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THE GENERAL LINEAR GROUP AS A COMPLETE INVARIANT FOR C^*-ALGEBRAS

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ABSTRACT. In 1955 Dye proved that two von Neumann factors not of type I_{2n} are isomorphic if and only if their unitary groups are isomorphic as abstract groups. We consider an analogue for C^*-algebras and show that the topological general linear group is a classifying invariant for simple unital AH-algebras of slow dimension growth and of real rank zero, and that the abstract general linear group is a classifying invariant for unital Kirchberg algebras in the UCT class.

KEYWORDS: *Operator algebras, classification, general linear group.*

MSC (2010): 46L35, 46L05, 46L80.

INTRODUCTION

Since the introduction of the Elliott invariant as a classifying invariant for C^*-algebras, the classification program for C^*-algebras has been rapidly evolving. New invariants were introduced to enrich the program, some more general, and other tailored to specific applications. For a large class of simple, amenable, unital, separable C^*-algebras, Al-Rawashdeh, Booth and the first named author showed in [1] that their unitary group forms a classifying invariant: from an isomorphism of the unitary group of such algebras, they deduced an isomorphism of their Elliott invariant.

In this paper we look at the general linear group (i.e., the group of invertible elements) of unital C^*-algebras as an invariant. For each unital C^*-algebra A we will denote its general linear group by $GL(A)$ and its set of idempotents by $\mathcal{I}(A)$ (see Notation 1.3). Given two unital C^*-algebras A and B, and a group isomorphism $\varphi : GL(A) \to GL(B)$ between their general linear groups, the formula

$$1 - 2\theta_\varphi(e) = \varphi(1 - 2e), \quad e \in \mathcal{I}(A),$$

induces a bijection $\theta_\varphi : \mathcal{I}(A) \to \mathcal{I}(B)$ between the set of idempotents of A and B. This map is not in general an orthoisomorphism of idempotents (i.e., a bijective map which preserves orthogonality of commuting idempotents). However, it turns out that in many cases θ_φ is essentially an orthoisomorphism. More precisely, generalising the notion of oddly decomposability given in [1] (see Definition 2.1), we show in Theorem 2.9 that there exist a partitioning of the non-trivial elements of $\mathcal{I}(A)$ into two sets $\mathcal{I}_o, \mathcal{I}_{\bar{o}}$, such that the map $\widetilde{\theta}_\varphi : \mathcal{I}(A) \to \mathcal{I}(B)$ defined by

$$\widetilde{\theta}_\varphi(e) = \begin{cases} \theta_\varphi(e) & \text{if } e \in \mathcal{I}_o, \\ 1 - \theta_\varphi(e) & \text{if } e \in \mathcal{I}_{\bar{o}}, \\ 1 & \text{if } e = 1, \\ 0 & \text{if } e = 0, \end{cases}$$

is an orthoisomorphism. Using the maps $\widetilde{\theta}_\varphi$ and φ between the idempotents and invertibles of A and B we construct homomorphisms from $K_0(A)$ to $K_0(B)$ and from $K_1(A)$ to $K_1(B)$ and invoke on classification to show A and B are isomorphic. By investigating which C^*-algebras are oddly decomposable we prove the following two main results:

(i) Let A and B be simple, unital AH-algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their general linear groups are topologically isomorphic.

(ii) Let A and B be unital Kirchberg algebras in the UCT class. Then A and B are isomorphic if and only if their general linear groups are isomorphic as abstract groups.

In the case the algebras A and B are simple and finite dimensional we refer to [21] by Schreier and Van der Waerden (see also [12], [15], and [22] for related results).

1. PROPERTIES OF THE INDUCED MAP θ_φ

Let A and B be two unital C^*-algebras. If $\varphi : GL(A) \to GL(B)$ is a group homomorphism between the general linear groups of A and B, then φ defines a map $\theta = \theta_\varphi : \mathcal{I}(A) \to \mathcal{I}(B)$ by setting

$$1 - 2\theta_\varphi(e) = \varphi(1 - 2e), \quad e \in \mathcal{I}(A).$$

A simple computation shows that $\theta_\varphi(e)$ is an idempotent for each $e \in \mathcal{I}(A)$ making θ_φ well defined. If φ is moreover a bijection — or more generally if φ restricts to a bijection of symmetries (elements whose square equals the unit) — it follows that θ_φ is a bijection of idempotents. The following additional properties of the map θ can be easily checked by adapting the arguments in [1] and [8] to the present situation.

PROPOSITION 1.1. *Let A and B be unital C^*-algebras, $\varphi : GL(A) \to GL(B)$ be a group isomorphism and θ be the induced map between idempotents. Then*

(i) $\theta(ueu^{-1}) = \varphi(u)\theta(e)\varphi(u)^{-1}$;

(ii) $\theta(0) = 0$;
(iii) *if* $e, f \in \mathcal{I}(A)$ *commute, then so do* $\theta(e)$ *and* $\theta(f)$ *in* $\mathcal{I}(B)$;
(iv) $\theta(e \triangle f) = \theta(e) \triangle \theta(f)$, *where* $\triangle$ *denotes the symmetric difference of commuting idempotents, i.e.,* $e \triangle f = e + f - 2ef$.

If the center $\mathcal{Z}(B)$ of a unital C^*-algebra B is reduced to the scalars, and $\varphi : GL(A) \to GL(B)$ is as above, then $\varphi(-1) = -1$: indeed, note that -1 is a central element which is not 1, but its product with itself equals 1. The same is true for $\varphi(-1)$. As a consequence we get the following lemma, see also [1] and [8].

LEMMA 1.2. *Let* A *and* B *be unital* C^**-algebras, whose center* $\mathcal{Z}(B) = \mathbb{C}1$. *Let* $\varphi : GL(A) \to GL(B)$ *be a group isomorphism and* $\theta : \mathcal{I}(A) \to \mathcal{I}(B)$ *be as above. Then* $\theta(1) = 1$, *and for each* $e \in \mathcal{I}(A)$, $\theta(1-e) = 1 - \theta(e)$.

To simplify notation, let us introduce the following.

NOTATION 1.3. (i) The quadruple (A, B, φ, θ) will denote a pair of simple unital C^*-algebras A and B, a group isomorphism $\varphi : GL(A) \to GL(B)$, and the induced bijection $\theta : \mathcal{I}(A) \to \mathcal{I}(B)$.
(ii) Let A be a unital C^*-algebra. Denote by $\mathcal{I}(A)$ the set of idempotents in A, and by $\mathcal{I}(A)^{\sim}$ the set $\mathcal{I}(A) \setminus \{0, 1\}$ of *non-trivial* idempotents in A. Denote by $GL(A)$ the general linear group of invertible elements in A.

DEFINITION 1.4. Let A be a unital C^*-algebra. We say that two idempotents $e, f \in \mathcal{I}(A)$ are *similar*, denoted $e \sim_s f$, if there exist $u \in GL(A)$ such that

$$f = ueu^{-1}.$$

The following lemma is a generalisation of Lemma 10 in [8] to simple, unital C^*-algebras.

LEMMA 1.5. *Let* (A, B, φ, θ) *be as in* (1.3). *Then for each fixed* $e \in \mathcal{I}(A)$,

$$\varphi(\lambda e + 1 - e) \in \mathbb{C}\theta(e) + \mathbb{C}\theta(1-e), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Proof. Fix $\lambda \in \mathbb{C} \setminus \{0\}$ and set $x := \varphi(\lambda e + 1 - e)$. Since every idempotent in B is similar to a projection (by Proposition 4.6.2 in [2]) we can choose $u \in GL(B)$ such that $q = u\theta(e)u^{-1}$ is a projection. For any subset $S \subseteq B$, let S' denote its relative commutant in B, and S'' its (relative) bicommutant.

We show that $uxu^{-1} \in \{q\}'$. Since $q = u\theta(e)u^{-1}$ we just need to show that $x\theta(e) = \theta(e)x$. This follows from $x\varphi(1-2e) = \varphi(1-2e)x$.

We show $uxu^{-1} \in \{q\}''$. Fix $b \in \{q\}'$. Since q is selfadjoint $\{q\}'$ is a C^*-subalgebra of B and contains unitary elements $\varphi(u_1), \ldots, \varphi(u_4)$ that span b, for some $u_1, \ldots, u_4 \in GL(A)$. Using $\varphi(u_i) \in \{q\}'$ commutes with $q = u\theta(e)u^{-1}$ we have that $u^{-1}\varphi(u_i)u$ commutes with $\theta(e)$ and with $\varphi(1-2e)$. This implies that $\varphi^{-1}(u^{-1})u_i\varphi^{-1}(u)$ commutes with $1-2e$, with e and with $\lambda e + 1 - e$. We now have that $u^{-1}\varphi(u_i)u$ commutes with $x = \varphi(\lambda e + 1 - e)$. Therefore $\varphi(u_i)$

commutes with uxu^{-1}, and b commutes with uxu^{-1}. Since b was an arbitrary element in $\{q\}'$ we conclude that uxu^{-1} commutes with every element in $\{q\}'$, i.e., $uxu^{-1} \in \{q\}''$.

Since q is a projection, $\{q\}'' \cap \{q\}' = \mathbb{C}q + \mathbb{C}(1-q)$, using the fact that B is simple so the hereditary C^*-subalgebra qBq is simple and consequently has centre $\mathbb{C}q$ (similarly for $1-q$). Multiplying uxu^{-1} on the left by u^{-1} and on the right by u we see that $x \in \mathbb{C}\theta(e) + \mathbb{C}\theta(1-e)$. ∎

In the following we let $\mathbb{C}^*$ denote the group $(\mathbb{C}\backslash\{0\}, \cdot)$ of non-zero complex numbers with multiplication as the group operation.

LEMMA 1.6. *Let (A, B, φ, θ) be as in* (1.3). *Then for each fixed $e \in \mathcal{I}(A)^{\sim}$ there exist group homomorphisms $a_e, b_e : \mathbb{C}^* \to \mathbb{C}^*$ such that*

$$\varphi(\lambda e + 1 - e) = a_e(\lambda)\theta(e) + b_e(\lambda)\theta(1-e), \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

Proof. Fix $\lambda \in \mathbb{C}\backslash\{0\}$. Since $e \in \mathcal{I}(A)^{\sim}$ the elements $\theta(e)$ and $\theta(1-e)$ are nonzero, and by Lemma 1.2 they are linearly independent. Using Lemma 1.5 we therefore have unique coefficients $a, b \in \mathbb{C}$ such that

$$\varphi(\lambda e + 1 - e) = a\theta(e) + b\theta(1-e).$$

Assuming $b = 0$ we obtain $\varphi(\lambda e + 1 - e)^2 = a^2\theta(e) = a(a\theta(e)) = a\varphi(\lambda e + 1 - e)$. Hence $a\theta(e) = \varphi(\lambda e + 1 - e) = a1 = a\theta(1)$. Since $\varphi(\lambda e + 1 - e)$ is invertible we get $\theta(e) = \theta(1)$, contradicting the injectivity of θ. By symmetry both $a, b \in \mathbb{C}\backslash\{0\}$.

It is easy to see that a_e, b_e are multiplicative and unital using that the map $\lambda \mapsto \varphi(\lambda e + 1 - e)$ is multiplicative and unital. We conclude both maps are group homomorphisms. ∎

Since the maps a_e, b_e are group homomorphism of $\mathbb{C}^*$ we will use their (multiplicative) inverses without any further explanation. To each $e \in \mathcal{I}(A)^{\sim}$, we associate the pair of maps (a_e, b_e) and the group homomorphism $c_e := a_e b_e^{-1}$ of $\mathbb{C}^*$. Moreover, we denote by $\sim_c$ the equivalence relation on $\mathcal{I}(A)^{\sim}$, given by:

$$e \sim_c f \quad \text{if and only if} \quad c_e = c_f.$$

The following proposition is essentially Proposition 2.7 in [1]. The proof can be adapted to the present situation and it is left to the reader.

PROPOSITION 1.7. *Let (A, B, φ, θ) be as in* (1.3). *Then for each $e \in \mathcal{I}(A)^{\sim}$:*

(i) *If $f \in \mathcal{I}(A)^{\sim}$ is similar to e then $e \sim_c f$;*

(ii) *$c_e(\lambda)^2 \neq 1$, for every $\lambda \in \mathbb{C}\backslash\{-1, 0, 1\}$;*

(iii) $c_e = c_{1-e}$.

DEFINITION 1.8. Two or more idempotents in a C^*-algebra A are *orthogonal* provided that any two of these idempotents commute and their product is equal to zero.

LEMMA 1.9. *Let* (A, B, φ, θ) *be as in* (1.3). *Suppose that* $e, f \in \mathcal{I}(A)^{\sim}$ *are two orthogonal idempotents in* A. *Then*

$$\theta(e+f)=\theta(e)\theta(1-f)+\theta(1-e)\theta(f), \quad \theta(1-e-f)=\theta(e)\theta(f)+\theta(1-e)\theta(1-f).$$

Proof. The proof of Lemma 2.3 in [1] does not directly generalise, so we include a short proof: using Proposition 1.1 and Lemma 1.2 we have

$$\theta(e+f) = \theta(e)\triangle\theta(f) = \theta(e) + \theta(f) - 2\theta(e)\theta(f) = \theta(e)\theta(1-f) + \theta(1-e)\theta(f).$$

The second equality follows by subtracting both sides of the above equality from 1. ∎

REMARK 1.10. The proof of Theorem 1.11, Corollary 1.12, and Corollary 1.13 corresponds to Proposition 2.8 and Theorem 2.9 in [1], but our proof includes a new characterisation of when $c_e = c_f, \ldots, c_e = c_{e+f}^{-1}$ in terms of the equations (1.5)–(1.8). This observation is essential in the subsequent proofs of Corollary 1.12, Corollary 1.13 used to prove Lemma 1.15 and Lemma 1.16.

THEOREM 1.11. *Let* (A, B, φ, θ) *be as in* (1.3). *Suppose that* $e, f \in \mathcal{I}(A)^{\sim}$ *are two orthogonal idempotents in* A *not adding to one. Then*

$$\begin{aligned}
\theta(e)\theta(f) = 0 \quad &\Leftrightarrow \quad c_e = c_f = c_{e+f},\\
\theta(1-e)\theta(1-f) = 0 \quad &\Leftrightarrow \quad c_e = c_f = c_{e+f}^{-1},\\
\theta(1-e)\theta(f) = 0 \quad &\Leftrightarrow \quad c_e = c_f^{-1} = c_{e+f},\\
\theta(e)\theta(1-f) = 0 \quad &\Leftrightarrow \quad c_e = c_f^{-1} = c_{e+f}^{-1}.
\end{aligned}$$

Proof. Since $\varphi(\lambda e + 1 - e)\varphi(\lambda f + 1 - f) = \varphi(\lambda(e+f) + 1 - (e+f))$ for $\lambda \neq 0$, Lemma 1.6 ensures that

$$(a_e\theta(e) + b_e\theta(1-e))(a_f\theta(f) + b_f\theta(1-f)) = a_{e+f}\theta(e+f) + b_{e+f}\theta(1-(e+f)).$$

Multiplying each side of the above equality by $\theta(e)\theta(f), \theta(1-e)\theta(f), \theta(1-e)\theta(f)$ or $\theta(e)\theta(1-f)$ and then using Lemma 1.9, we obtain the following four equations:

$$a_e a_f\theta(e)\theta(f) = b_{e+f}\theta(e)\theta(f), \tag{1.1}$$

$$b_e b_f\theta(1-e)\theta(1-f) = b_{e+f}\theta(1-e)\theta(1-f), \tag{1.2}$$

$$b_e a_f\theta(1-e)\theta(f) = a_{e+f}\theta(1-e)\theta(f), \tag{1.3}$$

$$a_e b_f\theta(e)\theta(1-f) = a_{e+f}\theta(e)\theta(1-f). \tag{1.4}$$

Consider the following properties:

$$a_e a_f = b_{e+f}, \tag{1.5}$$

$$b_e b_f = b_{e+f}, \tag{1.6}$$

$$b_e a_f = a_{e+f}, \tag{1.7}$$

$$a_e b_f = a_{e+f}. \tag{1.8}$$

We claim that

$$\begin{aligned}
(1.7),(1.8) &\quad\Leftrightarrow\quad c_e = c_f,\\
(1.5),(1.6) &\quad\Leftrightarrow\quad c_e = c_f^{-1},\\
(1.6),(1.8) &\quad\Leftrightarrow\quad c_e = c_{e+f},\\
(1.5),(1.7) &\quad\Leftrightarrow\quad c_e = c_{e+f}^{-1}.
\end{aligned}$$

Going from left to right is straightforward. To go from right to left one simply adds two of the equations (1.1)–(1.4), possibly with coefficients. For example if $c_e = c_f$ then $a_e b_f = b_e a_f$. By Lemma 1.9 the equality (1.3)+(1.4), where we add each side separately, reduces to

$$a_e b_f \theta(e+f) = b_e a_f \theta(e+f) = a_{e+f}\theta(e+f).$$

Hence (1.7) and (1.8) both hold. The remaining three equivalences are obtained similarly using (1.1)+(1.2), $a_e\cdot$ (1.2)+$b_e\cdot$ (1.3), and $b_e\cdot$ (1.1)+$a_e\cdot$ (1.3). We obtain

$$\begin{aligned}
(1.6),(1.7),(1.8) &\quad\Leftrightarrow\quad c_e = c_f = c_{e+f},\\
(1.5),(1.7),(1.8) &\quad\Leftrightarrow\quad c_e = c_f = c_{e+f}^{-1},\\
(1.5),(1.6),(1.8) &\quad\Leftrightarrow\quad c_e = c_f^{-1} = c_{e+f},\\
(1.5),(1.6),(1.7) &\quad\Leftrightarrow\quad c_e = c_f^{-1} = c_{e+f}^{-1}.
\end{aligned}$$

We now show that $\theta(e)\theta(f) = 0$ if and only if $c_e = c_f = c_{e+f}$. The other three equivalences follow from similar calculations.

Suppose first that $\theta(e)\theta(f) = 0$. Assume that $c_e = c_f = c_{e+f}$ does not hold. We derive a contradiction. Using the observation above, one of (1.6), (1.7), or (1.8) does not hold. It follows that $\theta(1-e)\theta(1-e) = 0$, $\theta(1-e)\theta(f) = 0$, or $\theta(e)\theta(1-f) = 0$. Adding this to $\theta(e)\theta(f) = 0$, Lemma 1.9 gives $\theta(1-e-f) = 0$, $\theta(f) = 0$, or $\theta(e) = 0$. Contradiction.

Conversely suppose that the first equation $c_e = c_f = c_{e+f}$ above holds. Since $c_g^2 \neq 1$ for $g = e, f, e+f$, by Proposition 1.7(ii), we obtain that all the other three equations are false. Since (1.6), (1.7), and (1.8) hold, (1.5) must fail. We conclude that $\theta(e)\theta(f) = 0$. ∎

COROLLARY 1.12. *Let (A, B, φ, θ) be as in* (1.3). *Suppose that $e, f \in \mathcal{I}(A)^{\sim}$ are two orthogonal, $\sim_c$-equivalent, idempotents in A not adding to one. Then precisely one of $\theta(e)\theta(f)$, $\theta(1-e)\theta(1-f)$ is zero.*

Proof. If both terms are zero then $c_{e+f} = c_{e+f}^{-1}$. If both terms are non-zero then (1.5) and (1.6) are true: if (1.5) fails then $\theta(e)\theta(f) = 0$ by (1.1), and similarly for (1.6). Hence $c_e = c_f^{-1}$ and by $\sim_c$-equivalence also $c_f = c_f^{-1}$. ∎

COROLLARY 1.13. *Let (A, B, φ, θ) be as in* (1.3). *Suppose that $e, f \in \mathcal{I}(A)^{\sim}$ are two orthogonal idempotents in A not adding to one. Then precisely one of $c_e = c_{e+f}$, $c_e = c_{e+f}^{-1}$ is true.*

Proof. Consider the four equations from Theorem 1.11:

$$c_e = c_f = c_{e+f}, \quad c_e = c_f = c_{e+f}^{-1}, \quad c_e = c_f^{-1} = c_{e+f}, \quad c_e = c_f^{-1} = c_{e+f}^{-1}.$$

It is clear that at least one of (1.5)–(1.8) is false. Consequently at least one of the four equations above is true: if (1.5) fails then $\theta(e)\theta(f) = 0$, so $c_e = c_f = c_{e+f}$. Similarly for the remaining cases. We obtain $c_e = c_{e+f}$ or $c_e = c_{e+f}^{-1}$ is true. Both can not be true because this contradicts Proposition 1.7(ii). ∎

DEFINITION 1.14. Let (A, B, φ, θ) be as in (1.3). For any subset $S \subseteq \mathcal{I}(A)$ we say that $\theta : \mathcal{I}(A) \to \mathcal{I}(B)$ *preserves orthogonality* (respectively *flips orthogonality*) on S if $\theta(e)\theta(f) = 0$ (respectively $\theta(1-e)\theta(1-f) = 0$) for any two orthogonal idempotents e and f in S.

LEMMA 1.15. *Let (A, B, φ, θ) be as in* (1.3). *Suppose that $e, f, g \in \mathcal{I}(A)^{\sim}$ are three orthogonal, $\sim_c$-equivalent, idempotents in A not adding to one. If θ preserves (respectively flips) orthogonality on a subset of $\{e, f, g\}$ of size two, then θ preserves (respectively flips) orthogonality on all of $\{e, f, g\}$.*

Proof. The proof of Lemma 2.14 in [1] does not generalise nicely, so we include a short proof:

Step 1. Suppose that $\theta(e)\theta(f) = 0$, $\theta(e)\theta(g) = 0$. Assume for contradiction that $\theta(1-f)\theta(1-g) = 0$. By Lemma 1.9

$$\theta(e)\theta(f+g) = \theta(e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Theorem 1.11 implies that $c_e = c_{f+g} = c_{e+f+g}$ and $c_f = c_g = c_{f+g}^{-1}$. Hence $c_{f+g} = c_{f+g}^{-1}$ which is false. We get $\theta(1-f)\theta(1-g) \neq 0$. By Corollary 1.12 we conclude $\theta(f)\theta(g) = 0$.

Step 2. Now suppose that $\theta(1-e)\theta(1-f) = 0$, $\theta(1-e)\theta(1-g) = 0$. Assume for contradiction that $\theta(f)\theta(g) = 0$. By Lemma 1.9,

$$\theta(1-e)\theta(f+g) = \theta(1-e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Theorem 1.11 implies that $c_e = c_{f+g}^{-1} = c_{e+f+g}$ and $c_f = c_g = c_{f+g}$. We obtain $c_{f+g} = c_{f+g}^{-1}$ which is false. Consequently, we have that $\theta(f)\theta(g) \neq 0$. By Corollary 1.12, $\theta(1-f)\theta(1-g) = 0$. It is evident that Steps 1 and 2 suffice to complete the proof. ∎

LEMMA 1.16. *Let (A, B, φ, θ) be as in* (1.3). *Suppose that $e, f, g \in \mathcal{I}(A)^{\sim}$ are three orthogonal, $\sim_c$-equivalent, idempotents in A not adding to one. Then*

$$e \sim_c f \sim_c g \sim_c e + f + g.$$

Proof. The proof of Lemma 2.16 in [1] can be adapted to the present situation. We include the details for completeness: by Corollary 1.12 precisely one of $\theta(f)\theta(g)$, $\theta(1-f)\theta(1-g)$ is zero. If $\theta(f)\theta(g) = 0$, then by Lemma 1.9 and Lemma 1.15

$$\theta(e)\theta(f+g) = \theta(e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Using Theorem 1.11, $c_e = c_{f+g} = c_{e+f+g}$. Similarly, if $\theta(1-f)\theta(1-g) = 0$, then by Lemma 1.9 and Lemma 1.15

$$\theta(1-e)\theta(f+g) = \theta(1-e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Using Theorem 1.11, $c_e = c_{f+g}^{-1} = c_{e+f+g}$. ∎

2. ODDLY DECOMPOSABLE C^*-ALGEBRAS

Let (A, B, φ, θ) be as in (1.3), and let $\sim_c$ be the equivalence relation on $\mathcal{I}(A)^{\sim}$ introduced in Section 1. We now introduce a sufficient condition on the C^*-algebra A, such that $\mathcal{I}(A)^{\sim}/\sim_c$ has at most two elements.

DEFINITION 2.1. A unital C^*-algebra A is said to be *oddly decomposable* if for every pair of idempotents $e, f \in \mathcal{I}(A)^{\sim}$ there is an odd integer $n \geqslant 3$ and n orthogonal idempotents $g_1, \ldots, g_n \in \mathcal{I}(A)^{\sim}$ adding to f, such that each g_i is similar to some $g_i' \in \mathcal{I}(A)^{\sim}$ with $g_i'e = eg_i' = g_i' \neq e$.

REMARK 2.2. Oddly decomposable C^*-algebras where introduced in [1], but with a definition in terms of projections and unitary equivalence rather than idempotents and similarity. In [1] a unital C^*-algebra A was called oddly decomposable if for every pair of projections $p, q \in A\backslash\{0,1\}$ there exist an odd integer $n \geqslant 3$ and n orthogonal non-zero projections $r_1, \ldots, r_n \in A$ adding to q, such that each r_i is unitary equivalent to some projection $r_i' \in A$ with $r_i' < p$. Let us outline why the two definitions coincide.

Fix a pair of idempotents $e, f \in \mathcal{I}(A)^{\sim}$. Find projections $p, q \in A$ and invertible elements $u, v \in GL(A)$ such that $e = upu^{-1}$, and $f = vqv^{-1}$ (see Lemma 3.2). Assuming odd decomposability in sense of [1] there exist an odd integer $n \geqslant 3$ and a decomposition of q as a sum $q = \sum_{i=1}^{n} r_i$ of pairwise nonzero orthogonal projections r_i of A, such that each r_i is unitarily equivalent to some projection $r_i' < p$. Define

$$g_i := vr_iv^{-1}, \quad g_i' := ur_i'u^{-1}, \quad i = 1, \ldots, n.$$

It follows that $g_i, g_i' \in \mathcal{I}(A)^{\sim}$ have the properties needed to make A oddly decomposable in sense of Definition 2.1.

Conversely, fix a pair of projections $p, q \in A\backslash\{0,1\}$. Assuming odd decomposability in sense of Definition 2.1 there exist an odd integer $n \geqslant 3$ and n orthogonal idempotents $g_1, \ldots, g_n \in \mathcal{I}(A)^{\sim}$ adding to q, such that each g_i is similar to some $g_i' \in \mathcal{I}(A)^{\sim}$ with $g_i'p = pg_i' = g_i' \neq p$. We can select $w_1, \dots, w_n \in GL(A)$ such that

$$r_i := w_i g_i w_i^{-1}, \quad i = 1, \dots, n$$

are orthogonal projections adding to q. We can select projections $r_1', \dots, r_n'$ in A such that each r_i' is similar to g_i' with $r_i' < p$. It follows that each r_i is similar and hence also unitarily equivalent (see Lemma 3.2) to r_i', making A oddly decomposable in sense of [1].

NOTATION 2.3. Let A be a unital C^*-algebra and $e \in \mathcal{I}(A)^{\sim}$. We define

$$\mathcal{I}_{c_e} := \{f \in \mathcal{I}(A)^{\sim} : c_f = c_e\}, \quad \mathcal{I}_{\bar{c}_e} := \{f \in \mathcal{I}(A)^{\sim} : c_f = c_e^{-1}\}.$$

LEMMA 2.4. *Let (A, B, φ, θ) be as in* (1.3) *with A oddly decomposable. Let e, f be two non-trivial idempotents in A. Then there exist idempotents $e', f' \in \mathcal{I}(A)^{\sim}$ and $u \in GL(A)$ such that*

$$e', f' \in \mathcal{I}_{c_f}, \quad e'e = ee' = e' \neq e, \quad f'f = ff' = f' \neq f, \quad e' = uf'u^{-1}.$$

Proof. One can adapt the proof of Corollary 2.17 in [1] to the present situation. We include the details for completeness: find an odd integer $n \geqslant 3$ and n commuting, orthogonal, idempotents $g_1, \dots, g_n \in \mathcal{I}(A)^{\sim}$ adding to f, such that each g_i is similar to some $g_i' \in \mathcal{I}(A)^{\sim}$ with the property that $g_i'e = eg_i' = g_i' \neq e$.

For each $i = 1, \dots, n$ set $e_i := f - g_i$. By Corollary 1.13 we get that for each i either $c_{g_i} = c_f$ or $c_{g_i} = c_f^{-1}$. If $c_{g_i} = c_f^{-1}$ for $i = 1, 2, 3$ then Lemma 1.16 ensures that $c_{g_1+g_2+g_3} = c_f^{-1}$. If $c_{g_i} = c_f^{-1}$ for all $i = 1, \dots, 5$ (if applicable) Lemma 1.16, used on $g_1 + g_2 + g_3$, g_4, g_5, ensures that $c_{g_1+\dots+g_5} = c_f^{-1}$. By induction we conclude that $c_{g_1+\dots+g_n} = c_f^{-1}$ if $c_{g_i} = c_f^{-1}$ for all i. Knowing that $c_{g_1+\dots+g_n} = c_f \neq c_f^{-1}$ we conclude that $c_{g_m} = c_f$ for some $m \in \{1, \dots, n\}$. Define $f' := g_m$ and $e' := g_m'$. Since f' is similar to e' there exists $u \in GL(A)$ such that $e' = uf'u^{-1}$. Using Proposition 1.7(i), $e' \in I_{c_f}$. We conclude that

$$e', f' \in \mathcal{I}_{c_f}, \quad e'e = ee' = e' \neq e, \quad f'f = ff' = f' \neq f, \quad e' = uf'u^{-1}. \quad \blacksquare$$

LEMMA 2.5. *Let (A, B, φ, θ) be as in* (1.3) *with A oddly decomposable. Then for each $e \in \mathcal{I}(A)^{\sim}$*

$$\mathcal{I}(A)^{\sim} = \mathcal{I}_{c_e} \cup \mathcal{I}_{\bar{c}_e}.$$

Proof. The proof of Remark 3.2 in [1] does not generalise nicely, so we include a short proof: fix any $f \in \mathcal{I}(A)^{\sim}$. Lemma 2.4 provides an idempotent $e' \in \mathcal{I}(A)^{\sim}$ such that

$$e' \in \mathcal{I}_{c_f}, \quad e'e = ee' = e' \neq e.$$

Applying Corollary 1.13 to e' and $e - e'$ we get that either $c_{e'} = c_e$ or $c_{e'} = c_e^{-1}$. Hence $c_f = c_e$ or $c_f = c_e^{-1}$. $\blacksquare$

REMARK 2.6. Borrowing material from a forthcoming paper [13] let us mention the following result: let (A, B, φ, θ) be as in (1.3) with φ continuous. Then for each $e \in \mathcal{I}(A)^{\sim}$

$$\mathcal{I}(A)^{\sim} = \mathcal{I}_{c_e} \cup \mathcal{I}_{\bar{c}_e}.$$

LEMMA 2.7. *Let* (A, B, φ, θ) *be as in* (1.3) *with* A *oddly decomposable. Let* e, f *be two non-trivial orthogonal idempotents in* A *not adding to one. Suppose that* θ *preserves (respectively flips) orthogonality on* $\{e, f\}$*. Then* θ *preserves (respectively flips) orthogonality on all of* $\mathcal{I}_{c_e}$*.*

Proof. The proof of Lemma 3.4 in [1] can be adapted to the present situation. We include the details for completeness: let $g, h \in \mathcal{I}_{c_e}$ be any two commuting, orthogonal idempotents. It is enough to show θ preserves (respectively flips) orthogonality on $\{g, h\}$. If $g + h = 1$ then $\theta(g)\theta(h) = \theta(g)\theta(1 - g) = 0$, hence assume $g + h \neq 1$. Using Lemma 2.4 select $x, x', y', z' \in \mathcal{I}(A)^{\sim}$ and $u \in GL(A)$ such that

$$\begin{aligned}
&x, x' \in \mathcal{I}_{c_{1-e-f}}, \quad x(1-g-h) = (1-g-h)x = x \neq 1-g-h,\\
&x'(1-e-f) = (1-e-f)x' = x' \neq 1-e-f, \quad x = ux'u^{-1},\\
&y' \in \mathcal{I}_{c_e}, \quad y'x' = x'y' = y' \neq x',\\
&z' \in \mathcal{I}_{c_e}, \quad z'(x'-y') = (x'-y')z' = z' \neq x'-y'.
\end{aligned}$$

By assumption θ preserves (respectively flips) orthogonality on $\{e, f\} \subseteq \mathcal{I}_{c_e}$. Using Lemma 1.15 we get that θ preserves (respectively flips) orthogonality on the evidently commuting, orthogonal idempotents

$$\{z', y', e, f\} \subseteq \mathcal{I}_{c_e}.$$

Define $y := uy'u^{-1}$ and $z := uz'u^{-1}$. Since $y', z' \in \mathcal{I}_{c_e}$ we obtain that $y, z \in \mathcal{I}_{c_e}$, see Proposition 1.7. By Proposition 1.1, $\theta(z)\theta(y) = \varphi(u)\theta(z')\theta(y')\varphi(u)^{-1} = 0$ (respectively $\theta(1-z)\theta(1-y) = \varphi(u)\theta(1-z')\theta(1-y')\varphi(u)^{-1} = 0$). Now Lemma 1.15 ensures that θ preserves (respectively flips) orthogonality on the clearly commuting, orthogonal idempotents

$$\{z, y, g, h\} \subseteq \mathcal{I}_{c_e}. \quad ∎$$

LEMMA 2.8. *Let* (A, B, φ, θ) *be as in* (1.3) *with* A *oddly decomposable. Let* e, f *be two non-trivial idempotents that are not* $\sim_c$*-equivalent. If* θ *preserves (respectively flips) orthogonality on one of the sets* $\mathcal{I}_{c_e}, \mathcal{I}_{c_f}$*, then* θ *flips (respectively preserves) orthogonality on the other set.*

Proof. One can adapt the proof of Proposition 3.6 in [1] to the present situation. We include the details for completeness: it suffices to show that

$$\theta \text{ can not preserve orthogonality on } \mathcal{I}_{c_e} \cup \mathcal{I}_{c_f}; \tag{2.1}$$

$$\theta \text{ can not flip orthogonality on } \mathcal{I}_{c_e} \cup \mathcal{I}_{c_f}. \tag{2.2}$$

Let us argue why (2.1)–(2.2) suffice: suppose θ preserves (respectively flips) orthogonality on $\mathcal{I}_{c_e}$. Using Lemma 2.4 select orthogonal idempotents $g, h \in \mathcal{I}_{c_f}$ not adding to one. By Corollary 1.12 and Lemma 2.7 we obtain that θ either preserves or flips orthogonality on all of $\{g, h\}$, and hence on all of $\mathcal{I}_{c_f}$. We conclude θ flips (respectively preserves) orthogonality on $\mathcal{I}_{c_f}$ by (2.1)–(2.2).

(2.1). By Lemma 2.5, $c_e = c_f^{-1}$. Using Lemma 2.4 choose $x, y, z \in \mathcal{I}(A)^\sim$ such that:

$$\begin{aligned} &x \in \mathcal{I}_{c_f}, \quad x(1-e) = (1-e)x = x \neq 1-e, \\ &y \in \mathcal{I}_{c_f}, \quad ye = ey = y \neq e, \\ &z \in \mathcal{I}_{c_e}, \quad zx = xz = z \neq x. \end{aligned}$$

If $1-e-x \in \mathcal{I}_{c_f}$ then $\{x, y, 1-e-x\} \subseteq \mathcal{I}_{c_f}$ are commuting, orthogonal $\sim_{\mathrm{c}}$-equivalent idempotents in A not adding to one. By Lemma 1.16 we have that $1-e+y \in \mathcal{I}_{c_f}$. Hence $e-y \in \mathcal{I}_{c_f}$, see Proposition 1.7. In particular

$$c_y = c_{e-y} = c_e^{-1}.$$

If $1-e-x \in \mathcal{I}_{c_e}$ then $\{e, z, 1-e-x\} \subseteq \mathcal{I}_{c_e}$ are commuting, orthogonal, $\sim_{\mathrm{c}}$-equivalent idempotents in A not adding to one. It follows that $1-x+z \in \mathcal{I}_{c_e}$ and $x-z \in \mathcal{I}_{c_e}$. In particular

$$c_z = c_{x-z} = c_x^{-1}.$$

We conclude that θ can not preserve orthogonality on $\mathcal{I}_{c_e} \cup \mathcal{I}_{c_f}$.

(2.2). By Lemma 2.5, $c_e = c_f^{-1}$. Using Lemma 2.4 choose $x, y, z \in \mathcal{I}(A)^\sim$ such that:

$$\begin{aligned} &x \in \mathcal{I}_{c_f}, \quad x(1-e) = (1-e)x = x \neq 1-e, \\ &y \in \mathcal{I}_{c_e}, \quad ye = ey = y \neq e, \\ &z \in \mathcal{I}_{c_f}, \quad zx = xz = z \neq x. \end{aligned}$$

If $1-e-x \in \mathcal{I}_{c_f}$ then $c_{e-y} \neq c_y$ implies that $c_{e-y} = c_y^{-1} = c_f$, see Lemma 2.5. Hence $\{e-y, x, 1-e-x\} \subseteq \mathcal{I}_{c_f}$ are commuting, orthogonal, $\sim_{\mathrm{c}}$-equivalent, idempotents in A not adding to one. It follows that $1-y \in \mathcal{I}_{c_f}$ and $y \in \mathcal{I}_{c_f}$: contradiction. In particular

$$c_y = c_{e-y} = c_e.$$

If $1-e-x \in \mathcal{I}_{c_e}$ then $c_{x-z} \neq c_z$ implies that $c_{x-z} = c_z^{-1} = c_e$, see Lemma 2.5. Hence $\{x-z, e, 1-e-x\} \subseteq \mathcal{I}_{c_e}$ are commuting, orthogonal, $\sim_{\mathrm{c}}$-equivalent, idempotents in A not adding to one. It follows that $1-z \in \mathcal{I}_{c_e}$ and $z \in \mathcal{I}_{c_e}$: contradiction. In particular

$$c_z = c_{x-z} = c_x.$$

We conclude that θ can not flip orthogonality on $\mathcal{I}_{c_e} \cup \mathcal{I}_{c_f}$. ∎

THEOREM 2.9. *Let (A, B, φ, θ) be as in* (1.3). *If A is oddly decomposable then φ induces an orthoisomorphism between the sets of idempotents $\mathcal{I}(A)$ and $\mathcal{I}(B)$, which preserves similarity of idempotents.*

Proof. The proof of Theorem 2.21 in [1] can be adapted to the present situation. We include the details for completeness: we may assume $\mathcal{I}(A)^{\sim}$ is nonempty. Using Lemma 2.4 select two non-trivial orthogonal, $\sim_{\mathrm{c}}$-equivalent, idempotents $e, f \in \mathcal{I}(A)^{\sim}$ not adding to one. Define $o := c_{e+f}$. By Corollary 1.13 either $c_e = o$ or $c_e = o^{-1}$. If $c_e = o^{-1}$ then $c_e = c_f = c_{e+f}^{-1}$ and θ flips orthogonality on $\mathcal{I}_{c_e}$. If $c_e = o$ then $c_e = c_f = c_{e+f}$ and θ preserves orthogonality on $\mathcal{I}_{c_e}$. In any case Lemma 2.8 ensures that θ preserves orthogonality on $\mathcal{I}_o$ and flips orthogonality on $\mathcal{I}_{\overline{o}}$. Define

$$\widetilde{\theta}(g) = \begin{cases} \theta(g) & \text{if } g \in \mathcal{I}_o, \\ 1 - \theta(g) & \text{if } g \in \mathcal{I}_{\overline{o}}, \\ 1 & \text{if } g = 1, \\ 0 & \text{if } g = 0. \end{cases}$$

Fix two commuting, orthogonal idempotents $g, h \in \mathcal{I}(A)$. If g or h is equal to zero then obviously $\widetilde{\theta}(g)\widetilde{\theta}(h) = 0$. If h, g are both nonzero and add to one then $c_h = c_g$, see Proposition 1.7. Hence $\widetilde{\theta}$ restricts to either θ or $1 - \theta$ on $\{g, h\}$, implying $\widetilde{\theta}(g)\widetilde{\theta}(h) = 0$. We may assume $g, h \in \mathcal{I}(A)^{\sim}$ with $g + h \neq 1$. Then

$$\begin{aligned} g, h \in \mathcal{I}_o \Rightarrow \quad & \theta(g)\theta(h) = 0; \Rightarrow & \\ g, h \in \mathcal{I}_{\overline{o}} \Rightarrow \quad & \theta(1-g)\theta(1-h) = 0 \Rightarrow & \quad \widetilde{\theta}(g)\widetilde{\theta}(h) = 0; \\ g \in \mathcal{I}_o, h \in \mathcal{I}_{\overline{o}} \Rightarrow \quad & \theta(g)\theta(1-h) = 0. \Rightarrow & \end{aligned}$$

The fact that $\theta(g)\theta(1-h) = 0$ above follows indirectly: since $c_g \neq c_h$ either $\theta(1-g)\theta(h) = 0$ or $\theta(g)\theta(1-h) = 0$ (by Theorem 1.11). The first equality implies that $c_g = c_h^{-1} = c_{g+h}$, hence $g, 1-g-h \in \mathcal{I}_o$. But then $\theta(g)\theta(1-g-h) = 0$, meaning that $c_g = c_{1-g-h} = c_{1-h}$: contradiction. We conclude that

$$\widetilde{\theta} : \mathcal{I}(A) \to \mathcal{I}(B)$$

preserves orthogonality on $\mathcal{I}(A)$. Surjectivity of $\widetilde{\theta}$ follows from Lemma 1.2. Injectivity of $\widetilde{\theta}$ follows from the fact that if $g \in \mathcal{I}_o$, $h \in \mathcal{I}_{\overline{o}}$ then $\widetilde{\theta}(g) \neq \widetilde{\theta}(h)$, since $\widetilde{\theta}(g) = \widetilde{\theta}(h)$ implies $g = 1 - h$. Finally, for any $g \in \mathcal{I}(A)^{\sim}$ and $u \in GL(A)$, we have $g \sim_{\mathrm{c}} ugu^{-1}$, by Proposition 1.7, so

$$\begin{aligned} g \in \mathcal{I}_o \qquad & \widetilde{\theta}(ugu^{-1}) = \theta(ugu^{-1}) = \varphi(u)\theta(g)\varphi(u)^{-1} = \varphi(u)\widetilde{\theta}(g)\varphi(u)^{-1}, \\ g \in \mathcal{I}_{\overline{o}} \qquad & \widetilde{\theta}(ugu^{-1}) = 1 - \theta(ugu^{-1}) = \cdots = \varphi(u)\widetilde{\theta}(g)\varphi(u)^{-1}. \end{aligned}$$

We conclude that θ preserves similarity of idempotents. ∎

3. THE CASE OF SIMPLE AH-ALGEBRAS

3.1. FROM AN ORTHOISOMORPHISM TO A K_0-ORDER ISOMORPHISM. In this subsection, we prove that an (abstract) isomorphism $GL(A) \cong GL(B)$ between the general linear groups of certain stably finite C^*-algebras A and B of real rank zero (including the simple AH-algebras of slow dimension growth) induces an isomorphism between their ordered K_0-groups. In particular, we have that if A and B are either two simple unital AF-algebras, or two irrational rotation algebras, then A is $*$-isomorphic to B if and only if their general linear groups are isomorphic (as abstract groups). Our approach uses ideas of [1], but with proofs that are somehow different, and some clarifications are given.

NOTATION 3.1. (i) Let $\mathcal{F}$ denote the class of simple, unital, separable C^*-algebras of real rank zero with cancellation (or equivalently with stable rank one, see Corollary 6.5.7 in [2]). Let $\mathcal{F}_1$ denote the class of C^*-algebras A in $\mathcal{F}$ for which $K_0(A)$ is noncyclic and weakly unperforated.

(ii) Let A be a unital C^*-algebra. Denote by $\mathcal{P}(A)$ the set of projections in A, and by $\mathcal{P}(A)^{\sim}$ the set $\mathcal{P}(A)\backslash\{0,1\}$ of *non-trivial* projections in A.

The following lemma is well known, see Proposition 4.6.2 and Proposition 4.6.5 in [2].

LEMMA 3.2. *Let A be a unital C^*-algebra. Every idempotent in A is similar to a projection in A. Every pair of projections is A are similar if and only if they are unitarily equivalent.*

PROPOSITION 3.3. *Each C^*-algebra in $\mathcal{F}_1$ is oddly decomposable.*

The result follows immediately from Proposition 4.2 in [1] in combination with Remark 2.2.

Following [2] and [19] an *ordered (abelian) group G* is an abelian group with a distinguished *positive cone*, i.e., a subset $G_+ \subseteq G$ fulfilling that

$$G_+ + G_+ \subseteq G_+, \quad G_+ \cap (-G_+) = \{0\}, \quad G_+ - G_+ = G.$$

The set G_+ induces a translation-invariant partial ordering on G given by $x \leqslant y$ if $y - x \in G_+$.

Essentially as in [9], (Effros presumes the group is unperforated (respectively is a dimension group) in his definition of an ordered group (respectively a scaled dimension group); we have removed these two constraints and changed the terminology accordingly) a *scaled ordered group G* is an ordered group with a distinguished *scale*, i.e., a subset $\Gamma = \Gamma(G)$ of G_+, which is generating, hereditary and directed, i.e.,

(i) For each $a \in G_+$, there exist $a_1, \dots, a_r \in \Gamma$ with $a = \sum\limits_{i=1}^{r} a_i$.

(ii) If $0 \leqslant a \leqslant b \in \Gamma$, then $a \in \Gamma$.

(iii) Given $a, b \in \Gamma$, there exists $c \in \Gamma$ with $a, b \leqslant c$.

A scale Γ has a partially defined addition; in fact $a \geqslant b$ in Γ if and only if $a = b + c$ for some $c \in \Gamma$. Following [9], a group homomorphism of scaled ordered groups $\alpha : G \to G'$ is a *contraction* if $\alpha(\Gamma(G)) \subseteq \Gamma(G')$. If Γ and Γ' are scales of two scaled ordered groups, then a map $\alpha : \Gamma \to \Gamma'$ is a *scale homomorphism* (respectively a *scale isomorphism*) if $a = b + c$ in Γ implies that (respectively is equivalent to) $\alpha(a) = \alpha(b) + \alpha(c)$ in Γ'.

PROPOSITION 3.4 (Effros). *Let G and G' be two scaled ordered groups with Riesz interpolation. Any scale homomorphism $\alpha : \Gamma(G) \to \Gamma(G')$ extends to a unique contraction $\widetilde{\alpha} : G \to G'$. If α is a scale isomorphism, then $\widetilde{\alpha}$ is an isomorphism of the scaled ordered groups G and G'.*

Notice that the proof of Lemma 7.3 and Corollary 7.4 in [9] does not use perforation nor countability of the groups involved.

LEMMA 3.5. *If $A \in \mathcal{F}$, then $K_0(A)$ is a simple scaled ordered group with Riesz interpolation and scale $\Sigma(A) := \{[p] : p \in \mathcal{P}(A)\}$.*

Proof. Since A is stably finite and simple, the group $K_0(A)$ is a simple scaled ordered group with Riesz interpolation by Proposition 3.3.7 and Theorem 3.3.18 in [17], and $\Sigma(A)$ is hereditary and directed, see p. 38 in [2].

For sake of completeness we show $\Sigma(A)$ is generating: fix any x in $K_0(A)_+$. Recall that $x = [p]$ for some projection p in $M_n(A)$ (with $n \in \mathbb{N}$). Let 1_n denote the unit of $M_n(A)$. Cleary, $1_n = \sum_{i=1}^{n} e_{ii}$, where $e_{ii} \in M_n(A)$ is the matrix with 1 at entry (i, i) and zero otherwise, and $p \leqslant 1_n$. Since $M_n(A)$ has real rank zero it follows from Corollary 3.3.17 in [17] that there exist projections $p_i \in M_n(A)$ such that $[p_i] \leqslant [e_{ii}]$ and $\sum_{i=1}^{n} p_i = p$. Hence $x = \sum_{i=1}^{n} [p_i]$ and $[p_i] \in \Sigma(A)$, using the characterisation $\Sigma(A) = \{x \in K_0(A)_+ : x \leqslant [1]\}$, see [2]. ∎

THEOREM 3.6. *Let A and B be two C^*-algebras in $\mathcal{F}_1$. If $GL(A)$ and $GL(B)$ are isomorphic (as abstract groups), then $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups.*

Proof. Let $\widetilde{\theta} : \mathcal{I}(A) \to \mathcal{I}(B)$ be the orthoisomorphism preserving similarity of idempotents given by Theorem 2.9 and Proposition 3.3 with $\widetilde{\theta}(1) = 1$. Let

$$\widetilde{\theta}_* : \Sigma(A) \to \Sigma(B)$$

be given by $\widetilde{\theta}_*([p]) = [p']$, where $p' = u\widetilde{\theta}(p)u^{-1}$ for some $u \in GL(B)$ such that p' is a projection in B, see Lemma 3.2.

We show $\widetilde{\theta}_*$ is well defined: fix any two projections p, q in $\mathcal{P}(A)$. Assume $[p] = [q]$. By Proposition 3.1.7(iv) in [19] also $[1 - p] = [1 - q]$. Since A has cancellation p and q are unitary equivalent, see Definition 7.3.1 and Proposition 2.2.2 in [19]. Since $\widetilde{\theta}$ preserves similarity $\widetilde{\theta}(p)$ and $\widetilde{\theta}(q)$ are similar. We conclude that

p' and q' are similar and (by Lemma 3.2) unitarily equivalent. We obtain that $[p'] = [q']$.

We show $\widetilde{\theta}_*$ is a scale homomorphism: fix $x, y, z \in \Sigma(A)$ satisfying the equality $x + y = z$. Find $p, q \in \mathcal{P}(A)$ such that $x = [p]$ and $y = [q]$. Select $q_1 \in \mathcal{P}(A)$ such that

$$[q_1] = [p] \quad \text{and} \quad q_1 \leqslant 1 - q$$

as follows: if $z = [1]$ set $q_1 := 1 - q$ implying $[q_1] = z - y = [p]$. If $z < [1]$ use $[p] = z - [q] < [1 - q]$ and Corollary 6.9.2 in [2] to deduce that p is Murray–von Neumann equivalent to a subprojection, say q_1, of $1 - q$. (Here we have used A is simple, unital, stably finite, of real rank zero, with cancellation and weakly unperforated $K_0(A)$.) We obtain that

$$\widetilde{\theta}(q_1) + \widetilde{\theta}(q) = \widetilde{\theta}(q_1 + q) \quad \text{and} \quad q_1 q = 0.$$

Find $v \in GL(B)$ such that $v\widetilde{\theta}(q_1)v^{-1}$ and $v\widetilde{\theta}(q)v^{-1}$ are orthogonal projections in B (easy exercise, see Remark 2.2). Hence

$$\begin{aligned}
\widetilde{\theta}_*(z) &= \widetilde{\theta}_*([p] + [q]) = \widetilde{\theta}_*([q_1] + [q]) = \widetilde{\theta}_*([q_1 + q]) \\
&= [v\widetilde{\theta}(q_1 + q)v^{-1}] = [v\widetilde{\theta}(q_1)v^{-1}] + [v\widetilde{\theta}(q)v^{-1}] \\
&= \widetilde{\theta}_*([q_1]) + \widetilde{\theta}_*([q]) = \widetilde{\theta}_*(x) + \widetilde{\theta}_*(y).
\end{aligned}$$

We show $\widetilde{\theta}_*$ is a scale isomorphism: as $\widetilde{\theta} : \mathcal{I}(A) \to \mathcal{I}(B)$ is an orthoisomorphism, its inverse induces a scale homomorphism $(\widetilde{\theta}^{-1})_* : \Sigma(B) \to \Sigma(A)$. For $p \in \mathcal{P}(A)$ we have that

$$(\widetilde{\theta}^{-1})_*(\widetilde{\theta}_*([p])) = (\widetilde{\theta}^{-1})_*[u\widetilde{\theta}(p)u^{-1}] = [v\widetilde{\theta}^{-1}(u\widetilde{\theta}(p)u^{-1})v^{-1}] = [vwpw^{-1}v^{-1}] = [p],$$

for appropriate $u \in GL(B)$, and $v, w \in GL(A)$, using that $\widetilde{\theta}^{-1}$ maps $u\widetilde{\theta}(p)u^{-1}$ to an idempotent similar to p. By symmetry both $(\widetilde{\theta}^{-1})_* \circ \widetilde{\theta}_*$ and $\widetilde{\theta}_* \circ (\widetilde{\theta}^{-1})_*$ are identity maps. Hence $(\widetilde{\theta}^{-1})_* = (\widetilde{\theta})_*^{-1}$.

Using Proposition 3.4 and Lemma 3.5 we obtain that $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups. ∎

LEMMA 3.7. *Every infinite-dimensional, simple, unital* AH-*algebra of slow dimension growth and of real rank zero belongs to the class* $\mathcal{F}_1$.

Proof. Cancellation: we refer to Theorem 1 in [3] and Proposition 6.5.1 in [2]. Weakly unperforated: see p. 2 in [23]. Noncyclic: see Remark 2.7 in [6]. ∎

COROLLARY 3.8. *If A and B are simple, unital* AH-*algebras of slow dimension growth and of real rank zero, with isomorphic general linear groups (as abstract groups), then*

$$(K_0(A), K_0(A)_+, [1_A]) \quad \textit{and} \quad (K_0(B), K_0(B)_+, [1_B])$$

are order isomorphic by a map preserving the distinguished order units.

Proof. If A is infinite-dimensional then so is B. (By Lemma 3.7 and Proposition 3.3 A is oddly decomposable. We can therefore find arbitrary many orthogonal idempotents (e_i) in $\mathcal{I}(A)^{\sim}$. The isomorphism of $GL(A)$ and $GL(B)$ induces an orthoisomorphism $\widetilde{\theta} : \mathcal{I}(A) \to \mathcal{I}(B)$, by Theorem 2.9. The orthogonal idempotents $\widetilde{\theta}(e_i)$ in $\mathcal{I}(B)^{\sim}$ ensure B is infinite-dimensional.) The desired result follows now from Theorem 3.6. If both A and B are finite dimensional we refer to [12]. ∎

Using H. Lin's characterization of TAF-algebras (see [17] or Theorem 3.3.5 in [20]) we can also state Corollary 3.8 as follows.

COROLLARY 3.9. *Let A and B be two simple, unital, nuclear, separable* TAF-*algebras of real rank zero, in the* UCT-*class $\mathcal{N}$ with isomorphic general linear groups (as abstract groups). Then*

$$(K_0(A), K_0(A)_+, [1_A]) \quad \textit{and} \quad (K_0(B), K_0(B)_+, [1_B])$$

are order isomorphic by a map preserving the distinguished order units.

COROLLARY 3.10. *If A and B are either two simple unital* AF-*algebras, or two irrational rotation algebras, then A is $*$-isomorphic to B if and only if their general linear groups are isomorphic (as abstract groups).*

Proof. Both the class of unital simple AF-algebras and the class of irrational rotation algebras are classified by $(K_0, K_{0+}, [1])$, see Theorem 7.3.4 in [19] and Corollary VI.5.3 in [7].

Any unital simple AF-algebra is a nuclear TAF-algebra of real rank zero, in the UCT-class $\mathcal{N}$, and any irrational rotation algebra is an AH-algebra of slow dimension growth and of real rank zero, see [10]. ∎

3.2. FROM A GENERAL LINEAR GROUP ISOMORPHISM TO A C^*-ISOMORPHISM. For simple AH-algebras of real rank zero, let us recall the classification theorem, provided independently by Gong in [14] and Dadarlat in [5], whose proof uses Elliott-Gong's classification in [11] (see for example Theorem 3.3.1 in [20]).

THEOREM 3.11 (Dadarlat, Gong, Elliott). *Let A and B be simple, unital, AH-algebras of slow dimension growth and of real rank zero. It follows that A is $*$-isomorphic to B if and only if*

$$(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B]), \quad K_1(A) \cong K_1(B).$$

NOTATION 3.12. Let A be a unital C^*-algebra. We equip the general linear group $GL(A)$ with the topology induced by the norm on A. Denote by $GL_0(A)$ the connected component $\{u : u \sim_{\mathrm{h}} 1\}$ of the identity element in $GL(A)$.

THEOREM 3.13. *Let A and B be simple, unital* AH-*algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their general linear groups are topologically isomorphic.*

Proof. If A and B are isomorphic then their general linear groups are topologically isomorphic. Conversely let $\varphi : GL(A) \to GL(B)$ be a topological isomorphism from $GL(A)$ onto $GL(B)$. By continuity of φ, $\varphi(GL(A)_0) = GL(B)_0$. It follows that $u + GL(A)_0 \mapsto \varphi(u) + GL(B)_0$ is an isomorphism of $GL(A)/GL(A)_0$ and $GL(B)/GL(B)_0$ (with inverse $v + GL(B)_0 \mapsto \varphi^{-1}(v) + GL(A)_0$). Recall that for a unital C^*-algebra C of stable rank one, $K_1(C)$ is isomorphic to the group $GL(C)/GL(C)_0$, see Theorem 2.10 in [18]. Consequently, we conclude that $K_1(A)$ is isomorphic to $K_1(B)$. ∎

4. THE CASE OF KIRCHBERG ALGEBRAS

4.1. FROM AN ORTHOISOMORPHISM TO A K_0-ISOMORPHISM.

In this subsection, inspired by [1], we show that an isomorphism between the general linear groups of simple, unital, purely infinite C^*-algebras induces an isomorphism between their K_0-groups.

THEOREM 4.1. *Every simple, unital, purely infinite C^*-algebra is oddly decomposable.*

The result follows immediately from Theorem 5.2 in [1] in combination with Remark 2.2.

Recall that if A is a purely infinite simple C^*-algebra, then every nonzero projection in A is infinite, and $K_0(A) = \{[p] : p \in \mathcal{P}(A), p \neq 0\}$ (see p. 73–85 in [19]). If A, in addition, is unital then 1 is an infinite projection and therefore Murray–von Neumann equivalent to a subprojection $q < 1$. Hence $[q] = [1]$, and $K_0(A) = \{[p] : p \in \mathcal{P}(A)^{\sim}\}$.

THEOREM 4.2. *If A and B are two unital, simple, purely infinite C^*-algebras, whose general linear groups are isomorphic (as abstract groups), then there is an isomorphism from $K_0(A)$ to $K_0(B)$, sending $[1_A]$ to $[1_B]$.*

Proof. Let $\widetilde{\theta} : \mathcal{I}(A) \to \mathcal{I}(B)$ be the orthoisomorphism preserving similarity of idempotents given by Theorem 2.9 and Theorem 4.1 with $\widetilde{\theta}(1) = 1$. Let

$$\widetilde{\theta}_* : K_0(A) \to K_0(B)$$

be given by $\widetilde{\theta}_*([p]) = [p']$, where $p' = u\widetilde{\theta}(p)u^{-1}$ for some $u \in GL(B)$ such that p' is a projection in B, see Lemma 3.2.

We show $\widetilde{\theta}_*$ is well defined and injective: fix any two projections p, q in $\mathcal{P}(A)^{\sim}$. Assume $[p] = [q]$. Since p, q are infinite the assumption is equivalent to p, q being unitarily equivalent, see Corollary 6.11.9 in [2]. By Lemma 3.2 the assumption is equivalent to p, q being similar. Since $\widetilde{\theta}$ and $\widetilde{\theta}^{-1}$ preserves similarity the assumption is equivalent to $\widetilde{\theta}(p), \widetilde{\theta}(q)$ being similar. By definition of p', q'

the assumption is equivalent to p', q' being similar, and hence unitarily equivalent (see Lemma 3.2). Using Corollary 6.11.9 in [2] once more we obtain that the assumption $[p] = [q]$ is equivalent to $[p'] = [q']$.

We show $\widetilde{\theta}_*$ is unital and surjective: since $\widetilde{\theta}(1) = 1$ is a projection we get that $\widetilde{\theta}_*([1]) = [\widetilde{\theta}(1)] = [1]$. Fix a projection $p \in \mathcal{P}(B)^\sim$. Find an idempotent $e \in \mathcal{I}(A)^\sim$ such that $\widetilde{\theta}(e) = p$. By Lemma 3.2 there exist a $u \in GL(A)$ such that ueu^{-1} is a projection. Now

$$\widetilde{\theta}_*([ueu^{-1}]) = [v\widetilde{\theta}(ueu^{-1})v^{-1}],$$

for an appropriate $v \in GL(B)$. Since e is similar to ueu^{-1} then $\widetilde{\theta}(e)$ is similar to $v\widetilde{\theta}(ueu^{-1})v^{-1}$. By Lemma 3.2 similar projections are unitarily equivalent. Hence Corollary 6.11.9 in [2] ensures $\widetilde{\theta}_*([ueu^{-1}]) = [p]$.

We show $\widetilde{\theta}_*$ is a homomorphism: fix any $p, q \in \mathcal{P}(A)^\sim$. Since $1 - q, q$ are (full and properly) infinite we can find projections $r \leqslant 1 - q$ and $s \leqslant q$ in A such that p (respectively q) is Murray–von Neumann equivalent to r (respectively s), see p. 75 in [19]. In particular $[p] = [r]$ and $[q] = [s]$. Since r and s are orthogonal $[r+s] = [r] + [s]$. Find $v \in GL(B)$ such that $v\widetilde{\theta}(r)v^{-1}$ and $v\widetilde{\theta}(s)v^{-1}$ are orthogonal projections in B (see Remark 2.2). Hence

$$\begin{aligned}\widetilde{\theta}_*([p]+[q]) &= \widetilde{\theta}_*([r]+[s]) = \widetilde{\theta}_*([r+s]) = [v\widetilde{\theta}(r+s)v^{-1}] = [v\widetilde{\theta}(r)v^{-1} + v\widetilde{\theta}(s)v^{-1}] \\ &= [v\widetilde{\theta}(r)v^{-1}] + [v\widetilde{\theta}(s)v^{-1}] = \widetilde{\theta}_*([r]) + \widetilde{\theta}_*([s]) = \widetilde{\theta}_*([p]) + \widetilde{\theta}_*([q]).\end{aligned}$$

This shows that $\widetilde{\theta}_*$ is the desired isomorphism. ∎

In [4], J. Cuntz proved that for $2 \leqslant n < \infty$, $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Hence, we have:

COROLLARY 4.3. *Two Cuntz algebras are isomorphic if and only if their general linear groups are isomorphic (as abstract groups).*

4.2. FROM A GENERAL LINEAR GROUP ISOMORPHISM TO A C^*-ISOMORPHISM.

Recall that a *Kirchberg algebra* is a purely infinite, simple, nuclear, separable C^*-algebra, see Definition 4.3.1 in [20], and that the following result of Kirchberg and Phillips essentially classifies such algebras.

THEOREM 4.4 (Kirchberg, Phillips). *Let A and B be unital Kirchberg algebras in the* UCT-*class $\mathcal{N}$. Then A and B are $*$-isomorphic if and only if there exist isomorphisms $\alpha_0 : K_0(A) \to K_0(B)$ and $\alpha_1 : K_1(A) \to K_1(B)$ with $\alpha_0([1_A]) = [1_B]$.*

NOTATION 4.5. Let A be a unital C^*-algebra. As usual, the topology on the unitary group $\mathcal{U}(A)$ is inherited from $GL(A)$. Denote by $\mathcal{U}_0(A)$ the connected component $\{u : u \sim_{\mathrm{h}} 1\}$ of the identity element in $\mathcal{U}(A)$.

THEOREM 4.6. *If A and B are two unital, simple, purely infinite C^*-algebras, whose general linear groups are isomorphic (as abstract groups), then the groups $K_1(A)$ and $K_1(B)$ are isomorphic.*

Proof. Let $\varphi : GL(A) \to GL(B)$ denote the isomorphism from $GL(A)$ into $GL(B)$. Since φ preserves symmetries (i.e., if $s^2 = 1$ in $GL(A)$ then $\varphi(s)^2 = 1$ in $GL(B)$) and symmetries generate the connected component of the identity (by Theorem 3.7 in [16]) we have that $\varphi(GL_0(A)) = GL_0(B)$. Let

$$\widetilde{\varphi}_* : \mathcal{U}(A)/\mathcal{U}_0(A) \to \mathcal{U}(B)/\mathcal{U}_0(B)$$

be given by $\widetilde{\varphi}_*([u]) = [u']$, where $u' = \omega(\varphi(u))$ and ω is the map from Proposition 2.1.8 in [19] turning invertible elements into unitaries.

We show $\widetilde{\varphi}_*$ is well defined and injective. Fix any two unitaries u, v in $\mathcal{U}(A)$. Assume $[u] = [v]$. Recall that $u \sim_{\mathrm{h}} v$ in $\mathcal{U}(A)$ if and only if $u \sim_{\mathrm{h}} v$ in $GL(A)$, see Proposition 2.1.8 in [19]. In particular the assumption is equivalent to $\varphi(u) \sim_{\mathrm{h}} \varphi(v)$ in $GL(B)$ (recalling $\varphi(GL_0(A)) = GL_0(B)$). By Proposition 2.1.8 in [19] both $\varphi(u) \sim_{\mathrm{h}} \omega(\varphi(u))$ and $\varphi(v) \sim_{\mathrm{h}} \omega(\varphi(v))$ in $GL(B)$. Hence the assumption is equivalent to $u' \sim_{\mathrm{h}} v'$ in $GL(B)$, and hence also to $[u'] = [v']$.

We show $\widetilde{\varphi}_*$ is surjective. Fix an unitary $u \in \mathcal{U}(B)$. Find an invertible element $v \in GL(A)$ such that $\varphi(v) = u$. Similarly to a previous argument we have that $\omega(v) \sim_{\mathrm{h}} v = \varphi^{-1}(u)$ in $GL(A)$ and $\varphi(\omega(v)) \sim_{\mathrm{h}} \varphi(v) = u$ in $GL(B)$. Using that $\varphi(\omega(v)) \sim_{\mathrm{h}} \omega(\varphi(\omega(v)))$ we obtain that $[\omega(\varphi(\omega(v)))] = [u]$. Hence $\widetilde{\varphi}_*([\omega(v)]) = [u]$.

We show $\widetilde{\varphi}_*$ is a homomorphism. Fix any $u, v \in \mathcal{U}(A)$. Using the equivalences $\varphi(u) \sim_{\mathrm{h}} \omega(\varphi(u))$ and $\varphi(v) \sim_{\mathrm{h}} \omega(\varphi(v))$ in $GL(B)$ we obtain that

$$\varphi(u)\varphi(v) \sim_{\mathrm{h}} \varphi(u)\omega(\varphi(v)) \sim_{\mathrm{h}} \omega(\varphi(u))\omega(\varphi(v)) \quad \text{in } GL(B).$$

We also have that $\omega(\varphi(uv)) \sim_{\mathrm{h}} \varphi(uv)$ in $GL(B)$. Combining these relations we have that $[\omega(\varphi(uv))] = [\omega(\varphi(u))\omega(\varphi(v))]$. We conclude that

$$\widetilde{\varphi}_*([u])\widetilde{\varphi}_*([v]) = [\omega(\varphi(u))\omega(\varphi(v))] = [\omega(\varphi(uv))] = \widetilde{\varphi}_*([uv]).$$

This shows that $\widetilde{\varphi}_*$ is the desired isomorphism. Recall that for a unital purely infinite simple C^*-algebra C, $K_1(C)$ is isomorphic to $\mathcal{U}(C)/\mathcal{U}_0(C)$ by Theorem 1.9 in [4]. Consequently, we conclude that $K_1(A)$ is isomorphic to $K_1(B)$. ∎

Thanks to Theorems 4.2 and 4.6, we have the following conclusion.

COROLLARY 4.7. *Let A and B be two unital Kirchberg algebras in the* UCT*-class $\mathcal{N}$. Then A and B are isomorphic if and only if their general linear groups are isomorphic (as abstract groups).*

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