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QUASI-MULTIPLIERS AND ALGEBRIZATIONS OF AN OPERATOR SPACE. II. EXTREME POINTS AND QUASI-IDENTITIES

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Dedicated to Vern I. Paulsen on the occasion of his 65th birthday

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ABSTRACT. We study extreme points of the unit ball of an operator space by introducing the new notion "(approximate) quasi-identities". More specifically, we characterize an operator algebra having a contractive (approximate) quasi- (respectively, left, right, two-sided) identity in terms of quasi-multipliers and extreme points of the unit ball (of the weak*-closure) of the underlying operator space. Furthermore, we give a necessary and sufficient condition for a given operator space to be qualified to become a C^*-algebra or a one-sided ideal in a C^*-algebra in terms of quasi-multipliers.

KEYWORDS: *Quasi-multipliers, extreme points, quasi-identities, abstract operator algebras, approximate identities, injective operator spaces, dual operator spaces, C^*-algebras, ideals, ternary rings of operators.*

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1. INTRODUCTION

The notion of quasi-multipliers of operator spaces was introduced in Definition 2.2 in [20] and its relation with operator algebra products was discussed in Theorem 2.6 in [20]. That is, for a given operator space X, the possible operator algebra products that X can be equipped with are precisely the bilinear mappings on X that are implemented by contractive quasi-multipliers. Moreover, in Theorem 3.3.1 in [16] and Theorem 4.1 in [17], the author gave a geometric characterization of operator algebra products in terms of only matrix norms using the Haagerup tensor product. These results were presented in the Great Plains Operator Theory Symposium (GPOTS) held at the University of Illinois at Urbana-Champaign in 2003. After the author's talk, G.K. Pedersen asked the question

"How can the extreme points of the unit ball of a quasi-multiplier space be characterized?" This question gave rise to a further direction to study quasi-multipliers. Through investigation, it turned out that what should be characterized is the "engagement" between quasi-multipliers and extreme points of the unit ball of the operator space, and not extreme points of the unit ball of the quasi-multiplier space. In order to accomplish characterization, the author introduces the new notion "(approximate) quasi-identities".

In Section 2 we briefly review a construction of an injective envelope and a triple envelope of an operator space, and recall the definition of quasi-multipliers and their correspondence to operator algebra products. Furthermore, we define important classes of extreme points: local isometries, local co-isometries, and local unitaries, which actually become isometries, co-isometries, and unitaries, respectively, with certain embeddings.

In Section 3 we give alternative definitions of one-sided and quasi-multipliers which are used to characterize operator algebras with approximate identities in Section 4.

Section 4 is the main part of this paper. We introduce the new notion "(approximate) quasi-identities" for normed algebras. We see that at least in the operator algebra case, contractive (approximate) quasi-identities are a natural (and "minimal" in a certain sense) generalization of contractive (approximate) one-sided identities. Then we characterize an operator algebra having a contractive (approximate) quasi- (respectively, left, right, two-sided) identity in terms of its associated quasi-multiplier and extreme points of the unit ball (of the weak*-closure) of the underlying operator space (Theorem 4.7, Corollary 4.9, Corollary 4.11, Corollary 4.12, Theorem 4.13). As seen from these theorems and corollaries, the unit ball of such an operator algebra and the unit ball of its quasi-multiplier space are coupled with each other like two gears mesh.

In Section 5 we give an operator space characterization of C^*-algebras and their one-sided ideals in a clear and simple manner in terms of quasi-multipliers.

Section 6 is just an addendum and it is devoted to showing that if an operator space has an operator space predual, then so has its quasi-multiplier space.

This paper is a revision and an enlargement of the author's manuscripts titled "Extreme points of the unit ball of a quasi-multiplier space" and "Extreme points of the unit ball of an operator space" which had been circulated since 2004.

2. PRELIMINARIES

We begin by recalling the construction of an injective envelope of an operator space due to Z.-J. Ruan ([24], [25]) and M. Hamana ([12], [13], also see [14]), independently. The reader unfamiliar with this subject is referred to Chapter 15 of [22], or Chapter 6 of [10], or [7], or [2], for example.

Let $X \subset \mathbb{B}(\mathcal{H})$ be a nonzero operator space, and consider the *Paulsen operator system*

$$\mathcal{S}_X := \begin{bmatrix} \mathbb{C}1_{\mathcal{H}} & X \\ X^* & \mathbb{C}1_{\mathcal{H}} \end{bmatrix} \subset \mathbb{M}_2(\mathbb{B}(\mathcal{H})).$$

One then takes a minimal (with respect to a certain ordering) completely positive $\mathcal{S}_X$-projection Φ on $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$, where an $\mathcal{S}_X$*-projection* is an idempotent that fixes each element of $\mathcal{S}_X$. Then the image $\mathrm{Im}\Phi$ turns out to be an injective envelope $I(\mathcal{S}_X)$ of $\mathcal{S}_X$. By a well-known result of M.-D. Choi and E.G. Effros ([8]), $\mathrm{Im}\Phi$ is a unital C^*-algebra with the product $\odot$ (which is called the *Choi–Effros product*) defined by $\xi \odot \eta := \Phi(\xi\eta)$ for $\xi, \eta \in \mathrm{Im}\Phi$ and with the other algebraic operations and norm taken to be the original ones in $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$. The C^*-algebraic structure of $I(\mathcal{S}_X)$ does not depend on a particular embedding $X \subset \mathbb{B}(\mathcal{H})$. By a well-known trick one may decompose Φ into

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}.$$

Accordingly, one may write

$$\mathrm{Im}\Phi = I(\mathcal{S}_X) = \begin{bmatrix} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{bmatrix} \subset \mathbb{M}_2(\mathbb{B}(\mathcal{H})), \tag{2.1}$$

where $I(X)$ is an injective envelope of X, and $I_{11}(X)$ and $I_{22}(X)$ are injective C^*-algebras (hence unital (see Proposition 2.8 in [7])). We denote the identities of $I_{11}(X)$ and $I_{22}(X)$ by 1_{11} and 1_{22}, respectively. Note that the last inclusion in expression (2.1) is not as a subalgebra since the multiplication in $I(\mathcal{S}_X)$ and the multiplication in $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ are not same in general. The new product $\odot$ induces a new product $\bullet$ between elements of $I_{11}(X)$, $I_{22}(X)$, $I(X)$, and $I(X)^*$. For instance, $x \bullet y^* = \psi_1(xy^*)$ for $x, y \in I(X)$. Note that the associativity of $\bullet$ is guaranteed by that of $\odot$.

The following property is often useful.

LEMMA 2.1 (Blecher–Paulsen, Corollary 1.3 in [7]). (i) *If $a \in I_{11}(X)$, and if $a \bullet x = 0$, $\forall x \in X$, then $a = 0$.*

(ii) *If $b \in I_{22}(X)$, and if $x \bullet b = 0$, $\forall x \in X$, then $b = 0$.*

One may write the C^*-subalgebra, denoted by $C^*(\partial X)$ of $I(\mathcal{S}_X)$ (with the new product) generated by

$$\begin{bmatrix} O & X \\ O & O \end{bmatrix}$$

as

$$C^*(\partial X) = \begin{bmatrix} \mathcal{E}(X) & \mathcal{T}(X) \\ \mathcal{T}(X)^* & \mathcal{F}(X) \end{bmatrix} \subset \begin{bmatrix} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{bmatrix},$$

where $\mathcal{T}(X)$ is a triple envelope of X, i.e., a "minimal" TRO that contains X completely isometrically. Here an operator space X being a *ternary ring of operators*

(*TRO* for short) or a *triple system* means that there is a complete isometry ι from X into a C^*-algebra such that $\iota(x)\iota(y)^*\iota(z) \in \iota(X)$, $\forall x, y, z \in X$.

We call the embedding $i : X \to \begin{bmatrix} O & X \\ O & O \end{bmatrix} \subset C^*(\partial X) \subset I(\mathcal{S}_X)$ the *Shilov embedding* of X, and often denote $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ $(x \in X)$ simply by x. Similarly, we often write X for $\begin{bmatrix} O & X \\ O & O \end{bmatrix}$, and 1_{11} for $\begin{bmatrix} 1_{11} & 0 \\ 0 & 0 \end{bmatrix}$, etc. The involution on $\mathbb{B}(\mathcal{H})$ induces an involution on $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ in an obvious way, and it is still denoted by *. For example, for $x \in X$, $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 \\ x^* & 0 \end{bmatrix}$.

In this paper, all operator spaces are assumed to be norm-complete. Whenever infinite-dimensional vector spaces are involved in products, we take the norm-closure of the linear span. For instance, $X \bullet z \bullet X := \overline{\text{span}}\{x \bullet z \bullet y : x, y \in X\}$, where $z \in I(X)^*$.

Now we are ready to recall the definition of one-sided and quasi-multipliers. We remark that the one-sided multipliers were first introduced by D.P. Blecher in [2]. The following definition (items (i) and (ii)) is an equivalent but more manageable version appearing in [7].

DEFINITION 2.2. Let X be a nonzero operator space.

(i) (Blecher–Paulsen, Definition 1.4 in [7]) The *left-multiplier algebra* of X is the operator algebra

$$\mathcal{LM}(X) := \{a \in I_{11}(X) : a \bullet X \subset X\}.$$

We call an element of $\mathcal{LM}(X)$ a *left multiplier* of X.

(ii) The *right multiplier algebra* of X is the operator algebra

$$\mathcal{RM}(X) := \{b \in I_{22}(X) : X \bullet b \subset X\}.$$

We call an element of $\mathcal{RM}(X)$ a *right multiplier* of X.

(iii) (Kaneda–Paulsen, Definition 2.2 in [20]) The *quasi-multiplier space* of X is the operator space

$$\mathcal{QM}(X) := \{z \in I(X)^* : X \bullet z \bullet X \subset X\}.$$

We call an element of $\mathcal{QM}(X)$ a *quasi-multiplier* of X.

Note that $\mathcal{LM}(X)$ and $\mathcal{RM}(X)$ are denoted by $IM_{\mathrm{l}}(X)$ and $IM_{\mathrm{r}}(X)$, respectively, in [7].

The following theorem characterizes operator algebra products in terms of quasi-multipliers and matrix norms. Especially, (iii) tells us that the operator algebra products (algebraic property) a given operator space can be equipped with are completely determined only by its underlying matrix norm structure (geometric property), which can be regarded as the "quasi" version of the τ-trick theorem by Blecher–Effros–Zarikian (Theorems 1.1 and 4.6 in [4]).

THEOREM 2.3 (Kaneda, Theorem 3.3.1 in [16], Theorem 4.1 in [17]). *Let X be a nonzero operator space with a bilinear mapping $\varphi : X \times X \to X$, and let $I(\mathcal{S}_X)$ be as above and 1 be its identity. We regard X as a subspace of $I(\mathcal{S}_X)$ by the Shilov embedding defined above. Let*

$$\begin{array}{cccc} & \mathbb{M}_2(I(\mathcal{S}_X) \overset{\mathrm{h}}{\otimes} I(\mathcal{S}_X)) & & \mathbb{M}_2(X) \\ & \cup & & \cup \\ \Gamma_\varphi : & \begin{bmatrix} X \overset{\mathrm{h}}{\otimes} \mathbb{C}1 & X \overset{\mathrm{h}}{\otimes} X \\ O & \mathbb{C}1 \overset{\mathrm{h}}{\otimes} X \end{bmatrix} & \to & \begin{bmatrix} X & X \\ O & X \end{bmatrix} \end{array}$$

be defined by

$$\Gamma_\varphi\left(\begin{bmatrix} x_1 \otimes 1 & x \otimes y \\ 0 & 1 \otimes x_2 \end{bmatrix}\right) := \begin{bmatrix} x_1 & \varphi(x,y) \\ 0 & x_2 \end{bmatrix}$$

and their linear extension and norm closure, where $\overset{\mathrm{h}}{\otimes}$ is the Haagerup tensor product. Then, the following are equivalent:

(i) *(X, φ) is an abstract operator algebra (i.e., there is a completely isometric homomorphism from X into $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, hence, in particular, φ is associative).*

(ii) *There exists a $z \in$ Ball$(\mathcal{QM}(X)) := \{z \in \mathcal{QM}(X) : \|z\| \leqslant 1\}$ such that $\forall x, y \in X$, $\varphi(x,y) = x \bullet z \bullet y$.*

(iii) *Γ_φ is completely contractive.*

Moreover, such a z is unique.

When the above conditions hold, we denote φ by m_z, and call (X, m_z) the operator algebra *corresponding to* the quasi-multiplier z, or the *algebrization* of X by the quasi-multiplier z. On the other hand, for a given operator algebra $\mathcal{A}$, we call z given in (ii) the quasi-multiplier *associated with* $\mathcal{A}$.

We denote by ext(Ball$(I(X))$) the set of the extreme points of the unit ball of $I(X)$. The following are particularly important subsets of ext(Ball$(I(X))$). We will see in Corollary 2.10 that these are actually subsets of ext(Ball$(I(X))$).

DEFINITION 2.4. Let X be a nonzero operator space, and let S be a subset of $I(X)$.

(i) $\mathcal{UR}_{\mathrm{loc}}(S) := \{x \in S : x^* \bullet x = 1_{22}\}$.

(ii) $\mathcal{UL}_{\mathrm{loc}}(S) := \{x \in S : x \bullet x^* = 1_{11}\}$.

(iii) $\mathcal{U}_{\mathrm{loc}}(S) := \mathcal{UR}_{\mathrm{loc}}(S) \cap \mathcal{UL}_{\mathrm{loc}}(S)$.

We call an element of $\mathcal{UR}_{\mathrm{loc}}(I(X))$ (respectively, $\mathcal{UL}_{\mathrm{loc}}(I(X))$, $\mathcal{U}_{\mathrm{loc}}(I(X))$) a *local isometry* (or, *local right unitary*) (respectively, *local co-isometry* (or, *local left unitary*), *local unitary*).

Item (iii) of the following proposition tells us that with certain embeddings, local isometries (respectively, local co-isometries, local unitaries) actually become isometries (respectively, co-isometries, unitaries).

LEMMA 2.5. *Let X be a nonzero operator space.*

(i) *If $\mathcal{UR}_{\mathrm{loc}}(I(X)) \neq \varnothing$, then there exists a commutative diagram*

$$\begin{array}{ccc} I_{11}(X) & \xleftarrow{\sigma_1} & I(X) \\ \sigma_1^* \uparrow & \circlearrowleft & \uparrow \rho_1 \\ I(X)^* & \xleftarrow{\rho_1^*} & I_{22}(X) \end{array}$$

such that $\rho_1^*(b) := \rho_1(b^*)^*$, $\forall b \in I_{22}(X)$, $\sigma_1^*(x^*) := \sigma_1(x)^*$, $\forall x \in I(X)$, and ρ_1, σ_1 (hence ρ_1^*, σ_1^*) are complete isometries, and $\sigma_1 \circ \rho_1$ (hence $\sigma_1^* \circ \rho_1^*$) is a $*$-monomorphism, and $\forall x \in \mathcal{UR}_{\mathrm{loc}}(I(X))$, $\sigma_1(x)$ is a partial isometry in the C^*-algebra $I_{11}(X)$.

(ii) *If $\mathcal{UL}_{\mathrm{loc}}(I(X)) \neq \varnothing$, then there exists a commutative diagram*

$$\begin{array}{ccc} I_{11}(X) & \xrightarrow{\sigma_2} & I(X) \\ \sigma_2^* \downarrow & \circlearrowright & \downarrow \rho_2 \\ I(X)^* & \xrightarrow{\rho_2^*} & I_{22}(X) \end{array}$$

such that $\rho_2^*(x^*) := \rho_2(x)^*$, $\forall x \in I(X)$, $\sigma_2^*(a) := \sigma_2(a^*)^*$, $\forall a \in I_{11}(X)$, and ρ_2, σ_2 (hence ρ_2^*, σ_2^*) are complete isometries, and $\rho_2 \circ \sigma_2$ (hence $\rho_2^* \circ \sigma_2^*$) is a $*$-monomorphism, and $\forall x \in \mathcal{UL}_{\mathrm{loc}}(I(X))$, $\rho_2(x)$ is a partial isometry in the C^*-algebra $I_{22}(X)$.

(iii) *If $\mathcal{U}_{\mathrm{loc}}(I(X)) \neq \varnothing$, then in* (i) *and* (ii)*, one can take ρ_1, σ_1, ρ_2, σ_2 (hence ρ_1^*, σ_1^*, ρ_2^*, σ_2^*) to be onto such that $\rho_2 = \rho_1^{-1}$, $\sigma_2 = \sigma_1^{-1}$ (hence $\rho_2^* = (\rho_1^*)^{-1}$, $\sigma_2^* = (\sigma_1^*)^{-1}$). Moreover,*

(a) *$\forall x \in \mathcal{UR}_{\mathrm{loc}}(I(X))$, $\sigma_1(x) = (\sigma_2)^{-1}(x)$ and $\rho_2(x) = (\rho_1)^{-1}(x)$ are isometries in the C^*-algebras $I_{11}(X)$ and $I_{22}(X)$, respectively;*

(b) *$\forall x \in \mathcal{UL}_{\mathrm{loc}}(I(X))$, $\sigma_1(x) = (\sigma_2)^{-1}(x)$ and $\rho_2(x) = (\rho_1)^{-1}(x)$ are co-isometries in the C^*-algebras $I_{11}(X)$ and $I_{22}(X)$, respectively;*

(c) *$\forall x \in \mathcal{U}_{\mathrm{loc}}(I(X))$, $\sigma_1(x) = (\sigma_2)^{-1}(x)$ and $\rho_2(x) = (\rho_1)^{-1}(x)$ are unitaries in the C^*-algebras $I_{11}(X)$ and $I_{22}(X)$, respectively.*

Proof. Once we define mappings ρ_1, ρ_2, σ_1, and σ_2 as follows, then the assertions are straightforward.

(i) Pick $v \in \mathcal{UR}_{\mathrm{loc}}(I(X))$, and define $\rho_1(b) := v \bullet b, \forall b \in I_{22}(X)$; $\sigma_1(x) := x \bullet v^*, \forall x \in I(X)$.

(ii) Pick $v \in \mathcal{UL}_{\mathrm{loc}}(I(X))$, and define $\rho_2(x) := v^* \bullet x, \forall x \in I(X)$; $\sigma_2(a) := a \bullet v, \forall a \in I_{11}(X)$.

(iii) Pick $v \in \mathcal{U}_{\mathrm{loc}}(I(X))$, and define ρ_1, ρ_2, σ_1, and σ_2, as (i) and (ii). ∎

A refinement of the following lemma will be given in Theorem 4.13 using the alternative definitions of quasi-multipliers given in Section 3.

LEMMA 2.6. *Let $\mathcal{A}$ be a nonzero operator algebra and z be the quasi-multiplier associated with $\mathcal{A}$.*

(i) *If $\mathcal{A}$ has a contractive approximate right identity, then $z^* \in \mathcal{UR}_{\mathrm{loc}}(I(\mathcal{A}))$.*

(ii) *If $\mathcal{A}$ has a contractive approximate left identity, then $z^* \in \mathcal{UL}_{\mathrm{loc}}(I(\mathcal{A}))$.*

(iii) *If $\mathcal{A}$ has a contractive approximate two-sided identity, then $z^* \in \mathcal{U}_{\mathrm{loc}}(I(\mathcal{A}))$.*

Proof. (i) follows from noticing that in the proof of Theorem 2.3 in [5], it is seen that v^* is the quasi-multiplier associated with $\mathcal{A}$ and $v \in \mathcal{UR}_{\mathrm{loc}}(I(\mathcal{A}))$, although the terminology "quasi-multiplier" does not appear there.

(ii) is similar by symmetry.

(iii) follows from (i) and (ii). ∎

It follows from Lemma 2.6 together with Lemma 2.5 that if $\mathcal{A}$ is an operator algebra with a contractive approximate right identity, then $I(\mathcal{A})$ and $I(\mathcal{A})^*$ are embedded in $I_{11}(\mathcal{A})$ completely isometrically, and $I_{22}(\mathcal{A})$ is embedded in $I_{11}(\mathcal{A})$ $*$-monomorphically. Part of this fact is already seen in Theorem 2.3 in [5].

Similar embeddings hold when $\mathcal{A}$ has a contractive approximate left identity.

Also note that in Lemma 3.2.2 in [16] and Lemma 3.4 in [17], assuming that $\mathcal{A}$ has a contractive approximate two-sided identity we embedded $I(\mathcal{A})$ and $I(\mathcal{A})^*$ in $I_{11}(\mathcal{A})$, accordingly we showed that our definition of quasi-multipliers (Definition 2.2) coincides with the classical one for C^*-algebras (Section 3.12 of [23]) in the sense that they are completely isometrically quasi-isomorphic (Theorem 3.2.3 in [16] and Theorem 3.5 in [17]).

Recall that if $\mathcal{A}$ is an operator algebra with a contractive approximate two-sided identity, then its injective envelope $I(\mathcal{A})$ is a unital C^*-algebra which contains $\mathcal{A}$ as a subalgebra (see Corollary 4.2.8(1) in [6], for example).

DEFINITION 2.7. (i) $\mathcal{LM}^1(\mathcal{A}) := \{a \in I(\mathcal{A}) : a\mathcal{A} \subset \mathcal{A}\}$.
(ii) $\mathcal{RM}^1(\mathcal{A}) := \{b \in I(\mathcal{A}) : \mathcal{A}b \subset \mathcal{A}\}$.
(ii) $\mathcal{QM}^1(\mathcal{A}) := \{z \in I(\mathcal{A}) : \mathcal{A}z\mathcal{A} \subset \mathcal{A}\}$.

$\mathcal{LM}^1(\mathcal{A})$, $\mathcal{RM}^1(\mathcal{A})$, and $\mathcal{QM}^1(\mathcal{A})$ are equivalent to $\mathcal{LM}(\mathcal{A})$, $\mathcal{RM}(\mathcal{A})$, and $\mathcal{QM}(\mathcal{A})$, respectively in the sense of the following lemma.

LEMMA 2.8. *Let $\mathcal{A}$ be a nonzero operator algebra with a contractive approximate two-sided identity. Then the following assertions hold:*

(i) *There is a multiplicative complete isometry λ from $\mathcal{LM}^1(\mathcal{A})$ onto $\mathcal{LM}(\mathcal{A})$ such that $ax = \lambda(a) \bullet x$, $\forall a \in \mathcal{LM}^1(\mathcal{A}), \forall x \in \mathcal{A}$.*

(ii) *There is a multiplicative complete isometry ρ from $\mathcal{RM}^1(\mathcal{A})$ onto $\mathcal{RM}(\mathcal{A})$ such that $xb = x \bullet \rho(b)$, $\forall b \in \mathcal{RM}^1(\mathcal{A}), \forall x \in \mathcal{A}$.*

(iii) *There is a complete isometry κ from $\mathcal{QM}^1(\mathcal{A})$ onto $\mathcal{QM}(\mathcal{A})$ such that $xzy = x \bullet \kappa(z) \bullet y$, $\forall z \in \mathcal{QM}^1(\mathcal{A}), \forall x, y \in \mathcal{A}$.*

Proof. To see (iii), first note that by Lemma 2.6(iii) and Lemma 2.5(iii), $I_{11}(\mathcal{A})$ is an injective envelope of $\mathcal{A}$. By the uniqueness of an injective envelope C^*-algebra up to $*$-isomorphism fixing each element of $\mathcal{A}$, $I(\mathcal{A})$ in the definition of $\mathcal{QM}^1(\mathcal{A})$ can be taken to be $I_{11}(\mathcal{A})$. Now the assertion follows from Lemma 3.2.2 in [16] (or Lemma 3.4 in [17]).

Items (i) and (ii) are similar by establishing a lemma corresponding to Lemma 3.2.2 in [16] (or Lemma 3.4 in [17]). The details are left to the reader. ∎

We close this preliminary section with Kadison's characterization of the extreme points of the unit ball of a C^*-algebra (Theorem 1 in [15]). We use the following version in Pedersen's book (Proposition 1.4.8 in [23]) and Sakai's book (Proposition 1.6.5 in [26]). This theorem motivated our definition of quasi-identities (Definition 4.2) and plays a key role in the proof of the characterization theorems (Theorems 4.7 and 4.13).

LEMMA 2.9 (Kadison). *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be orthogonal projections in $\mathcal{A}$. Then $x \in p\mathcal{A}q$ is an extreme point of* $\mathrm{Ball}(p\mathcal{A}q)$ *if and only if*

$$(p - xx^*)\mathcal{A}(q - x^*x) = \{0\}.$$

In this case, x is a partial isometry.

The following corollary is immediate from the lemma above.

COROLLARY 2.10. *Let X be a nonzero operator space. Then the sets $\mathcal{UR}_{\mathrm{loc}}(I(X))$, $\mathcal{UL}_{\mathrm{loc}}(I(X))$, and $\mathcal{U}_{\mathrm{loc}}(I(X))$ are subsets of* $\mathrm{ext}(\mathrm{Ball}(I(X)))$.

3. ALTERNATIVE DEFINITIONS OF ONE-SIDED AND QUASI-MULTIPLIERS

In this section we provide alternative definitions of one-sided and quasi-multipliers of an operator space X which are equivalent to the ones presented in Definition 2.2.

First, we give a definition of one-sided and quasi-multipliers of X using the second dual of $C^*(\partial X)$. Denote the second dual of $\mathcal{E}(X)$, $\mathcal{F}(X)$, and $\mathcal{T}(X)$ by $\mathcal{E}(X)''$, $\mathcal{F}(X)''$, and $\mathcal{T}(X)''$, respectively, and we regard them as the corners of the second dual of $C^*(\partial X)$ in the usual way:

$$C^*(\partial X)'' = \begin{bmatrix} \mathcal{E}(X)'' & \mathcal{T}(X)'' \\ \mathcal{T}(X)^{*\prime\prime} & \mathcal{F}(X)'' \end{bmatrix}.$$

(To avoid confusion with the adjoint and to distinguish $\mathcal{QM}''(\mathcal{A})$ from $\mathcal{QM}^{**}(\mathcal{A})$ in item (I) on page 86 of [16] and item (I) on page 351 of [17], in this paper we denote the dual by the prime instead of the star.) The Arens product on $C^*(\partial X)''$ induces a product between elements of $\mathcal{E}(X)''$, $\mathcal{F}(X)''$, $\mathcal{T}(X)''$, and $\mathcal{T}(X)^{*\prime\prime}$, which is an extension of $\bullet$ defined in Section 2 and still denoted by $\bullet$. Denote by $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ the identities of the W^*-algebras $\mathcal{E}(X)''$ and $\mathcal{F}(X)''$, respectively. Let $\widehat{\ } : C^*(\partial X) \to C^*(\partial X)''$ be the canonical embedding.

DEFINITION 3.1. (i) $\mathcal{LM}''(X) := \{a \in \mathcal{E}(X)'' : a \bullet \widehat{X} \subset \widehat{X}\}$.
(ii) $\mathcal{RM}''(X) := \{b \in \mathcal{F}(X)'' : \widehat{X} \bullet b \subset \widehat{X}\}$.
(iii) $\mathcal{QM}''(X) := \{z \in \mathcal{T}(X)^{*\prime\prime} : \widehat{X} \bullet z \bullet \widehat{X} \subset \widehat{X}\}$.

By Proposition 3.2(iii) and Theorem 2.3(ii)$\Rightarrow$(i), the operator space X together with the bilinear mapping φ defined by $\widehat{\varphi(x,y)} := \widehat{x} \bullet z \bullet \widehat{y}$, $\forall x, y \in X$ with a fixed $z \in \mathrm{Ball}(\mathcal{QM}''(X))$ is an operator algebra, and hence φ is denoted by m_z (instead of $m_{\kappa_1^{-1}(z)}$ which is consistent with the notation defined at the end of Theorem 2.3) and we call (X, m_z) the operator algebra *corresponding to z*.

$\mathcal{LM}''(X)$, $\mathcal{RM}''(X)$, and $\mathcal{QM}''(X)$ are equivalent to $\mathcal{LM}(X)$, $\mathcal{RM}(X)$, and $\mathcal{QM}(X)$, respectively in the sense of the following proposition which we shall prove shortly.

PROPOSITION 3.2. (i) *There is a multiplicative complete isometry λ_1 from $\mathcal{LM}(X)$ onto $\mathcal{LM}''(X)$ such that $\widehat{a \bullet x} = \lambda_1(a) \bullet \widehat{x}$, $\forall a \in \mathcal{LM}(X), \forall x \in X$.*

(ii) *There is a multiplicative complete isometry ρ_1 from $\mathcal{RM}(X)$ onto $\mathcal{RM}''(X)$ such that $\widehat{x \bullet b} = \widehat{x} \bullet \rho_1(b)$, $\forall b \in \mathcal{RM}(X), \forall x \in X$.*

(iii) *There is a complete isometry κ_1 from $\mathcal{QM}(X)$ onto $\mathcal{QM}''(X)$ such that*

$$\widehat{x \bullet z \bullet y} = \widehat{x} \bullet \kappa_1(z) \bullet \widehat{y}, \quad \forall z \in \mathcal{QM}(X), \forall x, y \in X.$$

Next, we give a definition of one-sided and quasi-multipliers of X using a representation of $C^*(\partial X)$ on a Hilbert space. Represent $C^*(\partial X)$ by a $*$-monomorphism π on the direct sum of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ nondegenerately so that $[\mathcal{T}(X)\mathcal{H}_2] := \overline{\operatorname{span}}\{x\xi : x \in \mathcal{T}(X), \xi \in \mathcal{H}_2\} = \mathcal{H}_1$ and $[\mathcal{T}(X)^*\mathcal{H}_1] = \mathcal{H}_2$. Denote by $1_{\mathcal{H}_1}$ and $1_{\mathcal{H}_2}$ the orthogonal projections onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

DEFINITION 3.3. (i) $\mathcal{LM}_\pi(X) := \{a \in \mathbb{B}(\mathcal{H}_1) : a\pi(X) \subset \pi(X)\}$.

(ii) $\mathcal{RM}_\pi(X) := \{b \in \mathbb{B}(\mathcal{H}_2) : \pi(X)b \subset \pi(X)\}$.

(iii) $\mathcal{QM}_\pi(X) := \{z \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2) : \pi(X)z\pi(X) \subset \pi(X)\}$.

By Proposition 3.4(iii) and Theorem 2.3(ii)⇒(i), the operator space X together with the bilinear mapping φ defined by $\pi(\varphi(x,y)) := \pi(x)z\pi(y)$, $\forall x, y \in X$ with a fixed $z \in \operatorname{Ball}(\mathcal{QM}_\pi(X))$ is an operator algebra, and hence φ is denoted by m_z (instead of $m_{\kappa_2^{-1}(z)}$ which is consistent with the notation defined at the end of Theorem 2.3) and we call (X, m_z) the operator algebra *corresponding to z*.

$\mathcal{LM}_\pi(X)$, $\mathcal{RM}_\pi(X)$, and $\mathcal{QM}_\pi(X)$ are equivalent to $\mathcal{LM}(X)$, $\mathcal{RM}(X)$, and $\mathcal{QM}(X)$, respectively in the sense of the following proposition.

PROPOSITION 3.4. (i) *There is a multiplicative complete isometry λ_2 from $\mathcal{LM}(X)$ onto $\mathcal{LM}_\pi(X)$ such that*

$$\pi(a \bullet x) = \lambda_2(a)\pi(x), \quad \forall a \in \mathcal{LM}(X), \forall x \in X.$$

(ii) *There is a multiplicative complete isometry ρ_2 from $\mathcal{RM}(X)$ onto $\mathcal{RM}_\pi(X)$ such that*

$$\pi(x \bullet b) = \pi(x)\rho_2(b), \quad \forall b \in \mathcal{RM}(X), \forall x \in X.$$

(iii) *There is a complete isometry κ_2 from $\mathcal{QM}(X)$ onto $\mathcal{QM}_\pi(X)$ such that*

$$\pi(x \bullet z \bullet y) = \pi(x)\kappa_2(z)\pi(y), \quad \forall z \in \mathcal{QM}(X), \forall x, y \in X.$$

Proof of Propositions 3.2 *and* 3.4. To see Proposition 3.4(iii), first note that the C^*-algebra $I(\mathcal{S}_X)$ is an injective envelope of $C^*(\partial X)$, and use $I(\mathcal{S}_X)$ to define

$\mathcal{QM}^1(C^*(\partial X))$ (see Definition 2.7(iii)). Since a C^*-algebra has a contractive approximate two-sided identity, by Lemma 2.8(iii) together with Lemma 3.2.2 and Theorem 3.2.3 in [16] (or, Lemma 3.4 and Theorem 3.5 in [17]) we have a sequence of completely isometric quasi-isomorphisms

$$\mathcal{QM}^1(C^*(\partial X)) \overset{\sim}{\to} \mathcal{QM}(C^*(\partial X)) \overset{\sim}{\to} \mathcal{QM}^{\widetilde{\psi}}(C^*(\partial X)) \overset{\sim}{\to} \mathcal{QM}^{\pi}(C^*(\partial X)), \tag{3.1}$$

where $\mathcal{QM}^{\widetilde{\psi}}(C^*(\partial X))$ and $\mathcal{QM}^{\pi}(C^*(\partial X))$ are as defined in Lemma 3.2.2 and item (II) on page 86 of [16] (or, Lemma 3.4 and item (II) on page 351 of [17]). Note that $\mathcal{QM}^{\pi}(C^*(\partial X))$ with "superscript π" appearing here is different from $\mathcal{QM}_{\pi}(C^*(\partial X))$ with "subscript π" defined in Definition 3.3 of the present paper, although they are quasi-isomorphic in the sense of Definition 3.1.1(2) in [16] and Definition 2.1(2) in [17]. Let us denote the composition of the sequence of completely isometric quasi-isomorphisms in (3.1) by κ, then

$$\pi(\xi\zeta\eta) = \pi(\xi)\kappa(\zeta)\pi(\eta), \quad \forall\zeta \in \mathcal{QM}^1(C^*(\partial X)), \forall\xi, \eta \in C^*(\partial X),$$

and it is not hard to see that the restriction of κ to $\mathcal{QM}(X)$ gives κ_2. Proposition 3.2(iii) is similar. Items (i) of Propositions 3.2 and 3.4 are also similar, but use Theorem 6.1 in [3] instead of Theorem 3.2.3 in [16] (or Theorem 3.5 in [17]), and the equivalence of $\mathcal{LM}(C^*(\partial X))$ and $M_l(C^*(\partial X))$ (Theorem 1.9(i) in [7]). Items (ii) of Propositions 3.2 and 3.4 are similar to items (i) of these propositions by symmetry. ∎

Finally, we define the following sets.

DEFINITION 3.5. Let X be a nonzero operator space.

(i) Let S be a subset of $\mathcal{T}(X)''$.

(a) $\mathcal{UR}_{\mathrm{loc}}(S) := \{x \in S : x^* \bullet x = 1_{\mathcal{F}}\}$.

(b) $\mathcal{UL}_{\mathrm{loc}}(S) := \{x \in S : x \bullet x^* = 1_{\mathcal{E}}\}$.

(c) $\mathcal{U}_{\mathrm{loc}}(S) := \mathcal{UR}_{\mathrm{loc}}(S) \cap \mathcal{UL}_{\mathrm{loc}}(S)$.

(ii) Let S be a subset of $\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$.

(a) $\mathcal{UR}_{\mathrm{loc}}(S) := \{x \in S : x^*x = 1_{\mathcal{H}_2}\}$.

(b) $\mathcal{UL}_{\mathrm{loc}}(S) := \{x \in S : xx^* = 1_{\mathcal{H}_1}\}$.

(c) $\mathcal{U}_{\mathrm{loc}}(S) := \mathcal{UR}_{\mathrm{loc}}(S) \cap \mathcal{UL}_{\mathrm{loc}}(S)$.

One may rewrite Lemma 2.5 using Definition 3.5 and appropriate algebraic operations in $C^*(\partial X)''$ or $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with $\mathcal{T}(X)''$ or $\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ in place of $I(X)$. The details are left to the reader. A similar remark works for Lemma 2.6.

The following corollary immediately follows from Kadison's theorem (Lemma 2.9).

COROLLARY 3.6. (i) *$\mathcal{UR}_{\mathrm{loc}}(\mathcal{T}(X)'')$, $\mathcal{UL}_{\mathrm{loc}}(\mathcal{T}(X)'')$, and $\mathcal{U}_{\mathrm{loc}}(\mathcal{T}(X)'')$ are subsets of* $\mathrm{ext}(\mathrm{Ball}(\mathcal{T}(X)''))$.

(ii) *$\mathcal{UR}_{\mathrm{loc}}(\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1))$, $\mathcal{UL}_{\mathrm{loc}}(\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1))$, and $\mathcal{U}_{\mathrm{loc}}(\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1))$ are subsets of* $\mathrm{ext}(\mathrm{Ball}(\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)))$.

Hereafter, we omit the symbol $\widehat{\ }$ or π, and we regard $C^*(\partial X)$ as a C^*-subalgebra of $C^*(\partial X)''$ or $\mathbb{B}(\mathcal{H})$. We also omit the symbol $\odot$ or $\bullet$ unless there is a possibility of confusion.

4. QUASI-IDENTITIES AND CHARACTERIZATION THEOREMS

Throughout this section, the following elementary lemma, which follows from the polarization identity, is useful.

LEMMA 4.1. (i) *Let $a \in \mathbb{B}(\mathcal{H})$. If $a^2 = a$ and $\|a\| \leqslant 1$, then $a^* = a$, i.e., a is an orthogonal projection.*

(ii) *Let $p \in \mathbb{B}(\mathcal{H})$ be an orthogonal projection, i.e., $p = p^* = p^2$, and let $b, c \in \mathbb{B}(\mathcal{H})$ such that $c^*b = p$ and $\|b + c\| \leqslant 2$. Then $\ker p \subset \ker b \cap \ker c$ if and only if $b = c$. In this case, $\ker p = \ker b = \ker c$.*

Proof. (i) Although this is standard, we include a proof for the convenience of the reader. Let $\xi \in \mathcal{H}$. Then by the polarization identity,

$$\begin{aligned}\|a\xi\|^2 &= \langle a\xi, a\xi\rangle = \langle a\xi, a^2\xi\rangle = \langle a^*a\xi, a\xi\rangle \\ &= \frac{1}{4}(\|(a^*a + a)\xi\|^2 - \|(a^*a - a)\xi\|^2) \leqslant \|a\xi\|^2 - \frac{1}{4}\|(a^*a - a)\xi\|^2.\end{aligned}$$

Since $\xi \in \mathcal{H}$ is arbitrary, $a = a^*a = a^*$.

(ii) Assume that $\ker p \subset \ker b \cap \ker c$, and let $\eta \in \mathcal{H}$. Then by the polarization identity,

$$\begin{aligned}\|p\eta\|^2 &= \langle c^*bp\eta, p\eta\rangle = \langle bp\eta, cp\eta\rangle \\ &= \frac{1}{4}(\|(b + c)p\eta\|^2 - \|(b - c)p\eta\|^2) \leqslant \|p\eta\|^2 - \frac{1}{4}\|(b - c)p\eta\|^2\end{aligned}$$

from which it follows that $b = c$. The converse direction and the last assertion are obvious and very basic facts. ∎

We introduce the new notion "(approximate) quasi-identities".

DEFINITION 4.2. (i) Let $\mathcal{R}$ be a ring. A *quasi-identity* of $\mathcal{R}$ is an element $e \in \mathcal{R}$ such that

$$r = er + re - ere, \quad \forall r \in \mathcal{R}.$$

(ii) Let $\mathcal{R}$ be a topological ring. An *approximate quasi-identity* of $\mathcal{R}$ is a net $\{e_\alpha\} \subset \mathcal{R}$ such that

$$r = \lim_\alpha (e_\alpha r + re_\alpha - e_\alpha re_\alpha), \quad \forall r \in \mathcal{R}.$$

(iii) Let $\mathcal{A}$ be a locally convex topological algebra. A *weak approximate quasi-identity* of $\mathcal{A}$ is a net $\{e_\alpha\} \subset \mathcal{A}$ such that

$$a = \text{w-}\lim_\alpha (e_\alpha a + ae_\alpha - e_\alpha ae_\alpha), \quad \forall a \in \mathcal{A}.$$

It is quite essential in the definition of an approximate quasi-identity that the limit is taken "at once". In fact, a bounded approximate left identity $\{e_\alpha\}$ of a normed algebra $\mathcal{A}$ is easily seen to be an approximate quasi-identity. However, $\lim_\alpha ae_\alpha$ need not exist for all $a \in \mathcal{A}$ as seen in Example 4.15.

PROPOSITION 4.3. (i) *A separable normed algebra with an approximate quasi-identity admits an approximate quasi-identity which is a sequence.*

(ii) *A finite-dimensional normed algebra with an approximate quasi-identity with a bound M has a quasi-identity of norm equal to or less than M.*

Proof. Item (i) can be proved in a similar way to showing a separable C^*-algebra admits an approximate identity which is a sequence (see Remark 3.1.1 in [21], for example), and the details are left to the reader.

To see (ii), let $\{e_\alpha\}$ be a bounded approximate quasi-identity of a finite-dimensional normed algebra $\mathcal{A}$ such that $\sup_\alpha \|e_\alpha\| \leqslant M$, and let e be an accumulation point of $\{e_\alpha\}$ in $\mathcal{A}$. Then $\|e\| \leqslant M$, and one can take a sequence $\{e_{\alpha_n}\} \subset \{e_\alpha\}$ such that $\lim_{n\to\infty} e_{\alpha_n} = e$. Therefore $\forall a \in \mathcal{A}$,

$$\begin{aligned}\|a-(ea+ae-eae)\| \leqslant{}& \|a - (e_{\alpha_n}a + ae_{\alpha_n} - e_{\alpha_n}ae_{\alpha_n})\| + \|e_{\alpha_n} - e\|\,\|a\| \\ &+\|a\|\,\|e_{\alpha_n}-e\|+\|e_{\alpha_n}-e\|\,\|a\|\,\|e_{\alpha_n}\|+\|e\|\,\|a\|\,\|e_{\alpha_n}-e\|\to 0\end{aligned}$$

as $n \to \infty$. ∎

The author believes that assuming boundedness in (ii) is redundant for the existence of a quasi-identity, and would like to leave the following as an open problem.

QUESTION 1. Does a finite-dimensional normed algebra with a (possibly unbounded) approximate quasi-identity admit a quasi-identity?

However, it is easy to see that a (possibly unbounded) approximate right (respectively, left, two-sided) identity of a finite-dimensional normed algebra can be replaced by a bounded one, and hence its accumulation point is a right (respectively, left, two-sided) identity. To see this, let $\mathcal{A}$ be a non-zero finite-dimensional normed algebra with a (possibly unbounded) approximate right identity $\{e_i\}_{i=1}^\infty$ (as remarked in the proof of Proposition 4.3(i), one can take a sequential approximate right identity), and let n be the dimension of the vector space $\operatorname{span}\{e_i : i \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers. We shall show that the approximate right identity $\{e_i\}_{i=1}^\infty$ can be replaced by a bounded one by mathematical induction on n. If $n = 1$, this is trivial. Indeed, pick $e \neq 0$ from $\operatorname{span}\{e_i : i \in \mathbb{N}\}$, then for each $i \in \mathbb{N}$, there is a unique scalar λ_i such that

$$e_i = \lambda_i e, \quad \text{and} \quad e = \lim_{i\to\infty} ee_i = \lim_{i\to\infty} \lambda_i e^2,$$

which forces $\{\lambda_i\}_{i=1}^\infty$, hence $\{e_i\}_{i=1}^\infty$ to be bounded (there is no need to replace in this case). Now let $n \geqslant 2$, and assume that the assertion is true if the dimension of span$\{e_i : i \in \mathbb{N}\}$ is $n-1$. Suppose that $\{e_i\}_{i=1}^{\infty}$ is unbounded and that the dimension of span$\{e_i : i \in \mathbb{N}\}$ is n. One may assume that $\lim_{i\to\infty} \|e_i\| = \infty$ and $e_i \neq 0$, $\forall i \in \mathbb{N}$ by passing to a subsequence if necessary. Let v be an accumulation point of the sequence $\{e_i/\|e_i\|\}_{i=1}^{\infty}$. Passing to a subsequence again if necessary, we can make $v = \lim_{i\to\infty} e_i/\|e_i\|$. Since for every $a \in \mathcal{A}$, $a = \lim_{i\to\infty} ae_i$, dividing the right-hand side by $\|e_i\|$ before taking the limit yields that $0 = av$, $\forall a \in \mathcal{A}$. By adjoining vectors $v_1, \ldots, v_{n-1}$ appropriately, we obtain a basis $\{v, v_1, \ldots, v_{n-1}\}$ of span$\{e_i : i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let c_i be the coefficient of v when e_i is expanded with respect to this basis, and put $\widetilde{e_i} := e_i - c_i v$ ($\in$ span$\{v_1, \ldots, v_{n-1}\}$). Clearly, the sequence $\{\widetilde{e_i}\}_{i=1}^{\infty}$ serves as an approximate right identity of $\mathcal{A}$ and the dimension of span$\{\widetilde{e_i} : i \in \mathbb{N}\}$ is $n-1$. Thus by the induction hypothesis, $\{\widetilde{e_i}\}_{i=1}^{\infty}$ can be replaced by a bounded right approximate identity, which completes the proof.

Identities, left identities, right identities of rings are quasi-identities. We shall see in Proposition 4.4 that in the operator algebra case, a contractive quasi-identity is unique if it exists. Moreover, it is necessarily idempotent and Hermitian (if the operator algebra is embedded in any C^*-algebra by any multiplicative complete isometry).

Bounded approximate left (respectively, right, two-sided) identities of normed algebras are approximate quasi-identities. Many normed algebras do not have an (approximate) two-sided or one-sided identity, but do have an (approximate) quasi-identity. Perhaps the following example illustrates a typical situation. Let $\mathcal{A}$ be a normed algebra which has a bounded left approximate identity $\{e_\alpha\}$ but does not have a right approximate identity, and let $\mathcal{B}$ be a normed algebra which has a bounded right approximate identity $\{f_\beta\}$ but does not have a left approximate identity. Then $\mathcal{A} \overset{p}{\oplus} \mathcal{B}$ with $1 \leqslant p \leqslant \infty$ has neither left nor right approximate identity, but does have a bounded approximate quasi-identity $\{e_\alpha \oplus f_\beta\}_{(\alpha,\beta)}$, where $\{(\alpha,\beta)\}$ is a directed set by the ordering defined by

$$(\alpha_1, \beta_1) \leqslant (\alpha_2, \beta_2) \quad \text{if and only if } \alpha_1 \leqslant \alpha_2 \text{ and } \beta_1 \leqslant \beta_2.$$

But if one can always decompose a normed algebra into the direct sum of two normed algebras one of which has a left approximate identity and the other has a right approximate identity, then it will not be so meaningful to define (approximate) quasi-identities since one can always reduce to the case of normed algebras with a one-sided approximate identity. In fact, a TRO with a predual always can be decomposed in such a way ([18]). We thank Takeshi Katsura for asking for such an example that cannot be decomposed into the direct sum of two normed algebras with a one-sided identity. The following simple example answers the question. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3(\mathbb{C})$ supported on the $(1,1)$-, $(1,2)$-, $(1,3)$-, $(2,3)$-, and $(3,3)$-entries only, where $\mathbb{M}_3(\mathbb{C})$ is equipped with the usual matrix operations. Then $\mathcal{A}$ has neither left nor right identity, and cannot be decomposed into the direct sum of any two algebras, but does have a quasi-identity

$E_1 + E_3$, where E_i denotes the matrix whose (i,i)-entry is 1 and all other entries are 0's.

The following proposition convinces us that the notion of (approximate) quasi-identities is a natural and in a certain sense "minimal" generalization of (approximate) identities or (approximate) one-sided identities at least in the operator algebra case.

PROPOSITION 4.4. (i) *If e is a quasi-identity of a ring, then so is e^n for each $n \in \mathbb{N}$.*

(ii) *A contractive quasi-identity of a normed algebra is an idempotent, and hence its norm is* 1 *unless the algebra is trivial.*

(iii) *A contractive quasi-identity of an operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ is unique if it exists, and is Hermitian (hence an orthogonal projection).*

Proof. (i) Let e be a quasi-identity of a ring $\mathcal{R}$. For brevity of writing, we add an identity 1 to $\mathcal{R}$ if it does not have one. Then

$$\Big(\sum_{k=1}^{n}(1-e)e^{k-1}\Big)r\Big(\sum_{l=1}^{n}e^{l-1}(1-e)\Big) = 0, \quad \forall r \in \mathcal{R}$$

since $(1-e)\mathcal{R}(1-e) = \{0\}$. But each series is a "telescoping series", and the equation is simplified to $(1-e^n)r(1-e^n) = 0, \forall r \in \mathcal{R}$, which means that e^n is a quasi-identity of $\mathcal{R}$.

(ii) Let e be a contractive quasi-identity of a normed algebra. Then $e = 2e^2 - e^3$, that is, $e(e-e^2) = e - e^2$. Therefore inductively $e^n(e-e^2) = e-e^2, \forall n \in \mathbb{N}$. Thus

$$e - e^{n+1} = \sum_{k=1}^{n}(e^k - e^{k+1}) = \sum_{k=1}^{n} e^{k-1}(e-e^2) = n(e-e^2),$$

and so $1 \geqslant \|e^{n+1}\| \geqslant n\|e-e^2\| - \|e\|$. Since this is true for all $n \in \mathbb{N}$, we have that $e^2 = e$.

(iii) Let e and e' be contractive quasi-identities of an operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$, hence they are idempotents by (ii), and hence they are Hermitian by Lemma 4.1(i). Since e is a quasi-identity, $e(e'-e'e) = e'-e'e$. Multiplying both sides by e' on the left and right yields that $e'e(e'-e'ee') = e'-e'ee'$ since e' is an idempotent. Therefore inductively $(e'e)^n(e'-e'ee') = e'-e'ee', \forall n \in \mathbb{N}$. Thus

$$e' - (e'e)^n e' = \sum_{k=1}^{n}(e'e)^{k-1}(e'-e'ee') = n(e'-e'ee'),$$

and hence $1 \geqslant \|(e'e)^n e'\| \geqslant n\|e'-e'ee'\| - \|e'\|, \forall n \in \mathbb{N}$. Therefore

$$e' = e'ee', \tag{4.1}$$

and so $e'e = (e'e)^2$, that is, $e'e$ is an idempotent. By Lemma 4.1(i), $e'e = (e'e)^* = ee'$. Thus by equation (4.1), $e' = (ee')e' = ee'$ since e' is an idempotent. By symmetry, $e = e'e$, and hence $e' = ee' = e'e = e$. ∎

In particular, if an operator algebra has a contractive one-sided or two-sided identity, then it is the only contractive quasi-identity.

As the following proposition shows, if a C^*-algebra has a quasi-identity (contractiveness is not assumed a priori), then it is necessarily an identity.

PROPOSITION 4.5. *If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ possesses a quasi-identity if and only if $\mathcal{A}$ is unital. In this case, the identity is the only quasi-identity.*

Proof. Let $\mathcal{A}$ be a nonzero C^*-algebra, and let $\{e_\alpha\}$ be an approximate identity of $\mathcal{A}$. Suppose that $\mathcal{A}$ has a quasi-identity e. We may assume that $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ nondegenerate, and denote the identity of $\mathbb{B}(\mathcal{H})$ by 1. Then $(1-e)a(1-e) = 0$, $\forall a \in \mathcal{A}$. In particular, for $\xi \in \mathcal{H}$, $(1-e)(e_\alpha - e)^*(1-e)\xi = 0$. By taking the limit with respect to α, we have that $(1-e)(1-e)^*(1-e)\xi = 0$. Since $\xi \in \mathcal{H}$ is arbitrary, $(1-e)(1-e)^*(1-e) = 0$, so that $(1-e)^*(1-e)(1-e)^*(1-e) = 0$, and thus $1 = e \in \mathcal{A}$. ∎

COROLLARY 4.6. *If J is a left (respectively, right) ideal in a C^*-algebra, then J possesses a contractive quasi-identity if and only if J has a contractive right (respectively, left) identity. In this case, the contractive right (respectively, left) identity is the only contractive quasi-identity.*

Proof. The second statement was already observed after Proposition 4.4, and the "if" direction of the first statement is trivial. Assume that a left ideal J in a C^*-algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ has a contractive quasi-identity e. Then e is also a quasi-identity of the weak*-closure $\overline{J}^{w^*}$ of J in $\mathbb{B}(\mathcal{H})$. Let f be the identity of the von Neumann algebra $\overline{J^*J}^{w^*}$ which is a subalgebra of $\overline{J}^{w^*}$. Then f is a contractive right identity of $\overline{J}^{w^*}$. Obviously, fef is a contractive quasi-identity of $\overline{J^*J}^{w^*}$, and hence by Proposition 4.5, $fef = f$. Thus $e = ef = (ef)^* = fe = fef = f$, where we used the fact that e and f are Hermitian. ∎

We are now in a position to present the characterization theorems.

THEOREM 4.7. *Let X be a nonzero operator space, $z \in$ Ball$(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra.*

(i) *(X, m_z) has a quasi-identity of norm 1 if $z^* \in$ ext(Ball(X)).*

(ii) *(X, m_z) has a right identity of norm 1 if and only if $z^* \in \mathcal{UR}_{\rm loc}(X)$.*

(iii) *(X, m_z) has a left identity of norm 1 if and only if $z^* \in \mathcal{UL}_{\rm loc}(X)$.*

(iv) *(X, m_z) has a two-sided identity of norm 1 if and only if $z^* \in \mathcal{U}_{\rm loc}(X)$.*

In each statement, z^* is the quasi- (respectively, right, left, two-sided) identity of norm 1.

Proof. To see (i), assume that $z^* \in$ ext(Ball(X)), then $\|z\| = 1$. Let TER$(X) := X \cap \mathcal{QM}(X)^*$ as in Definition 4.6 in [20]. Then TER(X) is a TRO, and $z^* \in$ ext(Ball(TER(X))). If $\mathcal{A}$ is the C*-algebra generated by $\begin{bmatrix} O & \mathrm{TER}(X) \\ O & O \end{bmatrix}$ and $\begin{bmatrix} 1_{11} & 0 \\ 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 0 \\ 0 & 1_{22} \end{bmatrix}$ in $I(\mathcal{S}(X))$, then $\mathrm{TER}(X) = 1_{11}\mathcal{A}1_{22}$. Thus by Kadison's theorem (Lemma 2.9), $(1_{11} - z^*z)\mathrm{TER}(X)(1_{22} - zz^*) = \{0\}$, and z is a partial isometry. We claim that $(1_{11} - z^*z)X(1_{22} - zz^*) = \{0\}$. Suppose the contrary, and pick $x_0 \in X$ with $\|x_0\| \leqslant 1$ such that $x_0 = (1_{11} - z^*z)x_0(1_{22} - zz^*) \neq 0$. Then

$$\|z^* \pm x_0\|^2 = \|zz^* + x_0^*x_0\| = \max\{\|z\|^2, \|x_0\|^2\} = 1,$$

and so $z^* \pm x_0 \in \mathrm{Ball}(X)$, and

$$z^* = \frac{1}{2}(z^* + x_0) + \frac{1}{2}(z^* - x_0).$$

This contradicts the fact that $z^* \in \mathrm{ext}(\mathrm{Ball}(X))$. Thus $(1_{11} - z^*z)X(1_{22} - zz^*) = \{0\}$ as claimed, i.e., $x = z^*zx + xzz^* - z^*zxzz^* \in X, \forall x \in X$, which tells us that $z^* \in X$ is a quasi-identity, and (i) has been shown. (ii) was observed in Proposition 2.10 in [20], and (iii) is similar by symmetry, and (iv) follows from (ii) and (iii). ∎

REMARK 4.8. (i) The converse direction in (i) of the above theorem does not hold. In fact, as we saw earlier in this section (before Proposition 4.4), let

$$X := \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ O & O & \mathbb{C} \\ O & O & \mathbb{C} \end{bmatrix} \subset \mathbb{M}_3(\mathbb{C})$$

with the usual matrix norm inherited from the operator norm of $\mathbb{M}_3(\mathbb{C})$. Then

$$\mathcal{QM}(X) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \\ O & \mathbb{C} & \mathbb{C} \end{bmatrix}.$$

Let $z = I_3 \in \mathcal{QM}(X)$, where I_3 denotes the identity matrix. Then (X, m_z) has a contractive quasi-identity $E_1 + E_3$, but z^* is not in X.

(ii) By Corollary 2.10, z^* in (ii)–(iv) of the above theorem is an extreme point of $\mathrm{Ball}(I(X))$, and hence an extreme point of $\mathrm{Ball}(\mathcal{QM}(X)^*)$, $\mathrm{Ball}(\mathcal{T}(X))$, $\mathrm{Ball}(X)$, $\mathrm{Ball}(\mathcal{T}(X) \cap \mathcal{QM}(X)^*)$ and $\mathrm{Ball}(\mathrm{TER}(X))$ too. However, in (i) of the theorem, z^* is not an extreme point of $\mathrm{Ball}(\mathcal{QM}(X)^*)$ (hence not an extreme point of $\mathrm{Ball}(I(X))$) or $\mathrm{Ball}(\mathcal{T}(X))$ in general, though it is an extreme point of $\mathrm{Ball}(\mathrm{TER}(X))$ as observed in the proof of Theorem 4.7. In fact, let X be as in the example in item (i) of this remark. Then $\mathcal{T}(X) = \mathbb{M}_3(\mathbb{C})$. Let $z = E_1 + E_3 \in \mathcal{QM}(X)$. Then $z^* \in \mathrm{ext}(\mathrm{Ball}(X))$. But z^* is not an extreme point of $\mathrm{Ball}(\mathcal{QM}(X)^*)$ or $\mathrm{Ball}(\mathcal{T}(X))$.

(iii) That $z \in \mathrm{ext}(\mathrm{Ball}(\mathcal{QM}(X)))$ does not imply that (X, m_z) has an approximate quasi-identity. To see this, let

$$X := \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \\ O & \mathbb{C} & \mathbb{C} \end{bmatrix},$$

which is a "dual" of the example above. Then

$$\mathcal{QM}(X) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ O & O & \mathbb{C} \\ O & O & \mathbb{C} \end{bmatrix}.$$

Let $z = E_1 + E_3 \in \mathcal{QM}(X)$. Then $z \in \text{ext}(\text{Ball}(\mathcal{QM}(X)))$, but (X, m_z) does not have an approximate quasi-identity.

Looking at (i) of Theorem 4.7 from a different point of view by not restricting X to a particular algebrization, we can construct an "if and only if" statement as follows.

COROLLARY 4.9. *Let X be a nonzero operator space. Then some algebrization of X admits a quasi-identity e of norm* 1 *if and only if* $e \in \text{ext}(\text{Ball}(X)) \cap \mathcal{QM}(X)^*$.

Proof. The direction "$\Leftarrow$" follows from Theorem 4.7(i), so we shall prove the converse "$\Rightarrow$". Let $e \in X$ with $\|e\| = 1$ be a quasi-identity of (X, m_z) for some $z \in \text{Ball}(\mathcal{QM}(X))$. Then by Proposition 4.4(ii), $eze = m_z(e, e) = e$ and so ez and ze are idempotents, and hence they are Hermitian by Lemma 4.1(i). Thus by the same argument as the one after equation (4.6) in the proof of Corollary 4.12, we obtain that $e^*e = ze$. Similarly, $ee^* = ez$. Therefore $\forall x, y \in X, xe^*y = xe^*z^*e^*y = xe^*ezy = xzezy \in X$, and hence $e^* \in \mathcal{QM}(X)$. Since e is a quasi-identity of (X, m_z), $(1_{11} - ez)X(1_{22} - ze) = \{0\}$. But since $ez = ee^*$ and $ze = e^*e$, we have that $(1_{11} - ee^*)X(1_{22} - e^*e) = \{0\}$. Now we can show that $e \in \text{ext}(\text{Ball}(X))$ by the same way as proving Kadison's theorem (Lemma 2.9), although X is not a TRO in general. See the proof of Proposition 1.4.7 in [23], for example. ∎

REMARK 4.10. (i) It is important to note that in the proof above, z need not be e^* in general. See the example in Remark 4.8(i). However, re-choosing $z = e^*$ also allows (X, m_z) to have a quasi-identity e by Theorem 4.7(i).

(ii) It is worth noting that this corollary implies that a quasi-identity, if it exists, must be an extreme point of the unit ball of the underlying operator space, but need not be an extreme point of the unit ball of the adjoint of its quasi-multiplier space as the example in Remark 4.8(ii) indicated.

The author thanks the referee for asking a question which resulted in adding the following corollary. This corollary suggests that adding the more restrictive condition "z^* is a quasi-identity" on the left-hand side of (i) of Theorem 4.7 also yields an "if and only if" statement.

COROLLARY 4.11. *Let X be a nonzero operator space, $z \in$ Ball$(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra. Then z^* is a quasi-identity of (X, m_z) if and only if $z^* \in$ ext(Ball(X)). In this case, the quasi-identity z^* is necessarily of norm* 1.

Proof. The direction "$\Leftarrow$" has been proved in Theorem 4.7(i). The converse "$\Rightarrow$" can be observed as follows. That z^* is a quasi-identity of (X, m_z) means $(1_{11} - z^*z)X(1_{22} - zz^*) = \{0\}$. Although X is not a TRO in general, this equality tells us that z^* is an extreme point of $\mathrm{Ball}(X)$ as remarked at the last stage of the proof of Corollary 4.9. ∎

Extreme points best match quasi-identities (actually they are equal) when an operator space is injective as the following corollary shows. This fact convinces us that our attempt to characterize extreme points in terms of quasi-identities is a correct direction, and that defining multipliers (especially, quasi-multipliers) with the use of injective envelopes is the most plausible way. However, we remark that the alternative definitions $\mathcal{QM}''(X)$ and $\mathcal{QM}_\pi(X)$ defined in Section 3 become useful when we deal with "approximate" identities as we will see in Theorem 4.13, and that it is also possible to present Theorem 4.7, Corollary 4.9, Corollary 4.11, and Corollary 4.12 in terms of $\mathcal{QM}''(X)$ or $\mathcal{QM}_\pi(X)$ in place of $\mathcal{QM}(X)$ using Definition 3.5.

COROLLARY 4.12. *If X is injective in addition to the assumptions of Theorem* 4.7, *then the converse of* (i) *in that theorem also holds, that is, (X, m_z) has a quasi-identity of norm* 1 *if and only if $z^* \in \mathrm{ext}(\mathrm{Ball}(X))$. In this case, z^* is the quasi-identity of norm* 1.

Proof. To show "⇒", note that if an operator space X is injective, then $X = I(X) = \mathcal{QM}(X)^*$, and hence $z^* \in X$. Let e be a quasi-identity of norm 1. Then

$$(1_{11} - ez)X(1_{22} - ze) = \{0\}. \tag{4.2}$$

By multiplying both sides by z on the right, we have that $(1_{11} - ez)Xz(1_{11} - ez) = \{0\}$. By choosing $(1_{11} - ez)^*z^* \in X$ and multiplying both sides by $(1_{11} - ez)^*$ on the right, we have that $(1_{11} - ez)(1_{11} - ez)^*z^*z(1_{11} - ez)(1_{11} - ez)^* = 0$, which implies that $z(1_{11} - ez)(1_{11} - ez)^* = 0$. Hence $z(1_{11} - ez)(1_{11} - ez)^*z^* = 0$, and accordingly, $z(1_{11} - ez) = 0$, so that

$$z = zez. \tag{4.3}$$

Thus

$$ez = ezez \quad \text{and} \quad ze = zeze, \tag{4.4}$$

which means that ez and ze are idempotents. (Equations (4.4) is actually an immediate consequence of Proposition 4.4(ii). But we deduced equation (4.3) since we use it toward the end of the proof.) Therefore by Lemma 4.1(i), ez and ze are orthogonal projections, and hence

$$ez = (ez)^* = z^*e^* \quad \text{and} \quad ze = (ze)^* = e^*z^*. \tag{4.5}$$

Hence equation (4.2) is rewritten to $(1_{11} - z^*e^*)X(1_{22} - e^*z^*) = \{0\}$, and by repeating the argument above equation (4.3), we obtain that

$$e^* = e^*z^*e^*. \tag{4.6}$$

(Alternatively, one can obtain equation (4.6) from Proposition 4.4(ii). In fact, $e = m_z(e,e) = eze$.) Thus $e^*e \geqslant e^*z^*ze \geqslant e^*z^*e^*eze = e^*e$, and together with equations (4.5) and (4.4), we have that $e^*e = e^*z^*ze = zeze = ze$. Hence together with equations (4.6), (4.5), and (4.3), we have that $e^* = e^*z^*e^* = e^*ez = zez = z$.

Thus equation (4.2) becomes $(1_{11} - z^*z)X(1_{22} - zz^*) = \{0\}$, which tells that $z^* \in \mathrm{ext}(\mathrm{Ball}(X))$ by Kadison's theorem (Lemma 2.9). ∎

In light of the above corollary, it is natural to ask the following question.

QUESTION 2. If an operator space X is injective, then always $\mathrm{ext}(\mathrm{Ball}(X)) \neq \varnothing$?

If the answer is yes, then any injective operator space can be made into an operator algebra with a quasi-identity of norm 1 by (i) of the above corollary. More generally, one may ask the following question.

QUESTION 3. For any operator space X, $\mathrm{ext}(\mathrm{Ball}(\mathcal{QM}(X))) \neq \varnothing$?

If this question has an affirmative answer, then so does Question 2 since $X = \mathcal{QM}(X)^*$ for an injective operator space X.

The alternative definitions of multipliers which we defined in Section 3 work in the "approximate" version of characterization.

THEOREM 4.13. *Let X be a nonzero operator space, z be in* $\mathrm{Ball}(\mathcal{QM}''(X))$ *(respectively,* $\mathrm{Ball}(\mathcal{QM}_\pi(X))$*), and (X, m_z) be the corresponding operator algebra. Then the following implications hold, where the weak*-closure is taken in $C^*(\partial X)''$ (respectively, $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$).*

(1) (i) (X, m_z) *has a contractive weak approximate quasi-identity;*
$\Leftarrow$ (ii) $z^* \in \mathrm{ext}(\mathrm{Ball}(\overline{X}^{w^*}))$.
(2) (i) (X, m_z) *has a contractive approximate right identity;*
$\Leftrightarrow$ (ii) $z^* \in \mathcal{UR}_{\mathrm{loc}}(\overline{X}^{w^*})$.
(3) (i) (X, m_z) *has a contractive approximate left identity;*
$\Leftrightarrow$ (ii) $z^* \in \mathcal{UL}_{\mathrm{loc}}(\overline{X}^{w^*})$.
(4) (i) (X, m_z) *has a contractive approximate two-sided identity;*
$\Leftrightarrow$ (ii) $z^* \in \mathcal{U}_{\mathrm{loc}}(\overline{X}^{w^*})$.

Proof. First, we shall show (1) in the case that $z \in \mathrm{Ball}(\mathcal{QM}''(X))$. We consider $C^*(\partial X)$ to be universally represented on the Hilbert space $\mathcal{H}_{\mathrm{u}}$, so that the second dual $C^*(\partial X)''$ is the strong closure of $C^*(\partial X)$ in $\mathbb{B}(\mathcal{H}_{\mathrm{u}})$. Suppose that $z^* \in \mathrm{ext}(\mathrm{Ball}(\overline{X}^{w^*})) \cap \mathrm{Ball}(\mathcal{QM}''(X)^*)$. Then $\|z\| = 1$, and z^* is an extreme point of the unit ball of the weak*-closed TRO $\mathrm{WTER}(X) := \overline{X}^{w^*} \cap \overline{\mathcal{QM}''(X)^*}^{w^*}$ as well. Thus by Kadison's theorem (Lemma 2.9), $(1_{\mathcal{E}} - z^*z)\mathrm{WTER}(X)(1_{\mathcal{F}} - zz^*) = \{0\}$. Then by the same argument as in the proof of Theorem 4.7(i), we obtain that

$$x = z^*zx + xzz^* - z^*zxzz^*, \quad \forall x \in \overline{X}^{w^*}. \tag{4.7}$$

Pick a net $\{e_\alpha\} \subset X$ of contractions such that $\text{SOT-}\lim_\alpha e_\alpha = z^*$. Then, clearly $e_\alpha zx \xrightarrow{\text{SOT}} z^*zx$ and $xze_\alpha \xrightarrow{\text{SOT}} xzz^*$, and also $\forall \xi, \eta \in \mathcal{H}_{\mathrm{u}}$,

$$|\langle (e_\alpha zxze_\alpha - z^*zxzz^*)\xi, \eta\rangle| \leqslant |\langle (e_\alpha zxze_\alpha - e_\alpha zxzz^*)\xi, \eta\rangle| + |\langle (e_\alpha zxzz^* - z^*zxzz^*)\xi, \eta\rangle|$$

$$\leqslant \|(e_\alpha - z^*)\xi\|\|\eta\| + |\langle (e_\alpha - z^*)zxzz^*\xi, \eta\rangle| \xrightarrow{\alpha} 0.$$

Thus in light of (4.7) we have that $e_\alpha zx + xze_\alpha - e_\alpha zxze_\alpha \to x$ in the WOT, and hence in the weak* topology (= σ-weak operator topology) since the convergence is on a bounded set. In particular, for $x \in X$ the convergence is also in the weak topology, and (1) has been proved in the case that $z \in \mathrm{Ball}(\mathcal{QM}''(X))$.

(1) in the case that $z \in \mathrm{Ball}(\mathcal{QM}_\pi(X))$ is the same except that $C^*(\partial X)$ is represented on $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and that the weak* topology on $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is concerned.

Next, we shall show "(i)$\Leftarrow$(ii)" of (2) in the case that $z \in \mathrm{Ball}(\mathcal{QM}''(X))$ (the case $z \in \mathrm{Ball}(\mathcal{QM}_\pi(X))$ is similar). Suppose that $z^* \in \mathcal{UR}_{\mathrm{loc}}(\overline{X}^{\mathrm{w}^*})$, then $xzz^* = x$, $\forall x \in \overline{X}^{\mathrm{w}^*}$. Pick a net $\{e_\alpha\} \subset \mathrm{Ball}(X)$ such that $\mathrm{w}^*\text{-}\lim_\alpha e_\alpha = z^*$. By the separate weak*-continuity of the product in $C^*(\partial X)''$, we have that

$$\mathrm{w}^*\text{-}\lim_\alpha xze_\alpha = xzz^* = x, \quad \forall x \in \overline{X}^{\mathrm{w}^*},$$

and thus

$$\mathrm{w}\text{-}\lim_\alpha xze_\alpha = x, \quad \forall x \in X. \tag{4.8}$$

Now we adopt a technique employed in the proof of Theorem 2.2 in [9]. Let $\mathcal{F}$ be the collection of the finite subsets of X, and let $\Lambda := \mathcal{F} \times \mathbb{N}$. Then Λ is a directed set under the ordering "$(F_1, n_1) \leqslant (F_2, n_2)$ if and only if $F_1 \subset F_2$ and $n_1 \leqslant n_2$". Given $F = \{x_1, \ldots, x_m\} \in \mathcal{F}$, let

$$V_F := \{(x_1ze - x_1, \ldots, x_mze - x_m) : e \in \mathrm{Ball}(X)\} \subset X^m,$$

where X^m is given the supremum norm. It follows from (4.8) that $\vec{0} := (0_1, \ldots, 0_m)$ lies in the weak closure of V_F in X^m, and hence it lies in the norm closure of V_F in X^m since V_F is convex. Therefore for a given $n \in \mathbb{N}$,

$$V_F \cap \{\vec{x} \in X^m; \|\vec{x}\| < 1/n\} \neq \varnothing.$$

The argument above tells us that for a given $(F, n) \in \Lambda$, we may choose $e_\lambda \in \mathrm{Ball}(X)$ with $\|x_kze_\lambda - x_k\| < 1/n$ for $k = 1, \ldots, m$. Hence we have obtained a contractive net $\{e_\lambda\}$ such that $\lim_\lambda xze_\lambda = x \in X$, $\forall x \in X$, and "(i)$\Leftarrow$(ii)" of (2) has been shown.

"(i)$\Leftarrow$(ii)" of (3)–(4) are similar.

To see "(i)$\Rightarrow$(ii)" of (2)–(4), we show "(i)$\Rightarrow$(ii)" of (3) in the case that $z \in \mathrm{Ball}(\mathcal{QM}''(X))$. The others are similar. Let $\{e_\alpha\} \subset X$ be a contractive approximate left identity of (X, m_z), and let e be its weak* accumulation point in $\overline{X}^{\mathrm{w}^*}$. Then $(1_{\mathcal{E}} - ez)X = \{0\}$, so that $ez = 1_{\mathcal{E}}$. Thus by Lemma 4.1(ii), $e = z^*$. ∎

REMARK 4.14. (i) The implication (i)$\Rightarrow$(ii) of (1) does not hold. See the example in Remark 4.8(i).

(ii) In the case that $z \in \mathrm{Ball}(\mathcal{QM}''(X))$, by Corollary 3.6(i), z^* in (ii) of (2)–(4) is an extreme point of $\mathrm{Ball}(\mathcal{T}(X)'')$, and hence an extreme point of

$$\mathrm{Ball}(\overline{\mathcal{QM}''(X)^*}^{\mathrm{w}^*}), \quad \mathrm{Ball}(\mathcal{QM}''(X)^*), \quad \mathrm{Ball}(\overline{X}^{\mathrm{w}^*}),$$
$$\mathrm{Ball}(\mathrm{WTER}(X)), \quad \text{and} \quad \mathrm{Ball}(\overline{X}^{\mathrm{w}^*} \cap \mathcal{QM}''(X)^*),$$

too, where $\mathrm{WTER}(X)$ is as defined in the proof of Theorem 4.13(i). However, in (ii) of (1), z^* is not an extreme point of $\mathrm{Ball}(\mathcal{QM}''(X)^*)$ (hence not an extreme point of $\mathrm{Ball}(\overline{\mathcal{QM}''(X)^*}^{\mathrm{w}^*})$ or $\mathrm{Ball}(\mathcal{T}(X)''))$ in general, though it is an extreme point of $\mathrm{Ball}(\mathrm{WTER}(X))$ (and hence an extreme point of $\mathrm{Ball}(\overline{X}^{\mathrm{w}^*} \cap \mathcal{QM}''(X)^*))$ as observed in the proof above. See the example in Remark 4.8(ii). Here all the weak* closures are taken in $C^*(\partial X)''$.

(iii) In the case that $z \in \mathrm{Ball}(\mathcal{QM}_\pi(X))$, by Corollary 3.6(ii), z^* in (ii) of (2)–(4) is an extreme point of $\mathrm{Ball}(\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1))$, and hence an extreme point of

$$\mathrm{Ball}(\overline{\mathcal{QM}_\pi(X)^*}^{\mathrm{w}^*}), \quad \mathrm{Ball}(\mathcal{QM}_\pi(X)^*), \quad \mathrm{Ball}(\overline{\mathcal{T}(X)}^{\mathrm{w}^*}), \quad \mathrm{Ball}(\overline{X}^{\mathrm{w}^*}),$$
$$\mathrm{Ball}(\overline{\mathcal{T}(X)}^{\mathrm{w}^*} \cap \overline{\mathcal{QM}_\pi(X)^*}^{\mathrm{w}^*}), \quad \mathrm{Ball}(\overline{\mathcal{T}(X)}^{\mathrm{w}^*} \cap \mathcal{QM}_\pi(X)^*),$$
$$\mathrm{Ball}(\mathrm{WTER}(X)), \quad \text{and} \quad \mathrm{Ball}(\overline{X}^{\mathrm{w}^*} \cap \mathcal{QM}_\pi(X)^*),$$

too, where $\mathrm{WTER}(X) := \overline{X}^{\mathrm{w}^*} \cap \overline{\mathcal{QM}_\pi(X)^*}^{\mathrm{w}^*}$ with the weak* closures taken in $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. However, in (ii) of (1), z^* is not an extreme point of $\mathrm{Ball}(\mathcal{QM}_\pi(X)^*)$ (hence not an extreme point of $\mathrm{Ball}(\overline{\mathcal{QM}_\pi(X)^*}^{\mathrm{w}^*}))$ or $\mathrm{Ball}(\overline{\mathcal{T}(X)}^{\mathrm{w}^*})$ in general, though it is an extreme point of $\mathrm{Ball}(\mathrm{WTER}(X))$ and hence an extreme point of $\mathrm{Ball}(\overline{X^*}^{\mathrm{w}^*} \cap \mathcal{QM}_\pi(X)^*)$. See the example in Remark 4.8(ii). Here all the weak* closures are taken in $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

In Theorem 4.13(1), we do not know whether a weak approximate quasi-identity can be replaced by an approximate quasi-identity. The difficulty lies in the following point. When one attempts to adopt a technique employed in Theorem 2.2 in [9] as we did in the proof of "(i)⇐(ii)" in (2) of Theorem 4.13, one comes up with the set

$$V := \{(ezx_1 + x_1ze - ezx_1ze - x_1, \ldots, ezx_m + x_mze - ezx_mze - x_m) : e \in \mathrm{Ball}(X)\} \subset X^m,$$

where $x_1, \ldots, x_m \in X$ are fixed. The problem is that this set does not seem to be convex. Thus we leave this as an open problem.

QUESTION 4. Does $z^* \in \mathrm{ext}(\mathrm{Ball}(\overline{X}^{\mathrm{w}^*}))$ imply that (X, m_z) admits a contractive approximate quasi-identity?

Related to Question 2 which is about injective operator spaces, the following question will be reasonable in light of the fact that an injective operator space is a TRO.

QUESTION 5. Does a TRO always have an algebrization which admits a contractive (weak) approximate quasi-identity?

Now, as mentioned after Definition 4.2, we provide an example of an operator algebra $\mathcal{A}$ which admits a contractive approximate left identity $\{e_\alpha\}$, but does not possess one for which $\lim_\alpha ae_\alpha$ exists for all $a \in \mathcal{A}$. We thank David P. Blecher for the basic idea of the example.

EXAMPLE 4.15. Let us canonically identify $\mathbb{B}\Big(\bigoplus_{n=1}^{\infty} l^2(\mathbb{N})\Big)$ with a subset of the set $\mathbb{M}(\mathbb{B}(l^2(\mathbb{N})))$ of $\aleph_0 \times \aleph_0$ matrices with entries in $\mathbb{B}(l^2(\mathbb{N}))$, and $\mathbb{B}(l^2(\mathbb{N}))$ with a subset of the set $\mathbb{M}(\mathbb{C})$ of $\aleph_0 \times \aleph_0$ matrices with entries in $\mathbb{C}$. Let

$$\mathcal{B} := \Big\{ [a_{i,j}] \in \mathbb{B}\Big(\bigoplus_{n=1}^{\infty} l^2(\mathbb{N})\Big) \subset \mathbb{M}(\mathbb{B}(l^2(\mathbb{N}))) : \exists i_0 \in \mathbb{N} \text{ such that } a_{i,j} = 0, \forall i \geqslant i_0, \forall j \in \mathbb{N} \Big\},$$

and let $\mathcal{A}$ be the norm closure of $\mathcal{B}$ in $\mathbb{B}\Big(\bigoplus_{n=1}^{\infty} l^2(\mathbb{N})\Big)$. Then $\mathcal{A}$ is a right ideal of the von Neumann algebra $\mathbb{B}\Big(\bigoplus_{n=1}^{\infty} l^2(\mathbb{N})\Big)$, so it has a contractive approximate left identity. (For example, the sequence $\{e_n\}$ with $e_n \in \mathcal{A}$ whose first n diagonal entries are the identity operators on $l^2(\mathbb{N})$ and all the other entries are zero operators, is a contractive approximate left identity of $\mathcal{A}$.) For each $j \in \mathbb{N}$, let us denote by E_j the element of $\mathbb{B}(l^2(\mathbb{N})) \subset \mathbb{M}(\mathbb{C})$ whose (j,j)-entry is 1 and other entries are 0's. Define $a = [a_{i,j}] \in \mathcal{B}$ as follows: $a_{1,j} := E_j, \forall j \in \mathbb{N}$; $a_{i,j} = 0, \forall i \geqslant 2, \forall j \in \mathbb{N}$. Also for each $i \in \mathbb{N}$, define $p_i \in \mathbb{M}(\mathbb{B}(l^2(\mathbb{N})))$ as follows: the (i,i)-entry of p_i is the identity operator on $\mathbb{B}(l^2(\mathbb{N}))$, and all other entries of p_i are zero operators. Note that both ap_i and $(ap_i)^*$ are in $\mathcal{A}$. Let $\{e_\alpha\}$ be "any" contractive approximate left identity of $\mathcal{A}$. By Lemma 2.2(1) in [3], $\lim_\alpha e_\alpha^* b = b, \forall b \in \mathcal{A}$. In particular, $\lim_\alpha e_\alpha^*(ap_i)^* = (ap_i)^*$, and hence $\lim_\alpha ap_i e_\alpha = ap_i$. Let e_{α_0} be any element in $\{e_\alpha\}$, and pick $e \in \mathcal{B}$ such that $\|e - e_{\alpha_0}\| < 1/3$. Then there exists an $m_0 \in \mathbb{N}$ such that the entries in the m-th row of e are all 0's for all $m \geqslant m_0$, and in particular, $p_{m_0} e = 0$. Choose e_{α_1} so that $\alpha_1 \geqslant \alpha_0$ and $\|ap_{m_0} e_{\alpha_1}\| > 2/3$. Now it follows that

$$\begin{aligned}\|ap_{m_0}e_{\alpha_1} - ap_{m_0}e_{\alpha_0}\| &\geqslant \|ap_{m_0}e_{\alpha_1} - ap_{m_0}e\| - \|ap_{m_0}e - ap_{m_0}e_{\alpha_0}\| \\ &\geqslant \|ap_{m_0}e_{\alpha_1}\| - \|ap_{m_0}\|\|e - e_{\alpha_0}\| > \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.\end{aligned}$$

Since e_{α_0} is any element in $\{e_\alpha\}$, $\lim_\alpha ap_{m_0}e_\alpha$ does not exist.

As a byproduct of the above example, we obtain the following proposition.

PROPOSITION 4.16. *There exists an operator space X for which there is a non-degenerate representation of the C^*-algebra $I(\mathcal{S}_X)$ on a Hilbert space $\mathcal{H}$ such that the C^*-subalgebra $C^*(\partial X)$ is degenerate on $\mathcal{H}$.*

Proof. Let X be the underlying operator space of the operator algebra $\mathcal{A}$ defined in Example 4.15, $z \in \operatorname{Ball}(\mathcal{QM}(X))$ be the quasi-multiplier associated with $\mathcal{A}$, and $\{e_\alpha\}$ be a contractive approximate left identity of $\mathcal{A} = (X, m_z)$. Take a nondegenerate representation of $I(\mathcal{S}_X)$ on a Hilbert space $\mathcal{H}$. In the proof, the weak* topology concerned is the one on $\mathbb{B}(\mathcal{H})$. Let e be a weak* accumulation point of $\{e_\alpha\}$. Then $x = \lim_\alpha e_\alpha zx = ezx, \forall x \in X$, and hence $(1_{11} - ez)\mathcal{T}(X) = \{0\}$. Suppose that $C^*(\partial X)$ is nondegenerate on $\mathcal{H}$. Then $1_{11} - ez = 0$, and so $e = z^*$ by Lemma 4.1(ii). Thus $xzz^* \in \overline{X}^{w^*} \cap I(\mathcal{S}_X) = X$. Now one can choose a contractive approximate left identity $\{e_\lambda\}$ of $\mathcal{A} = (X, m_z)$ so that $\lim_\lambda m_z(x, e_\lambda) = \lim_\lambda xze_\lambda$ exists in X for all $x \in X$ as in the proof of (i)$\Leftarrow$(ii) of (2) of Theorem 4.13. This contradicts the fact observed in Example 4.15. ∎

We close this section by recalling two examples from [20]. The quasi-multiplier space of the operator space X in Example 2.13 of that paper is $\{0\}$. So the "zero product" is the only possible operator algebra product that X can be equipped with. Hence, there is no algebrization for X to have a quasi-identity. A more interesting example is Example 2.11 of the same paper. Elementary but tedious calculations show that all points on the sphere (i.e., the set of points with norm 1) of $\mathcal{X}$ and $\mathcal{QM}(\mathcal{X})$ are extreme points, and $\mathcal{X}$ can have a quasi-identity for a certain algebrization, however, there is no algebrization for $\mathcal{X}$ to have a "contractive" quasi-identity.

5. C^*-ALGEBRAS AND THEIR ONE-SIDED IDEALS

In this section, we give an operator space characterization of C^*-algebras and their one-sided ideals in terms of quasi-multipliers. The characterization gives the "shapes" of the operator spaces that can be made into C^*-algebras or their one-sided ideals. Although we use $\mathcal{QM}(X)$ and algebraic operations in the injective C^*-algebra $I(\mathcal{S}_X)$, the reader should keep in mind that these characterizations can be reformulated using alternative definitions $\mathcal{QM}''(X)$ or $\mathcal{QM}_\pi(X)$, and algebraic operations in $C^*(\partial X)''$ or $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and Definition 3.5 by modifying the proofs appropriately.

First we characterize one-sided ideals in C^*-algebras. Another characterization of such ideals is given on page 2108 of [5]: left ideals in C^*-algebras are exactly the operator algebras A with a r.c.a.i. that are also abstract triple systems.

THEOREM 5.1. *Let X be a nonzero operator space, and $z \in \operatorname{Ball}(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra. Then there is a completely isometric homomorphism from (X, m_z) onto a left (respectively, right) ideal in some C^*-algebra if and only if $z^* \in \mathcal{UR}_{\mathrm{loc}}(I(X))$, $X^*X \subset zX$, and $Xz \subset XX^*$ (respectively, $z^* \in \mathcal{UL}_{\mathrm{loc}}(I(X))$, $XX^* \subset Xz$, and $zX \subset X^*X$).*

Proof. We prove the left ideal case. The right ideal case is similar by symmetry.

$\Rightarrow$. Assume that (X, m_z) is a left ideal in a C^*-algebra. Then it is a TRO and has a contractive approximate right identity $\{e_\alpha\}$. As in the proof of Theorem 2.3 in [5], there is a $v \in I(X)$ such that $xv^* = \lim_\alpha xe_\alpha^* \in XX^*, \forall x \in X$ and $v^*v = 1_{22}$, where the products are taken in the injective C^*-algebra $I(\mathcal{S}_X)$. Thus

$$v^*X = v^*XX^*X \supset \lim_\alpha v^*e_\alpha X^*X = v^*vX^*X = X^*X.$$

By the first sentence in the proof of Lemma 2.6 of the present paper, $z = v^*$. Hence $Xz \subset XX^*$, $z^* \in \mathcal{UR}_{\text{loc}}(I(X))$, and $X^*X \subset zX$.

$\Leftarrow$. That $X^*X \subset zX$ implies that $XX^*X \subset XzX \subset X$ since $z \in \mathcal{QM}(X)$. So X is a TRO, and hence XX^* is a C^*-algebra. Define $\psi : X \to XX^*$ by $\psi(x) := xz, \forall x \in X$. Then ψ is a completely contractive mapping from X into the C^*-algebra XX^*. In fact ψ is a complete isometry since the right multiplication by the contractive element z^* gives the inverse mapping of ψ, i.e., $\psi(x)z^* = xzz^* = x, \forall x \in X$ since $z^* \in \mathcal{UR}_{\text{loc}}(I(X))$. That $\psi(m_z(x_1, x_2)) = x_1zx_2z = \psi(x_1)\psi(x_2), \forall x_1, x_2 \in X$ shows that ψ is a homomorphism. Since $XX^*Xz \subset Xz$, $\psi(X) = Xz$ is a left ideal in the C^*-algebra XX^*. ∎

Now we give an operator space characterization of C^*-algebras in terms of quasi-multipliers. The characterization is quite simple and it makes a beautiful contrast with the one-sided ideal case above. We remark that the "$\Leftarrow$" directions of the theorem below was essentially first observed by Vern I. Paulsen assuming that X is a unital C^*-algebra and using the classical definition of quasi-multipliers (Section 3.12 of [23]), in which case $\mathcal{QM}(X) = X$. We thank him for letting us know his observation. In the following theorem and its proof we revive the symbols $\odot$ and $\bullet$ defined in Section 2 to avoid confusion.

THEOREM 5.2. *Let X be a nonzero operator space, and $z \in \mathrm{Ball}(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra. Then (X, m_z) is a C^*-algebra with a certain involution $\sharp$ if and only if $z^* \in \mathcal{U}_{\text{loc}}(I(X))$ and $X \bullet z = z^* \bullet X^*$ (or, equivalently, $z \bullet X = X^* \bullet z^*$, or $z^* \bullet X^* \bullet z^* = X$). The involution $\sharp$ is uniquely given by $x^\sharp = z^* \bullet x^* \bullet z^*$, $\forall x \in X$, which implies that for a given operator algebra there exists at most one involution that makes the operator algebra a C^*-algebra. Moreover, all such C^*-algebras are $*$-isomorphic, which recovers the no doubt well-known fact that for a given operator space there exists at most one C^*-algebra structure up to $*$-isomorphism.*

Proof. First note that the equivalence between $X \bullet z = z^* \bullet X^*$, $z \bullet X = X^* \bullet z^*$, and $z^* \bullet X^* \bullet z^* = X$ is straightforward from $z^* \bullet z = 1_{11}$ and $z \bullet z^* = 1_{22}$.

$\Leftarrow$. Let $z^* \in \mathcal{U}_{\text{loc}}(I(X)) \cap \mathrm{Ball}(\mathcal{QM}(X)^*)$ and $z \bullet X = X^* \bullet z^*$. Define an involution $\sharp$ by $x^\sharp := z^* \bullet x^* \bullet z^*$, $\forall x \in X$, where $*$ is the involution on the injective envelope C^*-algebra $I(\mathcal{S}_X)$. Since $z \bullet X = X^* \bullet z^*$, $z^* \bullet x^* \bullet z^*$ is certainly in X. Clearly, $\sharp$ is conjugate linear. And also $(x^\sharp)^\sharp = z^* \bullet z \bullet x \bullet z \bullet z^* = 1_{11} \bullet x \bullet$

$1_{22} = x$. Hence $\sharp$ is a well-defined involution. Then

$$\begin{aligned}\|m_z(x^\sharp, x)\| &= \|z^* \bullet x^* \bullet z^* \bullet z \bullet x\| = \|z^* \bullet x^* \bullet 1_{11} \bullet x\| = \|z^* \bullet x^* \bullet x\| \\ &\geqslant \|z \bullet z^* \bullet x^* \bullet x\| = \|1_{22} \bullet x^* \bullet x\| = \|x^* \bullet x\| = \|x\|^2\end{aligned}$$

shows that $(X, m_z, \sharp)$ is a C^*-algebra.

$\Rightarrow$. Assume that $(X, m_z, \sharp)$ is a C^*-algebra. Then it is a two-sided ideal in itself, so that $z^* \in \mathcal{UR}_{\text{loc}}(I(X)) \cap \mathcal{UL}_{\text{loc}}(I(X)) = \mathcal{U}_{\text{loc}}(I(X))$ by the "$\Rightarrow$" direction of Theorem 5.1. To check that $X \bullet z = z^* \bullet X^*$, we may assume that $(X, m_z, \sharp) \subset \mathbb{B}(\mathcal{K})$ as a C^*-subalgebra for some Hilbert space $\mathcal{K}$. Let $\mathcal{S}'_X := \begin{bmatrix} \mathbb{C}1_{\mathcal{K}} & X \\ X^\sharp & \mathbb{C}1_{\mathcal{K}} \end{bmatrix} \subset \mathbb{M}_2(\mathbb{B}(\mathcal{K}))$ (actually $X^\sharp = X$) be the Paulsen operator system, and $C^*(X) = \mathbb{M}_2(X)$ be the C^*-algebra generated by $\begin{bmatrix} O & X \\ O & O \end{bmatrix}$ in $\mathbb{M}_2(\mathbb{B}(\mathcal{K}))$. By using Hamana's theorem (Corollary 4.2 in [11]) it is easily seen that there is a $*$-homomorphism $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$, which is factored by a well-known trick, from $C^*(X)$ onto $C^*(\partial X)$ such that $\Psi_{12}(x) = x$ (and hence $\Psi_{21}(x^\sharp) = (\Psi_{12}(x))^* = x^*$), $\forall x \in X$, where $C^*(\partial X)$ is as in Section 2. Let $\{e_\alpha\}$ be a contractive approximate two-sided identity of the C^*-algebra $(X, m_z, \sharp)$. Then

$$\begin{aligned}\begin{bmatrix} 0 & m_z(x,y) \\ 0 & 0 \end{bmatrix} &= \lim_\alpha \Psi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e_\alpha^\sharp & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \right) \\ &= \lim_\alpha \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 \\ e_\alpha^* & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} = \lim_\alpha \begin{bmatrix} 0 & x \bullet e_\alpha^* \bullet y \\ 0 & 0 \end{bmatrix}, \quad \forall x, y \in X,\end{aligned}$$

so that $\lim\limits_\alpha x \bullet e_\alpha^* \bullet y = x \bullet z \bullet y$, $\forall x, y \in X$. Now

$$\begin{aligned}\lim_\alpha \begin{bmatrix} x \bullet e_\alpha^* & 0 \\ 0 & 0 \end{bmatrix} &= \lim_\alpha \Psi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) \odot \Psi\left(\begin{bmatrix} 0 & 0 \\ e_\alpha^\sharp & 0 \end{bmatrix} \right) \\ &= \lim_\alpha \Psi\left(\begin{bmatrix} x e_\alpha^\sharp & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \Psi_{11}(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall x \in X.\end{aligned}$$

Thus $\Psi_{11}(x) \bullet y = x \bullet z \bullet y$, $\forall x, y \in X$, and hence by Lemma 2.1(i), $\Psi_{11}(x) = x \bullet z$, $\forall x \in X$, so that $\Psi_{11}(X) = X \bullet z$. On the other hand,

$$\begin{aligned}\lim_\alpha \begin{bmatrix} e_\alpha \bullet x^* & 0 \\ 0 & 0 \end{bmatrix} &= \lim_\alpha \Psi\left(\begin{bmatrix} 0 & e_\alpha \\ 0 & 0 \end{bmatrix} \right) \odot \Psi\left(\begin{bmatrix} 0 & 0 \\ x^\sharp & 0 \end{bmatrix} \right) \\ &= \lim_\alpha \Psi\left(\begin{bmatrix} e_\alpha x^\sharp & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \Psi_{11}(x^\sharp) & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall x \in X.\end{aligned}$$

Thus $y^* \bullet \Psi_{11}(x^\sharp) = y^* \bullet z^* \bullet x^*$, $\forall x, y \in X$, since $\lim\limits_\alpha y^* \bullet e_\alpha \bullet x^* = y^* \bullet z^* \bullet x^*$, $\forall x, y \in X$. Hence by Lemma 2.1(i) again, $\Psi_{11}(x^\sharp) = z^* \bullet x^*$, $\forall x \in X$, so that

$\Psi_{11}(X) = z^* \bullet X^*$ noting that $X^\sharp = X$. Therefore, $X \bullet z = \Psi_{11}(X) = z^* \bullet X^*$. It also follows that $z^* \bullet x^* \bullet z^* = \Psi_{11}(x^\sharp) \bullet z^* = x^\sharp \bullet z \bullet z^* = x^\sharp$, $\forall x \in X$.

Finally, we show that all C^*-algebras which have the same underlying operator space X are $*$-isomorphic. Since this fact is no doubt well known, and a simpler proof (or observation) is possible, it might be redundant to present the proof. However, it would be instructive to show how two quasi-multipliers work out, so we include the proof. Let $z'^* \in \mathcal{U}_{\mathrm{loc}}(I(X))$, and assume that $(X, m_{z'}, \natural)$ is also a C^*-algebra. Then the involution $\natural$ is given by $x^\natural = z'^* \bullet x^* \bullet z'^*$. Define a linear mapping $\pi : (X, m_z, \sharp) \to (X, m_{z'}, \natural)$ by $x \mapsto x \bullet z \bullet z'^*$. We must check that the image is certainly in X. Note that $\Psi_{11}(x) = x \bullet z$ which is one-to-one, and $(\Psi_{11})^{-1}(a) = a \bullet z^*$, $\forall a \in \Psi_{11}(X)$. Considering $(X, m_{z'}, \natural) \subset \mathbb{B}(\mathcal{K}')$ as a C^*-subalgebra for some Hilbert space $\mathcal{K}'$, we can define Ψ'_{11} as we defined Ψ_{11}. Then $(\Psi'_{11})^{-1}(a) = a \bullet z'^*$, $\forall a \in \Psi'_{11}(X)$. By noting that $\Psi_{11}(X) = \Psi'_{11}(X) = \mathcal{E}(X)$, where $\mathcal{E}(X)$ is as in Section 2, we have that for $x \in X$, $x \bullet z \bullet z'^* = (\Psi'_{11})^{-1}(\Psi_{11}(x)) \in X$, so that $\mathrm{Im}\pi \subset X$. Similarly, we have that $x \bullet z' \bullet z^* = (\Psi_{11})^{-1}(\Psi'_{11}(x)) \in X$, $\forall x \in X$. Thus $x = x \bullet z' \bullet z^* \bullet z \bullet z'^* = \pi(x \bullet z' \bullet z^*)$, $\forall x \in X$, which shows that π is onto. π being one-to-one follows from $x = \pi(x) \bullet z' \bullet z^*$, $\forall x \in X$. Furthermore,

$$\pi(m_z(x,y)) = x \bullet z \bullet y \bullet z \bullet z'^* = x \bullet z \bullet z'^* \bullet z' \bullet y \bullet z \bullet z'^* = m_{z'}(\pi(x), \pi(y))$$

and

$$\begin{aligned} \pi(x^\sharp) &= z^* \bullet x^* \bullet z^* \bullet z \bullet z'^* = z^* \bullet x^* \bullet z'^* = z'^* \bullet z' \bullet z^* \bullet x^* \bullet z'^* \\ &= z'^* \bullet (x \bullet z \bullet z'^*)^* \bullet z'^* = \pi(x)^\natural \end{aligned}$$

show that $\pi : (X, m_z, \sharp) \to (X, m_{z'}, \natural)$ is a $*$-homomorphism. ∎

One may expect that the quasi-multiplier space of a C^*-algebra always can be a C^*-algebra for some algebrization, or the quasi-multiplier space of a TRO is a TRO. However, neither of them is true in general. The following example shows that the quasi-multiplier space of a C^*-algebra may not even be a TRO, hence may not be completely isometric to a one-sided ideal in any C^*-algebra.

EXAMPLE 5.3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathbb{K}(\mathcal{H})^1$ denote the unitization of $\mathbb{K}(\mathcal{H})$ by the identity 1 of $\mathbb{B}(\mathcal{H})$, where $\mathbb{K}(\mathcal{H})$ is the set of the compact operators on $\mathcal{H}$. Define $X := \begin{bmatrix} \mathbb{K}(\mathcal{H}) & \mathbb{K}(\mathcal{H}) \\ \mathbb{K}(\mathcal{H}) & \mathbb{K}(\mathcal{H})^1 \end{bmatrix}$. Give X the canonical operator space structure as a subspace of $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$, then X is a C^*-algebra with the product on $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. It is easy to see that the product $\bullet$ defined in Section 2 is the same as the original product on $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ knowing that $I(\mathbb{K}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$ (see Corollary 5.4 in [19]), and $\mathcal{LM}(X) = \begin{bmatrix} \mathbb{B}(\mathcal{H}) & \mathbb{K}(\mathcal{H}) \\ \mathbb{B}(\mathcal{H}) & \mathbb{K}(\mathcal{H})^1 \end{bmatrix}$, and accordingly $\mathcal{QM}(X) = \begin{bmatrix} \mathbb{B}(\mathcal{H}) & \mathbb{B}(\mathcal{H}) \\ \mathbb{B}(\mathcal{H}) & \mathbb{K}(\mathcal{H})^1 \end{bmatrix}$ which is easily seen not to be a TRO.

6. THE QUASI-MULTIPLIER SPACE OF A DUAL OPERATOR SPACE

In this supplementary section, we prove the following theorem. The argument is parallel to that of Corollary 3.2(1) in [1].

THEOREM 6.1. *If X is a nonzero operator space with an operator space predual, then so is $\mathcal{QM}(X)$. Thus by the Banach–Alaoglu theorem* $\mathrm{Ball}(\mathcal{QM}(X))$ *is compact in the weak* topology, and hence by the Krein–Milman theorem* $\mathrm{Ball}(\mathcal{QM}(X))$ *is the weak*-closure of the convex hull of the extreme points of* $\mathrm{Ball}(\mathcal{QM}(X))$.

To prove this, we need a couple of lemmas. Note that if X is an operator space with an operator space predual X_*, then $\mathbb{M}_n(X)$ also has an operator space predual which is given by the operator space projective tensor product $\mathbb{T}_n \widehat{\otimes} X_*$, where $\mathbb{T}_n$ is the set of $n \times n$ trace-class matrices, i.e., $\mathbb{T}_n = \mathbb{M}_n(\mathbb{C})$ as vector spaces, but $\mathbb{T}_n$ is given an operator space structure by the identification $\mathbb{T}_n \cong \mathbb{M}_n(\mathbb{C})'$ with the pairing $\langle \alpha, \beta \rangle := \sum_{i,j} \alpha_{i,j}\beta_{i,j}$, $\forall \alpha = [\alpha_{i,j}] \in \mathbb{T}_n$, $\forall \beta = [\beta_{i,j}] \in \mathbb{M}_n(\mathbb{C})$.

LEMMA 6.2 (Lemma 3.1 in [1]). *Let X and Y be operator spaces, with Y a dual operator space, and let $T : X \to Y$ be a one-to-one linear mapping. Then the following are equivalent:*

(i) *X has an operator space predual such that T is weak*-continuous;*

(ii) $T^{(n)}(\mathrm{Ball}(\mathbb{M}_n(X)))$ *is weak*-compact for every positive integer n;*

where $T^{(n)} : \mathbb{M}_n(X) \to \mathbb{M}_n(Y)$ is defined by $T^{(n)}([x_{i,j}]) := [T(x_{i,j})]$, $\forall [x_{i,j}] \in \mathbb{M}_n(X)$.

LEMMA 6.3. *Let X be a nonzero operator space. Then*

$$\mathbb{M}_n(\mathcal{QM}(X)) \cong \mathcal{QM}(\mathbb{M}_n(X)),$$

completely isometrically.

Proof. The assertion easily follows from $\mathbb{M}_n(I(X)) \cong I(\mathbb{M}_n(X))$ completely isometric, and the definition of the quasi-multipliers (Definition 2.2(iii)). ∎

Proof of Theorem 6.1. It suffices to show that the completely contractive one-to-one mapping $\iota : \mathcal{QM}(X) \to \mathcal{CB}(X \overset{h}{\otimes} X, X)$ defined uniquely by $\iota(z)(x \otimes y) := xzy$ satisfies (ii) of Lemma 6.2. We need to show that if $\{\varphi^\lambda\}$ is a net in $\iota^{(n)}(\mathrm{Ball}(\mathbb{M}_n(\mathcal{QM}(X))))$ converging in the weak* topology to $\varphi \in \mathbb{M}_n(\mathcal{CB}(X \overset{h}{\otimes} X, X))$, then $\iota^{(n)}(\varphi) \in \mathrm{Ball}(\mathbb{M}_n(\mathcal{QM}(X)))$, where we are identifying $\mathcal{CB}(X \overset{h}{\otimes} X, X)$ with $((X \overset{h}{\otimes} X)\widehat{\otimes} X_*)^*$ completely isometrically. But by Lemma 6.3 and the canonical identification of $\mathbb{M}_n(CB(X \overset{h}{\otimes} X, X)) \cong CB(X \overset{h}{\otimes} X, \mathbb{M}_n(X))$ (completely isometric) together with the remark before Lemma 6.2, it is enough to show that if $\{\varphi^\lambda\}$ is a net in $\iota(\mathrm{Ball}(\mathcal{QM}(X)))$ converging in the weak* topology to $\varphi \in \mathcal{CB}(X \overset{h}{\otimes} X, X)$, then $\varphi \in \iota(\mathrm{Ball}(\mathcal{QM}(X)))$. Let $[x_{p,q}], [y_{p,q}], [v_{p,q}], [w_{p,q}] \in$

$\mathbb{M}_m(X)$, $1 \leqslant p, q \leqslant m$. By Theorem 2.3(iii)⇒(ii), we are done if we have shown that

$$\left\| \begin{bmatrix} v_{p,q} & \sum_{k_{p,q}} \varphi(x_{p,q}^{(k_{p,q})}, y_{p,q}^{(k_{p,q})}) \\ 0 & w_{p,q} \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} v_{p,q} \otimes 1 & \sum_{k_{p,q}} x_{p,q}^{(k_{p,q})} \otimes y_{p,q}^{(k_{p,q})} \\ 0 & 1 \otimes w_{p,q} \end{bmatrix} \right\|, \tag{6.1}$$

where each matrix is $2m \times 2m$. However we do know by Theorem 2.3(ii)⇒(iii) that

$$\left\| \begin{bmatrix} v_{p,q} & \sum_{k_{p,q}} \varphi^{\lambda}(x_{p,q}^{(k_{p,q})}, y_{p,q}^{(k_{p,q})}) \\ 0 & w_{p,q} \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} v_{p,q} \otimes 1 & \sum_{k_{p,q}} x_{p,q}^{(k_{p,q})} \otimes y_{p,q}^{(k_{p,q})} \\ 0 & 1 \otimes w_{p,q} \end{bmatrix} \right\|. \tag{6.2}$$

Let us denote the matrix of the right-hand side of inequality (6.1) or (6.2) by $[\xi_{r,s}]$, and the matrix of the left-hand side of inequality (6.1) by $[x_{r,s}]$, and the matrix of the left-hand side of equation (6.2) by $[x^{\lambda}_{r,s}]$. Let $G \in \mathrm{Ball}(\mathbb{M}_{2m}(X)_*)$ which can be identified with $[g_{r,s}] \in \mathrm{Ball}(\mathbb{T}_{2m} \widehat{\otimes} X_*)$. Then

$$|\langle [x^{\lambda}_{r,s}], G \rangle| = \Big| \sum_{r,s=1}^{2m} \langle x^{\lambda}_{r,s}, g_{r,s} \rangle \Big| \leqslant \|[\xi_{r,s}]\|. \tag{6.3}$$

Since $\varphi^{\lambda} \to \varphi$ in the weak* topology on $((X \overset{\mathrm{h}}{\otimes} X) \widehat{\otimes} X_*)^*$, we have that $\varphi^{\lambda}(x, y) \to \varphi(x, y)$, $\forall x, y \in X$ in the weak* topology on X. Indeed, $\varphi^{\lambda} \overset{\mathrm{w}^*}{\to} \varphi$ means that $\forall x, y \in X$, $\forall f \in X_*$, $\langle \varphi^{\lambda}, (x \otimes y) \otimes f \rangle \to \langle \varphi, (x \otimes y) \otimes f \rangle$. But $\langle \varphi^{\lambda}, (x \otimes y) \otimes f \rangle = \langle \varphi^{\lambda}(x, y), f \rangle$ and $\langle \varphi, (x \otimes y) \otimes f \rangle = \langle \varphi(x, y), f \rangle$, thus $\langle \varphi^{\lambda}(x, y), f \rangle \to \langle \varphi(x, y), f \rangle$, $\forall x, y \in X$, $\forall f \in X_*$. Therefore, by taking the limit with respect to λ in (6.3), we have that

$$|\langle [x_{r,s}], G \rangle| = \Big| \sum_{r,s=1}^{2m} \langle x_{r,s}, g_{r,s}, \rangle \Big| \leqslant \|[\xi_{r,s}]\|.$$

Since $G \in \mathrm{Ball}(\mathbb{M}_{2m}(X)_*)$ is arbitrary, $\|[x_{r,s}]\| \leqslant \|[\xi_{r,s}]\|$, i.e., inequality (6.1) has been shown. ∎

From the proof above and "(ii)⇒(i)" of Lemma 6.2, the following corollary immediately follows.

COROLLARY 6.4. *Let X be a nonzero dual operator space, and $\{z_\alpha\} \subset \mathcal{QM}(X)$ be a bounded net, and $z \in \mathcal{QM}(X)$. Then $z_\alpha \to z$ in the weak* topology on $\mathcal{QM}(X)$ if and only if $x z_\alpha y \to x z y$, $\forall x, y \in X$ in the weak* topology on X.*

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REFERENCES

[1] D.P. BLECHER, Multipliers and dual operator algebras, *J. Funct. Anal.* **183**(2001), 498–525.

[2] D.P. BLECHER, The Shilov boundary of an operator space and the characterization theorems, *J. Funct. Anal.* **182**(2001), 280–343.

[3] D.P. BLECHER, One-sided ideals and approximate identities in operator algebras, *J. Austral. Math. Soc.* **76**(2004), 425–447.

[4] D.P. BLECHER, E.G. EFFROS, V. ZARIKIAN, One-sided M-ideals and multipliers in operator spaces. I, *Pacific J. Math.* **206**(2002), 287–319.

[5] D.P. BLECHER, M. KANEDA, The ideal envelope of an operator algebra, *Proc. Amer. Math. Soc.* **132**(2004), 2103–2113.

[6] D.P. BLECHER, C. LE MERDY, *Operator Algebras and Their Modules — An Operator Space Approach*, London Math. Soc. Monogr. (N.S.), vol. 30, The Clarendon Press, Oxford Univ. Press, Oxford 2004.

[7] D.P. BLECHER, V.I. PAULSEN, Multipliers of operator spaces, and the injective envelope, *Pacific J. Math.* **200**(2001), 1–17.

[8] M.-D. CHOI, E.G. EFFROS, Injectivity and operator spaces, *J. Funct. Anal.* **24**(1977), 156–209.

[9] E.G. EFFROS, Z.-J. RUAN, On non-self-adjoint operator algebras, *Proc. Amer. Math. Soc.* **110**(1990), 915–922.

[10] E.G. EFFROS, Z.-J. RUAN, *Operator Spaces*, London Math. Soc. Monogr. (N.S.), vol. 23, The Clarendon Press, Oxford Univ. Press, New York 2000.

[11] M. HAMANA, Injective envelopes of operator systems, *Publ. Res. Inst. Math. Sci.* **15**(1979), 773–785.

[12] M. HAMANA, Injective envelopes of dynamical systems, in *Operator Algebras and Operator Theory (Craiova, 1989)*, Pitman Res. Notes Math. Ser., vol. 271, Longman Sci. Tech., Harlow 1992, pp. 69–77.

[13] M. HAMANA, Injective envelopes of dynamical systems, preprint, 1991.

[14] M. HAMANA, Triple envelopes and Šilov boundaries of operator spaces, *Math. J. Toyama Univ.* **22**(1999), 77–93.

[15] R.V. KADISON, Isometries of operator algebras, *Ann. of Math.* **54**(1951), 325–338.

[16] M. KANEDA, Multipliers and algebrizations of operator spaces, Ph.D. Dissertation, Univ. of Houston, Houston 2003.

[17] M. KANEDA, Quasi-multipliers and algebrizations of an operator space, *J. Funct. Anal.* **251**(2007), 346–359.

[18] M. KANEDA, Ideal decompositions of a ternary ring of operators with predual, *Pacific J. Math.* **266**(2013), 297–303.

[19] M. KANEDA, V.I. PAULSEN, Characterizations of essential ideals as operator modules over C^*-algebras, *J. Operator Theory* **49**(2003), 245–262.

[20] M. KANEDA, V.I. PAULSEN, Quasi-multipliers of operator spaces, *J. Funct. Anal.* **217**(2004), 347–365.

[21] G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Inc., Boston 1990.

[22] V.I. PAULSEN, *Completely Bounded Maps and Operator Algebras*, Cambridge Stud. Adv. Math., vol. 78, Cambridge Univ Press, Cambridge 2002.

[23] G.K. PEDERSEN, *C^*-Algebras and their Automorphism Groups*, London Math. Soc. Monogr., vol. 14, Academic Press, Inc. [Harcourt Brace Jovanovich, Publ.], London-New York 1979.

[24] Z.-J. RUAN, On matricially normed spaces associated with operator algebras, Ph.D. Dissertation, Univ. of California, Los Angeles 1987.

[25] Z.-J. RUAN, Injectivity of operator spaces, *Trans. Amer. Math. Soc.* **315**(1989), 89–104.

[26] S. SAKAI, *C^*-Algebras and W^*-Algebras*, Ergeb. Math. Grenzgeb., vol. 60, Springer-Verlag, New York-Heidelberg 1971.

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