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TOEPLITZ OPERATORS AND TOEPLITZ ALGEBRA WITH SYMBOLS OF VANISHING OSCILLATION

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ABSTRACT. We study the C^*-algebra generated by Toeplitz operators with symbols of vanishing (mean) oscillation on the Bergman space of the unit ball. We show that the index calculation for Fredholm operators in this C^*-algebra can be easily and completely reduced to the classic case of Toeplitz operators with symbols that are continuous on the closed unit ball. Moreover, in addition to a number of other properties, we show that this C^*-algebra has uncountably many Fredholm components.

KEYWORDS: *Fredholm index, Toeplitz algebra, vanishing oscillation.*

MSC (2010): 46L35, 47A53, 47B35.

1. INTRODUCTION

Let $\mathbb{B}$ denote the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ in $\mathbb{C}^n$. Let $\mathrm{d}v$ be the volume measure on $\mathbb{B}$ with the normalization $v(\mathbb{B}) = 1$. The Bergman space $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$ is the subspace

$$\{h \in L^2(\mathbb{B}, \mathrm{d}v) : h \text{ is analytic on } \mathbb{B}\}$$

of $L^2(\mathbb{B}, \mathrm{d}v)$. Let P be the orthogonal projection from $L^2(\mathbb{B}, \mathrm{d}v)$ onto $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$. For each $f \in L^\infty(\mathbb{B}, \mathrm{d}v)$, the Toeplitz operator T_f with symbol f is defined by the formula

$$T_f h = P(fh), \quad h \in L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v).$$

For any non-empty subset G of $L^\infty(\mathbb{B}, \mathrm{d}v)$, we will write $\mathcal{T}(G)$ for the norm-closed algebra generated by the Toeplitz operators $\{T_f : f \in G\}$. If G is closed under complex conjugation, then $\mathcal{T}(G)$ is a C^*-algebra. In the case $G = L^\infty(\mathbb{B}, \mathrm{d}v)$, we will simply write $\mathcal{T}$ in place of $\mathcal{T}(L^\infty(\mathbb{B}, \mathrm{d}v))$. Often, this $\mathcal{T}$ is called the *full* Toeplitz algebra on the Bergman space. The main interest of this paper, however, will be on operators in $\mathcal{T}(G)$ for a particular G that has been introduced in previous investigations.

In fact, the G that we are interested in consists of functions of *vanishing oscillation* on $\mathbb{B}$, which were first introduced by Berger, Coburn and Zhu in [1] (see also [2], [17]). These functions are defined in terms of the *Bergman metric* on $\mathbb{B}$. For each $z \in \mathbb{B}\backslash\{0\}$, we have the Möbius transform φ_z given by the formula

$$\varphi_z(\zeta) = \frac{1}{1-\langle\zeta,z\rangle}\Big\{z - \frac{\langle\zeta,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\Big(\zeta - \frac{\langle\zeta,z\rangle}{|z|^2}z\Big)\Big\}$$

([12], page 25). In the case $z = 0$, we define $\varphi_0(\zeta) = -\zeta$. Then the formula

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z,w \in \mathbb{B},$$

gives us the Bergman metric on $\mathbb{B}$. Recall that a function g on $\mathbb{B}$ is said to have vanishing oscillation if it satisfies the following two conditions: (1) g is continuous on $\mathbb{B}$; (2) the limit

$$\lim_{|z|\uparrow 1}\sup_{\beta(z,w)\leqslant 1}|g(z)-g(w)| = 0$$

holds. We will write VO for the collection of functions of vanishing oscillation on $\mathbb{B}$. Moreover, we set

$$\mathrm{VO}_{\mathrm{bdd}} = \mathrm{VO}\cap L^\infty(\mathbb{B},\mathrm{d}v). \tag{1.1}$$

That is, $\mathrm{VO}_{\mathrm{bdd}}$ denotes the collection of functions of vanishing oscillation that are also bounded on $\mathbb{B}$.

Let us denote the collection of compact operators on $L^2_{\mathrm{a}}(\mathbb{B},\mathrm{d}v)$ by $\mathcal{K}$. It was shown in [1] that

$$\mathcal{T}(\mathrm{VO}_{\mathrm{bdd}}) = \{T_g : g \in \mathrm{VO}_{\mathrm{bdd}}\} + \mathcal{K}. \tag{1.2}$$

In fact, $\mathcal{T}(\mathrm{VO}_{\mathrm{bdd}})$ has a representation in terms of the more familiar notion of vanishing *mean* oscillation. Recall that the normalized reproducing kernel for the Bergman space is given by the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle\zeta,z\rangle)^{n+1}}, \quad z,\zeta \in \mathbb{B}.$$

Also recall that a function $f \in L^2(\mathbb{B},\mathrm{d}v)$ is said to have vanishing mean oscillation if

$$\lim_{|z|\uparrow 1}\|(f - \langle fk_z,k_z\rangle)k_z\| = 0.$$

Let VMO denote the collection of functions of vanishing mean oscillation on $\mathbb{B}$ defined as above. In the same spirit as (1.1), let us denote

$$\mathrm{VMO}_{\mathrm{bdd}} = \mathrm{VMO}\cap L^\infty(\mathbb{B},\mathrm{d}v).$$

That is, $\mathrm{VMO}_{\mathrm{bdd}}$ denotes the collection of functions of vanishing *mean* oscillation that are also bounded on $\mathbb{B}$. Then we have $\mathrm{VMO}_{\mathrm{bdd}} \supset \mathrm{VO}_{\mathrm{bdd}}$ [1].

It was shown in [1] that, in addition to (1.2), the equality

$$\mathcal{T}(\mathrm{VO}_{\mathrm{bdd}}) = \mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}) = \{T_g : g \in \mathrm{VMO}_{\mathrm{bdd}}\} + \mathcal{K} \tag{1.3}$$

also holds. In fact, it was shown in [1] that if $g \in \mathrm{VMO_{bdd}}$, then both operators $(1-P)M_g P$ and $PM_g(1-P)$ are compact. Therefore for all $f, g \in \mathrm{VMO_{bdd}}$ we have $T_f T_g - T_{fg} \in \mathcal{K}$.

For any non-empty $G \subset L^\infty(\mathbb{B}, \mathrm{d}v)$ and any $k \in \mathbb{N}$, let $M_k(G)$ denote the collection of $k \times k$ matrices whose entries belong to G. Then each $f \in M_k(G)$ gives rise to a Toeplitz operator T_f of *matrix symbol* on the Bergman space $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$ of vector-valued functions. That is, for each $h \in L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$, considered as a column vector, we define

$$T_f h = P_k(fh),$$

where P_k denotes the orthogonal projection from $L^2(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$ to $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$. For $G \subset L^\infty(\mathbb{B}, \mathrm{d}v)$ and $k \in \mathbb{N}$, let $\mathcal{T}(M_k(G))$ denote the norm-closed algebra generated by the Toeplitz operators $\{T_f : f \in M_k(G)\}$ of matrix symbols. From (1.2) and (1.3) we obtain

$$\begin{cases} \mathcal{T}(M_k(\mathrm{VO_{bdd}})) = \{T_g : g \in M_k(\mathrm{VO_{bdd}})\} + \mathcal{K}_k, \\ \mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \{T_g : g \in M_k(\mathrm{VMO_{bdd}})\} + \mathcal{K}_k, \quad \text{and} \\ \mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}})), \end{cases} \tag{1.4}$$

for every $k \in \mathbb{N}$, where $\mathcal{K}_k$ is the collection of compact operators on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$.

Let C denote the collection of continuous functions on the *closed* unit ball $\overline{\mathbb{B}}$. The starting point of this paper is the fact that the Fredholm index theory is well established for operators $B \in \mathcal{T}(M_k(C))$. See [3], [14]. We will show that index calculation for Fredholm operators in $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$ can be easily reduced to index calculation for operators in $\mathcal{T}(M_k(C))$. Thus, through this reduction we establish the Fredholm index theory for operators in $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$.

But sometimes numerical index does not tell the whole story. In fact, we will see later in the paper that sometimes the story numerical index tells is far from being whole. To further explain, let us introduce some useful notation.

All Hilbert spaces in this paper are assumed to be separable. For a Hilbert space $\mathcal{H}$, let us write $\mathrm{Fred}(\mathcal{H})$ for the collection of Fredholm operators on $\mathcal{H}$. We know that $\mathrm{Fred}(\mathcal{H})$ is the union of connected components

$$\mathrm{Fred}_m(\mathcal{H}) = \{B \in \mathrm{Fred}(\mathcal{H}) : \mathrm{index}(B) = m\},$$

$m \in \mathbb{Z}$. In fact, the Fredolm index theory is one of the best-understood theories concerning operators. We are all accustomed to the thinking that each Fredolm component is identified with the corresponding index. But there is one interesting phenomenon that has not really been noticed in the literature. Namely, for certain C^*-subalgebras $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, the intersection

$$\mathrm{Fred}_m(\mathcal{H}) \cap \mathcal{A} \tag{1.5}$$

can actually have uncountably many connected components. In such a situation, the index m hardly tells us anything about the above intersection. For each complex dimension n, there is a smallest $k(n) \in \mathbb{N}$ such that if $k \geqslant k(n)$, then $\mathcal{T}(M_k(C))$ contains a Fredholm operator of index 1 [14]. (We know, of course, that $k(1) = 1$.) We will see that if $k \geqslant k(n)$, then the C^*-algebra $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$ is an example of $\mathcal{A}$ for which (1.5) has uncountably many components.

The phenomenon that (1.5) has uncountably many components also occurs when $\mathcal{A}$ is the C^*-algebra generated by the Toeplitz operators with the familiar QC symbols on the *Hardy space* H^2 of the unit circle $\mathbb{T}$ [11]. Technically, the Bergman space case and the Hardy space case are quite different. In fact, we will have more to say about the difference between the Toeplitz algebra on the Bergman space and that on the Hardy space later.

To conclude the introduction, let us describe the organization of the paper. In Section 2 we present the index theory for $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$. In Section 3 we give a precise representation for scalar symbols $f \in \mathrm{VO_{bdd}}$ for which the Toeplitz operator T_f is Fredholm. We then show in Section 4 that $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$ has uncountably many Fredholm components. Finally, in Section 5 we present a number of properties that are forced on the structure of the full Toeplitz algebra $\mathcal{T}$ by $\mathcal{T}(\mathrm{VO_{bdd}})$.

2. INDEX THEORY

In this section we study the index theory in

$$\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}})).$$

Conventionally, one computes Fredolm index using paths that are continuous in the operator norm topology. That is, if A_t is a Fredholm operator for every $t \in [a, b]$, and if the map $t \mapsto A_t$ is continuous on $[a, b]$ with respect to the operator norm, then the integer-valued function $\mathrm{index}(A_t)$ remains constant on $[a, b]$. But in practical terms, the requirement that the map $t \mapsto A_t$ be continuous with respect to the operator norm may not always be met. This is because, in order to compute index, the path A_t has to be given by a practical formula, which does not always lead to norm continuity. This forces one to compute index under weaker conditions. Our first proposition offers a more practical substitute for the norm-continuity requirement of the map $t \mapsto A_t$.

PROPOSITION 2.1. *Suppose that A_t is a Fredholm operator on a Hilbert space $\mathcal{H}$ for every $t \in [a, b]$. If the maps*

$$t \mapsto A_t^* A_t \quad \text{and} \quad t \mapsto A_t A_t^* \tag{2.1}$$

from $[a, b]$ into $\mathcal{B}(\mathcal{H})$ are continuous with respect to the operator norm, then the function $t \mapsto \mathrm{index}(A_t)$ remains constant on $[a, b]$.

Proof. For each t, let E_t and F_t be the spectral measures for the self-adjoint operators $A_t^*A_t$ and $A_tA_t^*$ respectively. Consider the point b. Since A_b is Fredholm, there is a $c > 0$ such that the essential spectra of $A_b^*A_b$ and $A_bA_b^*$ are contained in $[c, \infty)$. Hence there are $0 < c_1 < c_2 < c$ such that

$$\mathrm{sp}(A_b^*A_b) \cap (c_1, c_2) = \emptyset = \mathrm{sp}(A_bA_b^*) \cap (c_1, c_2).$$

Since the maps (2.1) are continuous with respect to the operator norm, there are $\varepsilon_1 > 0$ and $c_1 < d_1 < d_2 < c_2$ such that

$$\text{(2.2)} \quad \mathrm{sp}(A_t^*A_t) \cap (d_1, d_2) = \emptyset = \mathrm{sp}(A_tA_t^*) \cap (d_1, d_2) \quad \text{whenever } t \in (b - \varepsilon_1, b].$$

Let $\varphi : [0, \infty) \to [0, 1]$ be a continuous function such that $\varphi = 1$ on $[0, d_1]$ and $\varphi = 0$ on $[d_2, \infty)$. Then it follows from (2.2) that

$$\text{(2.3)} \quad E_t([0, d_1]) = \varphi(A_t^*A_t) \quad \text{and} \quad F_t([0, d_1]) = \varphi(A_tA_t^*) \quad \text{whenever } t \in (b - \varepsilon_1, b].$$

Since φ is continuous on $[0, \infty)$, (2.1) and (2.3) together imply that the maps

$$t \mapsto E_t([0, d_1]) \quad \text{and} \quad t \mapsto F_t([0, d_1])$$

are continuous on $(b - \varepsilon_1, b]$ with respect to the operator norm. Therefore there is an $\varepsilon \in (0, \varepsilon_1)$ such that

$$\text{(2.4)} \qquad \|E_b([0, d_1]) - E_t([0, d_1])\| < 1 \quad \text{and} \quad \|F_b([0, d_1]) - F_t([0, d_1])\| < 1$$

if $t \in (b - \varepsilon, b]$. Since $d_1 < c$, the ranks of the projections $E_b([0, d_1])$ and $F_b([0, d_1])$ are finite. By (2.4), for every $t \in (b - \varepsilon, b]$, $E_t([0, d_1])$ and $F_t([0, d_1])$ are also finite-rank projections with $\mathrm{rank}(E_t([0, d_1])) = \mathrm{rank}(E_b([0, d_1]))$ and $\mathrm{rank}(F_t([0, d_1])) = \mathrm{rank}(F_b([0, d_1]))$. For a finite-rank orthogonal projection, its rank is the same as its trace. Hence

$$\text{(2.5)} \qquad \mathrm{tr}(E_t([0, d_1])) - \mathrm{tr}(F_t([0, d_1])) = \mathrm{tr}(E_b([0, d_1])) - \mathrm{tr}(F_b([0, d_1]))$$

for every $t \in (b - \varepsilon, b]$. On the other hand, for $t \in (b - \varepsilon, b]$, the polar decomposition of A_t gives us the identity $\mathrm{rank}(E_t((0, d_1])) = \mathrm{rank}(F_t((0, d_1]))$. From this and (2.5) we deduce

$$\mathrm{tr}(E_t(\{0\})) - \mathrm{tr}(F_t(\{0\})) = \mathrm{tr}(E_b(\{0\})) - \mathrm{tr}(F_b(\{0\}))$$

for every $t \in (b - \varepsilon, b]$. That is, the function $\mathrm{index}(A_t)$ is constant on $(b - \varepsilon, b]$.

By the same argument, $\mathrm{index}(A_t)$ is constant on $[a, a + \delta)$ for some $\delta > 0$. Similarly, if $s \in (a, b)$, then there is some $\eta > 0$ such that $\mathrm{index}(A_t)$ is constant on $(s - \eta, s + \eta) = (s - \eta, s] \cup [s, s + \eta)$. Thus we have shown that $\mathrm{index}(A_t)$ is locally constant on $[a, b]$. Since $[a, b]$ is connected, $\mathrm{index}(A_t)$ is constant on the entire interval $[a, b]$. ∎

For our specific situation, we will see that Proposition 2.1 enables us to calculate index with only $\|\cdot\|_{\mathrm{BMO}}$-continuity, rather than the traditional $\|\cdot\|_\infty$-continuity.

We will write C_0 for the collection of continuous functions f on $\mathbb{B}$ which vanish at the boundary, i.e.,

$$\lim_{|z|\uparrow 1} f(z) = 0.$$

Obviously, C_0 is an ideal in $\mathrm{VO}_{\mathrm{bdd}}$.

PROPOSITION 2.2. *Let $k \in \mathbb{N}$. For $f \in M_k(\mathrm{VO}_{\mathrm{bdd}})$, the Toeplitz operator T_f is Fredholm on $L^2_a(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$ if and only if the equivalence class $f + M_k(C_0)$ is an invertible element in the quotient algebra $M_k(\mathrm{VO}_{\mathrm{bdd}})/M_k(C_0)$.*

Proof. It was shown in [1] that the map $g + C_0 \mapsto T_g + \mathcal{K}$ is an isomorphism from $\mathrm{VO}_{\mathrm{bdd}}/C_0$ onto $\mathcal{T}(\mathrm{VO}_{\mathrm{bdd}})/\mathcal{K}$. By elementary C^*-algebra, this induces an isomorphism in the matrix case. That is, the map

$$f + M_k(C_0) \mapsto T_f + \mathcal{K}_k$$

is an isomorphism from $M_k(\mathrm{VO}_{\mathrm{bdd}})/M_k(C_0)$ onto $\mathcal{T}(M_k(\mathrm{VO}_{\mathrm{bdd}}))/\mathcal{K}_k$. Obviously, this implies the Fredholmness criterion stated in the proposition. ∎

Recall that for $g \in L^2(\mathbb{B}, \mathrm{d}v)$, we define

$$\|g\|_{\mathrm{BMO}} = \sup_{z \in \mathbb{B}} \|(g - \langle g k_z, k_z \rangle) k_z\|.$$

For an arbitrary function g on $\mathbb{B}$, let us define

$$\mathrm{diff}(g) = \sup\{|g(z) - g(w)| : \beta(z, w) \leqslant 1\}.$$

LEMMA 2.3. *There is a constant $C_{2.3}$ such that $\|g\|_{\mathrm{BMO}} \leqslant C_{2.3}\mathrm{diff}(\mathrm{g})$ for every $g \in L^2(\mathbb{B}, \mathrm{d}v)$.*

Proof. For $z \in \mathbb{B}$ and $r > 0$, we introduce $D(z, r) = \{w \in \mathbb{B} : \beta(z, w) < r\}$, the Bergman-metric ball. For $g \in L^2(\mathbb{B}, \mathrm{d}v)$, define

$$\|g\|_{\mathrm{BMO},1} = \sup_{z \in \mathbb{B}} \Big(\frac{1}{v(D(z,1))} \int_{D(z,1)} |g(w) - g_z|^2 \mathrm{d}v(w) \Big)^{1/2},$$

where

$$g_z = \frac{1}{v(D(z,1))} \int_{D(z,1)} g(u) \mathrm{d}v(u).$$

By Theorem 18 of [2] there is a constant $0 < C_1 < \infty$ such that

$$\|g\|_{\mathrm{BMO}} \leqslant C_1 \|g\|_{\mathrm{BMO},1}$$

for every $g \in L^2(\mathbb{B}, \mathrm{d}v)$. On the other hand, for every $z \in \mathbb{B}$ we have

$$\frac{1}{v(D(z,1))} \int_{D(z,1)} |g(w) - g_z|^2 \mathrm{d}v(w)$$
$$= \frac{1}{v(D(z,1))} \int_{D(z,1)} \Big| \frac{1}{v(D(z,1))} \int_{D(z,1)} (g(w) - g(u)) \mathrm{d}v(u) \Big|^2 \mathrm{d}v(w)$$

$$\leqslant \iint_{D(z,1)\times D(z,1)} \frac{|g(w)-g(u)|^2}{v^2(D(z,1))} \mathrm{d}v(w)\mathrm{d}v(u) \quad \text{(Cauchy–Schwarz inequality)}$$
$$\leqslant 4 \sup_{w\in D(z,1)} |g(w)-g(z)|^2 \leqslant 4(\mathrm{diff}(g))^2.$$

Hence the constant $C_{2.3} = 2C_1$ will do for the lemma. ∎

LEMMA 2.4. *Suppose that* $0 < \rho < 1$ *and let* $z, w \in \mathbb{B}$.
(i) *If* $\rho < |z| < 1$ *and* $\rho < |w| < 1$, *then* $\beta((\rho/|z|)z, (\rho/|w|)w) \leqslant \beta(z, w)$.
(ii) *If* $|w| \leqslant \rho < |z| < 1$, *then* $\beta(z, (\rho/|z|)z) \leqslant \beta(z, w)$.

Proof. (i) Since the function $x \mapsto \log\{(1+x)/(1-x)\}$ is increasing on $[0,1)$, it suffices to show that

$$|\varphi_{(\rho/|z|)z}((\rho/|w|)w)|^2 \leqslant |\varphi_z(w)|^2. \tag{2.6}$$

To prove (2.6), let us write $z = |z|\xi$ and $w = |w|\eta$, where ξ and η are unit vectors in $\mathbb{C}^n$. Thus $\langle \xi, \eta\rangle = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$ with $a^2 + b^2 \leqslant 1$. By Theorem 2.2.2 of [12], we have

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} = \frac{(1-|z|^2)(1-|w|^2)}{|1-|z||w|\langle \xi,\eta\rangle|^2}.$$

Let $c \in [0,1)$ be such that $c^2 = |z||w|$. Since $|z|^2|w|^2 = c^4$, a minimization on $|z|^2 + |w|^2$ gives us $(1-|z|^2)(1-|w|^2) \leqslant 1 - 2c^2 + c^4 = (1-c^2)^2$. Therefore

$$1 - |\varphi_z(w)|^2 \leqslant \frac{(1-c^2)^2}{|1-c^2\langle \xi,\eta\rangle|^2}.$$

On the other hand, it also follows from Theorem 2.2.2 of [12] that

$$1 - |\varphi_{(\rho/|z|)z}((\rho/|w|)w)|^2 = \frac{(1-\rho^2)^2}{|1-\rho^2\langle \xi,\eta\rangle|^2}.$$

We have $c^2 = |z||w| > \rho^2$ by assumption. Thus (2.6) follows from the assertion that

$$f(x) = \frac{(1-x)^2}{(1-ax)^2 + (bx)^2}$$

is a decreasing function on the interval $[0,1)$. This assertion itself, of course, is trivial if $a \leqslant 0$. In the case $a > 0$, to prove this assertion, it suffices to factor $f(x)$ in the form

$$f(x) = \Big(\frac{1-x}{1-ax}\Big)^2 \cdot \frac{1}{1 + \big(\frac{bx}{1-ax}\big)^2}$$

and observe that both factors are decreasing on $[0,1)$. This proves (2.6).

(ii) By Theorem 2.2.2 of [12], we have

$$|\varphi_w(z)|^2 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \geqslant 1 - \frac{(1-|z|^2)(1-|w|^2)}{(1-|z||w|)^2} = \Big(\frac{|z|-|w|}{1-|z||w|}\Big)^2,$$

$$|\varphi_{(\rho/|z|)z}(z)|^2 = 1 - \frac{(1-|z|^2)(1-\rho^2)}{(1-|z|\rho)^2} = \Big(\frac{|z|-\rho}{1-|z|\rho}\Big)^2.$$

Note that the function $x \mapsto (|z|-x)/(1-|z|x)$ is decreasing on the interval $[0,|z|]$. Since we now assume $|w| \leqslant \rho < |z|$, it follows that $|\varphi_{(\rho/|z|)z}(z)| \leqslant |\varphi_w(z)|$. Hence $\beta(z,(\rho/|z|)z) \leqslant \beta(z,w)$. ∎

Let f be a function on $\mathbb{B}$. For each $0 \leqslant \rho < 1$, we define the function f_ρ by the formula

$$f_\rho(r\xi) = \begin{cases} f(r\xi) & \text{if } 0 \leqslant r \leqslant \rho, \\ f(\rho\xi) & \text{if } \rho < r \leqslant 1, \end{cases} \qquad \text{for } \xi \in \mathbb{C} \text{ with } |\xi| = 1.$$

We emphasize that this defines f_ρ as a function on the closed ball $\overline{\mathbb{B}}$. In particular, if $f \in \mathrm{VO}$, then $f_\rho \in C$. For a matrix-valued function f on $\mathbb{B}$, we define f_ρ by the same formula.

LEMMA 2.5. *For every $f \in \mathrm{VO}$ we have*

$$\lim_{\rho\uparrow 1} \|f_\rho - f\|_{\mathrm{BMO}} = 0.$$

Proof. Let $f \in \mathrm{VO}$ be given. By Lemma 2.3, it suffices to show that

$$\lim_{\rho\uparrow 1} \mathrm{diff}(f_\rho - f) = 0. \tag{2.7}$$

To prove this, let $0 < \rho < 1$. Note that if $0 \leqslant |w| \leqslant \rho$, then $f_\rho(w) - f(w) = 0$. Thus

$$\mathrm{diff}(f_\rho - f) \leqslant \max\{a(\rho) + b(\rho), c(\rho)\},$$

where

$$a(\rho) = \sup\{|f(z) - f(w)| : \beta(z,w) \leqslant 1, \rho < |z| < 1, \rho < |w| < 1\},$$
$$b(\rho) = \sup\{|f((\rho/|z|)z) - f((\rho/|w|)w)| : \beta(z,w) \leqslant 1, \rho < |z| < 1, \rho < |w| < 1\},$$
$$c(\rho) = \sup\{|f((\rho/|z|)z) - f(z)| : \rho < |z| < 1, \beta(z,w) \leqslant 1 \text{ for some } w \text{ with } |w| \leqslant \rho\}.$$

Obviously, $a(\rho) \to 0$ as $\rho \uparrow 1$ by virtue of the fact that $f \in \mathrm{VO}$. By Lemma 2.4(i) and the fact that $f \in \mathrm{VO}$, we have $b(\rho) \to 0$ as $\rho \uparrow 1$. Finally, by Lemma 2.4(ii) and the fact that $f \in \mathrm{VO}$, we also have $c(\rho) \to 0$ as $\rho \uparrow 1$. This proves (2.7) and completes the proof. ∎

Our index calculation involves *Hankel operators* on the Bergman space. Recall that for each $f \in L^\infty(\mathbb{B}, dv)$, the Hankel operator $H_f : L^2_a(\mathbb{B}, dv) \to L^2(\mathbb{B}, dv) \ominus L^2_a(\mathbb{B}, dv)$ is defined by the formula $H_f h = (1-P)(fh)$, $h \in L^2_a(\mathbb{B}, dv)$. It is well known that there is a constant $0 < M < \infty$, which is determined by the complex dimension n, such that

$$\|H_f\| \leqslant M\|f\|_{\mathrm{BMO}}$$

for every $f \in L^\infty(\mathbb{B}, \mathrm{d}v)$ ([2], Theorem A). For a matrix symbol $f \in M_k(L^\infty(\mathbb{B}, \mathrm{d}v))$, we define the Hankel operator H_f by the formula $H_f h = (1 - P_k)(fh)$, $h \in L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$. For our purpose, the important thing is the following relation between Toeplitz operators and Hankel operators:

$$T_{gf} - T_g T_f = H^*_{g^*} H_f$$

for all $g, f \in M_k(L^\infty(\mathbb{B}, \mathrm{d}v))$.

With the above preparation, we can now present our index result, which says that index calculation in $\mathcal{T}(M_k(\mathrm{VMO_{bdd}})) = \mathcal{T}(M_k(\mathrm{VO_{bdd}}))$ can be reduced to the case of $M_k(C)$-symbols in a brutally simple way.

THEOREM 2.6. *Let $f \in M_k(\mathrm{VO_{bdd}})$ for some $k \in \mathbb{N}$. If the Toeplitz operator T_f is Fredholm on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$, then there is a $0 < c < 1$ such that the following two statements hold true:*

(i) *For every $c \leqslant \rho < 1$, the Toeplitz operator T_{f_ρ} is Fredholm.*

(ii) *We have* $\mathrm{index}(T_f) = \mathrm{index}(T_{f_\rho})$ *for every $c \leqslant \rho < 1$.*

Proof. Given $f \in M_k(\mathrm{VO_{bdd}})$, let us define $|f|(z) = (f^*(z)f(z))^{1/2}$, $z \in \mathbb{B}$. Suppose that T_f is Fredholm. Then by Proposition 2.2, the element $f + M_k(C_0)$ is invertible in the quotient algebra $M_k(\mathrm{VO_{bdd}})/M_k(C_0)$. This invertibility implies that there are $0 < c_1 < 1$ and $a > 0$ such that

$$f^*(z)f(z) \geqslant a \quad \text{for every } z \in \mathbb{B} \text{ satisfying the condition } |z| \geqslant c_1. \tag{2.8}$$

Obviously, we have $f^*f \in M_k(\mathrm{VO_{bdd}})$ and $f^*(z)f(z) \leqslant \|f\|^2_\infty$, $z \in \mathbb{B}$. Since $M_k(\mathrm{VO_{bdd}})$ is a C^*-algebra, it follows that $|f| \in M_k(\mathrm{VO_{bdd}})$. We claim that f admits a representation

$$f = u|f| + g$$

satisfying the following conditions:

(1) $u \in M_k(\mathrm{VO_{bdd}})$.

(2) There is a $c_1 < c < 1$ such that if $c \leqslant |z| < 1$, then $u(z)$ is a unitary matrix.

(3) $g \in M_k(C_0)$. In fact, $g(z) = 0$ whenever $c \leqslant |z| < 1$.

Indeed (2.8) implies that $|f|(z) \geqslant a^{1/2}$ if $c_1 \leqslant |z| < 1$. Therefore $\||f|^{-1}(z)\| \leqslant a^{-1/2}$ if $c_1 \leqslant |z| < 1$, where $\|\cdot\|$ denotes the matrix norm. Consequently,

$$\begin{aligned}\||f|^{-1}(z) - |f|^{-1}(w)\| &= \||f|^{-1}(z)(|f|(w) - |f|(z))|f|^{-1}(w)\| \\ &\leqslant a^{-1}\||f|(w) - |f|(z)\|\end{aligned}$$

for all z, w satisfying the conditions $c_1 \leqslant |z| < 1$ and $c_1 \leqslant |w| < 1$. Since $|f| \in M_k(\mathrm{VO_{bdd}})$, by a standard construction using continuous cutoff functions, we obtain a $q \in M_k(\mathrm{VO_{bdd}})$ and a $c_1 < c < 1$ such that $q(z) = |f|^{-1}(z)$ for every $z \in$

$\mathbb{B}$ satisfying the condition $c \leqslant |z| < 1$. Set $h = q|f| - 1$. Then $1 = q|f| - h$ and $h(z) = 0$ if $c \leqslant |z| < 1$. Let $g = -fh$. Then (3) holds and we have $f = fq|f| + g$. This tells us to set $u = fq$, which is in $M_k(\mathrm{VO_{bdd}})$, verifying (1). Now (2) simply follows from the polar decomposition of $f(z)$ and the fact that for $c \leqslant |z| < 1$, $q(z) = (f^*(z)f(z))^{-1/2}$.

By (2.8) and Proposition 2.2, $T_{|f|}$ is a Fredholm operator. Since $T_{|f|}$ is self-adjoint, we have $\mathrm{index}(T_{|f|}) = 0$. Since $g \in M_k(C_0)$, the operator T_g is compact. Hence $\mathrm{index}(T_f) = \mathrm{index}(T_u)$. Similarly, (2.8) implies that if $c \leqslant \rho < 1$, then $T_{|f|_\rho}$ is Fredholm. Also, (3) tells us that $g_\rho \in M_k(C_0)$ whenever $c \leqslant \rho < 1$. Hence for each $c \leqslant \rho < 1$, T_{f_ρ} is Fredholm with $\mathrm{index}(T_{f_\rho}) = \mathrm{index}(T_{u_\rho})$. Thus the proof will be complete if we can show that $\mathrm{index}(T_u) = \mathrm{index}(T_{u_\rho})$ for every $c \leqslant \rho < 1$.

To prove this, we define

$$A_\rho = \begin{cases} T_u & \text{if } \rho = 1, \\ T_{u_\rho} & \text{if } c \leqslant \rho < 1. \end{cases}$$

By Proposition 2.1, we will have $\mathrm{index}(T_u) = \mathrm{index}(T_{u_\rho})$, $c \leqslant \rho < 1$, if we can show that the maps

$$\rho \mapsto A_\rho^* A_\rho \quad \text{and} \quad \rho \mapsto A_\rho A_\rho^* \tag{2.9}$$

are continuous with respect to the operator norm on the closed interval $[c, 1]$. Since u is continuous on $\mathbb{B}$, the map $\rho \mapsto u_\rho$ is continuous on $[c, 1)$ with respect to the supremum norm on $\mathbb{B}$. This shows that the maps in (2.9) are continuous with respect to the operator norm on the half-open interval $[c, 1)$. Thus what remains is to show that these maps are also continuous with respect to the operator norm at the point $\rho = 1$.

To prove the continuity at $\rho = 1$, we use (2), which gives us

$$u^*(z)u(z) = 1 \quad \text{whenever } c \leqslant |z| < 1. \tag{2.10}$$

Thus it follows that

$$u_\rho^*(z)u_\rho(z) = 1 \quad \text{if } c \leqslant \rho < 1 \text{ and } c \leqslant |z| < 1. \tag{2.11}$$

On the other hand, by definition we have

$$u_\rho(z) = u(z) \quad \text{if } c \leqslant \rho < 1 \text{ and } |z| < c. \tag{2.12}$$

The combination of (2.10), (2.11) and (2.12) gives us the identity

$$u_\rho^* u_\rho = u^* u$$

on $\mathbb{B}$ for every $c \leqslant \rho < 1$. Thus, using the relation between Toeplitz operators and Hankel operators, for every $c \leqslant \rho < 1$ we have

$$\begin{aligned} A_\rho^* A_\rho - A_1^* A_1 &= T_{u_\rho^*} T_{u_\rho} - T_{u^*} T_u = \{T_{u^* u} - T_{u^*} T_u\} - \{T_{u_\rho^* u_\rho} - T_{u_\rho^*} T_{u_\rho}\} \\ &= H_u^* H_u - H_{u_\rho}^* H_{u_\rho}. \end{aligned} \tag{2.13}$$

On the other hand, there is a constant C_1 such that $\|H_u - H_{u_\rho}\| = \|H_{u-u_\rho}\| \leqslant C_1\|u - u_\rho\|_{\mathrm{BMO}}$. Thus, applying Lemma 2.5, we have

$$\lim_{\rho\uparrow 1} \|H_u - H_{u_\rho}\| = 0.$$

Combining this with (2.13), we find that

$$\lim_{\rho\uparrow 1} \|A_\rho^* A_\rho - A_1^* A_1\| = 0.$$

Thus we have shown that the map $\rho \mapsto A_\rho^* A_\rho$ is continuous with respect to the operator norm at the point $\rho = 1$. By a similar argument, the map $\rho \mapsto A_\rho A_\rho^*$ is also continuous with respect to the operator norm at the point $\rho = 1$. This completes the proof. ∎

DEFINITION 2.7. Let E be a subset of $L^2(\mathbb{B}, \mathrm{d}v)$ that is closed under complex conjugation. For each $k \in \mathbb{N}$, $\mathrm{SA}_k(E)$ denotes the collection of $h \in M_k(E)$ satisfying the condition $h^*(z) = h(z)$ for every $z \in \mathbb{B}$.

Let $h \in \mathrm{SA}_k(\mathrm{VMO})$ and $z \in \mathbb{B}$. For every self-adjoint $k \times k$ matrix B, the identity

$$\mathrm{e}^{\mathrm{i}h(z)} - \mathrm{e}^{\mathrm{i}B} = \mathrm{i}\int_0^1 \mathrm{e}^{\mathrm{i}th(z)}(h(z) - B)\mathrm{e}^{\mathrm{i}(1-t)B}\mathrm{d}t$$

yields the estimate

$$\|\mathrm{e}^{\mathrm{i}h(z)} - \mathrm{e}^{\mathrm{i}B}\| \leqslant \|h(z) - B\|. \tag{2.14}$$

From this we conclude that if $h \in \mathrm{SA}_k(\mathrm{VMO})$, then $\mathrm{e}^{\mathrm{i}h} \in M_k(\mathrm{VMO_{bdd}})$.

Recall from [1] that if $f, g \in \mathrm{VMO_{bdd}}$, then $T_f T_g - T_{fg} \in \mathcal{K}$. For each $h \in \mathrm{SA}_k(\mathrm{VMO})$, since $\mathrm{e}^{\mathrm{i}h} \in M_k(\mathrm{VMO_{bdd}})$, there are compact operators K_1 and K_2 such that

$$T_{\mathrm{e}^{-\mathrm{i}h}} T_{\mathrm{e}^{\mathrm{i}h}} = 1 + K_1 \quad \text{and} \quad T_{\mathrm{e}^{\mathrm{i}h}} T_{\mathrm{e}^{-\mathrm{i}h}} = 1 + K_2.$$

Hence if $h \in \mathrm{SA}_k(\mathrm{VMO})$, then the Toeplitz operator $T_{\mathrm{e}^{\mathrm{i}h}}$ on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$ is Fredholm.

THEOREM 2.8. *If $h \in \mathrm{SA}_k(\mathrm{VMO})$, then the index of the Toeplitz operator $T_{\mathrm{e}^{\mathrm{i}h}}$ equals* 0*.*

Proof. Let $h \in \mathrm{SA}_k(\mathrm{VMO})$ be given. For each $t \in [0, 1]$, define $A_t = T_{\mathrm{e}^{\mathrm{i}th}}$. Obviously, we have $\mathrm{index}(A_0) = 0$. Thus it suffices to show that $\mathrm{index}(A_t)$ is a constant on $[0, 1]$. By Proposition 2.1, we only need to show that the maps

$$t \mapsto A_t^* A_t \quad \text{and} \quad t \mapsto A_t A_t^*$$

are continuous on $[0, 1]$ with respect to the operator norm.

To prove this, we first note that, by (2.14), there is a constant C_1 such that $\|\mathrm{e}^{\mathrm{i}xh}\|_{\mathrm{BMO}} \leqslant C_1\|xh\|_{\mathrm{BMO}} = C_1|x|\|h\|_{\mathrm{BMO}}$ for every $x \in [-1, 1]$. Consequently there is a C_2 such that

$$\|H_{e^{ixh}}\| \leqslant C_2|x|\|h\|_{\mathrm{BMO}} \quad \text{for every } x \in [-1,1]. \tag{2.15}$$

Let $s, t \in [0,1]$ be given and write $x = t - s$. Then note that

$$T_{e^{ith}} = T_{e^{ixh}e^{ish}} = T_{e^{ixh}}T_{e^{ish}} + H^*_{(e^{ixh})^*}H_{e^{ish}}.$$

Therefore

$$\begin{aligned}
A_t^*A_t - A_s^*A_s &= T_{(e^{ith})^*}T_{e^{ith}} - T_{(e^{ish})^*}T_{e^{ish}} \\
&= \{T_{(e^{ish})^*}T_{(e^{ixh})^*} + H^*_{e^{ish}}H_{(e^{ixh})^*}\}\{T_{e^{ixh}}T_{e^{ish}} + H^*_{(e^{ixh})^*}H_{e^{ish}}\} - T_{(e^{ish})^*}T_{e^{ish}} \\
&= T_{(e^{ish})^*}(T_{(e^{ixh})^*}T_{e^{ixh}} - 1)T_{e^{ish}} + T_{(e^{ish})^*}T_{(e^{ixh})^*}H^*_{(e^{ixh})^*}H_{e^{ish}} \\
&\quad + H^*_{e^{ish}}H_{(e^{ixh})^*}T_{e^{ixh}}T_{e^{ish}} + H^*_{e^{ish}}H_{(e^{ixh})^*}H^*_{(e^{ixh})^*}H_{e^{ish}} \\
&= -T_{(e^{ish})^*}H^*_{e^{ixh}}H_{e^{ixh}}T_{e^{ish}} + T_{(e^{ish})^*}T_{(e^{ixh})^*}H^*_{(e^{ixh})^*}H_{e^{ish}} \\
&\quad + H^*_{e^{ish}}H_{(e^{ixh})^*}T_{e^{ixh}}T_{e^{ish}} + H^*_{e^{ish}}H_{(e^{ixh})^*}H^*_{(e^{ixh})^*}H_{e^{ish}}.
\end{aligned}$$

Combining this with (2.15), we see that $\|A_t^*A_t - A_s^*A_s\| \leqslant 4C_2|t-s|\|h\|_{\mathrm{BMO}}$. Hence the map $t \mapsto A_t^*A_t$ is continuous with respect to the operator norm. A similar argument shows that the map $t \mapsto A_tA_t^*$ is also continuous with respect to the operator norm. This completes the proof. ∎

3. SCALAR SYMBOLS

In this section we only consider Toeplitz operators with scalar symbols. Equivalently, this means $k = 1$. We will give a more precise representation for $f \in \mathrm{VO_{bdd}}$ for which the Toeplitz operator T_f is Fredholm on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$. But this involves the complex dimension of the underlying space $\mathbb{C}^n$. That is, there is a marked difference between the cases $n \geqslant 2$ and $n = 1$. We begin with the case $n \geqslant 2$, the simpler of the two.

PROPOSITION 3.1. *Suppose that $n \geqslant 2$. Let $f \in \mathrm{VO_{bdd}}$. If the Toeplitz operator T_f is Fredholm on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$, then there exist a real-valued function h in* VO *and a $g \in C_0$ such that $f = e^{ih}|f| + g$.*

Proof. As we showed at the beginning of the proof of Theorem 2.6, for $f \in \mathrm{VO_{bdd}}$, if T_f is Fredholm, then f admits a representation

$$f = u|f| + g_1, \tag{3.1}$$

where $g_1 \in C_0$ and u satisfies the following two conditions:

(1) $u \in \mathrm{VO_{bdd}}$.
(2) There is a $0 < c < 1$ such that if $c \leqslant |z| < 1$, then $|u(z)| = 1$.

Obviously, the proposition will follow if we can show that there exist a $g_2 \in C_0$ and a real-valued function h in VO such that

$$u = \mathrm{e}^{\mathrm{i}h} + g_2. \tag{3.2}$$

To prove this, let H denote the hollowed ball $\{z \in \mathbb{C}^n : c < |z| < 1\}$, where c is the constant in (2). Also, let $\mathbb{T}$ denote the unit circle $\{\tau \in \mathbb{C} : |\tau| = 1\}$. We can view u as a map from H to $\mathbb{T}$. Since $n \geqslant 2$, H is simply connected. Thus the continuous map $u : H \to \mathbb{T}$ lifts to a continuous map from H to $\mathbb{R}$, the universal covering of $\mathbb{T}$ ([10], Lemma 79.1). That is, there is a continuous $\zeta : H \to \mathbb{R}$ such that $u(z) = \mathrm{e}^{\mathrm{i}\zeta(z)}$ whenever $c < |z| < 1$.

Next we show that ζ has vanishing oscillation on H in the sense that

$$\lim_{|z|\uparrow 1} \sup_{\beta(z,w)\leqslant 1} |\zeta(z) - \zeta(w)| = 0. \tag{3.3}$$

If this did not hold, then there would be a $q > 0$ and sequences $\{z_j\}$ and $\{w_j\}$ with $|z_j| \uparrow 1$ as $j \to \infty$ such that $\beta(z_j, w_j) \leqslant 1$, $\{w \in \mathbb{B} : \beta(z_j, w) \leqslant 1\} \subset H$, and

$$|\zeta(z_j) - \zeta(w_j)| \geqslant q \quad \text{for every } j.$$

Pick a positive number $0 < p < 1$ such that $p \leqslant q$. We claim that for every j, there is a $w'_j \in H$ satisfying the conditions

$$\beta(z_j, w'_j) \leqslant \beta(z_j, w_j) \quad \text{and} \quad |\zeta(z_j) - \zeta(w'_j)| = p. \tag{3.4}$$

To find such a w'_j, recall that $\varphi_{z_j}(0) = z_j$ and $\varphi_{z_j}(\varphi_{z_j}(w_j)) = w_j$ ([12], Theorem 2.2.2). By the Möbius invariance of the Bergman metric, for every $r \in [0,1]$ we have

$$\begin{aligned}\beta(z_j, \varphi_{z_j}(r\varphi_{z_j}(w_j))) &= \beta(\varphi_{z_j}(0), \varphi_{z_j}(r\varphi_{z_j}(w_j))) = \beta(0, r\varphi_{z_j}(w_j)) \\ &= \frac{1}{2}\log\frac{1 + |r\varphi_{z_j}(w_j)|}{1 - |r\varphi_{z_j}(w_j)|} \leqslant \frac{1}{2}\log\frac{1 + |\varphi_{z_j}(w_j)|}{1 - |\varphi_{z_j}(w_j)|} = \beta(z_j, w_j).\end{aligned}$$

In particular, $\{\varphi_{z_j}(r\varphi_{z_j}(w_j)) : r \in [0,1]\} \subset H$. Define

$$f(r) = |\zeta(z_j) - \zeta(\varphi_{z_j}(r\varphi_{z_j}(w_j)))|, \quad r \in [0,1].$$

We have $f(0) = |\zeta(z_j) - \zeta(\varphi_{z_j}(0))| = |\zeta(z_j) - \zeta(z_j)| = 0$ and $f(1) = |\zeta(z_j) - \zeta(\varphi_{z_j}(\varphi_{z_j}(w_j)))| = |\zeta(z_j) - \zeta(w_j)| = q \geqslant p$. Therefore there is an $s \in [0,1]$ such that $f(s) = p$, i.e., $|\zeta(z_j) - \zeta(\varphi_{z_j}(s\varphi_{z_j}(w_j)))| = p$. Thus if we set $w'_j = \varphi_{z_j}(s\varphi_{z_j}(w_j))$, then $|\zeta(z_j) - \zeta(w'_j)| = p$. This proves (3.4). Now, by (3.4), we have

$$|u(z_j) - u(w'_j)| = |\mathrm{e}^{\mathrm{i}\zeta(z_j)} - \mathrm{e}^{\mathrm{i}\zeta(w'_j)}| = |\mathrm{e}^{\mathrm{i}p} - 1| \quad \text{for every } j.$$

Since $\beta(z_j, w'_j) \leqslant 1$, $j \geqslant 1$, and $\lim_{j\to\infty} |z_j| = 1$, this contradicts the fact that $u \in$ VO. Hence (3.3) holds.

Once (3.3) is proven, it is a standard exercise using an obvious cutoff function to produce a $c < d < 1$ and a real-valued $h \in$ VO such that $h(z) = \zeta(z)$

whenever $d \leqslant |z| < 1$. Therefore $u(z) = \mathrm{e}^{\mathrm{i}h(z)}$ whenever $d \leqslant |z| < 1$. That is, if we set $g_2 = u - \mathrm{e}^{\mathrm{i}h}$, then $g_2 \in C_0$. This proves (3.2) and completes the proof of the proposition. ∎

Next we consider the case where $n = 1$. We will write D for the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Write $\mathrm{d}A$ for the area measure on D with the normalization $A(D) = 1$. Fix a continuous function $0 \leqslant \alpha \leqslant 1$ on $[0,1]$ satisfying the conditions that $\alpha = 1$ on $[2/3, 1]$ and that $\alpha = 0$ on $[0, 1/3]$. For each $m \in \mathbb{Z}$, we define the function χ_m on D by the formula

$$\chi_m(r\tau) = \alpha(r)\tau^m \quad \text{for all } 0 \leqslant r < 1 \text{ and } \tau \in \mathbb{T}. \tag{3.5}$$

PROPOSITION 3.2. *Suppose that $n = 1$. Let $f \in \mathrm{VO}_{\mathrm{bdd}}$. If the Toeplitz operator T_f is Fredholm on $L^2_{\mathrm{a}}(D, \mathrm{d}A)$, then there exist an $m \in \mathbb{Z}$, a real-valued function h in* VO *and a $g \in C_0$ such that $f = \chi_m \mathrm{e}^{\mathrm{i}h}|f| + g$.*

Proof. As in the proof of Proposition 3.1, for $f \in \mathrm{VO}_{\mathrm{bdd}}$, if T_f is Fredholm, then f admits a representation (3.1), where $g_1 \in C_0$ and u satisfies conditions (1) and (2) listed there. We may, of course, assume that the constant c in (2) satisfies the condition $2/3 < c < 1$.

Again, the proposition will follow if we can show that there exist an $m \in \mathbb{Z}$, a real-valued function h in VO and a $g_2 \in C_0$ such that

$$u = \chi_m \mathrm{e}^{\mathrm{i}h} + g_2. \tag{3.6}$$

To prove this, for each $r \in (c, 1)$, consider the map $u^{(r)} : \mathbb{T} \to \mathbb{T}$ defined by the formula $u^{(r)}(\tau) = u(r\tau)$, $\tau \in \mathbb{T}$. Obviously, the winding number of $u^{(r)}$ is independent of $r \in (c, 1)$. Let m denote this common winding number.

Now consider the subset $B = [0, 2\pi] \times (c, 1)$ of $\mathbb{R}^2$, and define the continuous map $U : B \to \mathbb{T}$ by the formula

$$U(\theta, r) = u(r\mathrm{e}^{\mathrm{i}\theta}), \quad (\theta, r) \in B.$$

Since B is simply connected, U lifts to a continuous map Z from B to $\mathbb{R}$, the universal covering of $\mathbb{T}$. That is, there is a continuous $Z : B \to \mathbb{R}$ such that $U(\theta, r) = \mathrm{e}^{\mathrm{i}Z(\theta,r)}$ for every $(\theta, r) \in B$. Since the winding number of each $u^{(r)}$ equals m, we have

$$Z(2\pi, r) - Z(0, r) = 2\pi m \quad \text{for every } r \in (c, 1).$$

Hence if we define

$$\zeta(\theta, r) = Z(\theta, r) - m\theta, \quad (\theta, r) \in B,$$

then ζ is a continuous function on B satisfying the condition $\zeta(2\pi, r) = \zeta(0, r)$ for every $r \in (c, 1)$. Thus there is a real-valued continuous function ψ on $R = \{z \in \mathbb{C} : c < |z| < 1\}$ such that $\zeta(\theta, r) = \psi(r\mathrm{e}^{\mathrm{i}\theta})$ for every $(\theta, r) \in B$. A retracing of the definitions yields

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{\mathrm{i}\psi(r\mathrm{e}^{\mathrm{i}\theta})} \quad \text{for every } (\theta, r) \in B.$$

Equivalently, we have $u(r\tau) = \tau^m e^{i\psi(r\tau)}$ for all $\tau \in \mathbb{T}$ and $r \in (c, 1)$.

By an argument similar to the proof of (3.3), this ψ also has vanishing oscillation on R. Again, a standard exercise produces a $c < d < 1$ and a real-valued $h \in \mathrm{VO}$ such that $h(z) = \psi(z)$ whenever $d \leqslant |z| < 1$. Since $2/3 < c < d < 1$ and since $\alpha = 1$ on $[2/3, 1]$, we have $u(z) = \chi_m(z)e^{ih(z)}$ whenever $d \leqslant |z| < 1$. That is, if we set $g_2 = u - \chi_m e^{ih}$, then $g_2 \in C_0$. This proves (3.6) and completes the proof of the proposition. ∎

LEMMA 3.3. *Let η be a real-valued function in* VO. *If $1 - e^{i\eta} \in C_0$, then $\eta \in \mathrm{VO}_{\mathrm{bdd}}$.*

Proof. Note that $1 - e^{i\eta} \in C_0$ if and only if $1 - e^{-i\eta} \in C_0$. Suppose that $\eta \notin \mathrm{VO}_{\mathrm{bdd}}$. Then, replacing η by $-\eta$ if necessary, we may assume that there is a sequence $\{z_j\}_{j=J}^{\infty}$ in $\mathbb{B}$ such that $\eta(z_j) = (2j+1)\pi$ for every $j \geqslant J$. This, of course, implies that $|z_j| \uparrow 1$ as $j \to \infty$. But then $1 - e^{i\eta(z_j)} = 2$ for every $j \geqslant J$. Since $|z_j| \uparrow 1$ as $j \to \infty$, this is not reconcilable with the condition $1 - e^{i\eta} \in C_0$. ∎

Let $\mathrm{inv}(\mathrm{VO}_{\mathrm{bdd}}/C_0)$ be the collection of invertible elements in $\mathrm{VO}_{\mathrm{bdd}}/C_0$. Also, write $\mathrm{inv}_0(\mathrm{VO}_{\mathrm{bdd}}/C_0)$ for the connected component of $\mathrm{inv}(\mathrm{VO}_{\mathrm{bdd}}/C_0)$ that contains the identity element.

PROPOSITION 3.4. *Let h be a real-valued function in* VO. *If we have $e^{ih} + C_0 \in \mathrm{inv}_0(\mathrm{VO}_{\mathrm{bdd}}/C_0)$, then $h \in \mathrm{VO}_{\mathrm{bdd}}$.*

Proof. Since $\mathrm{VO}_{\mathrm{bdd}}/C_0$ is a commutative C^*-algebra, $\mathrm{inv}_0(\mathrm{VO}_{\mathrm{bdd}}/C_0)$ consists of the exponentials in $\mathrm{VO}_{\mathrm{bdd}}/C_0$. Thus if $e^{ih} + C_0 \in \mathrm{inv}_0(\mathrm{VO}_{\mathrm{bdd}}/C_0)$, then there is an $a \in \mathrm{VO}_{\mathrm{bdd}}/C_0$ such that $e^{ih} + C_0 = e^a$. If $a = f + C_0$ with $f \in \mathrm{VO}_{\mathrm{bdd}}$, then $e^a = e^f + C_0$. Hence there is a $g \in C_0$ such that $e^{ih} = e^f + g$. Write $f = f_1 + if_2$, where f_1 and f_2 are real-valued functions in $\mathrm{VO}_{\mathrm{bdd}}$. We have

$$e^{2f_1} = e^f \cdot \bar{e}^f = |e^f|^2 = |e^{ih} - g|^2 = 1 + g_1,$$

where $g_1 \in C_0$. Hence $f_1 \in C_0$, and consequently $e^{f_1} = 1 + g_2$ for some $g_2 \in C_0$. Thus

$$e^{ih} = e^{if_2} + g_3 \quad \text{for some } g_3 \in C_0.$$

Write $\eta = f_2 - h$. Then the above implies that $1 - e^{i\eta} \in C_0$. By Lemma 3.3, this means that $\eta = f_2 - h$ belongs to $\mathrm{VO}_{\mathrm{bdd}}$. Since f_2 is bounded, so is $h = f_2 - \eta$. ∎

PROPOSITION 3.5. *Let $f \in \mathrm{VO}_{\mathrm{bdd}}$ be such that the Toeplitz operator T_f is Fredholm on $L^2_a(\mathbb{B}, dv)$. If h is an unbounded, real-valued function in* VO, *then $f + C_0$ and $e^{ih}f + C_0$ do not belong to the same connected component of* $\mathrm{inv}(\mathrm{VO}_{\mathrm{bdd}}/C_0)$.

Proof. If $f + C_0$ and $e^{ih}f + C_0$ were contained in a single component of $\mathrm{inv}(\mathrm{VO}_{\mathrm{bdd}}/C_0)$, then so would $1 + C_0$ and $e^{ih} + C_0$. That is, we would have $e^{ih} + C_0 \in \mathrm{inv}_0(\mathrm{VO}_{\mathrm{bdd}}/C_0)$. By Proposition 3.4, this would imply $h \in \mathrm{VO}_{\mathrm{bdd}}$, which contradicts the assumption that h is unbounded. ∎

4. UNCOUNTABLY MANY FREDHOLM COMPONENTS

Considering the general case of matrix symbols, we will now show that for each $k \in \mathbb{N}$, the intersection

$$\operatorname{Fred}(L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k) \cap \mathcal{T}(M_k(\mathrm{VO_{bdd}})) \tag{4.1}$$

has uncountably many connected components, and the Fredholm index hardly tells us anything about these components.

THEOREM 4.1. *Let $f \in M_k(\mathrm{VO_{bdd}})$ for some $k \in \mathbb{N}$, and suppose that the Toeplitz operator T_f on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \otimes \mathbb{C}^k$ is Fredholm. Furthermore, suppose that h is an unbounded, real-valued function in* VO. *Then for every pair of $s \neq t$ in $\mathbb{R}$, the operators $T_{\mathrm{e}^{\mathrm{i}sh}f}$ and $T_{\mathrm{e}^{\mathrm{i}th}f}$ belong to distinct connected components of* (4.1). *On the other hand, we have* $\operatorname{index}(T_{\mathrm{e}^{\mathrm{i}th}f}) = \operatorname{index}(T_f)$ *for every $t \in \mathbb{R}$.*

Proof. If $\mathcal{A}$ is a unital C^*-algebra, then the connected component of $\operatorname{inv}(\mathcal{A})$ that contains 1 consists of elements of the form $\mathrm{e}^{a_1} \cdots \mathrm{e}^{a_\ell}$, $a_1, \ldots, a_\ell \in \mathcal{A}$ and $\ell \in \mathbb{N}$. From this it is easy to deduce that the component of (4.1) that contains 1 consists of elements of the form

$$T_{\mathrm{e}^{\varphi_1} \cdots \mathrm{e}^{\varphi_\ell}} + K,$$

where $\ell \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_\ell \in M_k(\mathrm{VO_{bdd}})$, and K is compact.

For $s \neq t$ in $\mathbb{R}$, if $T_{\mathrm{e}^{\mathrm{i}sh}f}$ and $T_{\mathrm{e}^{\mathrm{i}th}f}$ were in the same component of (4.1), then

$$T_{\mathrm{e}^{\mathrm{i}sh}f} = AT_{\mathrm{e}^{\mathrm{i}th}f} + K_1, \tag{4.2}$$

where A is in the component of (4.1) that contains 1 and K_1 is compact. We will show that this leads to a contradiction. Indeed by the preceding paragraph, (4.2) implies that there are $\varphi_1, \ldots, \varphi_\ell \in M_k(\mathrm{VO_{bdd}})$ such that $T_{\mathrm{e}^{\mathrm{i}sh}f} - T_{\mathrm{e}^{\varphi_1} \cdots \mathrm{e}^{\varphi_\ell} \mathrm{e}^{\mathrm{i}th}f}$ is compact. That is,

$$\mathrm{e}^{\mathrm{i}sh}f - \mathrm{e}^{\varphi_1} \cdots \mathrm{e}^{\varphi_\ell} \mathrm{e}^{\mathrm{i}th}f \in M_k(C_0).$$

The Fredholmness of T_f implies that $f + M_k(C_0)$ is invertible in the quotient $M_k(\mathrm{VO_{bdd}})/M_k(C_0)$. Hence the above implies that there is a $G \in M_k(C_0)$ such that

$$\mathrm{e}^{\varphi_1} \cdots \mathrm{e}^{\varphi_\ell} \mathrm{e}^{\mathrm{i}(t-s)h} = 1 + G.$$

Taking determinant on both sides, we find that

$$\mathrm{e}^{\varphi} \mathrm{e}^{\mathrm{i}k(t-s)h} = 1 + g$$

where $\varphi = \operatorname{tr}(\varphi_1) + \cdots + \operatorname{tr}(\varphi_\ell)$ and $g \in C_0$. Thus $\mathrm{e}^{\varphi}\mathrm{e}^{\mathrm{i}k(t-s)h} + C_0$ belongs to $\operatorname{inv}_0(\mathrm{VO_{bdd}}/C_0)$. Since $\varphi \in \mathrm{VO_{bdd}}$, $\mathrm{e}^{\varphi}\mathrm{e}^{\mathrm{i}k(t-s)h} + C_0$ and $\mathrm{e}^{\mathrm{i}k(t-s)h} + C_0$ belong to the same component of $\operatorname{inv}(\mathrm{VO_{bdd}}/C_0)$. Hence $\mathrm{e}^{\mathrm{i}k(t-s)h} + C_0 \in \operatorname{inv}_0(\mathrm{VO_{bdd}}/C_0)$. By Proposition 3.4, this forces $k(t-s)h$ to be bounded. Since $t \neq s$ and h is assumed to be unbounded, this is a contradiction. Hence $T_{\mathrm{e}^{\mathrm{i}sh}f}$ and $T_{\mathrm{e}^{\mathrm{i}th}f}$ are not in the same component of (4.1).

To show that $\operatorname{index}(T_{e^{ith}f}) = \operatorname{index}(T_f)$, $t \in \mathbb{R}$, it suffices to note that

$$\operatorname{index}(T_{e^{ith}f}) = \operatorname{index}(T_{e^{ith}I_k}) + \operatorname{index}(T_f),$$

where I_k denotes the $k \times k$ identity matrix. By Theorem 2.8, we have $\operatorname{index}(T_{e^{ith}I_k}) = \operatorname{index}(T_{e^{ith}I_k}) = 0$. Hence $\operatorname{index}(T_{e^{ith}f}) = \operatorname{index}(T_f)$. ∎

But even with Theorem 4.1 established, we still need to do one more thing before we can claim that (4.1) truly has uncountably many connected components. Namely, we need to produce at least one unbounded, real-valued function in VO.

EXAMPLE 4.2. We will now construct an unbounded, real-valued function h in VO. To define the desired function, we first set the value $\psi(1-2^{-\nu+1}) = \sum_{j=1}^{\nu} j^{-1}$ for every $\nu \in \mathbb{N}$. We then define ψ to be the increasing, continuous function on $[0,1)$ that is linear on each $[1-2^{-\nu+1}, 1-2^{-\nu}]$, $\nu \in \mathbb{N}$. With ψ so defined, we define $h(z) = \psi(|z|)$, $z \in \mathbb{B}$. Thus h is a radial function. Obviously, h is both continuous and unbounded on $\mathbb{B}$. What remains is to show that h has vanishing oscillation.

Let $z, w \in \mathbb{B}$. By Theorem 2.2.2 of [12], we have

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w, z\rangle|^2} \leqslant \frac{(1-|z|^2)(1-|w|^2)}{(1-|w||z|)^2}.$$

Elementary algebra then leads to

$$|\varphi_z(w)|^2 \geqslant 1 - \frac{(1-|z|^2)(1-|w|^2)}{(1-|w||z|)^2} = \Big(\frac{|w|-|z|}{1-|w||z|}\Big)^2.$$

Note that the function $x \mapsto \log\{(1+x)/(1-x)\}$ is increasing on the interval $[0,1)$. Therefore for $z, w \in \mathbb{B}$ satisfying the condition $|z| \leqslant |w|$, we have

$$\beta(w,z) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} \geqslant \frac{1}{2}\log\frac{(1+|w|)(1-|z|)}{(1-|w|)(1+|z|)}. \tag{4.3}$$

Hence if we have both $|z| \leqslant |w|$ and $\beta(z,w) \leqslant 1$, then $(1-|z|)/(1-|w|) \leqslant e^2 < 2^4$. For a pair of z, w satisfying these two conditions, there are $\nu \leqslant \nu'$ in $\mathbb{N}$ such that $|z| \in [1-2^{-\nu+1}, 1-2^{-\nu})$ and $|w| \in [1-2^{-\nu'+1}, 1-2^{-\nu'})$. From this we deduce

$$\frac{2^{-\nu}}{2^{-\nu'+1}} \leqslant \frac{1-|z|}{1-|w|} < 2^4.$$

That is, $\nu' < \nu + 5$. Using this inequality and definition of ψ, we have

$$\begin{aligned}|h(z) - h(w)| &= \psi(|w|) - \psi(|z|) \leqslant \psi(1-2^{-\nu'}) - \psi(1-2^{-\nu+1}) \\ &= \sum_{j=1}^{\nu'+1} j^{-1} - \sum_{j=1}^{\nu} j^{-1} = \sum_{j=\nu+1}^{\nu'+1} j^{-1} \leqslant \frac{5}{\nu+1} \leqslant \frac{5}{\log\frac{1}{1-|z|}}.\end{aligned}$$

Thus if we drop the condition $|z| \leqslant |w|$ but retain the requirement $\beta(z,w) \leqslant 1$, then

$$|h(z) - h(w)| \leqslant \frac{5}{\min\left\{ \log \frac{1}{1-|z|}, \log \frac{1}{1-|w|} \right\}}.$$

From this inequality one sees that $h \in \mathrm{VO}$.

In the special case of scalar symbols, we have a more precise description of Fredholm components. But for this more precise description, we again need to separate the cases of complex dimension $n \geqslant 2$ and complex dimension $n = 1$. Write $\mathrm{VO}^{(\mathrm{r})}$ (respectively, $\mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}$) for the collection of real-valued functions in VO (respectively, in $\mathrm{VO}_{\mathrm{bdd}}$).

THEOREM 4.3. *Suppose that $n \geqslant 2$. Let $f_1, f_2 \in \mathrm{VO}_{\mathrm{bdd}}$ be such that the Toeplitz operators T_{f_1} and T_{f_2} are Fredholm. Let $f_1 = \mathrm{e}^{\mathrm{i}h_1}|f_1| + g_1$ and $f_2 = \mathrm{e}^{\mathrm{i}h_2}|f_2| + g_2$ be the representations provided by Proposition* 3.1*, i.e., $g_1, g_2 \in C_0$ and $h_1, h_2 \in \mathrm{VO}^{(\mathrm{r})}$. Then T_{f_1} and T_{f_2} belong to the same connected component of*

$$\mathrm{Fred}(L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)) \cap \mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}) = \mathrm{Fred}(L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)) \cap \mathcal{T}(\mathrm{VO}_{\mathrm{bdd}}) \tag{4.4}$$

if and only if $h_2 - h_1 \in \mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}$. Consequently, there is a natural one-to-one correspondence between the connected components in (4.4) *and the elements in the quotient linear space $\mathrm{VO}^{(\mathrm{r})}/\mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}$.*

Proof. For a non-negative function φ in $\mathrm{VO}_{\mathrm{bdd}}$, if T_φ is Fredholm, then T_φ belongs to the component of (4.4) that contains the identity operator. Hence for each $\nu \in \{1,2\}$, T_{f_ν} and $T_{\mathrm{e}^{\mathrm{i}h_\nu}}$ belong to the same component of (4.4). Moreover, T_{f_1} and T_{f_2} belong to one single component of (4.4) if and only $T_{\mathrm{e}^{\mathrm{i}h_1}}$ and $T_{\mathrm{e}^{\mathrm{i}h_2}}$ belong to one single component of (4.4). If $h_2 - h_1 \in \mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}$, then obviously $T_{\mathrm{e}^{\mathrm{i}h_1}}$ and $T_{\mathrm{e}^{\mathrm{i}h_2}}$ belong to the same component of (4.4). On the other hand, we can write $f = \mathrm{e}^{\mathrm{i}h_1}$ and $h = h_2 - h_1$, consequently $\mathrm{e}^{\mathrm{i}h_2} = \mathrm{e}^{\mathrm{i}h} f$. Thus, by Proposition 3.5, if $T_{\mathrm{e}^{\mathrm{i}h_1}}$ and $T_{\mathrm{e}^{\mathrm{i}h_2}}$ belong to the same component of (4.4), then $h = h_2 - h_1$ is bounded. This completes the proof. ∎

Recall that for $m \in \mathbb{Z}$, the function χ_m was defined by (3.5). It is well known that the index of the Toeplitz operator T_{χ_m} on $L^2_{\mathrm{a}}(D, \mathrm{d}A)$ equals $-m$ ([3], [14]).

THEOREM 4.4. *Suppose that $n = 1$. Let $f_1, f_2 \in \mathrm{VO}_{\mathrm{bdd}}$ be such that the Toeplitz operators T_{f_1} and T_{f_2} are Fredholm. Let $f_1 = \chi_{m_1}\mathrm{e}^{\mathrm{i}h_1}|f_1| + g_1$ and $f_2 = \chi_{m_2}\mathrm{e}^{\mathrm{i}h_2}|f_2| + g_2$ be the representations provided by Proposition* 3.2*, i.e., $m_1, m_2 \in \mathbb{Z}$, $g_1, g_2 \in C_0$ and $h_1, h_2 \in \mathrm{VO}^{(\mathrm{r})}$. Then T_{f_1} and T_{f_2} belong to the same connected component of*

$$\mathrm{Fred}(L^2_{\mathrm{a}}(D, \mathrm{d}A)) \cap \mathcal{T}(\mathrm{VMO}_{\mathrm{bdd}}) = \mathrm{Fred}(L^2_{\mathrm{a}}(D, \mathrm{d}A)) \cap \mathcal{T}(\mathrm{VO}_{\mathrm{bdd}}) \tag{4.5}$$

if and only if both conditions $m_1 = m_2$ and $h_2 - h_1 \in \mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}$ are satisfied. Consequently, there is a natural one-to-one correspondence between the connected components in (4.5) *and the elements in the product set $\mathbb{Z} \times \{\mathrm{VO}^{(\mathrm{r})}/\mathrm{VO}^{(\mathrm{r})}_{\mathrm{bdd}}\}$.*

Using the fact that $\mathrm{index}(T_{\chi_m}) = -m$, the proof of Theorem 4.4 follows the same argument as in the proof of Theorem 4.3. Therefore we will not repeat the proof here.

5. VANISHING OSCILLATION AND ESSENTIAL CENTER

Recall that we write $\mathcal{T}$ for the *full* Toeplitz algebra on the Bergman space. In other words, $\mathcal{T}$ is the C^*-algebra on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$ generated by the full set of Toeplitz operators $\{T_f : f \in L^\infty(\mathbb{B}, \mathrm{d}v)\}$. In this section, we will consider $\mathcal{T}(\mathrm{VO}_{\mathrm{bdd}})$ from the view point of the *essential center* of $\mathcal{T}$. Recall that if $\mathcal{A}$ is a C^*-algebra of operators on a Hilbert space $\mathcal{H}$, then its essential center is defined to be

$$\mathrm{EssCen}(\mathcal{A}) = \{B \in \mathcal{A} : [B, A] \text{ is compact for every } A \in \mathcal{A}\}.$$

What particular interests us here is the inclusion relation

$$\mathrm{EssCen}(\mathcal{T}) \supset \mathcal{T}(\mathrm{VO}_{\mathrm{bdd}}), \tag{5.1}$$

which was established in [1]. We will see that much can be gleaned from this inclusion.

We begin with the main technical result of the section. Recall that a state φ on a unital C^*-algebra $\mathcal{A}$ is said to be *faithful* if it has the property that for every positive element $A \in \mathcal{A}$, the condition $A \neq 0$ implies $\varphi(A) > 0$.

PROPOSITION 5.1. *The quotient algebra $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$ does not admit any faithful state. Consequently, if $(X, \mathcal{M}, \mu)$ is any probability space, then $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$ is not isomorphic to any unital C^*-subalgebra of $L^\infty(X, \mathcal{M}, \mu)$.*

The technical part of the proof of Proposition 5.1 is the construction of certain radial functions in $\mathrm{VO}_{\mathrm{bdd}}$. For each integer $k \geqslant 2$, let u_k be a continuous function on $[0, 1)$ satisfying the following conditions:

(i) $0 \leqslant u_k \leqslant 1$ on $[0, 1)$;
(ii) $u_k = 0$ on $[0, 1 - 2^{-k}) \cup [1 - 2^{-4k}, 1)$;
(iii) $u_k = 1$ on $[1 - 2^{-2k}, 1 - 2^{-3k})$
(v) $\sup\{|u_k(r) - u_k(r')| : r, r' \in [1 - 2^{-j+1}, 1 - 2^{-j}]\} \leqslant 1/k$ for every $j \in \mathbb{N}$.

Such a continuous function exists; in fact we can obviously pick one that is linear on each $[1 - 2^{-j+1}, 1 - 2^{-j}]$, $j \in \mathbb{N}$. Next we pick a sequence of natural numbers

$$2 \leqslant k(1) < k(2) < k(3) < \cdots k(i) < \cdots$$

such that

(a) $4k(i) < k(i+1)$ for every $i \in \mathbb{N}$ and

(b) $k(i) \geqslant 2^i$ for every $i \in \mathbb{N}$.

Now for each $i \in \mathbb{N}$ we define the radial function

$$g_i(z) = u_{k(i)}(|z|), \quad z \in \mathbb{B},$$

on the unit ball. Moreover, for each subset E of $\mathbb{N}$ we define

$$g_E = \sum_{i \in E} g_i.$$

Obviously, it follows from (ii) and (a) that each g_E is a continuous function on $\mathbb{B}$. Furthermore, by (i), (ii) and (a) we have $\|g_E\|_\infty \leqslant 1$ for every $E \subset \mathbb{N}$.

LEMMA 5.2. *For every subset $E \subset \mathbb{N}$ we have $g_E \in \mathrm{VO}_{\mathrm{bdd}}$.*

Proof. By the boundedness mentioned above, it suffices to show that $g_E \in \mathrm{VO}$. Let us first estimate $\mathrm{diff}(g_i)$ for each $i \in \mathbb{N}$. We claim that

$$\mathrm{diff}(g_i) \leqslant \frac{5}{k(i)} \quad \text{for every } i \in \mathbb{N}. \tag{5.2}$$

Let $z, w \in \mathbb{B}$. Recall from (4.3) that if we have both $|z| \leqslant |w|$ and $\beta(z,w) \leqslant 1$, then $(1-|z|)/(1-|w|) \leqslant \mathrm{e}^2 < 2^4$. For a pair of z, w satisfying these two conditions, there are $j \leqslant j'$ in $\mathbb{N}$ such that $|z| \in [1-2^{-j+1}, 1-2^{-j})$ and $|w| \in [1-2^{-j'+1}, 1-2^{-j'})$. Hence

$$\frac{2^{-j}}{2^{-j'+1}} \leqslant \frac{1-|z|}{1-|w|} < 2^4.$$

That is, $j' < j+5$. Using this inequality and (v), the usual telescoping trick gives us

$$\begin{aligned}
&|g_i(z)-g_i(w)| \\
&\quad = |u_{k(i)}(|z|) - u_{k(i)}(|w|)| \\
&\quad \leqslant \sup\{|u_{k(i)}(r)-u_{k(i)}(s)| : r \in [1-2^{-j+1}, 1-2^{-j}), s \in [1-2^{-j'+1}, 1-2^{-j'})\} \\
&\quad \leqslant \frac{5}{k(i)},
\end{aligned}$$

which proves (5.2).

Let $E \subset \mathbb{N}$ be given. To show that $g_E \in \mathrm{VO}$, pick any $\varepsilon > 0$. Using (b), we can partition E in the form $E = F \cup E'$, where F is a finite set and $\sum_{i \in E'} (5/k(i)) \leqslant \varepsilon$. Since F is finite, by (ii) there is a $0 < t_0 < 1$ such that $g_i(\zeta) = 0$ for all $i \in F$ and $\zeta \in \mathbb{B}$ satisfying the condition $|\zeta| \geqslant t_0$. For this t_0, there is a $t_0 < t_1 < 1$ such that for any pair of $z, w \in \mathbb{B}$, if we have both $\beta(z,w) \leqslant 1$ and $|z| \geqslant t_1$, then $|w| \geqslant t_0$. Hence if $z, w \in \mathbb{B}$ are such that $\beta(z,w) \leqslant 1$ and $|z| \geqslant t_1$, then $g_i(z) = 0 = g_i(w)$ for every $i \in F$. Thus if $|z| \geqslant t_1$ and $\beta(z,w) \leqslant 1$, then

$$|g_E(z) - g_E(w)| \leqslant \sum_{i \in E'} \mathrm{diff}(g_i) \leqslant \sum_{i \in E'} \frac{5}{k(i)} \leqslant \varepsilon.$$

This proves that g_E has vanishing oscillation on $\mathbb{B}$. ∎

LEMMA 5.3. *Let $E \subset \mathbb{N}$. Then the Toeplitz operator T_{g_E} on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$ is compact if and only if the set E is finite.*

Proof. If E is a finite set, then by (ii) the support of g_E is obviously a compact subset of $\mathbb{B}$, and consequently the Toeplitz operator T_{g_E} is compact.

On the other hand, Lemma 5.2 tells us that $g_E \in \mathrm{VO}_{\mathrm{bdd}}$ for every $E \subset \mathbb{N}$. Thus if T_{g_E} is compact, then $g_E \in C_0$ (see Theorems B and A in [1]). By condition (iii), if $g_E \in C_0$, then E has to be a finite subset of $\mathbb{N}$. ∎

Proof of Proposition 5.1. We need the following fact: there is an uncountable family $\{E_\lambda : \lambda \in \Lambda\}$ of infinite subsets of $\mathbb{N}$ such that for every pair of $\lambda \neq \lambda'$ in Λ, the intersection $E_\lambda \cap E_{\lambda'}$ is finite. The construction of such a family is a standard exercise in textbooks. But instead a reference, it will be easier to simply give a proof: arrange the natural numbers as the vertices of a binary tree. That is, 1 has descendants 2 and 3, 2 has descendants 4 and 5, and 3 has descendants 6 and 7, and so on. Then we simply let Λ be the collection of all possible paths down the tree starting from 1. For each path $\lambda \in \Lambda$, let E_λ be the collection of natural numbers found along λ. The finite intersection property follows from the fact that any two distinct paths will diverge at some point.

By Lemma 5.2 and (5.1), in the quotient algebra $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$ we have the element

$$A_\lambda = T_{g_{E_\lambda}} + \mathcal{K}$$

for every $\lambda \in \Lambda$. Since $T_{g_{E_\lambda}}$ is obviously a positive operator on $L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v)$, A_λ is a positive element in $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$. Since each E_λ is an infinite set, Lemma 5.3 tells us that $A_\lambda \neq 0$. Suppose that there were a faithful state φ on $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$. Then $\varphi(A_\lambda) > 0$ for every $\lambda \in \Lambda$. Since the index set Λ is uncountable, there is a $c > 0$ such that the set

$$\Lambda_c = \{\lambda \in \Lambda : \varphi(A_\lambda) \geqslant c\}$$

is infinite. Since $c > 0$, we can pick an $m \in \mathbb{N}$ such that $mc > 1$. Now let

$$\lambda(1), \dots, \lambda(m)$$

be m distinct elements in Λ_c. For every pair of $1 \leqslant \ell < \nu \leqslant m$, the intersection $E_{\lambda(\ell)} \cap E_{\lambda(\nu)}$ is finite. Hence there is a finite subset F of $\mathbb{N}$ such that

$$\{E_{\lambda(\ell)} \backslash F\} \cap \{E_{\lambda(\nu)} \backslash F\} = \emptyset \quad \text{for every pair of } 1 \leqslant \ell < \nu \leqslant m. \tag{5.3}$$

For each $1 \leqslant \nu \leqslant m$, define $E_\nu = E_{\lambda(\nu)} \backslash F$. Also, define $\mathcal{E} = \bigcup_{\nu=1}^{m} E_\nu$. Then it follows from (5.3) that

$$g_{\mathcal{E}} = \sum_{\nu=1}^{m} g_{E_\nu}. \tag{5.4}$$

Since F is a finite set, by Lemma 5.3 we have $T_{g_{E_{\lambda(\nu)}}} - T_{g_{E_\nu}} = T_{g_{E_{\lambda(\nu)} \backslash E_\nu}} \in \mathcal{K}$ for every $1 \leqslant \nu \leqslant m$. Thus $A_{\lambda(\nu)} = T_{g_{E_\nu}} + \mathcal{K}$, $1 \leqslant \nu \leqslant m$. Combining this with (5.4), we find that

$$A_{\lambda(1)} + \cdots + A_{\lambda(m)} = T_{g_{\mathcal{E}}} + \mathcal{K}.$$

Properties (i), (ii) and (a) ensure that $\|g_{\mathcal{E}}\|_\infty \leqslant 1$. Hence $\|A_{\lambda(1)} + \cdots + A_{\lambda(m)}\| \leqslant 1$. Since φ is a state, it follows that

$$\varphi(A_{\lambda(1)} + \cdots + A_{\lambda(m)}) \leqslant 1.$$

On the other hand, by virtue of the membership $\lambda(\nu) \in \Lambda_c$, $1 \leqslant \nu \leqslant m$, we have

$$\varphi(A_{\lambda(1)} + \cdots + A_{\lambda(m)}) = \varphi(A_{\lambda(1)}) + \cdots + \varphi(A_{\lambda(m)}) \geqslant mc > 1.$$

The last two displayed inequalities obviously contradict each other. ∎

Suppose that $\mathcal{A}$ is a unital C^*-algebra of operators on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ contains $\mathcal{K}(\mathcal{H})$, the collection of compact operators on $\mathcal{H}$, then $\mathcal{K}(\mathcal{H})$ is the smallest nonzero, closed ideal in $\mathcal{A}$. In this case $\mathrm{EssCen}(\mathcal{A})$ can be alternately described as the collection of operators in $\mathcal{A}$ that commute with $\mathcal{A}$ modulo its smallest nonzero, closed ideal. In other words, if $\mathcal{A} \supset \mathcal{K}(\mathcal{H})$, then $\mathrm{EssCen}(\mathcal{A})$ is C^*-algebraically defined.

For the Toeplitz algebra $\mathcal{T}$ on the Bergman space and the the Toeplitz algebra $\mathcal{T}^{\mathrm{Hardy}}$ on the Hardy space, we have $\mathcal{T} \supset \mathcal{K}$ and $\mathcal{T}^{\mathrm{Hardy}} \supset \mathcal{K}^{\mathrm{Hardy}}$. Thus, in view of the comments in the preceding paragraph, it makes sense to compare the two essential centers $\mathrm{EssCen}(\mathcal{T})$ and $\mathrm{EssCen}(\mathcal{T}^{\mathrm{Hardy}})$, and that was the reason for establishing Proposition 5.1. But before we make such a comparison, it is necessary to recall the relevant definitions and notations in the Hardy-space case.

As usual, let S denote the unit sphere $\{z : |z| = 1\}$ in $\mathbb{C}^n$. Let $\mathrm{d}\sigma$ be the standard spherical measure on S. That is, $\mathrm{d}\sigma$ is the positive, regular Borel measure on S with $\sigma(S) = 1$ that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ which fix 0. Recall that the Hardy space $H^2(S)$ is just the norm closure of $\mathbb{C}[z_1, \ldots, z_n]$ in $L^2(S, \mathrm{d}\sigma)$. Given any $f \in L^\infty(S, \mathrm{d}\sigma)$, the Toeplitz operator T_f^{Hardy} on the Hardy space $H^2(S)$ is defined by the formula

$$T_f^{\mathrm{Hardy}} h = P^{\mathrm{Hardy}}(fh), \quad h \in H^2(S),$$

where P^{Hardy} is the orthogonal projection from $L^2(S, \mathrm{d}\sigma)$ onto $H^2(S)$. The Toeplitz algebra $\mathcal{T}^{\mathrm{Hardy}}$ on $H^2(S)$ is the C^*-algebra generated by $\{T_f^{\mathrm{Hardy}} : f \in L^\infty(S, \mathrm{d}\sigma)\}$. Let $\mathcal{K}^{\mathrm{Hardy}}$ denote the collection of compact operators on $H^2(S)$.

Suppose that $\mathcal{S}$ is a set of bounded operators on a Hilbert space $\mathcal{H}$. Recall that the *essential commutant* of $\mathcal{S}$ is defined to be

$$\mathrm{EssCom}(\mathcal{S}) = \{X \in \mathcal{B}(\mathcal{H}) : [X, A] \text{ is compact for every } A \in \mathcal{S}\}.$$

For a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, we always have $\mathrm{EssCom}(\mathcal{A}) \supset \mathrm{EssCen}(\mathcal{A})$.

Let $\mathrm{VMO}_{\mathrm{bdd}}(S)$ be the collection of bounded functions of vanishing mean oscillation on the sphere S. That is, $\mathrm{VMO}_{\mathrm{bdd}}(S)$ is the collection of $f \in L^\infty(S, \mathrm{d}\sigma)$

satisfying the condition

$$\lim_{|z|\uparrow 1} \|(f - \langle f k_z^{\mathrm{Hardy}}, k_z^{\mathrm{Hardy}}\rangle) k_z^{\mathrm{Hardy}}\| = 0,$$

where k_z^{Hardy} is the normalized reproducing kernel for the Hardy space $H^2(S)$. It is now a well-known fact that

$$\mathrm{EssCom}(\mathcal{T}^{\mathrm{Hardy}}) = \mathrm{EssCen}(\mathcal{T}^{\mathrm{Hardy}}) = \{T_g^{\mathrm{Hardy}} : g \in \mathrm{VMO}_{\mathrm{bdd}}(S)\} + \mathcal{K}^{\mathrm{Hardy}}. \tag{5.5}$$

Recall that (5.5) was first proved by Davidson in [4] for the case $n = 1$ and generalized to the case $n \geqslant 2$ by Ding, Guo and Sun in [6], [8]. For the latest development along this line, see [7]. By the symbol calculus on $\mathcal{T}^{\mathrm{Hardy}}$, we have

$$(\{T_g^{\mathrm{Hardy}} : g \in \mathrm{VMO}_{\mathrm{bdd}}(S)\} + \mathcal{K}^{\mathrm{Hardy}})/\mathcal{K}^{\mathrm{Hardy}} \cong \mathrm{VMO}_{\mathrm{bdd}}(S). \tag{5.6}$$

See [5] or Lemma 4.12 of [16].

The Bergman space Toeplitz algebra $\mathcal{T}$ is known to coincide with its ideal generated by the commutators [9], [13]. Consequently, there is no symbol calculus on the whole of the C^*-algebra $\mathcal{T}$. On the other hand, since there is a symbol calculus on $\mathcal{T}^{\mathrm{Hardy}}$, the commutator ideal in $\mathcal{T}^{\mathrm{Hardy}}$ is well known to be a proper ideal. Therefore the two Toeplitz algebras $\mathcal{T}$ and $\mathcal{T}^{\mathrm{Hardy}}$ are not isomorphic as C^*-algebras. Now we can show that the two essential centers are also not isomorphic as C^*-algebras.

PROPOSITION 5.4. *The C^*-algebras* $\mathrm{EssCen}(\mathcal{T})$ *and* $\mathrm{EssCen}(\mathcal{T}^{\mathrm{Hardy}})$ *are not isomorphic to each other.*

Proof. Suppose that there were an isomorphism

$$\psi : \mathrm{EssCen}(\mathcal{T}) \to \mathrm{EssCen}(\mathcal{T}^{\mathrm{Hardy}}).$$

Since the collection of compact operators is the smallest nonzero ideal in each essential center, ψ induces an isomorphism

$$\psi_* : \mathrm{EssCen}(\mathcal{T})/\mathcal{K} \to \mathrm{EssCen}(\mathcal{T}^{\mathrm{Hardy}})/\mathcal{K}^{\mathrm{Hardy}}$$

between the quotient algebras. Combining this with (5.5) and (5.6), we would have

$$\mathrm{EssCen}(\mathcal{T})/\mathcal{K} \cong \mathrm{VMO}_{\mathrm{bdd}}(S).$$

Since $\mathrm{VMO}_{\mathrm{bdd}}(S)$ is a unital C^*-subalgebra of $L^\infty(S, \mathrm{d}\sigma)$, this contradicts Proposition 5.1. ∎

REMARK 5.5. Because $\mathcal{T} \supset \mathcal{K}$ and $\mathcal{T}^{\mathrm{Hardy}} \supset \mathcal{K}^{\mathrm{Hardy}}$, Proposition 5.4 actually gives us a new proof of the fact that the two Toeplitz algebras $\mathcal{T}$ and $\mathcal{T}^{\mathrm{Hardy}}$ are not isomorphic. The point is that this proof does not use [9], [13].

Actually, we can make a stronger statement than just that $\mathcal{T} \not\cong \mathcal{T}^{\mathrm{Hardy}}$.

PROPOSITION 5.6. *Let $\mathcal{A}$ be a C^*-algebra of bounded operators on the Hardy space $H^2(S)$. If $\mathcal{A} \supset \mathcal{T}^{\mathrm{Hardy}}$, then $\mathcal{A}$ is not isomorphic to $\mathcal{T}$.*

Proof. Suppose that $\mathcal{A} \supset \mathcal{T}^{\mathrm{Hardy}}$. Again, because $\mathcal{T}^{\mathrm{Hardy}} \supset \mathcal{K}^{\mathrm{Hardy}}$ and $\mathcal{T} \supset \mathcal{K}$, if it were true that $\mathcal{A} \cong \mathcal{T}$, then we would have $\mathrm{EssCen}(\mathcal{A}) \cong \mathrm{EssCen}(\mathcal{T})$, which would further imply

$$\mathrm{EssCen}(\mathcal{A})/\mathcal{K}^{\mathrm{Hardy}} \cong \mathrm{EssCen}(\mathcal{T})/\mathcal{K}. \tag{5.7}$$

But since $\mathcal{A} \supset \mathcal{T}^{\mathrm{Hardy}}$, we have $\mathrm{EssCen}(\mathcal{A}) \subset \mathrm{EssCom}(\mathcal{T})^{\mathrm{Hardy}}$. Thus we conclude that $\mathrm{EssCen}(\mathcal{A})/\mathcal{K}^{\mathrm{Hardy}}$ is a unital C^*-subalgebra of

$$\mathrm{EssCom}(\mathcal{T})^{\mathrm{Hardy}}/\mathcal{K}^{\mathrm{Hardy}} \cong \mathrm{VMO}_{\mathrm{bdd}}(S).$$

Combining this with (5.7), we would have to conclude that $\mathrm{EssCen}(\mathcal{T})/\mathcal{K}$ is isomorphic to a unital C^*-subalgebra of $L^\infty(S, \mathrm{d}\sigma)$, which contradicts Proposition 5.1. ∎

In view of Proposition 5.6, one may wonder, if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{T}^{\mathrm{Hardy}}$, can $\mathcal{A}$ be isomorphic to $\mathcal{T}$? The answer is still negative if $1 \in \mathcal{A}$. This is because, by the symbol calculus, the conditions $\mathcal{A} \subset \mathcal{T}^{\mathrm{Hardy}}$ and $1 \in \mathcal{A}$ imply that the commutator ideal in $\mathcal{A}$ is a proper ideal. That is, for the proof of this fact, we do need to know that $\mathcal{T}$ coincides with its commutator ideal.

Finally, we can present Proposition 5.6 in a slightly different form.

PROPOSITION 5.7. *Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{T}$. If $\mathcal{A} \supset \mathcal{K}$, then $\mathcal{A}$ is not isomorphic to $\mathcal{T}^{\mathrm{Hardy}}$.*

Proof. Suppose that there were an isomorphism $\psi : \mathcal{A} \to \mathcal{T}^{\mathrm{Hardy}}$. Under the condition $\mathcal{A} \supset \mathcal{K}$, we have $\psi(\mathcal{K}) = \mathcal{K}^{\mathrm{Hardy}}$. Thus ψ is unitarily implemented. That is, there is a unitary operator $U : L^2_{\mathrm{a}}(\mathbb{B}, \mathrm{d}v) \to H^2(S)$ such that

$$\psi(A) = U^*AU \quad \text{for every } A \in \mathcal{A}.$$

Accordingly, $\mathcal{T}$ is isomorphic to $U^*\mathcal{T}U$, which is a C^*-subalgebra of $\mathcal{B}(H^2(S))$ containing $U^*\mathcal{A}U = \psi(\mathcal{A}) = \mathcal{T}^{\mathrm{Hardy}}$. But this contradicts Proposition 5.6. ∎

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