J. OPERATOR THEORY
76:1(2016), 57–65
doi: 10.7900/jot.2015jul24.2077

A CRITERION FOR THE $\mathbb{Z}_d$-SYMMETRY OF THE SPECTRUM OF A COMPACT OPERATOR

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Communicated by Hari Bercovici

ABSTRACT. If A is a compact operator in a Banach space and some power A^q is nuclear, we give a criterion of $\mathbb{Z}_d$-symmetry of its spectrum σA in terms of vanishing of the traces $\operatorname{Tr} A^n$ for all n, $n \geqslant 0$, $n \neq 0 \mod d$, sufficiently large.

KEYWORDS: *Nuclear operators, trace, Cauchy–Riesz formula.*

MSC (2010): Primary 47B10; Secondary 47B07, 47B40.

1. INTRODUCTION

In the case of matrices, or linear operators $T : X \to X$ in a finite-dimensional space, one can check (prove) that the following conditions are equivalent:

(a) the spectrum of T is symmetric, or $\mathbb{Z}_2$-symmetric, i.e., $\lambda \in \sigma(T) \to -\lambda \in \sigma(T)$ and their algebraic multiplicities $m(\lambda), m(-\lambda)$ are equal;

(b) $\operatorname{Tr} T^p = 0$ for all odd $p \in \mathbb{N}$.

M. Zelikin [12] observed and proved that this claim could be extended to $\mathfrak{S}_1$, the trace-class operators in a Hilbert space. We will show that such claims could be made

(i) for the $\mathbb{Z}_d$-symmetry of a spectrum, $d \geqslant 2$;

(ii) in general Banach spaces (although we assume in the sequel that our Banach spaces have the approximation property).

2. TECHNICAL PRELIMINARIES

Of course, we need to make sure that Tr is well-defined if we write conditions like (b). In addition, the formula for the trace, $\operatorname{Tr} A = \sum\limits_j \lambda_j(A)$, should be properly explained if we use it. We now recall a few notions and facts about nuclear operators (see more in [5]).

An operator $A : X \to Y$ between two Banach spaces is called *nuclear* if it has the representation

$$Ax = \sum_{k=1}^{q} a_k f_k(x) y_k, \quad q \leqslant \infty \tag{2.1}$$

where

$$a_k > 0, \quad a^* = \sum a_k < \infty, \quad \text{and} \quad \|f_k|_{X'}\| \leqslant 1, \quad \|y_k|_Y\| \leqslant 1, \quad \forall k. \tag{2.2}$$

Let $\mathcal{N}(X;Y)$ denote the Banach space of all nuclear operators $X \to Y$ with the norm

$$A_1 = \inf\{a^* : A \text{ has a representation as in (2.1), (2.2)}\}. \tag{2.3}$$

A linear functional Tr is well-defined on $\mathcal{N}(X;X)$ (for any Banach space X with the approximation property) by

$$\mathrm{Tr} A = \sum_{k=1}^{q} a_k f_k(y_k). \tag{2.4}$$

Of course,

$$\|\mathrm{Tr} A\| \leqslant A_1,$$

and $\|\mathrm{Tr}\| = 1$.

A. Grothendieck [3] showed that for the operators (2.1), $(X = Y)$,

$$\text{if } \sum_{k=1}^{q} a_k^{2/3} < \infty \text{ then } \sum |\lambda_j(A)| < \infty \tag{2.5}$$

where the points of the spectrum $\sigma(A)$ are enumerated taking into account their multiplicity, and

$$\mathrm{Tr} A = \sum \lambda_j(A). \tag{2.6}$$

The presentation (2.1) of A under the conditions (2.2) gives a factorization

$$A = JF, \quad X \xrightarrow{F} \ell_2(\mathbb{N}) \xrightarrow{J} X, \tag{2.7}$$

where

$$Fx = \sum_1^\infty a_k^{1/2} f_k(x) e_k, \quad \text{and} \tag{2.8}$$

$$J\xi = \sum_1^\infty a_k^{1/2} \xi_k y_k, \tag{2.9}$$

with

$$\|F\| \leqslant (a^*)^{1/2}, \quad \|J\| \leqslant (a^*)^{1/2}. \tag{2.10}$$

Moreover, the product FJ is a Hilbert–Schmidt operator, or of the Schatten class $\mathfrak{S}_2$ in a Hilbert space $\ell^2(\mathbb{N})$; see more in [2], [11]. Indeed,

$$\langle FJe_k, e_m\rangle = a_k^{1/2} a_m^{1/2} f_m(y_k) \quad \text{and} \tag{2.11}$$

$$\sum_{k,m=1}^{\infty} |\langle FJe_k, e_m\rangle|^2 = \sum_{k,m=1}^{\infty} a_k a_m |f_m(y_k)|^2 \leqslant (a^*)^2\,, \tag{2.12}$$

so $FJ_2 \leqslant a^*$.

By Hölder inequality for Schatten classes ([2] or [11]),

$$BCD_{2/3} \leqslant B_2 C_2 D_2$$

so $(FJ)^3 \in \mathfrak{S}_{2/3}$ and has a representation

$$(FJ)^3 = \sum_{k=1}^{\infty} c_k \langle \cdot, f_k\rangle h_k, \quad c > 0,$$

where $\|f_k\|, \|h_k\| \leqslant 1$ and

$$\sum_{k=1}^{\infty} c_k^{2/3} < \infty.$$

Therefore, the operator

$$A^4 = J(FJ)^3 F = \sum_{k=1}^{\infty} c_k \langle F(\cdot), f_k\rangle Jh_k$$

satisfies the condition (2.5), and

$$\sum_{j=1}^{\infty} |\lambda_j(A^q)| < \infty \quad \text{for all } q \geqslant 4, \tag{2.13}$$

with

$$\mathrm{Tr} A^q = \sum_{j=1}^{\infty} \lambda_j(A^q). \tag{2.14}$$

More careful geometric analysis, based on approximative characteristics of operators ([7], [8]) — if we use [4], or Theorem 4.a.6, p. 227 of [5] — shows that we can lower q in (2.13), (2.14) to 3. Indeed, $(FJ)^2$ is in $\mathfrak{S}_1(\ell^2(\mathbb{N}))$, so there are finite-dimensional operators G_n, Rank $G_n \leqslant n$, such that

$$\sum_n \alpha_n < \infty, \quad \text{where } \alpha_n := \|(FJ)^2 - G_n\|.$$

Then

$$\|A^3 - JG_nF\| \leqslant a^* \cdot \alpha_n$$

and by Theorem 4.a.6 of [5]

$$A^3 \text{ is nuclear,} \quad \sum_j |\lambda_j(A^3)| \leqslant 2a^* \sum_n \alpha_n < \infty, \quad \text{and} \quad \mathrm{Tr} A^3 = \sum_j \lambda_j(A^3).$$

But this remark will not improve our Theorem 4.1 (below) in an essential way (although it would help to say $p > p_* \geqslant 3q_*$ instead of $p \geqslant p_* \geqslant 4q_*$ in (4.2)).

In a Hilbert space $X = H$, by Lisdkiĭ theorem [6], for any trace-class operator $C \in \mathfrak{S}_1$,

$$\sum_{j=1}^{\infty} |\lambda_j(C)| < \infty \quad \text{and} \quad \mathrm{Tr} C = \sum_{j=1}^{\infty} \lambda_j(C).$$

M. Zelikin considered in Theorem 2 of [12] only Hilbert spaces, perhaps because he worked in the context of Hilbert spaces.

3. CAUCHY–RIESZ FORMULA

Before stating our main result, let us recall from Chapter VII, Sections 3 and 4 of [1] some elements of the Riesz theory of compact operators.

If $T : X \to X$ is compact its spectrum $\sigma(T)$ is discrete with 0 being the only accumulation point, and it has the following properties:

(i) for any $\rho > 0$, $\sigma(T) \cap \{z : |z| \geqslant \rho\}$ is a finite set;

(ii) if

$$\delta(\alpha) = \frac{1}{2} \min\{|\alpha - \lambda| : \lambda \in \sigma(T), \lambda \neq \alpha\}$$

(so $\delta(\alpha) > 0$ for any $\alpha \in \mathbb{C} \setminus \{0\}$) and

$$P(\alpha) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z-\alpha|=\delta(\alpha)} (z - T)^{-1} \, \mathrm{d}z,$$

then

$$m(\alpha) = \mathrm{Rank}\, P(\alpha) < \infty, \quad \alpha \in \mathbb{C} \setminus \{0\}$$

with

$$m(\alpha) = 0 \quad \text{if and only if } \alpha \notin \sigma(T).$$

For $\alpha \in \sigma(T) \setminus 0$, $m(\alpha)$ is the algebraic multiplicity of the eigenvalue α.

The operational calculus ([1], Chapter VII, Sections 3 and 4) explains that for any $\rho > 0$ such that

$$\sigma(T) \cap \{|z| = \rho\} = \emptyset,$$

we have

$$T = \sum_{|\alpha|>\rho} T(\alpha) + S, \quad \text{where } T(\alpha) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z-\alpha|=\delta(\alpha)} z(z - T)^{-1} \, \mathrm{d}z$$

is an operator of rank $m(\alpha)$ with

$$\sigma(T(\alpha)) = \{\alpha\}, \quad \text{and} \quad S = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=\rho} z(z - T)^{-1} \, \mathrm{d}z.$$

Moreover, for any entire function $F(z)$, (for instance, for polynomials),

$$F(T) = \sum_{|\alpha|>\rho} F(T(\alpha)) + F(S), \tag{3.1}$$

where by the Cauchy–Riesz formulae,

$$F(T(\alpha)) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z-\alpha|=\delta(\alpha)} F(z)(z-T)^{-1}\,\mathrm{d}z, \quad F(S) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=\rho} F(z)(z-T)^{-1}\,\mathrm{d}z.$$

It follows that

$$\mathrm{Tr} F(T(\alpha)) = F(\alpha)\cdot m(\alpha); \quad F(T(\alpha)) = 0 \quad \text{if } F^{(j)}(\alpha) = 0, 0 \leqslant j \leqslant m(\alpha). \tag{3.2}$$

4. MAIN RESULT AND ITS PROOF

Now we are ready to prove the main result.

THEOREM 4.1. *Let T be a compact operator in a Banach space X, and let some power T^{q_*} be a nuclear operator. Then $\sigma(T)$ is $\mathbb{Z}_d$-symmetric, i.e., for any $\beta \in \mathbb{C}\setminus\{0\}$,*

$$m(\beta\omega^k) = m(\beta) \quad \textit{for all } k = 0,1,\ldots d-1,\ \omega = \exp(\mathrm{i}2\pi/d) \tag{4.1}$$

if and only if

$$\mathrm{Tr} T^{dp+r} = 0, \quad 1 \leqslant r \leqslant d-1, \tag{4.2}$$

for all sufficiently large p, say $p \geqslant p_* \geqslant 4q_*$.

Of course, if $d = 2$, this is an extension of Theorem 2 in [12], to the Banach case.

Proof. We divide the proof into two parts, one for each implication.

Part 1. (4.1) $\Rightarrow$ (4.2). First we notice that the assumption $p \geqslant 4q_*$ guarantees that all operators T^n, $n = dp + r$, in (4.2) satisfy the condition (2.5). Indeed, by the Grothendieck theorem, (2.6) holds, i.e.

$$\mathrm{Tr} T^n = \sum_{j=1}^{\infty} \lambda_j(T^n).$$

The absolute convergence permits to rearrange the terms of the right sum as we wish, to write

$$\mathrm{Tr} T^n = \sum \mu \cdot m(\mu; T^n). \tag{4.3}$$

With

$$m(\mu; T^n) = 0 \quad \text{for } \mu \notin \sigma(T^n)$$

we can "add" the terms with $\mu \notin \sigma(T^n)$ and this does not change the right side in (4.3). For

$$n = dp + r, \quad 0 \leqslant r \leqslant d-1, \quad \text{define } g = \gcd\{r, d\} \tag{4.4}$$

so

$$r = ag, \quad d = bg, \quad (a,b) = 1, \tag{4.5}$$

and with $r \leqslant d-1$ we have $1 \leqslant a < b$. For any $\mu \in \mathbb{C} \setminus \{0\}$ take its $\mathbb{Z}_b$-orbit, i.e.,

$$\widetilde{\mu} = \{\mu \cdot \tau^j : 0 \leqslant j < b\}, \quad \tau = \omega^g = \exp(\mathrm{i}2\pi/b). \tag{4.6}$$

The sum in (4.3) could be written as

$$\sum_{\mathbb{Z}_b-\text{orbits}} \sum_{j=0}^{b-1} \mu\tau^j \cdot m(\mu\tau^j; T^n) \tag{4.7}$$

where, in order to be sure that μ is in the orbit (4.6), it is chosen as $\mu = |\mu| e^{\mathrm{i}\vartheta}$, $0 \leqslant \vartheta < 2\pi/b$. Now we will show that the sum in (4.7) over each orbit is equal to zero. With the numbers defined in (4.4) put $\kappa = \exp(\mathrm{i}2\pi/n)$ so $\kappa^n = 1$ and notice that if $\mu = \lambda^n$, we choose

$$\lambda = |\mu|^{1/n} e^{\mathrm{i}\vartheta'}, \quad \vartheta' = \frac{\vartheta}{n}, \tag{4.8}$$

so that

$$(\lambda\omega^k)^n = \mu\omega^{k(dp+r)} = \mu\omega^{kr} = \mu\tau^{ak} \quad \text{and} \tag{4.9}$$

$$\begin{aligned}
\sum_{j=0}^{b-1} \mu\tau^j m(\mu\tau^j; T^n) &\overset{(1)}{=} \frac{1}{g} \sum_{k=0}^{d-1} \mu\tau^{ak} m(\mu\tau^{ak}; T^n) \\
&\overset{(2)}{=} \frac{1}{g} \sum_{k=0}^{d-1} (\lambda\omega^k)^n \sum_{s=0}^{n-1} m(\lambda\omega^k\kappa^s; T) \\
&\overset{(3)}{=} \frac{1}{g} \sum_{s=0}^{n-1} \sum_{k=0}^{d-1} (\lambda\omega^k)^n m(\lambda\kappa^s \cdot \omega^k; T) \\
&\overset{(4)}{=} \frac{1}{g} \sum_{s=0}^{n-1} m(\lambda\kappa^s; T) \mu \sum_{k=0}^{d-1} \tau^{ak} \\
&\overset{(5)}{=} \mu \Big(\sum_{s=0}^{n-1} m(\lambda\kappa^s; T) \Big) \Big(\sum_{j=0}^{b-1} \tau^j \Big) = 0.
\end{aligned} \tag{4.10}$$

The steps in (4.10) are justified in the following way. (1) comes from (4.9). (2) is just the change of order of the double summation. (3) uses in an essential way the theorem's assumption (4.1) that $m(\beta\omega^k)$ is independent of k. (4) is based on the properties of the roots ω, τ, $\omega^d = 1$, $\tau = \omega^g$ under (4.5). Of course, in (5) $\sum_{j=0}^{b-1} \tau^j = 0$, and $\{\tau^{ak}\}_{k=0}^{d-1}$ runs g times over $\{\tau^j\}_{j=0}^{b-1}$. The implication (4.1) $\Rightarrow$ (4.2) is proven.

Part 2. (4.2) $\Rightarrow$ (4.1) Take $\lambda \neq 0$ and as before

$$n = dp_* + dp + r, \quad 1 \leqslant r \leqslant d-1, \ p \geqslant 0,$$

and $0 < \rho < |\lambda|$ is such that

$$\sigma(T) \cap \{z \in \mathbb{C} : |z| = \rho\} = \varnothing, \tag{4.11}$$

with

$$\widetilde{\lambda} = \{\lambda\omega^k : 0 \leqslant k \leqslant d-1\}$$

being the $\mathbb{Z}_d$-orbit of λ. Now we use (3.1) for the special choice $F = F_{pr}$ defined as

$$F_{pr}(z) = \left(\frac{z}{\lambda}\right)^{dp_*+dp+r} \varphi(z),$$

where

$$\varphi(z) = \prod_{|\alpha|\geqslant\rho,\, \alpha\in\sigma(T),\, \alpha\notin\widetilde{\lambda}} \left(\frac{z^d-\alpha^d}{\lambda^d-\alpha^d}\right)^{m(\alpha)} = \psi(z^d), \quad \text{and } \psi \text{ is a polynomial.}$$

Then by (3.2)

$$\varphi(T(\alpha)) = 0; \quad F_{pr}(T(\alpha)) = 0, \quad \forall\alpha \notin \widetilde{\lambda}, |\alpha| > \rho,$$

but for $\beta \in \widetilde{\lambda}$, i.e., $\beta = \lambda\omega^k$,

$$\mathrm{Tr}F_{pr}(T(\beta)) = m(\beta)F_{pr}(\beta) = m(\lambda\omega^k)\omega^{kr}.$$

Therefore,

$$\mathrm{Tr}F_{pr}(T) = \sum_{k=0}^{d-1} \omega^{kr} m(\lambda\omega^k) + \mathrm{Tr}F_{pr}(S) \tag{4.12}$$

where

$$F_{pr}(S) = \left(\frac{T}{\lambda}\right)^{dp_*} \cdot \frac{1}{2\pi\mathrm{i}} \int\limits_{|z|=\rho} \left(\frac{z}{\lambda}\right)^{dp+r} \varphi(z)(z-T)^{-1}\,\mathrm{d}z.$$

Put

$$\Phi = \max\{|\varphi(z)| : |z| \leqslant \rho\}$$

and with (4.11)

$$M = \max\{\|R(z;T)\| : |z| = \rho\} < \infty.$$

Then

$$F_{pr}(S)_1 \leqslant Ct^p \quad \text{for any } r, 1 \leqslant r \leqslant d-1, \tag{4.13}$$

where

$$C = \frac{\Phi \cdot M \cdot \rho \cdot T^{dp_*}{}_1}{|\lambda|^{dp*}} \quad \text{and} \tag{4.14}$$

$$t = \left(\frac{\rho}{|\lambda|}\right)^d < 1. \tag{4.15}$$

Now by (4.2) and (4.12)

$$0 = \sum_{k=0}^{d-1} \omega^{kr} m(\lambda\omega^k) + \xi_{pr} \quad \text{for any } p \geqslant 1 \text{ and } r, 1 \leqslant r \leqslant d-1. \tag{4.16}$$

The sum $\sum_{k=0}^{d-1}$ does not depend on p but the remaining terms ξ_{pr} by (4.13), (4.14), (4.15) have estimates

$$|\xi_{pr}| \leqslant Ct^p \quad \text{so } \xi_{pr} \to 0 \ (p \to \infty).$$

This implies by (4.16)

$$\sum_{k=0}^{d-1} \omega^{kr} m(\lambda\omega^k) = 0, \quad \forall r, 1 \leqslant r \leqslant d-1, \tag{4.17}$$

or

$$y_k = m(\lambda\omega^k), \quad 1 \leqslant k \leqslant d-1,$$

is a solution of the system

$$\sum_{k=1}^{d-1} \omega^{kr} y_k = -y_0, \quad 1 \leqslant r \leqslant d-1. \tag{4.18}$$

Its determinant is of Vandermonde type so

$$\det\{\omega^{kr}\}_{k,r=1}^{d-1} \neq 0,$$

and the identities

$$\sum_{k=0}^{d-1} (\omega^r)^k = 0, \quad \forall r, 1 \leqslant r \leqslant d-1,$$

show that by (4.18)

$$y_k = y_0, \text{ i.e., } m(\lambda\omega^k) = m(\omega), \quad \forall k, 1 \leqslant k \leqslant d-1.$$

This proves that the multiplicity function m is constant on $\mathbb{Z}_d$-orbits in $\mathbb{C} \setminus \{0\}$, and (4.1) is proven. ∎

It is worth to notice that Part 2 of the proof does not use any form of the Grothendieck or Lidskĭi thoerems but it uses only properties of a linear function Tr on $\mathcal{N}(X; X)$ and an elementary formula for TrK when K is an operator of finite rank.

Acknowledgements. I thank M. Zelikin who told me his results [12] in September 2007 during the 18th Crimean Autumn Mathematical School–Symposium, in Laspi–Batiliman, Crimea. At that time I realized that his result could be extended to the Banach space case. In the late 2007 I had useful discussions with P. Djakov, E. Gorin, H. König, M. Solomyak. Recently, during the Aleksander Pelczynski Memorial Conference, July 2014, Bedlewo, Poland, O. Reinov brought my attention to his papers [9], [10]; he advised me to publish my 2007 results. I thank C. Baker and O. Reinov for recent discussions.

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Received July 24, 2015.