{cases}$$

The equation (4.6) implies that

$$\frac{f_{a^{-1}sa}}{k_m} = \alpha^{-1}(f_s/k_m).$$

An easy induction gives

$$f_{a^{-n}sa^n} = \alpha^{-n}(f_s/k_m)k_m, \quad \forall n \in \mathbb{Z}, \forall s \in G \text{ with } \rho(s) = m. \tag{4.7}$$

We claim that $f_s = 0$ if s contains $b^{\pm 1}$ in its reduced form. To prove this statement, we will use the Furstenberg's multiple recurrence theorem which we quote below for the convenience of the reader.

THEOREM 4.4 ([2], Theorem 7.4). *Let $(X, \mathcal{B}, \mu)$ be a probability space and $\alpha : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving map, i.e., $\mu(\alpha^{-1}(B)) = \mu(B)$ for any $B \in \mathcal{B}$. If $B \in \mathcal{B}$ with $\mu(B) > 0$, then for any $k \in \mathbb{N}$,*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(B \cap \alpha^{-n}(B) \cap \alpha^{-2n}(B) \cap \cdots \cap \alpha^{-kn}(B)) > 0.$$

Since $\mu(\{x : k_m(x) = \infty\}) = 0$, we only need to show that $f_s/k_m = 0$. This can be proved by contradiction. If $h := f_s/k_m \neq 0$, note that $\mu(\{x : k_m(x) = 0\}) = 0$, then there exist two constants $c > 0$ and $\delta > 0$ such that the measure of the set

$$S = \left\{x : |h(x)| > c \text{ and } |km(x)| \geqslant \frac{\delta}{c}\right\}$$

is non zero.

By the Furstenberg's multiple recurrence theorem, there is $\varepsilon > 0$ such that

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(S \cap a^{-n}(S)) = \varepsilon > 0.$$

Let $\{n_i\}_i$ be a subsequence such that $\mu(S \cap a^{-n_i}(S)) \geqslant \varepsilon$. If $x \in S \cap a^{-n_i}(S)$, then $|f_{a^{-n_i}sa^{n_i}}(x)| = |h(a^{n_i}(x))|\,|k_m(x)| \geqslant \delta$ by equation (4.7). Therefore

$$\infty = \sum_i \delta^2\varepsilon \leqslant \sum_i \int\limits_{S\cap a^{-n_i}(S)} |f_{a^{-n_i}sa^{n_i}}|^2 \mathrm{d}\mu \leqslant \sum_i \int\limits_X |f_{a^{-n_i}sa^{n_i}}(x)|^2 \mathrm{d}\mu < \infty.$$

It is a contradiction and f_s must equals 0.

Hence, if $A = \sum_s \Psi(f_s)L_s$ commutes with T, then $f_s = 0$ if s contains $b^{\pm 1}$ in the reduced form. Now using the same argument as in proof of the hyperfinite case, we can easily deduce that $f_s = 0$ if s is not the unit of F_2 and A is a scalar. ∎

4.3. RELATIVE TRANSITIVE SUBSPACE LATTICES IN A II_1 FACTOR. For a subset $\mathcal{L}$ of $\mathcal{P}(\mathcal{H})$ where $\mathcal{P}(\mathcal{H})$ is the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$, let $\mathrm{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathcal{H}) : (I-P)TP = 0, \forall P \in \mathcal{L}\}$. If $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$, then $\mathcal{L}$ is called a *transitive family of projections*. It is easy to see that any pair of subspaces is not transitive. Halmos gave an example of a transitive lattice with 5 nontrivial projections in [5]. Harrison, Radjavi and Rosenthal presented an example of a transitive quadruple of projections in [6]. The existence of transitive triples, that is transitive family with only three nontrivial projections, is proved recently by V. Lomonosov and F. Nazarov in [9].

Let $\mathfrak{A}$ be a II_1 factor and $\mathcal{L} \subseteq \mathfrak{A}$ be a family of projections in $\mathfrak{A}$. $\mathcal{L}$ is said to be transitive relative to $\mathfrak{A}$ if the only elements in $\mathfrak{A}$ that leave all projections in $\mathcal{L}$ invariant are scalars. In [1], J. Bannon showed that if $\mathfrak{A}$ is a II_1 factor generated by two selfadjoint elements, then there is a transitive family of projections in $\mathfrak{A} \otimes M_2(\mathbb{C})$ with 5 nontrivial projections.

With the help of unbounded operators with trivial relative commutant, we can construct relative transitive quadruples of projections.

PROPOSITION 4.5. *Suppose $T = HU$ is a closed operator in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$ with trivial relative commutant in the II_1 factor $\mathfrak{A}$, where U is unitary and H is a positive element in $\mathscr{A}_{\mathfrak{F}}(\mathfrak{A})$. Then the following family of projections*

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad P_3 = \begin{pmatrix} I/2 & I/2 \\ I/2 & I/2 \end{pmatrix},$$
$$P_4 = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & I - U^*KU \end{pmatrix},$$

is a relative transitive quadruple of projections in $\mathfrak{A} \otimes M_2(\mathbb{C})$, where $K = H^2(I + H^2)^{-1}$.

Proof. An easy computation shows that if $A \in \mathfrak{A} \otimes M_2(\mathbb{C})$ such that $(I - P_i)AP_i = 0, i = 1,2,3$, then $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$ where $A_1 \in \mathfrak{A}$. Note that the range of P_4 is $S = \{(T\xi, \xi) : \xi \in \mathscr{D}(T)\}$. Thus $(I - P_4)AP_4 = 0$ implies that $((A_1T)\xi, A_1\xi) \in S$. This is true only if $A_1T = TA_1$. Since the relative commutant of T in $\mathfrak{A}$ is trivial, A_1 must be a scalar. ∎

Therefore, by Proposition 4.5 and the discussion in Section 4.1 and Section 4.2, we know that transitive quadruples of projections do exist in some II_1 factors.

We conclude this section by pointing out that any family of projections in a II_1 factor with less than four nontrivial elements is not transitive relative to the factor. If P_1 and P_2 are two nontrivial projections in a II_1 factor, then the partial isometry from E_1 onto E_2 leaves P_1 and P_2 invariant, where $E_1 \leqslant I - P_1$ and $E_2 \leqslant P_2$. For a family of projections with three nontrivial elements, we have the following fact.

PROPOSITION 4.6. *If $\mathcal{L} = \{P_1, P_2, P_3\}$ is a subset of three projections in a II_1 factor $\mathfrak{A}$, then $\mathcal{L}$ is not transitive relative to $\mathfrak{A}$.*

Proof. We first show that if $\mathcal{L}$ is transitive then $P_i \vee P_j = I$ and $P_i \wedge P_j = 0$, $i \neq j$. Without loss of generality, we may assume that $\tau(P_1) \leqslant 1/2$ and $\tau(P_2) \leqslant 1/2$, where τ is the faithful normal trace on $\mathfrak{A}$. Let $E = I - P_1 \vee P_2$. If $E \neq 0$, then it is not hard to check that any partial isometry V satisfying $V^*V \leqslant E$ and $VV^* \leqslant P_3$ is in $\mathrm{Alg}(\mathcal{L})$. As $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$, we have $E = 0$. By the Kaplansky formula $\tau(P_1) + \tau(P_2) = \tau(P_1 \vee P_2) + \tau(P_1 \wedge P_2)$, we have $\tau(P_1) = \tau(P_2) = 1/2$ and $\tau(P_1 \wedge P_2) = 0$. Hence, $P_1 \wedge P_2 = 0$.

If $\tau(P_3) > 1/2$, we may consider $I - \mathcal{L} = \{I - P_1, I - P_2, I - P_3\}$ instead (note that $\mathcal{L}$ is transitive if and only if $I - \mathcal{L}$ is transitive). And the exact same argument shows that $P_3 \wedge P_i = 0$ and $P_3 \vee P_i = I$, $i = 1, 2$. From Theorem 2.1. of [7], we have $\mathrm{Alg}(\mathcal{L}) \neq \mathbb{C}I$ and the proof is complete. ∎

Acknowledgements. This paper was partially supported by the National Science Foundation of China (NSFC) under grant numbers 11271390, 11371222, 11301511, 11321101 and 11371290.

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Received January 23, 2015.