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HOLOMORPHIC AUTOMORPHISMS OF NONCOMMUTATIVE POLYBALLS

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ABSTRACT. In this paper, we study free holomorphic functions on regular polyballs $\mathbf{B_n}$ and provide analogues of several classical results from complex analysis such as: Abel theorem, Hadamard formula, Cauchy inequality, Schwarz lemma, and maximum principle. These results are used together with a class of noncommutative Berezin transforms to obtain a complete description of the group $\mathrm{Aut}(\mathbf{B_n})$ of all free holomorphic automorphisms of the polyball $\mathbf{B_n}$. We also obtain a concrete description for the group of automorphisms of the tensor product $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ of Cuntz–Toeplitz algebras which leave invariant the tensor product $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$ of noncommutative disc algebras, which extends Voiculescu's result when $k = 1$.

KEYWORDS: *Noncommutative polyball, automorphism, Berezin transform, Fock space, creation operators, Cuntz–Toeplitz algebra.*

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INTRODUCTION

Recently (see [22], [24]), we have tried to unify the multivariable operator model theory for ball-like domains and commutative polydiscs, and extend it to a more general class of noncommutative polydomains (which includes the regular polyballs) and use it to develop a theory of free holomorphic functions. In general, one can view the free holomorphic functions as free noncommutative functions in the sense of [8]. What is remarkable for these polydomains is that they have universal models, in a certain sense, which are (weighted) creation operators acting on tensor products of full Fock spaces. The model theory and the free holomorphic function theory on these polydomains are related, via noncommutative Berezin transforms, to the study of the operator algebras generated by the universal models, as well as to the theory of functions in several complex variable ([9], [27], [28]). It is the interplay between these three fields that lead to a rich analytic function theory on these noncommutative polydomains. Our work on curvature invariant [23] and Euler characteristic [25] on noncommutative regular polyballs has led us to study the free holomorphic automorphisms of these polyballs, which is the goal of the present paper and continues work of Voiculescu [30], of Davidson and Pitts [6], of Helton, Klep, McCullough and Singled [7], of Benhida and Timotin [2], [3], and of the author in [19], [21]. In a related context we mention the work of Muhly and Solel [11], and of Power and Solel [26].

Throughout this paper, $B(\mathcal{H})$ stands for the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. We denote by $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, where $n_i \in \mathbb{N} := \{1, 2, \ldots\}$, the set of all tuples $\mathbf{X} := (X_1, \ldots, X_k)$ in $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$. Note that the operators $X_{s,1}, \ldots, X_{s,n_s}$ are not necessarily commuting. Let $\mathbf{n} := (n_1, \ldots, n_k)$ and define the polyball

$$\mathbf{P_n}(\mathcal{H}) := [B(\mathcal{H})^{n_1}]_1 \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1,$$

where

$$[B(\mathcal{H})^n]_1 := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \|X_1 X_1^* + \cdots + X_n X_n^*\| < 1\}, \quad n \in \mathbb{N}.$$

If A is a positive invertible operator, we write $A > 0$. The *regular polyball* on the Hilbert space $\mathcal{H}$ is defined by

$$\mathbf{B_n}(\mathcal{H}) := \{\mathbf{X} \in \mathbf{P_n}(\mathcal{H}) : \mathbf{\Delta_X}(I) > 0\},$$

where the *defect mapping* $\mathbf{\Delta_X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\mathbf{\Delta_X} := (\mathrm{id} - \Phi_{X_1}) \circ \cdots \circ (\mathrm{id} - \Phi_{X_k}),$$

and $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \quad Y \in B(\mathcal{H}).$$

We call the operator $\mathbf{\Delta_X}(I)$ the defect of $\mathbf{X}$. Note that if $k = 1$, then $\mathbf{B_n}(\mathcal{H})$ coincides with the noncommutative unit ball $[B(\mathcal{H})^{n_1}]_1$. We remark that the scalar representation of the (*abstract*) *regular polyball*

$$\mathbf{B_n} := \{\mathbf{B_n}(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$$

is $\mathbf{B_n}(\mathbb{C}) = \mathbf{P_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \ldots, e_{n_i}^i$. We consider the *full Fock space* of H_{n_i} defined by $F^2(H_{n_i}) := \mathbb{C}1 \oplus \bigoplus_{p \geqslant 1} H_{n_i}^{\otimes p}$, where $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i, \ldots, g_{n_i}^i$ and the identity g_0^i. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. The length of $\alpha \in \mathbb{F}_{n_i}^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0^i$ and $|\alpha| := p$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i$, where $j_1, \ldots, j_p \in \{1, \ldots, n_i\}$. We define the *left creation operator* $S_{i,j}$ acting on the Fock space $F^2(H_{n_i})$ by setting $S_{i,j}e^i_\alpha := e^i_{g^i_j\alpha}$, $\alpha \in \mathbb{F}^+_{n_i}$, and the operator $\mathbf{S}_{i,j}$ acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$\mathbf{S}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We introduce the noncommutative Hardy algebra $\mathbf{F}^\infty_{\mathbf{n}}$ (respectively the polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$) as the weakly closed (respectively norm closed) non-selfadjoint algebra generated by $\{\mathbf{S}_{i,j}\}$ and the identity.

We proved in [24] (in a more general setting) that $\mathbf{X} \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ is a *pure* element in the regular polyball $\mathbf{B_n}(\mathcal{H})^-$, i.e. $\lim_{q_i \to \infty} \Phi^{q_i}_{X_i}(I) = 0$ in the weak operator topology, if and only if there is a Hilbert space $\mathcal{K}$ and a subspace $\mathcal{M} \subset F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{K}$ invariant under each operator $\mathbf{S}_{i,j} \otimes I$ such that $X^*_{i,j} = (\mathbf{S}^*_{i,j} \otimes I)|_{\mathcal{M}^\perp}$ under an appropriate identification of $\mathcal{H}$ with $\mathcal{M}^\perp$. The k-tuple $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, where $\mathbf{S}_i := (\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i})$, is an element in the regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$ and plays the role of *universal model* for the abstract polyball $\mathbf{B}^-_{\mathbf{n}} := \{\mathbf{B_n}(\mathcal{H})^- : \mathcal{H} \text{ is a Hilbert space}\}$. The existence of the universal model will play an important role in our paper, since it will make the connection between noncommutative function theory, operator algebras, and complex function theory in several variables. The latter is due to the fact that the joint eingenvectors for the universal model are parameterized by the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$ via the Berezin transforms (see [22]).

In Section 1, we show that the regular polyball $\mathbf{B_n}$ is a logarithmically convex complete Reinhardt noncommutative domain, in an appropriate sense. We provide characterizations for free holomorphic functions on polyballs in terms of their universal models, obtain an analogue of Abel theorem from complex analysis, Cauchy type inequalities for the coefficients of free holomorphic functions, and an analogue of Liouville's theorem for entire functions. We prove that the largest regular polyball $\gamma\mathbf{B_n}$, $\gamma \in [0, \infty]$, which is included in the universal domain of convergence of a formal power series φ in indeterminates $\{Z_{i,j}\}$ and representation $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ with $A_{(\alpha)} \in B(\mathcal{K})$, is given by the relation

$$\frac{1}{\gamma} := \limsup_{(p_1,\ldots,p_k) \in \mathbb{Z}^k_+} \left\| \sum_{\alpha_i \in \mathbb{F}^+_{n_i}, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} A^*_{(\alpha)} A_{(\alpha)} \right\|^{1/2(p_1+\cdots+p_k)},$$

where $Z_{(\alpha)} := Z_{1,\alpha_1} \cdots Z_{k,\alpha_k}$ if $(\alpha) := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$ and $Z_{i,\alpha_i} := Z_{i,j_1} \cdots Z_{i,j_p}$ if $\alpha_i = g^i_{j_1} \cdots g^i_{j_p} \in \mathbb{F}^+_{n_i}$.

In Section 2, we prove a Schwarz type result ([28]) which states that if $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ is a bounded free holomorphic function with $\|F\|_\infty \leqslant 1$ and $F(0) = 0$, then

$$\|F(\mathbf{X})\| \leqslant m_{\mathbf{B_n}}(\mathbf{X}) < 1 \quad \text{and} \quad m_{\mathbf{B_n}}(\mathbf{X}) \leqslant \|\mathbf{X}\|, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where $m_{\mathbf{B}}$ is the Minkovski functional associated with the regular polyball $\mathbf{B_n}$. This result is used to prove a maximum principle for bounded free holomorphic functions on polyballs which states that if $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a bounded free holomorphic function and there exists $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that

$$\|F(\mathbf{X})\| \leqslant \|F(\mathbf{X}_0)\|, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

then F must be a constant. The results of Section 2 will play an important role in the next sections.

In Section 3, we give a complete description of the free holomorphic automorphisms of the polyball $\mathbf{B_n}$ (see Theorem 3.6), which extends Rudin's characterization of the holomorphic automorphisms of the polydisc [28], and prove some of their basic properties (see Theorem 3.9). We also present an analogue of Poincaré's result [9], that the open unit ball of $\mathbb{C}^n$ is not biholomorphic equivalent to the polydisk $\mathbb{D}^n$, for noncommutative regular polyballs. More precisely, if $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $\mathbf{m} = (m_1, \ldots, m_q) \in \mathbb{N}^q$, we show that there is a biholomorphic map between the polyballs $\mathbf{B_n}$ and $\mathbf{B_m}$ if and only if $k = q$ and there is a permutation σ of the set $\{1, \ldots, k\}$ such that $m_{\sigma(i)} = n_i$ for any $i \in \{1, \ldots, k\}$. Moreover, any free biholomorphic function $F : \mathbf{B_n} \to \mathbf{B_m}$ is up to a permutation of $(m_1, \ldots, m_k)$ an automorphism of the noncommutative regular polyball $\mathbf{B_n}$. This resembles the classical result of Ligocka [10] and Tsyganov [29] concerning biholomorphic automorphisms of product spaces with nice boundaries. The results of this section are used to show that

$$\mathbf{Aut}(\mathbf{B_n}) \simeq \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1).$$

More precisely, we prove that the map Λ defined by

$$\Lambda(\boldsymbol{\Psi})(\mathbf{z}) := (\boldsymbol{\mathcal{B}}_z[\widehat{\boldsymbol{\Psi}}_1], \ldots, \boldsymbol{\mathcal{B}}_z[\widehat{\boldsymbol{\Psi}}_k]) \quad \mathbf{z} \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1,$$

is a group isomorphism, where $\widehat{\boldsymbol{\Psi}} := \text{SOT-}\lim_{r\to 1} \boldsymbol{\Psi}(r\mathbf{S})$ is the boundary function of $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$ with respect to the universal model $\mathbf{S}$, and $\boldsymbol{\mathcal{B}}_z$ is the noncommutative Berezin transform at $\mathbf{z}$.

In Section 4, we prove that any automorphism Γ of the Cuntz–Toeplitz C^*-algebra $C^*(\mathbf{S})$, generated by the universal model $\mathbf{S} = \{\mathbf{S}_{i,j}\}$, which leaves invariant the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$, i.e. $\Gamma(\boldsymbol{\mathcal{A}}_\mathbf{n}) = \boldsymbol{\mathcal{A}}_\mathbf{n}$, has the form

$$\Gamma(g) := \boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[g] = \mathbf{K}_{\widehat{\Psi}}[g \otimes I_{\mathcal{D}_{\widehat{\Psi}}}]\mathbf{K}^*_{\widehat{\Psi}}, \quad g \in C^*(\mathbf{S}),$$

where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$ and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is the noncommutative Berezin transform at the boundary function $\widehat{\Psi}$. In this case, the noncommutative Berezin kernel $\mathbf{K}_{\widehat{\Psi}}$ is a unitary operator and Γ is a unitarily implemented automorphism of $C^*(\mathbf{S})$. Moreover, we have

$$\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S})) \simeq \mathrm{Aut}(\mathbf{B_n}),$$

where $\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}(C^*(\mathbf{S}))$ is the group of automorphisms of $C^*(\mathbf{S})$ which leave invariant the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$. As a consequence, we obtain a concrete description for the group of automorphisms of the tensor product $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ of Cuntz–Toeplitz algebras which leave invariant the tensor product $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$ of noncommutative disc algebras, which extends Voiculescu's result when $k = 1$. In particular, each holomorphic automorphism of the regular polyball $\mathbf{B_n}$ induces an automorphism of the tensor product of Cuntz algebras $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ which leaves invariant the non-self-adjoint subalgebra $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$.

In Section 5, we prove that any unitarily implemented automorphism of the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$ (respectively the noncommutative Hardy algebra $\mathbf{F}_{\mathbf{n}}^\infty$) is the Berezin transform of a boundary function $\widehat{\Psi}$, where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Moreover, we have

$$\mathrm{Aut}_{\mathrm{u}}(\boldsymbol{\mathcal{A}}_{\mathbf{n}}) \simeq \mathrm{Aut}_{\mathrm{u}}(\mathbf{F}_{\mathbf{n}}^\infty) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

When $k = 1$, we recover some of the results obtained by Davidson and Pitts [6] and the author [19]. Let $H^\infty(\mathbf{B_n})$ be the Hardy algebra of all bounded free holomorphic functions on the regular polyball. If $\Lambda : H^\infty(\mathbf{B_n}) \to H^\infty(\mathbf{B_n})$ is a unital algebraic homomorphism, it induces a unique homomorphism $\widetilde{\Lambda} : \mathbf{F}_{\mathbf{n}}^\infty \to \mathbf{F}_{\mathbf{n}}^\infty$ such that $\Lambda \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}} \widetilde{\Lambda}$, where $\boldsymbol{\mathcal{B}}$ is the noncommutative Berezin transform. We prove that $\widetilde{\Lambda}$ is a unitarily implemented automorphism of $\mathbf{F}_{\mathbf{n}}^\infty$ if and only if there is $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ such that

$$\Lambda(f) = f \circ \varphi, \quad f \in H^\infty(\mathbf{B_n}).$$

A similar result holds for the algebra $A(\mathbf{B_n})$ of all bounded free holomorphic functions on $\mathbf{B_n}(\mathcal{H})$ with continuous extension to $\mathbf{B_n}(\mathcal{H})^-$.

In Section 6, we prove that the free holomorphic automorphism group of the polyball $\mathbf{B_n}$ is a σ-compact, locally compact topological group with respect to the topology induced by the metric

$$d_{\mathbf{B_n}}(\phi, \psi) := \|\phi - \psi\|_\infty + \|\phi^{-1}(0) - \psi^{-1}(0)\|, \quad \phi, \psi \in \mathrm{Aut}(\mathbf{B_n}).$$

We also show that if $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, then the free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n})$ has $\mathrm{card}(\Sigma)$ path connected components, where

$$\Sigma := \{\sigma \in \mathcal{S}_k : (n_{\sigma(1)}, \ldots, n_{\sigma(k)}) = (n_1, \ldots, n_k)\}$$

and $\mathcal{S}_k$ is the symmetric group on the set $\{1, \ldots, k\}$. We mention that a map $\pi : \mathrm{Aut}(\mathbf{B_n}) \to \mathcal{U}(\mathcal{K})$, where $\mathcal{U}(\mathcal{K})$ is the unitary group on the Hilbert space $\mathcal{K}$, is called (unitary) projective representation if $\pi(\mathrm{id}) = I$,

$$\pi(\mathbf{\Phi})\pi(\mathbf{\Psi}) = c_{(\mathbf{\Phi}, \mathbf{\Psi})} \pi(\mathbf{\Phi} \circ \mathbf{\Psi}), \quad \mathbf{\Phi}, \mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n}),$$

where $c_{(\mathbf{\Phi}, \mathbf{\Psi})}$ is a complex number with $|c(\mathbf{\Phi}, \mathbf{\Psi})| = 1$, and the map $\mathrm{Aut}(\mathbf{B_n}) \ni \mathbf{\Phi} \mapsto \langle \pi(\mathbf{\Phi})\xi, \eta \rangle \in \mathbb{C}$ is continuous for each $\xi, \eta \in \mathcal{K}$. Using the structure of the free holomorphic automorphisms of the regular polyball $\mathbf{B_n}$, we conclude Section 6 by providing a concrete unitary projective representation of the topological group $\text{Aut}(\mathbf{B_n})$, with respect to the metric $d_{\mathbf{B_n}}$, in terms of noncommutative Berezin kernels associated with regular polyballs.

We mention that the techniques of the present paper will be used in a future one to study the structure of the automorphism groups associated with certain classes of noncommutative varieties in polyballs, including the case of commutative operatorial polyballs. We also expect some of our results to extend to more general noncommutative polydomains ([22], [24]).

1. NONCOMMUTATIVE POLYBALLS AND FREE HOLOMORPHIC FUNCTIONS

In this section, we show that the regular polyball $\mathbf{B_n}$ is a logarithmically convex complete Reinhardt noncommutative domain. We study free holomorphic functions on regular polyballs and provide analogues of several classical results from complex analysis such as: Abel theorem, Hadamard formula, Cauchy inequality, and Liouville theorem for entire functions.

First, we introduce a class of noncommutative Berezin transforms associated with regular polyballs. Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-$ with $X_i := (X_{i,1}, \ldots, X_{i,n_i})$. We use the notation $X_{i,\alpha_i} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha_i = g^i_{j_1} \cdots g^i_{j_p} \in \mathbb{F}^+_{n_i}$ and $X_{i,g^i_0} := I$. The *noncommutative Berezin kernel* associated with any element $\mathbf{X}$ in the noncommutative polyball $\mathbf{B_n}(\mathcal{H})^-$ is the operator

$$\mathbf{K_X} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\mathbf{\Delta_X}(I)(\mathcal{H})}$$

defined by

$$\mathbf{K_X} h := \sum_{\beta_i \in \mathbb{F}^+_{n_i},\, i=1,\ldots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes \mathbf{\Delta_X}(I)^{1/2} X^*_{1,\beta_1} \cdots X^*_{k,\beta_k} h,$$

where $\mathbf{\Delta_X}(I)$ was defined in the introduction. A very important property of the Berezin kernel is that $\mathbf{K_X} X^*_{i,j} = (\mathbf{S}^*_{i,j} \otimes I)\mathbf{K_X}$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. The *Berezin transform at* $X \in \mathbf{B_n}(\mathcal{H})$ is the map

$$\boldsymbol{\mathcal{B}}_{\mathbf{X}} : B\Big(\bigotimes_{i=1}^k F^2(H_{n_i})\Big) \to B(\mathcal{H})$$

defined by

$$\boldsymbol{\mathcal{B}}_{\mathbf{X}}[g] := \mathbf{K}^*_{\mathbf{X}}(g \otimes I_{\mathcal{H}})\mathbf{K_X}, \quad g \in B\Big(\bigotimes_{i=1}^k F^2(H_{n_i})\Big).$$

If g is in the C^*-algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$, we define the Berezin transform at $X \in \mathbf{B_n}(\mathcal{H})^-$, by

$$\boldsymbol{\mathcal{B}}_{\mathbf{X}}[g] := \lim_{r\to 1} \mathbf{K}^*_{r\mathbf{X}}(g \otimes I_{\mathcal{H}})\mathbf{K}_{r\mathbf{X}}, \quad g \in C^*(\mathbf{S}),$$

where the limit is in the operator norm topology. In this case, the Berezin transform at X is a unital completely positive linear map such that

$$\mathcal{B}_{\mathbf{X}}(\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^*) = \mathbf{X}_{(\alpha)}\mathbf{X}_{(\beta)}^*, \quad (\alpha),(\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

where $\mathbf{S}_{(\alpha)} := \mathbf{S}_{1,\alpha_1} \cdots \mathbf{S}_{k,\alpha_k}$ if $(\alpha) := (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. The Berezin transform will play an important role in this paper. More properties concerning noncommutative Berezin transforms and multivariable operator theory on noncommutative balls and polydomains, can be found in [17], [18], [19], [20], [22], and [24]. For basic results on completely positive (respectively bounded) maps we refer the reader to [12] and [13].

In what follows, we present some properties of the regular polyballs. Our first observation is that, in general, the inclusion $\mathbf{B}_{\mathbf{n}}(\mathcal{H}) \subset \mathbf{P}_{\mathbf{n}}(\mathcal{H})$ is strict. Indeed, consider the particular case $n_1 = \cdots = n_k = 1$. Let $\mathcal{M}$ be a Hilbert space, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}$, and $T_i := \begin{pmatrix} 0 & 0 \\ A_i & 0 \end{pmatrix}$, $i \in \{1,\ldots,k\}$, where $A_i \in B(\mathcal{M})$ and $\|A_i\| < 1$. It is clear that $T_iT_s = T_sT_i$ for $i,s \in \{1,\ldots,k\}$, and $\mathbf{\Delta}_{\mathbf{T}}(I) = \begin{pmatrix} I & 0 \\ 0 & I - A_1A_1^* - \cdots - A_kA_k^* \end{pmatrix}$. Consequently, $\mathbf{T} = (T_1,\ldots,T_k) \in \mathbf{B}_{(1,\ldots,1)}(\mathcal{H})$ if and only if $\|A_1A_1^* + \cdots + A_kA_k^*\| < 1$. This clearly proves our assertion. On the other hand, note that there is $r \in (0,1)$ such that $r\mathbf{P}_{\mathbf{n}}(\mathcal{H}) \subset \mathbf{B}_{\mathbf{n}}(\mathcal{H})$. Moreover, due to Proposition 1.3 from [24], one can easily see that $[B(\mathcal{H})^{n_1+\cdots+n_k}]_1 \subset \mathbf{B}_{\mathbf{n}}(\mathcal{H})$.

If $\mathbf{z} = (\mathbf{z}_1,\ldots,\mathbf{z}_k)$, where $\mathbf{z}_i = (z_{i,1},\ldots,z_{i,n_i}) \in \mathbb{C}^{n_i}$, and $\mathbf{X} := (\mathbf{X}_1,\ldots,\mathbf{X}_k)$ is in the cartesian product $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with $\mathbf{X}_i = (X_{i,1},\ldots,X_{i,n_i})$, we denote $\mathbf{zX} := (\mathbf{z}_1\mathbf{X}_1,\ldots,\mathbf{z}_k\mathbf{X}_k)$, where $\mathbf{z}_i\mathbf{X}_i := (z_{i,1}X_{i,1},\ldots,z_{i,k}X_{i,n_i})$. If $\mathbf{r} := (r_1,\ldots,r_k)$, $r_i > 0$, we set $\mathbf{rX} := (r_1\mathbf{X}_1,\ldots,r_k\mathbf{X}_k)$. When $r \in \mathbb{R}^+$, the notation $r\mathbf{X}$ is clear.

LEMMA 1.1. *If $\lambda_i \in \overline{\mathbb{D}}$, $i \in \{1,\ldots,k\}$, and $\mathbf{S} = (\mathbf{S}_1,\ldots,\mathbf{S}_k)$ is the universal model for the regular polyball $\mathbf{B}_{\mathbf{n}}^-$, then*

$$(\mathrm{id} - \Phi_{\lambda_1\mathbf{S}_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\lambda_k\mathbf{S}_k})^{p_k}(I) \geqslant \prod_{i=1}^k (1 - |\lambda_i|^2)^{p_i} I.$$

If $\mathbf{z} = (\mathbf{z}_1,\ldots,\mathbf{z}_k)$, where $\mathbf{z}_i = (z_{i,1},\ldots,z_{i,n_i}) \in \overline{\mathbb{D}}^{n_i}$, then

$$(\mathrm{id} - \Phi_{\mathbf{z}_1\mathbf{S}_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\mathbf{z}_k\mathbf{S}_k})^{p_k}(I) \geqslant (\mathrm{id} - \Phi_{\mathbf{S}_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\mathbf{S}_k})^{p_k}(I)$$

for any $p_i \in \{0,1\}$.

Proof. We recall that two operators $A, B \in B(\mathcal{H})$ are called doubly commuting if $AB = BA$ and $AB^* = B^*A$. Since the entries of $\mathbf{S}_i$ are doubly commuting with the entries of $\mathbf{S}_t$, whenever $i,t \in \{1,\ldots,k\}$, $i \neq t$, we have

$$(\mathrm{id} - \Phi_{\lambda_1\mathbf{S}_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\lambda_k\mathbf{S}_k})^{p_k}(I) = \prod_{i=1}^k (I - \Phi_{\lambda_i\mathbf{S}_i}(I))^{p_i}.$$

Taking into account that $I - \Phi_{\lambda_i \mathbf{S}_i}(I) \geqslant (1-|\lambda_i|^2)I$, the first inequality follows. Similarly, using the inequality $I - \Phi_{\mathbf{z}_i \mathbf{S}_i}(I) \geqslant I - \Phi_{\mathbf{S}_i}(I)$, one can deduce the second inequality. ∎

DEFINITION 1.2. Let G be a subset of $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$.

(i) G is a *complete Reinhardt set* if $\mathbf{z}\mathbf{X} \in G$ for any $\mathbf{X} \in G$ and $\mathbf{z} \in \overline{\mathbb{D}}^{n_1+\cdots+n_k}$.

(ii) G is a *logarithmically convex set* if

$$\{(\log\|X_1\|,\ldots,\log\|X_k\|) : (X_1,\ldots,X_k) \in G, X_i \neq 0\}$$

is a convex subset of $\mathbb{R}^k$.

PROPOSITION 1.3. *The following properties hold:*

(i) *the regular polyball $\mathbf{B_n}(\mathcal{H})$ is relatively open in $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, and its closure in the operator norm topology is equal to*

$$\{\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \mathbf{\Delta}_{\mathbf{X}}^{\mathbf{p}}(I) \geqslant 0 \text{ for } \mathbf{p} = (p_1,\ldots,p_k), p_i \in \{0,1\}\},$$

where $\mathbf{\Delta}_{\mathbf{X}}^{\mathbf{p}} := (\mathrm{id} - \Phi_{X_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{X_k})^{p_k}$ *and* $(\mathrm{id} - \Phi_{X_i})^0 := \mathrm{id}$;

(ii) $\mathbf{B_n}(\mathcal{H})$ *is a complete Reinhardt domain such that*

$$\mathbf{B_n}(\mathcal{H}) = \bigcup_{\mathbf{z}\in\overline{\mathbb{D}}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H}) = \bigcup_{\mathbf{z}\in\mathbb{D}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H})^- = \bigcup_{\mathbf{z}\in\mathbb{D}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H}), \quad \textit{and}$$

$$\mathbf{B_n}(\mathcal{H}) = \bigcup_{0\leqslant r<1} r\mathbf{B_n}(\mathcal{H}) = \bigcup_{0\leqslant r<1} r\mathbf{B_n}(\mathcal{H})^-;$$

(iii) $\mathbf{B_n}(\mathcal{H})^-$ *is a complete Reinhardt set and*

$$\mathbf{B_n}(\mathcal{H})^- = \bigcup_{\mathbf{z}\in\overline{\mathbb{D}}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H})^- = \bigcup_{0\leqslant r\leqslant 1} r\mathbf{B_n}(\mathcal{H})^-.$$

Proof. If $\mathbf{X} = (X_1,\ldots,X_k) \in \mathbf{B_n}(\mathcal{H})$, then there is $c > 0$ such that $\mathbf{\Delta_X}(I) > cI$. Given $d \in (0,c)$, there is $\varepsilon > 0$ such that $-dI \leqslant \mathbf{\Delta_Y}(I) - \mathbf{\Delta_X}(I) \leqslant dI$ for any $\mathbf{Y} = (Y_1,\ldots,Y_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ with $\max\limits_{i\in\{1,\ldots,k\}} \|X_i - Y_i\| < \varepsilon$. Consequently, we have

$$\mathbf{\Delta_Y}(I) = (\mathbf{\Delta_Y}(I) - \mathbf{\Delta_X}(I)) + \mathbf{\Delta_X}(I) \geqslant (c-d)I > 0,$$

which proves that $\mathbf{B_n}(\mathcal{H})$ is relatively open in $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ with respect to the product topology. To prove the second part of item (i), set

$$\mathcal{D} := \{\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \mathbf{\Delta}_{\mathbf{X}}^{\mathbf{p}}(I) \geqslant 0 \text{ for } \mathbf{p} = (p_1,\ldots,p_k), p_i \in \{0,1\}\}.$$

We shall prove that $\mathbf{B_n}(\mathcal{H})^- = \mathcal{D}$. Since $\mathbf{B_n}(\mathcal{H})$ is open, if $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, then there is $r \in [0,1)$ such that $\frac{1}{r}\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. Applying the Berezin transform at $\frac{1}{r}\mathbf{X}$ to the first inequality of Lemma 1.1, when $\lambda_i = r$, we deduce that

$$\mathbf{\Delta}_{\mathbf{X}}^{\mathbf{p}}(I) = (\mathrm{id} - \Phi_{X_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{X_k})^{p_k}(I) \geqslant \prod_{i=1}^{k} (1-r^2)^{p_i} I.$$

Hence, if $\mathbf{Y} \in \mathbf{B_n}(\mathcal{H})^-$, a limiting process implies that $\boldsymbol{\Delta}_{\mathbf{Y}}^{\mathbf{p}}(I) \geqslant 0$ for any $\mathbf{p} = (p_1, \ldots, p_k)$ with $p_i \in \{0,1\}$. Therefore, $\mathbf{B_n}(\mathcal{H})^- \subseteq \mathcal{D}$. To prove the reverse inequality, let $\mathbf{Y} = (Y_1, \ldots, Y_k) \in \mathcal{D}$. In particular, we have $\|rY_i\| < 1$ for any $r \in [0,1)$. Due to Lemma 1.1 and using the Berezin transform at Y, we have $\boldsymbol{\Delta}_{r\mathbf{Y}}(I) \geqslant (1-r^2)^k I$, which shows that $r\mathbf{Y} \in \mathbf{B_n}(\mathcal{H})$. Since $r\mathbf{Y} \to \mathbf{Y}$, as $r \to 1$, we conclude that $\mathcal{D} \subseteq \mathbf{B_n}(\mathcal{H})^-$, which proves item (i).

If $\mathbf{z} \in \overline{\mathbb{D}}^{n_1+\cdots+n_k}$ and $\mathbf{T} \in \mathbf{B_n}(\mathcal{H})$, then applying the Berezin transform at $\mathbf{T}$ to the second inequality of Lemma 1.1 we obtain $\boldsymbol{\Delta}_{\mathbf{zT}}^{\mathbf{p}}(I) \geqslant \boldsymbol{\Delta}_{\mathbf{T}}^{\mathbf{p}}(I) > 0$ for any $\mathbf{p} = (p_1, \ldots, p_k)$ with $p_i \in \{0,1\}$. Consequently, we have

$$\mathbf{z}\mathbf{B_n}(\mathcal{H}) \subseteq \mathbf{B_n}(\mathcal{H}), \quad \mathbf{z} \in \overline{\mathbb{D}}^{n_1+\cdots+n_k},$$

which shows that $\mathbf{B_n}(\mathcal{H})$ is a complete Reinhardt domain. Moreover, we have $\mathbf{B_n}(\mathcal{H}) = \bigcup_{\mathbf{z} \in \overline{\mathbb{D}}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H})$.

Let $\mathbf{T} \in \mathbf{B_n}(\mathcal{H})^-$ and $\mathbf{z} \in \mathbb{D}^{n_1+\cdots+n_k}$. Then there is $r \in (0,1)$ such that $\frac{1}{r}\mathbf{z} \in \mathbb{D}^{n_1+\cdots+n_k}$. Applying the Berezin transform at $r\mathbf{T}$ to the first inequality of Lemma 1.1 when $\lambda_1 = \cdots = \lambda_k = r$, we deduce that $r\mathbf{T} \in \mathbf{B_n}(\mathcal{H})$. Therefore, $\mathbf{zT} \in \frac{1}{r}\mathbf{z}\mathbf{B_n}(\mathcal{H}) \in \mathbf{B_n}(\mathcal{H})$, which shows that

$$\mathbf{z}\mathbf{B_n}(\mathcal{H})^- \subseteq \mathbf{B_n}(\mathcal{H}), \quad \mathbf{z} \in \mathbb{D}^{n_1+\cdots+n_k}. \tag{1.1}$$

Since $\mathbf{B_n}(\mathcal{H})$ is open, for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, there is $r \in (0,1)$ such that $\mathbf{X} \in r\mathbf{B_n}(\mathcal{H})$. Consequently,

$$\mathbf{B_n}(\mathcal{H}) \subset \bigcup_{0 \leqslant r < 1} r\mathbf{B_n}(\mathcal{H}) \subset \bigcup_{\mathbf{z} \in \mathbb{D}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H}) \subseteq \bigcup_{\mathbf{z} \in \mathbb{D}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H})^- \quad \text{and} \tag{1.2}$$

$$\mathbf{B_n}(\mathcal{H}) \subset \bigcup_{0 \leqslant r < 1} r\mathbf{B_n}(\mathcal{H}) \subset \bigcup_{0 \leqslant r < 1} r\mathbf{B_n}(\mathcal{H})^-. \tag{1.3}$$

The relations (1.1) and (1.2) show that the first sequence of equalities in (ii) holds. Due to relation (1.1), for each $r \in [0,1)$, we have $r\mathbf{B_n}(\mathcal{H})^- \subseteq \mathbf{B_n}(\mathcal{H})$ which together with relation and (1.3) show that the second sequence of equalities in item (ii) holds. Now, one can easily see that item (iii) follows immediately from (ii). The proof is complete. ∎

We remark that if $\mathbf{r} := (r_1, \ldots, r_k)$, $r_i > 0$, then we also have $\mathbf{B_n}(\mathcal{H}) = \bigcup_{0 \leqslant r_i < 1} \mathbf{r}\mathbf{B_n}(\mathcal{H})^-$. Note also that the regular polyball $\mathbf{B_n}(\mathcal{H})$ is a logarithmically convex complete Reinhardt domain.

For each $i \in \{1, \ldots, k\}$, let $Z_i := (Z_{i,1}, \ldots, Z_{i,n_i})$ be an n_i-tuple of noncommuting indeterminates and assume that, for any $p, q \in \{1, \ldots, k\}$, $p \neq q$, the entries in Z_p are commuting with the entries in Z_q. We set $Z_{i,\alpha_i} := Z_{i,j_1} \cdots Z_{i,j_p}$ if $\alpha_i \in \mathbb{F}_{n_i}^+$ and $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i$, and $Z_{i,g_0^i} := 1$, where g_0^i is the identity in $\mathbb{F}_{n^i}^+$. Given $A_{(\alpha_1,\ldots,\alpha_k)} \in B(\mathcal{K})$ with $(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we consider formal power series

$$\varphi=\sum_{\alpha_1\in\mathbb{F}_{n_1}^+,\ldots,\alpha_k\in\mathbb{F}_{n_k}^+} A_{(\alpha_1,\ldots,\alpha_k)}\otimes Z_{1,\alpha_1}\cdots Z_{k,\alpha_k},\quad A_{(\alpha_1,\ldots,\alpha_k)}\in B(\mathcal{K}),$$

in indeterminates $Z_{i,j}$. In what follows, we set $(\alpha):=(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, $Z_{(\alpha)}:=Z_{1,\alpha_1}\cdots Z_{k,\alpha_k}$, and $A_{(\alpha)}:=A_{(\alpha_1,\ldots,\alpha_k)}$. We will also use the abbreviation $\varphi=\sum\limits_{(\alpha)} A_{(\alpha)}\otimes Z_{(\alpha)}$.

The next result is an analogue of Abel theorem from complex analysis in our noncommutative multivariable setting.

THEOREM 1.4. *If* $\varphi=\sum\limits_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\ \sum\limits_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes Z_{(\alpha)}$ *is a formal power series and* $\mathbf{r}=(r_1,\ldots,r_k)$, $r_i>0$, *then the following statements hold:*

(i) *If the set*

$$A:=\left\{\left\|r_1^{2p_1}\cdots r_k^{2p_k}\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^*A_{(\alpha)}\right\| : (p_1,\ldots,p_k)\in\mathbb{Z}_+^k\right\}$$

is bounded, then the series

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}\right\|$$

is convergent in $\mathbf{r}\mathbf{B_n}(\mathcal{H})$, *the regular polyball of polyradius* $\mathbf{r}=(r_1,\ldots,r_k)$, *and uniformly convergent on* $\mathbf{s}\mathbf{B_n}(\mathcal{H})^-$ *for* $\mathbf{s}=(s_1,\ldots,s_k)$ *with* $0\leqslant s_i<r_i$.

(ii) *If the set* A *is unbounded, then the series*

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}\right\|\quad\text{and}\quad\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\ \sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}$$

are divergent for some $\mathbf{X}\in\mathbf{r}\mathbf{B_n}(\mathcal{H})^-$ *and some Hilbert space* $\mathcal{H}$.

Proof. Let $s_i<r_i$, $i\in\{1,\ldots,k\}$, and $\mathbf{X}\in\mathbf{r}\mathbf{B_n}(\mathcal{H})$, and assume that there is $C>0$ such that

$$\left\|r_1^{2p_1}\cdots r_k^{2p_k}\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^*A_{(\alpha)}\right\|\leqslant C,\quad (p_1,\ldots,p_k)\in\mathbb{Z}_+^k.$$

Due to the von Neumann type inequality [24], we have

$$\begin{aligned}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}\right\|&\leqslant\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} s_1^{p_1}\cdots s_k^{p_k}A_{(\alpha)}\otimes\mathbf{S}_{(\alpha)}\right\|\\&=s_1^{p_1}\cdots s_k^{p_k}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^*A_{(\alpha)}\right\|^{1/2}\\&<\left(\frac{s_1}{r_1}\right)^{p_1}\cdots\left(\frac{s_k}{r_k}\right)^{p_k}C^{1/2}\end{aligned}$$

for any $\mathbf{X} \in s\mathbf{B_n}(\mathcal{H})^-$. On the other hand, due to Proposition 1.3, we have $\mathbf{rB_n}(\mathcal{H}) = \bigcup_{0\leqslant s_i<r_i} \mathbf{sB_n}(\mathcal{H})^-$. Now, one can easily complete the proof of part (i).

To prove (ii), assume that the set A is unbounded. Then, using the fact that the isometries $\mathbf{S}_{(\alpha)}$, with $(\alpha) = (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, $|\alpha_i| = p_i$, have orthogonal ranges, one can easily deduce that the series

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \left\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes r_1^{p_1}\cdots r_k^{p_k}\mathbf{S}_{(\alpha)} \right\| \quad \text{and}$$
$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes r_1^{p_1}\cdots r_k^{p_k}\mathbf{S}_{(\alpha)}$$

are divergent, and $\mathbf{rS} := (r_1\mathbf{S}_1,\ldots,r_k\mathbf{S}_k) \in \mathbf{rB_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. ∎

DEFINITION 1.5. A power series $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ is called *free holomorphic function* (with coefficients in $B(\mathcal{K})$) on the *abstract polyball* $\boldsymbol{\rho}\mathbf{B_n} := \{\boldsymbol{\rho}\mathbf{B_n}(\mathcal{H}) : \mathcal{H}$ is a Hilbert space$\}$, $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_k)$, $\rho_i > 0$, if the series

$$\varphi(\mathbf{X}) := \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes X_{(\alpha)}$$

is convergent in the operator norm topology for any $\mathbf{X} = \{X_{i,j}\} \in \boldsymbol{\rho}\mathbf{B_n}(\mathcal{H})$ with $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, and any Hilbert space $\mathcal{H}$. We denote by $\mathrm{Hol}(\boldsymbol{\rho}\mathbf{B_n})$ the set of all free holomorphic functions on $\boldsymbol{\rho}\mathbf{B_n}$ with scalar coefficients.

Using Theorem 1.4, one can easily deduce the following characterization for free holomorphic functions on regular polyballs.

COROLLARY 1.6. *Let $\mathbf{S}$ be the universal model associated with the abstract regular polyball $\mathbf{B_n}$. A formal power series $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ is a free holomorphic function (with coefficients in $B(\mathcal{K})$) on the abstract polyball $\boldsymbol{\rho}\mathbf{B_n}$, where $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_k)$, $\rho_i > 0$, if and only if the series*

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \left\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes r_1^{p_1}\cdots r_k^{p_k}\mathbf{S}_{(\alpha)} \right\|$$

converges for any $r_i \in [0,\rho_i)$, $i \in \{1,\ldots,k\}$.

Throughout the paper, we say that the abstract polyball $\mathbf{B_n}$ or a free holomorphic function F on $\mathbf{B_n}$ has a certain property, if the property holds for any Hilbert space representation of $\mathbf{B_n}$ and F, respectively. We remark that the coefficients of a free holomorphic function on a polyball are uniquely determined by its representation on an infinite dimensional Hilbert space. Indeed, assume that $F = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$, $A_{(\alpha)} \in B(\mathcal{K})$, is a free holomorphic function with $F(r\mathbf{S}) = 0$

for any $r \in [0,1)$. Then, for any $x, y \in \mathcal{K}$, we have

$$\langle F(r\mathbf{S})(x \otimes 1), (y \otimes \mathbf{S}_{(\alpha)} 1\rangle = r^{|\alpha_1|+\cdots+|\alpha_k|} \langle A_{(\alpha)} x, y\rangle = 0$$

for any $(\alpha) = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. Hence $A_{(\alpha)} = 0$, which proves our assertion.

COROLLARY 1.7. *If $\varphi = \sum_{(\alpha)} a_{(\alpha)} \otimes Z_{(\alpha)}$, $a_{(\alpha)} \in \mathbb{C}$ is a free holomorphic function on the abstract polyball $\boldsymbol{\rho}\mathbf{B_n}$, $\rho = (\rho_1, \ldots, \rho_k)$, then its representation on $\mathbb{C}$, i.e.*

$$\varphi(\lambda_1, \ldots, \lambda_k) = \sum_{(\alpha)} a_{(\alpha)} \otimes \lambda_{(\alpha)}, \quad \lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,n_i}),$$

is a holomorphic function on the scalar polyball $\boldsymbol{\rho}\mathbf{P_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_{\rho_1} \times \cdots \times (\mathbb{C}^{n_k})_{\rho_k}$

In what follows, we obtain Cauchy type inequalities for the coefficients of free holomorphic functions on regular polyballs.

THEOREM 1.8. *Let $F : \boldsymbol{\rho}\mathbf{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$ be a free holomorphic function with representation*

$$F(\mathbf{X}) = \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \ \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes X_{(\alpha)}.$$

Let $\mathbf{r} = (r_1, \ldots, r_k)$ be such that $0 < r_i < \rho_i$ and set $M(\mathbf{r}) := \sup_{\mathbf{X}\in \mathbf{r}\mathbf{B_n}(\mathcal{H})^-} \|F(\mathbf{X})\|$. Then, for each $(p_1, \ldots, p_k) \in \mathbb{Z}_+^k$, we have

$$\Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2} \leqslant \frac{1}{r_1^{p_1}\cdots r_k^{p_k}} M(\mathbf{r}).$$

Moreover, $M(\mathbf{r}) = \|F(\mathbf{rS})\|$, where $\mathbf{S}$ is the universal model of the regular polyball $\mathbf{B_n}$.

Proof. Using the fact that the isometries $\mathbf{S}_{(\alpha)}$, with $(\alpha) = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, $|\alpha_i| = p_i$, have orthogonal ranges, we deduce that

$$\begin{aligned}
&\Big|\Big\langle \Big(\sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^* \otimes \mathbf{S}_{(\alpha)}^* \Big) F(\mathbf{rS})(h\otimes 1), h\otimes 1 \Big\rangle\Big| \\
&\qquad \leqslant \Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^* \otimes \mathbf{S}_{(\alpha)}^* \| M(\mathbf{r}) \| h \Big\|^2 \\
&\qquad = \Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^* A_{(\alpha)} \|^{1/2} M(\mathbf{r}) \| h \Big\|^2
\end{aligned}$$

for any $h \in \mathcal{K}$. On the other hand, we have

$$\Big\langle \Big(\sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)}^* \otimes \mathbf{S}_{(\alpha)}^* \Big) F(\mathbf{rS})(h\otimes 1), h\otimes 1 \Big\rangle$$

$$= r_1^{p_1} \cdots r_k^{p_k} \left\langle \Big(\sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \otimes I \Big) (h \otimes 1), h \otimes 1 \right\rangle$$

$$= r_1^{p_1} \cdots r_k^{p_k} \Big\| \Big(\sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big)^{1/2} h \Big\|^2.$$

Hence, using the previous inequality, we deduce that

$$r_1^{p_1} \cdots r_k^{p_k} \Big\| \Big(\sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big)^{1/2} h \Big\|^2$$

$$\leqslant \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2} M(\mathbf{r}) \|h\|^2$$

for any $h \in \mathcal{K}$, and the inequality in the theorem follows. The fact that $M(\mathbf{r}) = \|F(\mathbf{r}\mathbf{S})\|$ is due to von Neumann inequality [17]. The proof is complete. ∎

We remark that due to the fact that there is $r \in (0,1)$ such that $r\mathbf{P_n}(\mathcal{H}) \subset \mathbf{B_n}(\mathcal{H})$, we have

$$B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} = \bigcup_{\rho > 0} \rho \mathbf{B_n}(\mathcal{H}).$$

We say that F is an *entire function* in $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ if F is free holomorphic on every regular polyball $\rho\mathbf{B_n}(\mathcal{H})$, $\rho > 0$.

Here is an analogue of Liouville's theorem for entire functions on $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$.

COROLLARY 1.9. *If $F : B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$ is an entire function with the property that there is a constant $C > 0$ and $(q_1, \dots, q_k) \in \mathbb{Z}_+^k$ such that*

$$\|F(\mathbf{X})\| \leqslant C \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = q_i,\, i \in \{1,\dots,k\}} \mathbf{X}_{(\alpha)} \mathbf{X}_{(\alpha)}^* \Big\|^{1/2}$$

for any $\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, then F is a polynomial of degree at most $q_1 + \cdots + q_k$. In particular, a bounded free holomorphic function must be constant.

Proof. Let F have the representation

$$F(\mathbf{X}) = \sum_{(p_1,\dots,p_k) \in \mathbb{Z}_+^k} \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes X_{(\alpha)}.$$

Due to the hypothesis, we have

$$\|F(\mathbf{r}\mathbf{S})\| \leqslant C r_1^{q_1} \cdots r_k^{q_k} \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = q_i,\, i \in \{1,\dots,k\}} \mathbf{S}_{(\alpha)} \mathbf{S}_{(\alpha)}^* \Big\|^{1/2} \leqslant C r_1^{q_1} \cdots r_k^{q_k}$$

for any $r_i > 0$. Hence, and using Theorem 1.8, we deduce that

$$\left\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \leqslant \frac{1}{r_1^{p_1} \cdots r_k^{p_k}} M(\mathbf{r}) \leqslant \frac{1}{r_1^{p_1} \cdots r_k^{p_k}} \|F(\mathbf{rS})\|$$
$$\leqslant C \frac{1}{r_1^{p_1 - q_1} \cdots r_k^{p_k - q_k}}$$

for any $r_i > 0$ and $i \in \{1, \ldots, k\}$. Consequently, if there is $s \in \{1, \ldots, k\}$ such that $p_s > q_s$, then taking $r_s \to \infty$ we obtain

$$\sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} A_{(\alpha)}^* A_{(\alpha)} = 0,$$

which implies $A_{(\alpha)} = 0$ for any $(\alpha) = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_i \in \mathbb{F}_{n_i}^+$ and $|\alpha_i| = p_i$ and any $p_i \in \mathbb{Z}^+$, $i \neq s$. Hence, we have

$$F(\mathbf{X}) = \sum_{(p_1,\ldots,p_k) \in \mathbb{Z}_+^k,\, p_i \leqslant q_i} \; \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} A_{(\alpha)} \otimes X_{(\alpha)}.$$

The proof is complete. ∎

Let Λ be equal to the set

$$\left\{ \mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{R}_+^k : \left\{ \left\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} r_1^{2p_1} \cdots r_k^{2p_k} A_{(\alpha)}^* A_{(\alpha)} \right\| \right\}_{(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} \text{is bounded} \right\}.$$

Given a formal power series $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$, we define the set

$$\mathbf{D}_\varphi(\mathcal{H}) := \bigcup_{\mathbf{r} \in \Lambda} \mathbf{r}\mathbf{B}_{\mathbf{n}}(\mathcal{H}).$$

We say that $\mathbf{D}_\varphi$ is logarithmically convex if Λ is log-convex, i.e. the set

$$\{(\log r_1, \ldots, \log r_k) : (r_1, \ldots, r_k) \in \Lambda, r_i > 0\}$$

is convex.

PROPOSITION 1.10. *Let* $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ *be a formal power series. The following statements hold:*

(i) φ *is free holomorphic on* $\mathbf{D}_\varphi$ *and*

$$\varphi(\mathbf{X}) = \sum_{(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} \; \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\ldots,k\}} A_{(\alpha)} \otimes X_{(\alpha)}, \quad \mathbf{X} \in \mathbf{D}_\varphi,$$

where the series is convergent in the operator norm.

(ii) $\mathbf{D}_\varphi$ *is a logarithmically convex complete Reinhardt domain.*

Proof. According to Theorem 1.4 and due to the uniqueness of the representation for free holomorphic functions on polyballs, φ is a free holomorphic function on $\mathbf{D}_\varphi(\mathcal{H}) := \bigcup_{\mathbf{r} \in \Lambda} \mathbf{r}\mathbf{B}_{\mathbf{n}}(\mathcal{H})$ and has the representation of item (i). To prove (ii), note first that, due to Proposition 1.3, $\mathbf{D}_\varphi$ is a complete Reinhardt domain. Now, let $(r_1,\ldots,r_k)$ and $(s_1,\ldots,s_k)$ be in Λ. Then there is a constant $C>0$ such that

$$\|r_1^{2p_1}\cdots r_k^{2p_k}Q_{\mathbf{p}}\|\leqslant C \quad \text{and} \quad \|s_1^{2p_1}\cdots s_k^{2p_k}Q_{\mathbf{p}}\|\leqslant C$$

for any $\mathbf{p}=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$, where $Q_{\mathbf{p}}:=\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\alpha)}^*A_{(\alpha)}$. Consequently, due to the spectral theorem for positive operators, we have

$$\begin{aligned}
\|(r_1^ts_1^{1-t})^{2p_1}\cdots(r_k^ts_k^{1-t})^{2p_k}Q_{\mathbf{p}}\| &= \|(r_1^{2p_1}\cdots r_k^{2p_k}Q_{\mathbf{p}})^t(s_1^{2p_1}\cdots s_k^{2p_k}Q_{\mathbf{p}})^{1-t}\|\\
&\leqslant \|(r_1^{2p_1}\cdots r_k^{2p_k}Q_{\mathbf{p}})^t\|\|(s_1^{2p_1}\cdots s_k^{2p_k}Q_{\mathbf{p}})^{1-t}\|\\
&\leqslant \|r_1^{2p_1}\cdots r_k^{2p_k}Q_{\mathbf{p}}\|^t\|s_1^{2p_1}\cdots s_k^{2p_k}Q_{\mathbf{p}}\|^{1-t}\\
&\leqslant C^tC^{1-t}=C.
\end{aligned}$$

Consequently, $(r_1^ts_1^{1-t},\ldots,r_k^tk_1^{1-t})\in\Lambda$, which proves that $\mathbf{D}_\varphi$ is logarithmically convex. The proof is complete. ∎

We remark that, due to Theorem 1.4, if $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_k)\notin\Lambda$, then the series $\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}$ is divergent for some $\mathbf{X}\in\boldsymbol{\rho}\mathbf{B_n}(\mathcal{H})^-$ and some Hilbert space $\mathcal{H}$. Indeed, take $\mathbf{X}=\boldsymbol{\rho}\mathbf{S}$ and use Theorem 1.4. We call the set $\mathbf{D}_\varphi$ the *universal domain of convergence* of the power series φ.

Our next task is to find the largest polyball $r\mathbf{B_n}(\mathcal{H})$, $r>0$, which is included in the universal domain of convergence of φ.

THEOREM 1.11. *Let $\varphi=\sum_{(\alpha)}A_{(\alpha)}\otimes Z_{(\alpha)}$ be a formal power series and define $\gamma\in[0,\infty]$ by setting*

$$\frac{1}{\gamma}:=\limsup_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\alpha)}^*A_{(\alpha)}\right\|^{1/2(p_1+\cdots+p_k)}.$$

Then the following statements hold:

(i) *The series*

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\left\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes X_{(\alpha)}\right\|,\quad \mathbf{X}\in\gamma\mathbf{B_n}(\mathcal{H}),$$

is convergent. Moreover, the convergence is uniform on $r\mathbf{B_n}(\mathcal{H})^-$ if $0\leqslant r<\gamma$.

(ii) *For any $s>\gamma$, there is a Hilbert space $\mathcal{H}$ and $\mathbf{Y}\in s\mathbf{B_n}(\mathcal{H})^-$ such that the series*

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k}\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\alpha)}\otimes Y_{(\alpha)}$$

is divergent in the operator norm topology.

Proof. Assume that $\gamma > 0$ and let $\mathbf{X} \in r\mathbf{B_n}(\mathcal{H})^-$, where $0 \leqslant r < \gamma$. Fix $\rho \in (r, \gamma)$ and note that

$$\Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2(p_1+\cdots+p_k)} < \frac{1}{\rho}$$

for all but finitely many $(p_1, \dots, p_k) \in \mathbb{Z}_+^k$. Consequently, due to the von Neumann type inequality [17], we have

$$\begin{aligned}
\Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes X_{(\alpha)} \Big\| &\leqslant \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes r^{p_1+\cdots+p_k} \mathbf{S}_{(\alpha)} \Big\| \\
&= r^{p_1+\cdots+p_k} \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2} \\
&< \Big(\frac{r}{\rho}\Big)^{p_1+\cdots+p_k}
\end{aligned}$$

for all but finitely many $(p_1, \dots, p_k) \in \mathbb{Z}_+^k$. Hence, item (i) holds and also implies that the series $\sum_{q=0}^{\infty} \Big\| \sum_{|\alpha_1|+\cdots+|\alpha_k| = q,\, \alpha_i \in \mathbb{F}_{n_i}^+} A_{(\alpha)} \otimes X_{(\alpha)} \Big\|$ is uniformly convergent on $r\mathbf{B_n}(\mathcal{H})^-$. The case when $\gamma = \infty$ can be treated in a similar manner. Now, assume that $\gamma < \rho < s$ and let $\mathbf{Y} := s\mathbf{S}$, where $\mathbf{S}$ is the universal model of $\mathbf{B_n}^-$. It is clear that $\mathbf{Y} \in s\mathbf{B_n}(\mathcal{H})^-$ and

$$\Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes Y_{(\alpha)} \Big\| = s^{p_1+\cdots+p_k} \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2}.$$

Since $\frac{1}{\rho} < \frac{1}{\gamma}$, there are infinitely many tuples $(p_1, \dots, p_k) \in \mathbb{Z}_+^k$ such that

$$\Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{1/2(p_1+\cdots+p_k)} > \frac{1}{\rho}$$

and, consequently, $\Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes Y_{(\alpha)} \Big\| > (\frac{s}{\rho})^{p_1+\cdots+p_k}$. This shows that item (ii) holds and, moreover, that the series

$$\sum_{(p_1,\dots,p_k) \in \mathbb{Z}_+^k} \Big\| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i,\, i \in \{1,\dots,k\}} A_{(\alpha)} \otimes Y_{(\alpha)} \Big\|$$

is divergent. ∎

The number γ satisfying properties (i) and (ii) in the theorem above is called the *polyball radius of convergence* for the power series φ.

COROLLARY 1.12. *Under the conditions of Theorem* 1.11, *the following statements hold:*

(i) *The series*

$$\sum_{q=0}^\infty \left\| \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\, \alpha_i\in\mathbb{F}_{n_i}^+} A_{(\alpha)}\otimes X_{(\alpha)}\right\|$$

is uniformly convergent on $r\mathbf{B_n}(\mathcal{H})^-$ *if* $0\leqslant r<\gamma$.

(ii) *For any* $s>\gamma$*, there is* $\mathbf{Y}\in s\mathbf{B_n}(\mathcal{H})^-$ *such that the series*

$$\sum_{q=0}^\infty \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\, \alpha_i\in\mathbb{F}_{n_i}^+} A_{(\alpha)}\otimes Y_{(\alpha)}$$

is divergent in the operator norm topology.

Proof. A closer look at the proof of Theorem 1.11 reveals that item (i) was already proved and the only thing that we need in order to complete the proof of item (ii) is that, under the condition $\gamma<\rho<s$,

$$\sum_{q=0}^\infty \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\, \alpha_i\in\mathbb{F}_{n_i}^+} A_{(\alpha)}\otimes s^q\mathbf{S}_{(\alpha)}$$

is divergent in the operator norm topology. Assume the contrary and apply the convergent series above to the vector $x\otimes 1$, where $x\in\mathcal{K}$. We deduce that $\sum_{q=0}^\infty \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\, \alpha_i\in\mathbb{F}_{n_i}^+} A_{(\alpha)}\otimes s^q e_{(\alpha)}$ is in the Hilbert space $\mathcal{K}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. Since the sequence $\{e_{(\alpha)}\}_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}$ is an orthonormal basis for $\bigotimes_{i=1}^k F^2(H_{n_i})$, we conclude that the series $\sum_{(\alpha)} A_{(\alpha)}^*A_{(\alpha)}$ is WOT-convergent. Let $r\in[0,1)$ and note that

$$\begin{aligned}
&\sum_{p=0}^\infty r^p \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k,\, p_1+\cdots+p_k=p} \left\| \sum_{\beta_i\in\mathbb{F}_{n_i}^+, |\beta_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\beta)}\otimes s^{p_1+\cdots+p_k}\mathbf{S}_{(\beta)}\right\|\\
&\leqslant \sum_{p=0}^\infty r^p \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k,\, p_1+\cdots+p_k=p} \left\| \sum_{\beta_i\in\mathbb{F}_{n_i}^+, |\beta_i|=p_i,\, i\in\{1,\ldots,k\}} s^{2(|\beta_1|+\cdots+|\beta_k|)} A_{(\beta)}^*A_{(\beta)}\right\|^{1/2}\\
&\leqslant \sum_{p=0}^\infty r^p \binom{p+k-1}{k-1} \left\| \sum_{\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\ldots,k\}} s^{2(|\beta_1|+\cdots+|\beta_k|)} A_{(\beta)}^*A_{(\beta)}\right\|^{1/2}.
\end{aligned}$$

Since the latter series is convergent for any $r\in[0,1)$, we deduce that

$$\sum_{p=0}^\infty \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k,\, p_1+\cdots+p_k=p} \left\| \sum_{\beta_i\in\mathbb{F}_{n_i}^+, |\beta_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\beta)}\otimes (rs)^{p_1+\cdots+p_k}\mathbf{S}_{(\beta)}\right\| <\infty,$$

which implies that

$$\sum_{p=0}^\infty \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k,\, p_1+\cdots+p_k=p} \left\| \sum_{\beta_i\in\mathbb{F}_{n_i}^+, |\beta_i|=p_i,\, i\in\{1,\ldots,k\}} A_{(\beta)}\otimes \mathbf{X}_{(\beta)}\right\| <\infty$$

for any $\mathbf{X} \in \rho\mathbf{B_n}(\mathcal{H})^-$, where $\rho \in (\gamma, s)$, which contradicts Theorem 1.11 (see the end of its proof). Therefore, item (ii) holds. ∎

A closer look at the proofs of Theorem 1.11 and Corollary 1.12 reveals the following result.

COROLLARY 1.13. *The radius of convergence of a power series* $\varphi = \sum\limits_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ *satisfies the relation*

$$\gamma = \sup\Big\{r \geqslant 0 : \sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+} A_{(\alpha)} \otimes r^q \mathbf{S}_{(\alpha)} \text{ is convergent in norm}\Big\}$$
$$= \sup\Big\{r \geqslant 0 : \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i,\ i\in\{1,\ldots,k\}} A_{(\alpha)} \otimes r^{p_1+\cdots+p_k} \mathbf{S}_{(\alpha)} \text{ is convergent in norm}\Big\}.$$

Moreover, we have the following characterization for free holomorphic functions on polyballs.

COROLLARY 1.14. *Let* $\mathbf{S}$ *be the universal model associated with the abstract regular polyball* $\mathbf{B_n}$. *A formal power series* $\varphi = \sum\limits_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ *is a free holomorphic function (with coefficients in* $B(\mathcal{K})$*) on the abstract polyball* $\boldsymbol{\rho}\mathbf{B_n}$, *where* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k)$, $\rho_i > 0$, *if and only if the series*

$$\sum_{q=0}^{\infty} \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} A_{(\alpha)} \otimes r^q \rho_1^{|\alpha_1|} \cdots \rho_k^{|\alpha_k|} \mathbf{S}_{(\alpha)}$$

is convergent in the operator norm topology for any $r \in [0, 1)$. *Moreover, the set* $\operatorname{Hol}(\boldsymbol{\rho}\mathbf{B_n})$ *of all free holomorphic functions with scalar coefficients on* $\boldsymbol{\rho}\mathbf{B_n}$ *is an algebra.*

2. MAXIMUM PRINCIPLE AND SCHWARZ TYPE RESULTS

In this section, we present some results concerning the composition of free holomorphic functions and study bounded free holomorphic functions with scalar coefficients on polyball. We prove a Schwarz lemma and a maximum principle in this setting. The results play an important role in the next sections.

Let $H^\infty(\mathbf{B_n})$ denote the set of all elements φ in $\operatorname{Hol}(\mathbf{B_n})$ such that

$$\|\varphi\|_\infty := \sup \|\varphi(\mathbf{X})\| < \infty,$$

where the supremum is taken over all $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. One can show that $H^\infty(\mathbf{B_n})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$. For each $p \in \mathbb{N}$, we define the norms $\|\cdot\|_p : M_{p\times p}(H^\infty(\mathbf{B_n})) \to [0, \infty)$ by setting

$$\|[\varphi_{st}]_{p\times p}\|_p := \sup \|[\varphi_{st}(\mathbf{X})]_{p\times p}\|,$$

where the supremum is taken over all $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_p$, $p \in \mathbb{N}$, determine an operator space structure on $H^\infty(\mathbf{B_n})$, in the sense of Ruan ([12], [13]).

Given $\varphi \in \mathbf{F_n^\infty}$ and a Hilbert space $\mathcal{H}$, the noncommutative Berezin transform associated with the abstract noncommutative polyball $\mathbf{B_n}$ generates a function whose representation on $\mathcal{H}$ is

$$\boldsymbol{\mathcal{B}}[\varphi] : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$$

defined by

$$\boldsymbol{\mathcal{B}}[\varphi](\mathbf{X}) := \boldsymbol{\mathcal{B}}_\mathbf{X}[\varphi], \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where $\boldsymbol{\mathcal{B}}_\mathbf{X} : B(\bigotimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ is the *Berezin transform at* X defined by

$$\boldsymbol{\mathcal{B}}_\mathbf{X}[g] := \mathbf{K}_\mathbf{X}^*\Big(g \otimes I_\mathcal{H}\Big)\mathbf{K}_\mathbf{X}, \quad g \in B(\bigotimes_{i=1}^k F^2(H_{n_i})),$$

where $F^2(H_{n_i})$ is the full Fock space on n_i generators and

$$\mathbf{K}_\mathbf{X} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\mathbf{\Delta}_\mathbf{X}(I)(\mathcal{H})}$$

is the *noncommutative Berezin kernel* associated with $\mathbf{X}$.

We call $\boldsymbol{\mathcal{B}}[\varphi]$ the *Berezin transform of* φ. In [24], we identified the noncommutative algebra $\mathbf{F_n^\infty}$ with the Hardy subalgebra $H^\infty(\mathbf{B_n})$ of bounded free holomorphic functions with scalar coefficients on $\mathbf{B_n}$. More precisely, we proved that the map $\Phi : H^\infty(\mathbf{B_n}) \to \mathbf{F}_n^\infty$ defined by

$$\Phi\Big(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\Big) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{S}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g := \sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}$ is a free holomorphic function with scalar coefficients on the abstract polyball $\mathbf{B_n}$, then the following statements are equivalent:

(i) $g \in H^\infty(\mathbf{B_n})$;

(ii) $\sup_{0\leqslant r<1} \|g(r\mathbf{S})\| < \infty$, where $g(r\mathbf{S}) := \sum_{q=0}^\infty \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} \mathbf{S}_{(\alpha)}$;

(iii) there exists $\varphi \in \mathbf{F_n^\infty}$ with $g = \boldsymbol{\mathcal{B}}[\varphi]$, where $\boldsymbol{\mathcal{B}}$ is the noncommutative Berezin transform associated with the abstract polyball $\mathbf{B_n}$.

In this case,

$$\Phi(g) = \text{SOT-}\lim_{r\to 1} g(r\mathbf{S}); \quad \Phi^{-1}(\varphi) = \boldsymbol{\mathcal{B}}[\varphi], \quad \varphi \in \mathbf{F_n^\infty}, \quad \text{and}$$
$$\|\Phi(g)\| = \sup_{0\leqslant r<1} \|g(r\mathbf{S})\| = \lim_{r\to 1} \|g(r\mathbf{S})\|.$$

We use the notation $\widehat{g} := \Phi(g)$ and call $\widehat{g}$ the (*model*) *boundary function* of g with respect to the universal model $\mathbf{S}$. We denote by $A(\mathbf{B_n})$ the set of all elements g in $\mathrm{Hol}(\mathbf{B_n})$ such that the mapping

$$\mathbf{B_n}(\mathcal{H}) \ni \mathbf{X} \mapsto g(\mathbf{X}) \in B(\mathcal{H})$$

has a continuous extension to $[\mathbf{B_n}(\mathcal{H})]^-$ for any Hilbert space $\mathcal{H}$. One can show that $A(\mathbf{B_n})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and it has an operator space structure under the norms $\|\cdot\|_p$, $p\in\mathbb{N}$. Moreover, we can identify the polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ with the subalgebra $A(\mathbf{B_n})$. We proved in [24] that the map $\Phi : A(\mathbf{B_n}) \to \boldsymbol{\mathcal{A}}_\mathbf{n}$ defined by

$$\Phi\Big(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\Big) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{S}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g := \sum\limits_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}$ is a free holomorphic function on the abstract polyball $\mathbf{B_n}$, then the following statements are equivalent:

(i) $g \in A(\mathbf{B_n})$;

(ii) $g(r\mathbf{S}) := \sum\limits_{q=0}^{\infty} \sum\limits_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} \mathbf{S}_{(\alpha)}$ is convergent in the norm topology as $r\to 1$;

(iii) there exists $\varphi \in \boldsymbol{\mathcal{A}}_n$ with $g = \boldsymbol{\mathcal{B}}[\varphi]$, where $\boldsymbol{\mathcal{B}}$ is the noncommutative Berezin transform associated with the abstract polyball $\mathbf{B_n}$.

In this case,

$$\Phi(g) = \lim_{r\to 1} g(r\mathbf{S}) \quad \text{and} \quad \Phi^{-1}(\varphi) = \boldsymbol{\mathcal{B}}[\varphi], \quad \varphi \in \boldsymbol{\mathcal{A}}_\mathbf{n},$$

where the limit is in the operator norm topology.

In what follows, we consider free holomorphic functions $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{m_1+\cdots+m_k}$ with $F := (F_1,\ldots,F_k)$ and $F_i = (F_{i,1},\ldots,F_{i,m_i})$, where each $F_{i,j}$ is a free holomorphic function with scalar coefficients on $\mathbf{B_n}(\mathcal{H})$. Note that $F(0)$ is always a scalar operator.

LEMMA 2.1. *Let $F : \mathbf{B_n}(\mathcal{H})^- \to B(\mathcal{H})^{m_1}\times\cdots\times B(\mathcal{H})^{m_q}$ be a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ and continuous on $\mathbf{B_n}(\mathcal{H})^-$. If $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})^-$ and $\widehat{F} \in \mathbf{B_m}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$ are pure elements, then so is $F(\mathbf{X}) \in \mathbf{B_m}(\mathcal{H})^-$.*

Proof. Let $f : \mathbf{B_n}(\mathcal{H})^- \to [B(\mathcal{H})]_1^-$ be a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ and continuous on $\mathbf{B_n}(\mathcal{H})^-$. If $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})^-$ is pure, we can apply the noncommutative Berezin transform and obtain

$$f(\mathbf{X})f(\mathbf{X})^* = \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\mathbf{X}}[\widehat{f}\widehat{f}^*] = \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widehat{f}_r\widehat{f}_r^*].$$

Since $\lim\limits_{r\to 1} \widehat{f}_r = \widehat{f}$ in norm and $\boldsymbol{\mathcal{B}}_\mathbf{X}$ is continuous in norm, we deduce that $f(\mathbf{X})f(\mathbf{X})^* = \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{f}\widehat{f}^*]$. In a similar manner, if $F=(F_1,\ldots,F_q)$ and $i\in\{1,\ldots,q\}$, we obtain

$$\sum_{\alpha\in\mathbb{F}_{m_i}^+, |\alpha|=p} F_{i,\alpha}(\mathbf{X})F_{i,\alpha}(\mathbf{X})^* = \boldsymbol{\mathcal{B}}_\mathbf{X}\Big[\sum_{\alpha\in\mathbb{F}_{m_i}^+, |\alpha|=p} \widehat{F}_{i,\alpha}\widehat{F}_{i,\alpha}^*\Big].$$

Since

$$\Big\| \sum_{\alpha\in\mathbb{F}_{m_i}^+,|\alpha|=p} \widehat{F}_{i,\alpha}\widehat{F}_{i,\alpha}^* \Big\| \leqslant 1 \quad \text{and} \quad \sum_{\alpha\in\mathbb{F}_{m_i}^+,|\alpha|=p} \widehat{F}_{i,\alpha}\widehat{F}_{i,\alpha}^* \to 0 \quad \text{strongly as } p\to\infty,$$

we deduce that $F_i(\mathbf{X})$ is pure and, therefore, so is $F(\mathbf{X})$. The proof is complete. ∎

PROPOSITION 2.2. *Let $G : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{m_1} \times \cdots \times B(\mathcal{H})^{m_q}$ be a free holomorphic function, where $\mathbf{n} = (n_1,\dots,n_k) \in \mathbb{N}^k$ and $\mathbf{m} = (m_1,\dots,m_q) \in \mathbb{N}^q$. Then* range $G \subseteq \mathbf{B_m}(\mathcal{H})$ *if and only if*

$$G(r\mathbf{S}) \in \mathbf{B_m}\Big(\bigotimes\nolimits_{i=1}^k F^2(H_{n_i})\Big), \quad r\in[0,1),$$

where $\mathbf{S}$ is the universal model of the regular polyball $\mathbf{B_n}$.

Proof. Since $r\mathbf{S} \in \mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r\in[0,1)$, the direct implication is obvious. To prove the converse, assume that $G = (G_1,\dots,G_q)$ has the property that $G(r\mathbf{S}) \in \mathbf{B_m}(\otimes_{i=1}^k F^2(H_{n_i}))$. Consequently, if $i,s\in\{1,\dots,q\}$, $i\neq s$, then each entry of $G_i(r\mathbf{S}) = (G_{i,1}(r\mathbf{S}),\dots,G_{i,m_i}(r\mathbf{S}))$ commutes with each entry of $G_s(r\mathbf{S}) = (G_{s,1}(r\mathbf{S}),\dots,G_{s,m_s}(r\mathbf{S}))$. Moreover, $G(r\mathbf{S})$ is a pure element with entries $\{G_{i,j}(r\mathbf{S})\}$ in the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ and, for each $r\in[0,1)$,

$$(\mathrm{id}-\Phi_{G_1(r\mathbf{S})})\circ\cdots\circ(\mathrm{id}-\Phi_{G_q(r\mathbf{S})})(I) > d_r I,$$

for some $d_r>0$. If $\mathbf{X} = (X_1,\dots,X_k) \in \mathbf{B_n}(\mathcal{H})$, then there is $t\in(0,1)$ such that $\mathbf{X}\in t\mathbf{B_n}(\mathcal{H})$. Since G is a free holomorphic function, it is continuous and $G(t\mathbf{S})$ has the entries in $\boldsymbol{\mathcal{A}}_\mathbf{n}$. Applying the noncommutative Berezin transform at $\frac{1}{t}\mathbf{X}$ to the relations mentioned above, when $r=t$, we deduce that the entries of $G_i(\mathbf{X})$ commute with the entries of $G_s(\mathbf{X})$, if $i,s\in\{1,\dots,q\}$, $i\neq s$, and

$$(\mathrm{id}-\Phi_{G_1(\mathbf{X})})\circ\cdots\circ(\mathrm{id}-\Phi_{G_q(\mathbf{X})})(I) > 0. \tag{2.1}$$

On the other hand, since $G_i(t\mathbf{S})$ is pure, Lemma 2.1 implies that $G_i(\mathbf{X})$ is pure. Hence, and using relation (2.1), we conclude that $G(\mathbf{X}) \in \mathbf{B_n}(\mathcal{H})$ for any $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})$. The proof is complete. ∎

Using Proposition 2.2 and the properties of the noncommutative Berezin transform, one can easily deduce the following result.

COROLLARY 2.3. *Let $G = (G_1,\dots,G_q)$, with $G_i : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{m_i}$, be a free holomorphic function such that, for each $r\in[0,1)$,*

(i) *$\|G_t(r\mathbf{S})\| < 1$, $t\in\{1,\dots,q\}$;*

(ii) *the entries of $G_t(r\mathbf{S})$ are commuting with the entries of $G_s(r\mathbf{S})$ for any $s,t\in\{1,\dots,q\}$ with $s\neq t$.*

Then range $G\subseteq \mathbf{B_m}(\mathcal{H})$ *if either one of the following conditions holds:*

(a) *$\mathbf{\Delta}_{G(r\mathbf{S})}(I) > 0$ for any $r\in[0,1)$;*

(b) *the entries of $G_t(r\mathbf{S})$ are doubly commuting with the entries of $G_s(r\mathbf{S})$ for any $s,t\in\{1,\dots,q\}$ with $s\neq t$.*

THEOREM 2.4. *Let* $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ *and* $\mathbf{m} = (m_1, \dots, m_q) \in \mathbb{N}^q$. *If* $G : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H})$ *and* $F : \mathbf{B_m}(\mathcal{H}) \to B(\mathcal{H})\overline{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ *are free holomorphic functions on regular polyballs, then* $F \circ G$ *is a free holomorphic function on* $\mathbf{B_n}(\mathcal{H})$.

Proof. If F has the Fourier representation

$$F(\mathbf{Y}) = \sum_{p=0}^{\infty} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} A_{(\gamma)} \otimes Y_{(\gamma)}, \quad \mathbf{Y} \in \mathbf{B_m}(\mathcal{H}),$$

then we have

$$(F \circ G)(\mathbf{X}) = \sum_{p=0}^{\infty} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} A_{(\gamma)} \otimes G_{(\gamma)}(\mathbf{X}), \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where the convergence is in the operator norm topology. Due to Proposition 2.2,

$$G(r\mathbf{S}) = \{G_{s,t}(r\mathbf{S})\} \in \mathbf{B_m}(F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})), \quad r \in [0,1),$$

where $s \in \{1, \dots, q\}$, $t \in \{1, \dots, m_s\}$ and $\mathbf{S}$ is the universal model of the regular polyball $\mathbf{B_n}$. Since $F : \mathbf{B_m}(\mathcal{H}) \to B(\mathcal{H})\overline{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a free holomorphic function, for each $r \in [0,1)$,

$$\Lambda_r := \sum_{p=0}^{\infty} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} A_{(\gamma)} \otimes G_{(\gamma)}(r\mathbf{S}), \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}), \tag{2.2}$$

is convergent in the operator norm topology. Taking into account that $G_{s,t}(\mathbf{S})$ is in the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$, we have $\Lambda_r \in B(\mathcal{E}, \mathcal{G}) \otimes \boldsymbol{\mathcal{A}}_\mathbf{n} \subset B(\mathcal{E}, \mathcal{G})\overline{\otimes} F_\mathbf{n}^\infty$. This implies that, for each $r \in [0,1)$, the operator Λ_r has the Fourier representation $\sum\limits_{q=0}^{\infty} \sum\limits_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} C_{(\alpha)}(r) \otimes r^{|\alpha_1|+\cdots+|\alpha_k|} \mathbf{S}_{(\alpha)}$ and

$$\Lambda_r = \text{SOT-}\lim_{\ell \to 1} \sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} C_{(\alpha)}(r) \otimes (r\ell)^{|\alpha_1|+\cdots+|\alpha_k|} \mathbf{S}_{(\alpha)}, \tag{2.3}$$

where the series converges in the operator topology. The next step in our proof is to show that $C_{(\alpha)}(r)$ does not depend on $r \in [0,1)$. Using relations (2.2) and (2.3), we have

$$\begin{aligned} &\langle C_{(\alpha)}(r)x, y\rangle \\ &\quad = \left\langle (I \otimes \mathbf{S}_{(\alpha)}^*) \frac{1}{r^{|\alpha_1|+\cdots+|\alpha_k|}} \Lambda_r(x \otimes 1), (y \otimes 1) \right\rangle \\ &\quad = \lim_{d \to \infty} \sum_{p=0}^{d} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} \langle A_{(\gamma)}x, y\rangle \left\langle \frac{1}{r^{|\alpha_1|+\cdots+|\alpha_k|}} \mathbf{S}_{(\alpha)}^* G_{(\gamma)}(r\mathbf{S})1, 1 \right\rangle \end{aligned}$$

for any $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ and for any $x \in \mathcal{E}$, $y \in \mathcal{G}$. On the other hand, the product $G_{(\gamma)}$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ and has a representation

$$G_{(\gamma)}(\mathbf{X}) = \sum_{p=0}^{\infty} \sum_{|\beta_1|+\cdots+|\beta_q|=p,\ \beta_i \in \mathbb{F}_{n_i}^+} d_{(\beta)}^{(\gamma)} X_{(\beta)}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

Consequently,

$$\left\langle \frac{1}{r^{|\alpha_1|+\cdots+|\alpha_k|}} \mathbf{S}_{(\alpha)}^* G_{(\gamma)}(r\mathbf{S})1, 1 \right\rangle = d_{(\alpha)}^{(\gamma)}, \quad r \in [0,1),$$

for any $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ and $(\gamma) \in \mathbb{F}_{m_1}^+ \times \cdots \times \mathbb{F}_{m_k}^+$. Therefore, $C_{(\alpha)}(r)$ does not depend on $r \in [0,1)$. We set $C_{(\alpha)}(r) = C_{(\alpha)}$, and note that relation (2.3) implies that

$$Q(\mathbf{X}) := \sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} C_{(\alpha)} \otimes X_{(\alpha)}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$. Moreover, since Q is continuous in the operator norm we deduce that

$$\Lambda_r := \sum_{p=0}^{\infty} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} A_{(\gamma)} \otimes G_{(\gamma)}(r\mathbf{S}) = \sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} C_{(\alpha)} \otimes r^{|\alpha_1|+\cdots+|\alpha_k|} \mathbf{S}_{(\alpha)}$$

for any $r \in [0,1)$. Now, if $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, then there is $r \in (0,1)$ such that $\mathbf{X} \in r\mathbf{B_n}(\mathcal{H})$. Applying the noncommutative Berezin transform at $\frac{1}{r}\mathbf{X}$ to the relation above, we deduce that

$$(F \circ G)(\mathbf{X}) = \sum_{p=0}^{\infty} \sum_{|\gamma_1|+\cdots+|\gamma_q|=p,\ \gamma_i \in \mathbb{F}_{m_i}^+} A_{(\gamma)} \otimes G_{(\gamma)}(\mathbf{X}) = Q(\mathbf{X})$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. The proof is complete. ∎

PROPOSITION 2.5. *Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{K}) \overline{\otimes}_{\min} B(\mathcal{H})$ be a bounded free holomorphic function with coefficients in $B(\mathcal{K})$ and representation*

$$\sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} A_{(\alpha)} \otimes X_{(\alpha)}.$$

If $\|F\|_\infty \leqslant 1$, then

$$\sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} A_{(\alpha)}^* A_{(\alpha)} \leqslant I - A_{(0)}^* A_{(0)}$$

for any $q \in \mathbb{N}$, where $A_{(0)} := F(0)$.

Proof. Let $\mathcal{M}$ be the subspace of $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ spanned by the vectors $1, e_{\alpha_1}^1 \otimes \cdots \otimes e_{\alpha_k}^k$, where $\alpha_i \in \mathbb{F}_{n_i}$ and $|\alpha_1| + \cdots + |\alpha_k| = q \in \mathbb{N}$. Note that the operator $C := P_{\mathcal{K}\otimes\mathcal{M}} F(\mathbf{S})|_{\mathcal{K}\otimes\mathcal{M}}$ is a contraction and, with respect to the decomposition

$$\mathcal{K}\otimes\mathcal{M} = \mathcal{K} \oplus \bigoplus_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+} e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k}\otimes\mathcal{K},$$

has the operator matrix representation

$$\left(\begin{matrix} A_{(0)} & [0 \ \cdots \ \cdots \ \cdots \ 0] \\ \left[\begin{matrix} A_{(\alpha)} \\ \vdots \\ |\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+ \end{matrix}\right] & \left[\begin{matrix} A_{(0)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & 0 & \cdots & 0 & A_{(0)} \end{matrix}\right] \end{matrix}\right),$$

where $A_{(0)}$ is on the main diagonal. Indeed, we have

$$\begin{aligned} &\langle C(x\otimes 1), y\otimes 1\rangle = \langle F(\mathbf{S})(x\otimes 1), y\otimes 1\rangle = \langle A_{(0)}x, y\rangle \quad \text{and} \\ &\langle C(x\otimes 1), y\otimes e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k}\rangle = \langle A_{(\alpha)}x, y\rangle \end{aligned}$$

for any $(\alpha) = (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$ with $|\alpha_1|+\cdots+|\alpha_k| = q$. If $|\alpha_1|+\cdots+|\alpha_k| = |\beta_1|+\cdots+|\beta_k| = q$, then we have

$$\langle C(x\otimes e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k}), y\otimes e^1_{\beta_1}\otimes\cdots\otimes e^k_{\beta_k}\rangle = \delta_{\alpha_1\beta_1}\cdots\delta_{\alpha_k\beta_k}A_{(0)}$$

for any $x, y\in\mathcal{K}$. This proves our assertion. Consequently, the column operator matrix

$$\left(\begin{matrix} A_{(0)} \\ \left[\begin{matrix} A_{(\alpha)} \\ \vdots \\ |\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+ \end{matrix}\right] \end{matrix}\right)$$

is a contraction, which completes the proof. ∎

We recall that $\mathbf{B_n}(\mathcal{H})$ is a complete Reinhardt domain and

$$B(\mathcal{H})^{n_1}\times_c\cdots\times_c B(\mathcal{H})^{n_k} = \bigcup_{\rho>0}\rho\mathbf{B_n}(\mathcal{H}).$$

We define the Minkovski functional associated with the regular polyball $\mathbf{B_n}(\mathcal{H})$ to be the function $m_{\mathbf{B_n}} : B(\mathcal{H})^{n_1}\times_c\cdots\times_c B(\mathcal{H})^{n_k}\to[0,\infty)$ given by

$$m_{\mathbf{B_n}}(\mathbf{X}) := \inf\{r>0 : \mathbf{X}\in r\mathbf{B_n}(\mathcal{H})\}.$$

PROPOSITION 2.6. *The Minkovski functional associated with the regular polyball $\mathbf{B_n}(\mathcal{H})$ has the following properties:*

(i) $m_{\mathbf{B_n}}(\lambda\mathbf{X}) = |\lambda| m_{\mathbf{B}}(\mathbf{X})$ *for* $\lambda\in\mathbb{C}$;

(ii) $m_{\mathbf{B_n}}$ *is upper semicontinuous;*

(iii) $\mathbf{B_n}(\mathcal{H}) = \{\mathbf{X}\in B(\mathcal{H})^{n_1}\times_c\cdots\times_c B(\mathcal{H})^{n_k} : m_{\mathbf{B_n}}(\mathbf{X}) < 1\}$;

(iv) $\mathbf{B_n}(\mathcal{H})^- = \{\mathbf{X}\in B(\mathcal{H})^{n_1}\times_c\cdots\times_c B(\mathcal{H})^{n_k} : m_{\mathbf{B_n}}(\mathbf{X}) \leqslant 1\}$;

(v) *there is a polyball* $r\mathbf{P_n}(\mathcal{H}) \subset \mathbf{B_n}(\mathcal{H})$ *for some* $r \in (0,1)$, *where* $m_{\mathbf{B_n}}$ *is continuous.*

Proof. To prove (i), we may assume that $\mathbf{X} \neq 0$ and $\lambda \neq 0$. It is clear that $m_{\mathbf{B_n}}(\lambda\mathbf{X}) = t > 0$ if and only if $\lambda\mathbf{X} \in c\mathbf{B_n}(\mathcal{H})$ for any $c > t$, and $\lambda\mathbf{X} \notin d\mathbf{B_n}(\mathcal{H})$ if $0 < d < t$. Taking into account that $\mathbf{B_n}(\mathcal{H}) = \mathrm{e}^{\mathrm{i}\theta}\mathbf{B_n}(\mathcal{H})$ for any $\theta \in \mathbb{R}$, we deduce that the latter conditions are equivalent to $\mathbf{X} \in \frac{c}{|\lambda|}\mathbf{B_n}(\mathcal{H})$ for any $c > t$ and $\mathbf{X} \notin \frac{d}{|\lambda|}\mathbf{B_n}(\mathcal{H})$ if $0 < d < t$. Hence, we obtain that $m_{\mathbf{B_n}}(\mathbf{X}) = \frac{t}{|\lambda|}$, which shows item (i). We skip the proof of item (ii), since it is due to (i) and a straightforward argument.

According to Proposition 1.3, we have $\mathbf{B_n}(\mathcal{H}) = \bigcup_{0<r<1} r\mathbf{B_n}(\mathcal{H})$. Using this result, one can easily deduce item (iii). As we saw in the proof of the same proposition, for any $r \in (0,1)$, we have $\mathbf{B_n}(\mathcal{H})^- \subseteq \frac{1}{r}\mathbf{B_n}(\mathcal{H})$. Consequently, $m_{\mathbf{B_n}}(\mathbf{X}) \leqslant 1$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$. Now, assume that $\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ is such that $m_{\mathbf{B_n}}(\mathbf{X}) = 1$. Then there is a sequence $\{t_m\}$ with $t_m > 1$ and $t_m \to 1$ such that $\mathbf{X} \in t_m\mathbf{B_n}(\mathcal{H})$ for any $m \in \mathbb{N}$. Taking $t_m \to 1$, we deduce that $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$. Hence, and using item (iii), one can see that item (iv) holds. To prove (v), note the fact that $r\mathbf{P_n}(\mathcal{H}) \subset \mathbf{B_n}(\mathcal{H})$ for some $r \in (0,1)$ is quite clear, while the continuity of $m_{\mathbf{B_n}}$ on $r\mathbf{P_n}(\mathcal{H})$ is due to the convexity of the latter polyball. The proof is complete. ∎

Let $\mathbb{C}\langle Z_{i,j}\rangle$ be the algebra of all polynomials in indeterminates $Z_{i,j}$, where $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$. We define the free partial derivation $\frac{\partial}{\partial Z_{i,j}}$ on $\mathbb{C}\langle Z_{i,j}\rangle$ as the unique linear operator on this algebra, satisfying the conditions

$$\frac{\partial I}{\partial Z_{i,j}} = 0, \quad \frac{\partial Z_{i,j}}{\partial Z_{i,j}} = I, \quad \frac{\partial Z_{i,j}}{\partial Z_{s,q}} = 0 \quad \text{if } (i,j) \neq (s,q) \quad \text{and}$$
$$\frac{\partial (fg)}{\partial Z_{i,j}} = \frac{\partial f}{\partial Z_{i,j}} g + f \frac{\partial g}{\partial Z_{i,j}}$$

for any $f, g \in \mathbb{C}\langle Z_{i,j}\rangle$. The same definition extends to formal power series in the noncommuting indeterminates $Z_{i,j}$. If $F := \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} A_{(\alpha)} \otimes Z_{(\alpha)}$ is a power series with operator-valued coefficients, then the free partial derivative of F with respect to $Z_{i,j}$ is the power series $\frac{\partial F}{\partial Z_{i,j}} := \sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)} \otimes \frac{\partial Z_{(\alpha)}}{\partial Z_{i,j}}$. One can prove that if F is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ then so is $\frac{\partial F}{\partial Z_{i,j}}$. We leave the proof to the reader.

The next result is an analogue of Schwarz lemma from complex analysis.

THEOREM 2.7. *Let* $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ *be a bounded free holomorphic function with* $\|F\|_\infty \leqslant 1$. *If* $F(0) = 0$, *then*

$$\|F(\mathbf{X})\| \leqslant m_{\mathbf{B_n}}(\mathbf{X}) < 1 \quad \textit{and} \quad m_{\mathbf{B_n}}(\mathbf{X}) \leqslant \|\mathbf{X}\|, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where $m_{\mathbf{B_n}}$ is the Minkovski functional associated with the regular polyball $\mathbf{B_n}(\mathcal{H})$. In particular, if $p = 1$, the free holomorphic function

$$\psi(\mathbf{X}) = \sum_{i=1}^{k} \sum_{j=1}^{n_j} \frac{\partial F}{\partial Z_{i,j}}(0) X_{i,j}, \quad \mathbf{X} = (X_{i,j}) \in \mathbf{B_n}(\mathcal{H}),$$

has the property that $\|\psi(\mathbf{X})\| \leqslant m_{\mathbf{B_n}}(\mathbf{X}) < 1$.

Proof. Fix $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ and let $t \in (0,1)$ be such that $m_{\mathbf{B_n}}(\mathbf{X}) < t < 1$. Since $\frac{1}{t}\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, Proposition 1.3 implies $\frac{\lambda}{t}\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ for any $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For each $x, y \in \mathcal{H}^{(p)}$ with $\|x\| \leqslant 1$ and $\|y\| \leqslant 1$, define the function $\varphi_{x,y} : \mathbb{D} \to \mathbb{C}$ by setting

$$\varphi_{x,y}(\lambda) := \left\langle F\left(\frac{\lambda}{t}\mathbf{X}\right)x, y\right\rangle, \quad \lambda \in \mathbb{D}.$$

Taking into account that F is free holomorphic on $\mathbf{B_n}(\mathcal{H})$ and $\|F\|_\infty \leqslant 1$, we deduce that $\varphi_{x,y}$ is a holomorphic function on the unit disc and $|\varphi_{x,y}(\lambda)| \leqslant 1$. Since $\varphi_{x,y}(0) = 0$, an application of the classical Schwarz lemma to $\varphi_{x,y}$ implies $|\varphi_{x,y}(\lambda)| \leqslant |\lambda|$ for any $\lambda \in \mathbb{D}$. Taking $\lambda = m_{\mathbf{B_n}}(\mathbf{X})$, we obtain

$$\varphi_{x,y}(\lambda) := \left\langle F\left(\frac{m_{\mathbf{B_n}}(\mathbf{X})}{t}\mathbf{X}\right)x, y\right\rangle \leqslant m_{\mathbf{B_n}}(\mathbf{X}), \quad \lambda \in \mathbb{D}.$$

for any $t \in (0,1)$ with $m_{\mathbf{B_n}}(\mathbf{X}) < t < 1$. Since F is continuous on $\mathbf{B_n}(\mathcal{H})$ and taking $t \to m_{\mathbf{B_n}}(\mathbf{X})$, we obtain $|\langle F(\mathbf{X})x, y\rangle| \leqslant m_{\mathbf{B_n}}(\mathbf{X})$ for any $x, y \in \mathcal{H}^{(p)}$ with $\|x\| \leqslant 1$ and $\|y\| \leqslant 1$. Consequently,

$$\|F(\mathbf{X})\| \leqslant m_{\mathbf{B_n}}(\mathbf{X}) < 1, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

According to Proposition 1.9 from [24], if $\|\mathbf{X}\| := \Phi_{X_1}(I) + \cdots + \Phi_{X_k}(I) \leqslant I$ then $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-$. Consequently, if $\mathbf{X} \in B_n(\mathcal{H})$, then $\frac{\mathbf{X}}{\|\mathbf{X}\|} \in \mathbf{B_n}(\mathcal{H})^-$, which implies $m_{\mathbf{B_n}}(\mathbf{X}) \leqslant t\|\mathbf{X}\|$ for any $t > 1$. taking $t \to 1$, we deduce that $m_{\mathbf{B_n}}(\mathbf{X}) \leqslant \|\mathbf{X}\|$.

Now, we consider the particular case when $p = 1$. Due to the classical Schwarz lemma, we also have $|\varphi'_{x,y}(0)| \leqslant 1$. Since $\varphi'_{x,y}(0) = \langle \frac{1}{t}\psi(\mathbf{X})x, y\rangle$, we deduce that $\|\psi(\mathbf{X})\| \leqslant t < 1$. Taking $t \to m_{\mathbf{B_n}}(\mathbf{X})$, we obtain $\|\psi(\mathbf{X})\| \leqslant m_{\mathbf{B_n}}(\mathbf{X}) < 1$. The proof is complete. ∎

We have all the ingredients to prove the following maximum principle.

THEOREM 2.8. *Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ be a bounded free holomorphic function. If there exists $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that*

$$\|F(\mathbf{X})\| \leqslant \|F(\mathbf{X}_0)\|, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

then F must be a constant.

Proof. Assume that $\|F\|_\infty = 1$ and that there exists $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that $\|F(\mathbf{X}_0)\| = 1$. Let F have the representation

$$\sum_{q=0}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+} a_{(\alpha)} X_{(\alpha)}.$$

According to Proposition 2.5, we have

$$\sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i\in\mathbb{F}_{n_i}^+} |a_{(\alpha)}|^2 \leqslant 1 - |F(0)|^2$$

for any $q \in \mathbb{N}$. Hence, if $|F(0)| = 1$, then $a_{(\alpha)} = 0$ for any $(\alpha) = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ with $|\alpha_1| + \cdots + |\alpha_k| \geqslant 1$, which implies $F = F(0)$.

Now, we assume that $|F(0)| < 1$ and set $\lambda := F(0)$. Note that if Ψ_λ is the corresponding automorphism of the open unit ball $[B(\mathcal{H})]_1$ (see the remarks preceding Theorem 3.6), then, due to Theorem 2.4, $G := \Psi_\lambda \circ F$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ with the property that $G(0) = 0$ and $\|G\|_\infty \leqslant 1$. Using Theorem 2.7, we have $\|G(\mathbf{X})\| < 1$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. Hence, $\|\Psi_\lambda(F(\mathbf{X}_0))\| < 1$. Since Ψ_λ is an involutive automorphism of the open unit ball $[B(\mathcal{H})]_1$, we deduce that

$$\|F(\mathbf{X})\| = \|\Psi_\lambda(\Psi_\lambda(F(\mathbf{X})))\| < 1$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, which contradicts our assumption that $\|F(\mathbf{X}_0)\| = 1$. The proof is complete. ∎

COROLLARY 2.9. *Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ be a nonconstant bounded free holomorphic function. Then the following statements hold:*

(i) $\|F(\mathbf{X})\| < \|F\|_\infty$ *for any* $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$;

(ii) *the map*

$$[0,1)^k \ni \mathbf{r} \mapsto \|F_\mathbf{r}\|_\infty, \quad \mathbf{r} = (r_1, \ldots, r_k)$$

is strictly increasing with respect to each r_i, where

$$F_\mathbf{r}(X_1, \ldots, X_k) := F(r_1X_1, \ldots, r_kX_k), \quad (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H}).$$

Proof. Without loss of generality, we may assume that $\|F\|_\infty = 1$. Part (i) is a consequence of Theorem 2.8. To prove part (ii), let $0 \leqslant r_1 < t_1 < 1$ and set $r := \frac{r_1}{t_1} \in [0,1)$. Since F is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$, the operator $F(r\mathbf{S}_1, r_2\mathbf{S}_2, \ldots, r_k\mathbf{S}_k)$ is in the polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ and $\|F_{(r,r_2,\ldots,r_k)}\|_\infty = \|F(r\mathbf{S}_1, r_2\mathbf{S}_2, \ldots, r_k\mathbf{S}_k)\|$. Applying part (i) to the bounded free holomorphic function $F_{(r,r_2,\ldots,r_k)}$ on $\mathbf{B_n}(\mathcal{H})$ and $\mathbf{X} = (r\mathbf{S}_1, r_2\mathbf{S}_2, \ldots, r_k\mathbf{S}_k)$, we obtain

$$\begin{aligned}\|F_{(r_1,r_2,\ldots,r_k)}\|_\infty = \|F_{(r_1,r_2,\ldots,r_k)}(\mathbf{S})\|_\infty &= \left\| F_{(t_1,r_2,\ldots,r_k)}\left(\frac{r_1}{t_1}\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k\right)\right\| \\ &< \|F_{(t_1,r_2,\ldots,r_k)}(\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k)\| = \|F_{(t_1,r_2,\ldots,r_k)}\|_\infty.\end{aligned}$$

The proof is complete. ∎

The next version of the maximum principle is needed in the next sections.

THEOREM 2.10. *Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ be a bounded free holomorphic function with $\|F(0)\| < \|F\|_\infty$. Then there is no $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that $\|F(\mathbf{X}_0)\| = \|F\|_\infty$.*

Proof. Without loss of generality we may assume that $\|F\|_\infty = 1$. If $F(0) = 0$, Theorem 2.7 implies $\|F(\mathbf{X})\| < 1$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, which completes the proof.

Now we consider the case when $0 \neq \|F(0)\| < 1$. Suppose that there is $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that $\|F(\mathbf{X}_0)\| = 1$. Since $\|F(0)\| < \|F\|_\infty = 1$, we have $\lambda := F(0) \in (\mathbb{C}^p)_1$. Let Ψ_λ be the automorphism of the open unit ball $[B(\mathcal{H})^p]_1$ (see the remarks preceding Theorem 3.6). We recall that Ψ_λ is a free holomorphic function on $[B(\mathcal{H})^p]_\gamma$, where $\gamma := \frac{1}{\|\lambda\|_2}$, and $\Psi_\lambda(\Psi_\lambda(X)) = X$ for any $X \in [B(\mathcal{H})^p]_\gamma$, where $\gamma := \frac{1}{\|\lambda\|_2}$. Using Theorem 2.4, we deduce that $G := \Psi_\lambda \circ F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ such that $G(0) = 0$ and $\|G\|_\infty \leqslant 1$. Due to the Schwarz type result of Theorem 2.7, we have $\|G(\mathbf{X})\| < 1$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. In particular, we have $\|\Psi_\lambda(F(\mathbf{X}_0))\| < 1$. Since Ψ_λ is an involutive automorphism of the open unit ball $[B(\mathcal{H})^p]_1$, we deduce that

$$\|F(\mathbf{X}_0)\| = \|\Psi_\lambda(\Psi_\lambda(F(\mathbf{X}_0)))\| < 1,$$

which is a contradiction. The proof is complete. ∎

3. FREE HOLOMORPHIC AUTOMORPHISMS OF NONCOMMUTATIVE POLYBALLS

In this section, we use noncommutative Berezin transforms to obtain a complete description of the group $\mathrm{Aut}(\mathbf{B_n})$ of all free holomorphic automorphisms of the polyball $\mathbf{B_n}$, which is an analogue of Rudin's characterization of the holomorphic automorphisms of the polydisc, and prove some of their basic properties. We show that $\mathrm{Aut}(\mathbf{B_n}) \simeq \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_1})_1)$ and obtain an analogue of Poincaré's classical result that the open unit ball of $\mathbb{C}^n$ is not biholomorphic equivalent to the polydisk $\mathbb{D}^n$, for noncommutative regular polyballs.

PROPOSITION 3.1. *If $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, then the following statements hold:*

(i) *If $C_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, are contractions, then $g : \mathbf{B_n}(\mathcal{H})^- \to \mathbf{B_n}(\mathcal{H})^-$, defined by*

$$g(\mathbf{X}) = (X_1 C_1, \ldots, X_k C_k), \quad \mathbf{X} := (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-,$$

is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$. In particular, if each C_i is a unitary operator, then $g|_{\mathbf{B_n}(\mathcal{H})} \in \mathrm{Aut}(\mathbf{B_n})$ and g is a homeomorphism of $\mathbf{B_n}(\mathcal{H})^-$.

(ii) *If σ is a permutation of the set $\{1, \ldots, k\}$ such that $n_{\sigma(i)} = n_i$, then $p_\sigma : \mathbf{B_n}(\mathcal{H})^- \to \mathbf{B_n}(\mathcal{H})^-$, defined by*

$$p_\sigma(\mathbf{X}) = (X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \quad \mathbf{X} := (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-,$$

is a homeomorphism of $\mathbf{B_n}(\mathcal{H})^-$ and $p_\sigma|_{\mathbf{B_n}(\mathcal{H})}$ a free holomorphic automorphism of $\mathbf{B_n}(\mathcal{H})$.

(iii) *If* $\varphi_i : [B(\mathcal{H})^{n_i}]_1 \to [B(\mathcal{H})^{n_i}]_1^-$, $i \in \{1, \dots, k\}$, *is a free holomorphic function, then* $G : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ *defined by*

$$G(\mathbf{X}) := (\varphi_1(X_1), \dots, \varphi_k(X_k)), \quad \mathbf{X} = (X_1, \dots, X_k) \in \mathbf{B_n}(\mathcal{H}),$$

is a free holomorphic function on the regular polyball and range $G \subseteq \mathbf{B_n}(\mathcal{H})$. *In particular, if each* φ_i *is a free holomorphic automorphism of the unit ball* $[B(\mathcal{H})^{n_i}]_1$, *then* $G \in \mathrm{Aut}(\mathbf{B_n})$.

The results are immediate consequences of Theorem 2.4 and Corollary 2.3.

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{n_1+\cdots+n_k}$ be a free holomorphic function with $F := (F_1, \dots, F_k)$ and $F_i = (F_{i,1}, \dots, F_{i,n_i})$, where each $F_{i,j}$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ with scalar coefficients. Note that we always have $F(0) \in \mathbb{C}^{n_1+\cdots+n_k}$. We define $F'(0)$ as the linear operator on $\mathbb{C}^{n_1+\cdots+n_k}$ having the matrix

$$\begin{bmatrix}
\frac{\partial F_{1,1}}{\partial Z_{1,1}}(0) & \cdots \frac{\partial F_{1,1}}{\partial Z_{1,n_1}}(0) & \cdots & \frac{\partial F_{1,1}}{\partial Z_{k,1}}(0) & \cdots \frac{\partial F_{1,1}}{\partial Z_{k,n_k}}(0) \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_{1,n_1}}{\partial Z_{1,1}}(0) & \cdots \frac{\partial F_{1,n_1}}{\partial Z_{1,n_1}}(0) & \cdots & \frac{\partial F_{1,n_1}}{\partial Z_{k,1}}(0) & \cdots \frac{\partial F_{1,n_1}}{\partial Z_{k,n_k}}(0) \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_{k,1}}{\partial Z_{1,1}}(0) & \cdots \frac{\partial F_{k,1}}{\partial Z_{1,n_1}}(0) & \cdots & \frac{\partial F_{k,1}}{\partial Z_{k,1}}(0) & \cdots \frac{\partial F_{k,1}}{\partial Z_{k,n_k}}(0) \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_{k,n_k}}{\partial Z_{1,1}}(0) & \cdots \frac{\partial F_{k,n_k}}{\partial Z_{1,n_1}}(0) & \cdots & \frac{\partial F_{k,n_k}}{\partial Z_{k,1}}(0) & \cdots \frac{\partial F_{k,n_k}}{\partial Z_{k,n_k}}(0)
\end{bmatrix}.$$

Now, we can prove the following noncommutative version of Cartan's uniqueness theorem [4], for free holomorphic functions on regular polyballs.

THEOREM 3.2. *Let* $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})$ *be a free holomorphic function such that* $F(0) = 0$ *and* $F'(0) = I$. *Then*

$$F(\mathbf{X}) = \mathbf{X}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

Proof. Let $\mathbf{X} = (X_{1,1}, \dots, X_{1,n_1}, \dots, X_{k,1}, \dots, X_{k,n_k}) \in \mathbf{B_n}(\mathcal{H})$ and let

$$F(\mathbf{X}) = (F_{1,1}(\mathbf{X}), \dots, F_{1,n_1}(\mathbf{X}), \dots, F_{k,1}(\mathbf{X}), \dots, F_{k,n_k}(\mathbf{X})),$$

where $F_{i,j}$ are free holomorphic functions on the regular polyball $\mathbf{B_n}(\mathcal{H})$, for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. We will also use the row matrix notation $\mathbf{X} = [X_{i,j};\ i,j]$, where the indices i, j are as above. Since $F(0) = 0$ and $F'(0) = I$, we must have

$$F_{i,j}(\mathbf{X}) = X_{i,j} + \sum_{q=2}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_s \in \mathbb{F}_{n_s}^+} a^{(ij)}_{\alpha_1,\dots,\alpha_k} X_{1,\alpha_1} \cdots X_{k,\alpha_k}, \quad a^{(ij)}_{\alpha_1,\dots,\alpha_k} \in \mathbb{C},$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Assume that at least one of the coefficients $a^{(ij)}_{\alpha_1,\dots,\alpha_k}$ is different from zero. Let $m \geqslant 2$ be the smallest natural number such that there exist $i_0 \in \{1, \dots, k\}$, $j_0 \in \{1, \dots, n_{i_0}\}$, and $\alpha_s^0 \in \mathbb{F}_{n_s}^+$ such that

$|\alpha_1^0|+\cdots+|\alpha_k^0|=m$ and $a^{(i_0j_0)}_{\alpha_1^0,\ldots,\alpha_k^0}\neq 0$. Then we have $F_{i,j}(\mathbf{X})=X_{i,j}+\sum\limits_{p=m}^{\infty}G_p^{(ij)}(\mathbf{X})$, where

$$G_p^{(ij)}(\mathbf{X}):=\sum_{|\alpha_1|+\cdots+|\alpha_k|=p,\ \alpha_s\in\mathbb{F}_{n_s}^+}a^{(ij)}_{\alpha_1,\ldots,\alpha_k}X_{1,\alpha_1}\cdots X_{k,\alpha_k}\tag{3.1}$$

for any $p\geqslant m$, $i\in\{1,\ldots,k\}$, and $j\in\{1,\ldots,n_i\}$. Due to Theorem 2.4, $G_p^{(ij)}\circ F$, $p\geqslant m$, is a free holomorphic function and

$$(G_m^{(ij)}\circ F)(\mathbf{X})=\sum_{|\alpha_1|+\cdots+|\alpha_k|=m,\ \alpha_s\in\mathbb{F}_{n_s}^+}a^{(ij)}_{\alpha_1,\ldots,\alpha_k}F_{1,\alpha_1}(\mathbf{X})\cdots F_{k,\alpha_k}(\mathbf{X})=G_m^{(ij)}(\mathbf{X})+K_{m+1}^{(ij)}(\mathbf{X}),$$

where $K_{m+1}^{(ij)}$ is a free holomorphic function containing only monomials of degree greater than or equal to $m+1$. Using now Theorem 2.4, we deduce that $F^{[2]}:=F\circ F$ is a free holomorphic function on the regular polyball $\mathbf{B_n}(\mathcal{H})$. Note that

$$F(\mathbf{X})=[X_{i,j}:i,j]+\Big[\sum_{p=m}^{\infty}G_p^{(ij)}(\mathbf{X}):i,j\Big]\quad\text{and}$$

$$\begin{aligned}F^{[2]}(\mathbf{X})&=[F_{i,j}(\mathbf{X}):i,j]+\Big[G_m^{(ij)}(F(\mathbf{X}))+\sum_{p=m+1}^{\infty}G_p^{(ij)}(F(\mathbf{X})):i,j\Big]\\&=\Big[X_{i,j}+G_m^{(ij)}(\mathbf{X})+\sum_{p=m+1}^{\infty}G_p^{(ij)}(\mathbf{X}):i,j\Big]+[G_m^{(ij)}(\mathbf{X})+\Omega_{m+1}^{(ij)}(\mathbf{X}):i,j]\\&=[X_{i,j}:i,j]+[2G_m^{(i,j)}(\mathbf{X}):i,j]+[\Gamma_{m+1}^{(ij)}(\mathbf{X}):i,j],\end{aligned}$$

where $\Omega_{m+1}^{(ij)}$ and $\Gamma_{m+1}^{(ij)}$ are free holomorphic functions containing only monomials of degree greater than or equal to $m+1$. Continuing this process, we obtain

$$F^{[n]}(\mathbf{X})=[X_{i,j}:i,j]+[nG_m^{(i,j)}(\mathbf{X}):i,j]+[\Lambda_{m+1}^{(ij)}(\mathbf{X}):i,j]\quad n\in\mathbb{N},\tag{3.2}$$

where $\Lambda_{m+1}^{(ij)}$ are free holomorphic functions containing only monomials of degree greater than or equal to $m+1$.

Recall that $\alpha_s^0\in\mathbb{F}_{n_s}^+$ and $|\alpha_1^0|+\cdots+|\alpha_k^0|=m$. Consequently, if $\beta_i\in\mathbb{F}_{n_i}^+$ with $|\beta_1|+\cdots+|\beta_k|=p\geqslant m$, then $S_{1,\beta_1}^*\otimes\cdots\otimes S_{k,\beta_k}^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0})\neq 0$ if and only if $p=m$ and $\beta_i=\alpha_i^0$ for any $i\in\{1,\ldots,k\}$. In this case, we have $S_{1,\alpha_1^0}^*\otimes\cdots\otimes S_{k,\alpha_k^0}^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0})=1$. Hence, and using relation (3.2) when $\mathbf{X}=\mathbf{S}$, we obtain

$$\begin{aligned}&F^{[n]}(r\mathbf{S})^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0})\\&\qquad=r[\mathbf{S}_{i,j}^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0}):i,j]+nr^m[G_m^{(ij)}(\mathbf{S})^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0}):i,j],\end{aligned}$$

where $G_m^{(ij)}$ are homogeneous polynomials of degree m (see relation (3.1)). Taking into account the latter relation and the fact that

$$\|[G_m^{(ij)}(\mathbf{S})^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0}) : i,j]\| \geqslant |a^{(i_0j_0)}_{\alpha_1^0,\ldots,\alpha_k^0}| > 0,$$

we deduce that

$$nr^m|a^{(i_0j_0)}_{\alpha_1^0,\ldots,\alpha_k^0}| \leqslant \|F^{[n]}(r\mathbf{S})^*(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0})\| + \|r[\mathbf{S}^*_{i,j}(e^1_{\alpha_1^0}\otimes\cdots\otimes e^k_{\alpha_k^0}) : i,j]\|$$

for any $n\in\mathbb{N}$. Since $F^{[n]}(r\mathbf{S})\in\mathbf{B_n}(F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k}))$, taking $n\to\infty$ in the inequality above, we obtain a contradiction. Therefore, we must have $F(\mathbf{X}) = \mathbf{X}$. The proof is complete. ∎

If $L := [a_{ij}]_{n\times n}$ is a bounded linear operator on $\mathbb{C}^n$, it generates a function $\boldsymbol{\Phi}_L : B(\mathcal{H})^n \to B(\mathcal{H})^n$ by setting

$$\boldsymbol{\Phi}_L(X_1,\ldots,X_n) := [X_1,\ldots,X_n]\mathbf{L} = \Big[\sum_{i=1}^n a_{i1}X_i,\ldots,\sum_{i=1}^n a_{in}X_i\Big]$$

where $\mathbf{L} := [a_{ij}I_{\mathcal{H}}]_{n\times n}$. By abuse of notation, we also write $\boldsymbol{\Phi}_L(X) = XL$.

A map $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})$ is called free biholomorphic if F is free holomorphic, one-to-one and onto, and has free holomorphic inverse. The automorphism group of $\mathbf{B_n}(\mathcal{H})$, denoted by $\mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$, consists of all free biholomorphic functions of $\mathbf{B_n}(\mathcal{H})$. It is clear that $\mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ is a group with respect to the composition of free holomorphic functions.

In what follows, we characterize the free biholomorphic functions with the property that $F(0) = 0$.

THEOREM 3.3. *Let $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})$ be a free biholomorphic function with $F(0) = 0$. Then there is an invertible bounded linear operator L on $\mathbb{C}^{n_1+\cdots+n_k}$ such that*

$$F(\mathbf{X}) = \boldsymbol{\Phi}_L(\mathbf{X}), \quad \mathbf{X}\in\mathbf{B_n}(\mathcal{H}).$$

Proof. Consider the set $\Lambda_{\mathbf{n}} := \{(i,j) : i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}\}$ with the lexicographic order. Since $F(0) = 0$, we have $F(\mathbf{X}) = [F_{s,t}(\mathbf{X}) : (s,t)\in\Lambda_{\mathbf{n}}]$ with

$$F_{s,t}(\mathbf{X}) = \sum_{(i,j)\in\Lambda_{\mathbf{n}}} a^{(i,j)}_{(s,t)}X_{i,j} + \Psi_{s,t}(\mathbf{X}), \tag{3.3}$$

where $\Psi_{s,t}$ is a free holomorphic function which contains only monomials of degree $\geqslant 2$. Therefore, we have

$$\Psi_{s,t}(\mathbf{X}) = \sum_{m=2}^\infty \sum_{|\alpha_1|+\cdots+|\alpha_k|=m,\ \alpha_s\in\mathbb{F}^+_{n_s}} c^{(s,t)}_{\alpha_1,\ldots,\alpha_k}X_{1,\alpha_1}\cdots X_{k,\alpha_k} \tag{3.4}$$

for some coefficients $c^{(s,t)}_{\alpha_1,\ldots,\alpha_k}\in\mathbb{C}$. Let $L := [a^{(i,j)}_{(s,t)}]_{((i,j),(s,t))\in\Lambda_{\mathbf{n}}\times\Lambda_{\mathbf{n}}}$ and note that

$$F(\mathbf{X}) = [X_{i,j} : (i,j)\in\Lambda_{\mathbf{n}}]L + [\Psi_{s,t}(\mathbf{X}) : (s,t)\in\Lambda_{\mathbf{n}}].$$

Since F is free biholomorphic function with $F(0)=0$, its inverse $G:\mathbf{B_n}(\mathcal{H})\to\mathbf{B_n}(\mathcal{H})$ is also a free holomorphic function with $G(0)=0$. As above, one can see that G must have a representation of the form

$$G(\mathbf{X})=[X_{s,t}:(s,t)\in\Lambda_{\mathbf{n}}]M+[\Gamma_{i,j}(\mathbf{X}):(i,j)\in\Lambda_{\mathbf{n}}],$$

where $M:=[b^{(s,t)}_{(i,j)}]_{((s,t),(i,j))\in\Lambda_{\mathbf{n}}\times\Lambda_{\mathbf{n}}}$ is a square matrix with complex coefficients and $\Gamma_{i,j}$ is a free holomorphic function which contains only monomials of degree $\geqslant 2$. Now, one can easily see that

$$\begin{aligned}(G\circ F)(\mathbf{X})&=[F_{s,t}(\mathbf{X}):(s,t)\in\Lambda_{\mathbf{n}}]M+[\Gamma_{i,j}(F(\mathbf{X})):(i,j)\in\Lambda_{\mathbf{n}}]\\&=[X_{i,j}:(i,j)\in\Lambda_{\mathbf{n}}]LM+[\Psi_{s,t}(\mathbf{X}):(s,t)\in\Lambda_{\mathbf{n}}]M+[\Gamma_{i,j}(F(\mathbf{X})):(i,j)\in\Lambda_{\mathbf{n}}]\\&=[X_{i,j}:(i,j)\in\Lambda_{\mathbf{n}}]LM+[Q_{i,j}(\mathbf{X}):(i,j)\in\Lambda_{\mathbf{n}}],\end{aligned}$$

where each $Q_{i,j}$ is a free holomorphic function which contains only monomials of degree $\geqslant 2$. Since $(G\circ F)(\mathbf{X})=\mathbf{X}$ and due to the uniqueness of the representation of free holomorphic functions, we deduce that $Q_{i,j}=0$ for any $(i,j)\in\Lambda_{\mathbf{n}}$ and $LM=I_{n_1+\cdots+n_k}$. In a similar manner, one can prove that $ML=I_{n_1+\cdots+n_k}$. Therefore, L is an invertible operator.

Since $\mathbf{B_n}(\mathcal{H})$ is a noncommutative Reinhardt domain (see Proposition 1.3), for each $\theta\in\mathbb{R}$, the map $\mathbf{X}\mapsto \mathrm{e}^{-\mathrm{i}\theta}F(\mathrm{e}^{\mathrm{i}\theta}\mathbf{X})$ is a free holomorphic function on the regular polyball $\mathbf{B_n}(\mathcal{H})$. Consequently, Theorem 2.4 implies that

$$H(\mathbf{X}):=G(\mathrm{e}^{-\mathrm{i}\theta}F(\mathrm{e}^{\mathrm{i}\theta}\mathbf{X})),\quad \mathbf{X}\in\mathbf{B_n}(\mathcal{H})$$

is a free holomorphic function with $H(0)=0$ and

$$H(\mathbf{X})=[X_{i,j}:(i,j)\in\Lambda_{\mathbf{n}}]LM+[P_{i,j}(\mathbf{X}):(i,j)\in\Lambda_{\mathbf{n}}],$$

where each $P_{i,j}$ is a free holomorphic function which contains only monomials of degree $\geqslant 2$. Since $LM=I_{n_1+\cdots+n_k}$, we can apply Theorem 3.2 and deduce that $H(\mathbf{X})=\mathbf{X}$. Due to the definition of H and using the fact that $F\circ G=\mathrm{id}$, we obtain $\mathrm{e}^{\mathrm{i}\theta}F(\mathbf{X})=F(\mathrm{e}^{\mathrm{i}\theta}\mathbf{X})$ for any $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})$, and $\theta\in\mathbb{R}$. Using relations (3.3), (3.4) and due to the uniqueness of the coefficients in the representation of free holomorphic functions, we deduce that

$$c^{(s,t)}_{\alpha_1,\ldots,\alpha_k}\mathrm{e}^{\mathrm{i}\theta(|\alpha_1|+\cdots+|\alpha_k|)}=\mathrm{e}^{\mathrm{i}\theta}c^{(s,t)}_{\alpha_1,\ldots,\alpha_k},\quad \theta\in\mathbb{R},$$

for any $\alpha_i\in\mathbb{F}^+_{n_i}$ with $|\alpha_1|+\cdots+|\alpha_k|\geqslant 2$, and $(s,t)\in\Lambda_{\mathbf{n}}$. Hence, $c^{(s,t)}_{\alpha_1,\ldots,\alpha_k}=0$ and, therefore, $\Psi_{s,t}=0$. Now, relation (3.3) implies $F(\mathbf{X})=\mathbf{X}L$, and the proof is complete. ∎

THEOREM 3.4. *Let $\mathbf{n}=(n_1,\ldots,n_k)\in\mathbb{N}^k$ and let $F:\mathbf{B_n}(\mathcal{H})\to\mathbf{B_n}(\mathcal{H})$ be a free biholomorphic function with $F(0)=0$. Then there are unitary operators $U_i\in B(\mathbb{C}^{n_i})$, $i\in\{1,\ldots,k\}$, and a permutation $\sigma\in\mathcal{S}_k$ with the property that $n_{\sigma^{-1}(i)}=n_i$ for $i\in\{1,\ldots,k\}$ such that*

$$(p_{\sigma^{-1}}\circ F)(\mathbf{X})=[X_1U_1,\ldots,X_kU_k],\quad \mathbf{X}=(X_1,\ldots,X_k)\in\mathbf{B_n}(\mathcal{H}).$$

Moreover, the converse is also true.

Proof. According to Theorem 3.3, there is an invertible bounded linear operator L on $\mathbb{C}^{n_1+\cdots+n_k}$ such that

$$F(\mathbf{X}) = [X_1, \ldots, X_k]\mathbf{L}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

Since $F \in \operatorname{Aut}(\mathbf{B_n})$, its scalar representation $f(\lambda_1, \ldots, \lambda_k) := [\lambda_1, \ldots, \lambda_k]\mathbf{L}$ is an automorphism of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$. Due to the classical result (see [10], [27], [29]), there is a permutation $\sigma \in \mathcal{S}_k$ such that $n_{\sigma^{-1}(i)} = n_i$ for $i \in \{1, \ldots, k\}$, such that

$$(p_{\sigma^{-1}} \circ f)(\lambda_1, \ldots, \lambda_k) = (g_1(\lambda_1), \ldots, g_k(\lambda_k))$$

for $(\lambda_1, \ldots, \lambda_k) \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$,, where $g_i \in \operatorname{Aut}((\mathbb{C}^{n_i})_1)$ with $g_i(0) = 0$ for any $i \in \{1, \ldots, k\}$. According to [28], each $g_i \in \operatorname{Aut}((\mathbb{C}^{n_i})_1)$ with $g_i(0) = 0$ has the form $g_i(\lambda_i) = \lambda_i U_i$, where $U_i \in B(\mathbb{C}^{n_i})$ is a unitary operator. Consequently, we obtain

$$(p_{\sigma^{-1}} \circ f)(\lambda_1, \ldots, \lambda_k) = [\lambda_1, \ldots, \lambda_k]\mathbf{U}, \quad (\lambda_1, \ldots, \lambda_k) \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1,$$

where the unitary operator $\mathbf{U} \in B(C^{n_1+\cdots+n_k})$ is the direct sum $\mathbf{U} = U_1 \oplus \cdots \oplus U_k$. Hence, we deduce that $(p_{\sigma^{-1}} \circ F)(\lambda_1, \ldots, \lambda_k) = [\lambda_1, \ldots, \lambda_k]\mathbf{U}$, which, due to the linearity of each component of F, implies

$$(p_{\sigma^{-1}} \circ F)(X_1, \ldots, X_k) = [X_1, \ldots, X_k]\mathbf{U}$$

for any $(X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})$.

To prove the converse, let $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, be unitary operators. Note that the map g_i defined by $g_i(X_i) := X_i U_i$, $X_i \in [B(\mathcal{H})^{n_i}]_1$, is a free holomorphic automorphism of the noncommutative ball $[B(\mathcal{H})^{n_i}]_1$. Hence, and using Proposition 3.1, we deduce that $g := (g_1, \ldots, g_k)$ and p_σ are holomorphic automorphisms of the regular polyball $\mathbf{B_n}$. Consequently, $F := p_\sigma \circ g \in \operatorname{Aut}(\mathbf{B_n})$ with $F(0) = 0$. The proof is complete. ∎

Under the conditions of Theorem 3.3, we consider the unitary operator $\mathbf{U} \in B(C^{n_1+\cdots+n_k})$ defined as the direct sum $\mathbf{U} = U_1 \oplus \cdots \oplus U_k$ and let $\boldsymbol{\Phi}_\mathbf{U} : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})$ be the free biholomorphic function defined by $\boldsymbol{\Phi}_\mathbf{U}(\mathbf{X}) := \mathbf{X}\mathbf{U}$. Then we have $F = p_\sigma \circ \boldsymbol{\Phi}_\mathbf{U}$.

THEOREM 3.5. *Let $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})$ be a free holomorphic function such that $F'(0)$ is a unitary operator on $\mathbb{C}^{n_1+\cdots+n_k}$. Then F is a free holomorphic automorphism of $\mathbf{B_n}$ and*

$$F(\mathbf{X}) = \mathbf{X}[F'(0)]^\tau, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where $^\tau$ denotes the transpose.

Proof. Assume that F has the representation

$$F(\mathbf{X}) := A_{(0)} + \sum_{q=1}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\ \alpha_i \in \mathbb{F}_{n_i}^+} A_{(\alpha)} \otimes X_{(\alpha)}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

where $A_{(\alpha)} \in \mathbf{P_n}(\mathbb{C})$ is written as a row operator with entries in $\mathbb{C}$. Note that

$$F'(0) = [A^{\tau}_{(\alpha)} : |\alpha_1| + \cdots + |\alpha_k| = 1, \quad \alpha_i \in \mathbb{F}^+_{n_i}].$$

Taking into account that $F'(0)$ is a co-isometry, we have $\sum_{|\alpha_1|+\cdots+|\alpha_k|=1,\, \alpha_i\in\mathbb{F}^+_{n_i}} A^*_{(\alpha)} A_{(\alpha)} = I$. Since F is a free holomorphic function with $\|F\|_\infty = 1$, we can apply Proposition 2.5. Consequently, we have

$$\sum_{|\alpha_1|+\cdots+|\alpha_k|=1,\, \alpha_i\in\mathbb{F}^+_{n_i}} A^*_{(\alpha)} A_{(\alpha)} \leqslant I - F(0)^* F(0),$$

which implies $F(0) = 0$. Therefore, since $[F'(0)]^\tau = \begin{bmatrix} A_{(\alpha)} \\ \vdots \\ |\alpha_1| + \cdots + |\alpha_k| = 1 \end{bmatrix}$, we have

$$F(\mathbf{X}) = \mathbf{X}[F'(0)]^\tau + \sum_{q=2}^{\infty} \sum_{|\alpha_1|+\cdots+|\alpha_k|=q,\, \alpha_i\in\mathbb{F}^+_{n_i}} A_{(\alpha)} \otimes X_{(\alpha)}. \tag{3.5}$$

On the other hand, since $F'(0)$ is an isometry, we have $F'(0)^\tau [F'(0)^\tau]^* = I$. Multiplying relation (3.5) to the right by $([F'(0)]^\tau)^*$, we obtain

$$H(\mathbf{X}) := F(\mathbf{X})([F'(0)]^\tau)^* = \mathbf{X} + [G_{i,j}(\mathbf{X}) : (i,j) \in \Lambda_{\mathbf{n}}],$$

where $\Lambda_{\mathbf{n}} := \{(i,j) : i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}\}$ and each $G_{i,j}$ is a free holomorphic function containing only monomials of degree $\geqslant 2$. Since H is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ with $H(0) = 0$ and $H'(0) = I_{n_1+\cdots+n_k}$, Theorem 3.2 implies $H(\mathbf{X}) = \mathbf{X}$. Consequently, we have $F(\mathbf{X})([F'(0)]^\tau)^* = \mathbf{X}$. Multiplying this relation to the right by $[F'(0)]^\tau$ and taking into account that $F'(0)$ is a co-isometry, we deduce that $F(\mathbf{X}) = \mathbf{X}[F'(0)]^\tau$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. This completes the proof. ∎

In [19], the theory of noncommutative characteristic functions for row contractions (see [14]) was used to find all the involutive free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$. They turned out to be of the form

$$\Psi_\lambda(Y_1,\ldots,Y_n) = -\Theta_\lambda(Y_1,\ldots,Y_n) := \lambda - \Delta_\lambda \Big(I_{\mathcal{K}} - \sum_{i=1}^{n} \overline{\lambda}_i Y_i \Big)^{-1} [Y_1 \cdots Y_n] \Delta_{\lambda^*},$$

for some $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{B}_n$, where Θ_λ is the characteristic function of the row contraction λ, and $\Delta_\lambda, \Delta_{\lambda^*}$ are the defect operators defined by $\Delta_\lambda = (1 - \|\lambda\|_2^2)^{1/2}$ and $\Delta_{\lambda^*} = (I_{\mathbb{C}^n} - \lambda^*\lambda)^{1/2}$. Moreover, we determined the group $\mathrm{Aut}([B(\mathcal{H})^n]_1)$ of all the free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$ and showed that if $\Psi \in \mathrm{Aut}([B(\mathcal{H})^n]_1)$ and $\lambda := \Psi^{-1}(0)$, then there is a unitary operator U on $\mathbb{C}^n$ such that

$$\Psi = \Psi_U \circ \Psi_\lambda,$$

where $\Psi_U(Y) := YU$ for any $Y \in [B(\mathcal{H})^n]_1$. Let $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n \backslash \{0\}$ and let $\gamma := \frac{1}{\|\lambda\|_2}$. Then $\Psi_\lambda := -\Theta_\lambda$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has the following properties:

(i) $\Psi_\lambda(0) = \lambda$ and $\Psi_\lambda(\lambda) = 0$;

(ii) the identity

$$I_{\mathcal{H}} - \Psi_\lambda(X)\Psi_\lambda(X)^* = \Delta_\lambda(I - X\lambda^*)^{-1}(I - XX^*)(I - \lambda X^*)^{-1}\Delta_\lambda$$

holds for all $X \in [B(\mathcal{H})^n]_\gamma$;

(iii) Ψ_λ is an involution, i.e., $\Psi_\lambda(\Psi_\lambda(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$;

(iv) Ψ_λ is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$;

(v) Ψ_λ is a homeomorphism of $[B(\mathcal{H})^n]_1^-$ onto $[B(\mathcal{H})^n]_1^-$.

Now, we can prove a structure theorem for holomorphic automorphisms of regular polyballs.

THEOREM 3.6. *Let* $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ *and let* $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$. *If* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, *then there are unique unitary operators* $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \dots, k\}$, *and a unique permutation* $\sigma \in \mathcal{S}_k$ *with* $n_{\sigma(i)} = n_i$ *such that*

$$\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}},$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$ *and* $\boldsymbol{\Psi}_{\boldsymbol{\lambda}} := (\Psi_{\lambda_1}, \dots, \Psi_{\lambda_k})$.

Proof. Let $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$. For each $i \in \{1, \dots, k\}$, $\lambda_i \in (\mathbb{C}^{n_i})_1$, and Ψ_{λ_i} is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^{n_i}]_1$. Moreover, $\Psi_{\lambda_i}(\Psi_{\lambda_i}(X)) = X$ for any $X \in [B(\mathcal{H})^{n_i}]_1$, $\Psi_{\lambda_i}(0) = \lambda_i$. Consequently, using Proposition 3.1 and Theorem 2.4, we deduce that $\boldsymbol{\Psi}_{\boldsymbol{\lambda}} := (\Psi_{\lambda_1}, \dots, \Psi_{\lambda_k})$ is a holomorphic automorphism of the regular polyball $\mathbf{B_n}$ with the property that

$$\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\mathbf{X})) = \mathbf{X}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

and $\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(0) = \boldsymbol{\lambda}$. Hence, $\boldsymbol{\Psi} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ and $(\boldsymbol{\Psi} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}})(0) = 0$. Applying Theorem 3.3, there are unitary operators $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \dots, k\}$, and a permutation $\sigma \in \mathcal{S}_k$ with the property that $n_{\sigma^{-1}(i)} = n_i$ for $i \in \{1, \dots, k\}$ such that

$$(p_{\sigma^{-1}} \circ (\boldsymbol{\Psi} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}))(\mathbf{X}) = [X_1 U_1, \dots, X_k U_k] \quad \mathbf{X} = (X_1, \dots, X_k) \in \mathbf{B_n}(\mathcal{H}).$$

Hence, taking into account that $\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\mathbf{X})) = \mathbf{X}$, we obtain $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$, which completes the proof. ∎

We remark that, unlike the classical case, the free holomorphic automorphism group is not transitive because 0 must be mapped to another scalar point.

COROLLARY 3.7. *Let* $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H})$ *be a bounded free holomorphic function and* $\mathbf{a} \in \mathbf{B_n}(\mathbb{C})$. *Then*

$$\|\boldsymbol{\Psi}_{F(\mathbf{a})}(F(\mathbf{X}))\| \leqslant m_{\mathbf{B_n}}(\boldsymbol{\Psi}_{\mathbf{a}}(\mathbf{X})) \leqslant \|\boldsymbol{\Psi}_{\mathbf{a}}(\mathbf{X})\|$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, *where* $m_{\mathbf{B_n}}$ *is the Minkovski functional.*

Proof. Consider the automorphisms $\mathbf{\Psi_a} \in \mathrm{Aut}(\mathbf{B_n})$ and $\mathbf{\Psi}_{F(\mathbf{a})} \in \mathrm{Aut}(\mathbf{B_m})$. Due to Theorem 2.4 and using the fact that $\mathbf{\Psi_a}(0) = \mathbf{a}$ and $\mathbf{\Psi}_{F(\mathbf{a})}(F(\mathbf{a})) = 0$, we deduce that $G := \mathbf{\Psi}_{F(\mathbf{a})} \circ F \circ \mathbf{\Psi_a}$ is a free holomorphic function from $\mathbf{B_n}(\mathcal{H})$ to $\mathbf{B_m}(\mathcal{H})$, and $G(0) = 0$. Applying Theorem 2.7 to G, we obtain

$$\|\mathbf{\Psi}_{F(\mathbf{a})} \circ F \circ \mathbf{\Psi_a}(\mathbf{Y})\| \leqslant m_{\mathbf{B_m}}(\mathbf{Y}) \leqslant \|\mathbf{Y}\|, \quad \mathbf{Y} \in \mathbf{B_m}(\mathcal{H}).$$

Setting $\mathbf{Y} = \mathbf{\Psi_a}(\mathbf{Y})$ and using the fact that $\mathbf{\Psi_a} \circ \mathbf{\Psi_a} = \mathrm{id}$, we complete the proof. ∎

In what follows, we present an analogue of Poincaré result that the open unit ball of $\mathbb{C}^n$ is not biholomorphic equivalent to the polydisk $\mathbb{D}^n$, for noncommutative regular polyballs.

We denote by $\mathrm{Bih}(\mathbf{B_n}(\mathcal{H}), \mathbf{B_m}(\mathcal{H}))$ the set of all biholomorphic functions from $\mathbf{B_n}(\mathcal{H})$ to $\mathbf{B_m}(\mathcal{H})$.

THEOREM 3.8. *Let* $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ *and* $\mathbf{m} = (m_1, \dots, m_q) \in \mathbb{N}^q$. *Then*

$$\mathrm{Bih}(\mathbf{B_n}(\mathcal{H}), \mathbf{B_m}(\mathcal{H})) \neq \emptyset$$

if and only if $k = q$ *and there is a permutation* $\sigma \in \mathcal{S}_k$ *such that* $m_{\sigma(i)} = n_i$ *for any* $i \in \{1, \dots, k\}$. *Moreover, any free biholomorphic function* $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H}))$ *is up to a permutation of* $(m_1, \dots, m_k)$ *an automorphism of the noncommutative regular polyball* $\mathbf{B_n}$.

Proof. Let $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H}))$ be a free biholomorphic function. Then its scalar representation

$$f : (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1 \to (\mathbb{C}^{m_1})_1 \times \cdots \times (\mathbb{C}^{m_q})_1,$$

defined by $f(\mathbf{z}) := F(\mathbf{z})$, $\mathbf{z} = \{z_{i,j}\} \in \mathbf{B_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$, is a scalar biholomorphic function. Using Browder's invariance of domain theorem, we deduce that $n_1 + \cdots + n_k = m_1 + \cdots + m_q$. On the other hand, according to the classical result of Ligocka and Tsyganov (which is a generalization of Rudin's characterization of the holomorphic automorphisms of the polydisc [27]), we must have $k = q$ and there is a permutation $\sigma \in \mathcal{S}_k$ such that $m_{\sigma(i)} = n_i$ for any $i \in \{1, \dots, k\}$. Using Proposition 3.1 and Theorem 2.4, we deduce that $p_\sigma \circ F \in \mathrm{Aut}(\mathbf{B_n})$, which completes the proof. ∎

Let $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$, $\lambda \neq 0$, and let $\widetilde{\Theta}_\lambda$ be the boundary function of the characteristic function with respect to the right creation operators $R_1, \dots, R_n$ on the Fock space $F^2(H_n)$, i.e., $\widetilde{\Theta}_\lambda := \text{SOT-}\lim_{r\to 1} \Theta_\lambda(rR_1, \dots, rR_n)$. We recall from [19], the following properties:

(i) the map Θ_λ is a free holomorphic function on the open ball $[B(\mathcal{H})^n]_\gamma$, where $\gamma := \frac{1}{\|\lambda\|_2}$;

(ii) $\widetilde{\Theta}_\lambda = \Theta_\lambda(R_1, \dots, R_n) = -\lambda + \Delta_\lambda \Big(I_{F^2(H_n)} - \sum_{i=1}^n \overline{\lambda}_i R_i \Big)^{-1} [R_1, \dots, R_n] \Delta_{\lambda^*}$;

(iii) $\widetilde{\Theta}_\lambda$ is a pure row isometry with entries in the noncommutative disc algebra generated by $R_1, \ldots, R_n$ and the identity;

(iv) rank $(I - \widetilde{\Theta}_\lambda \widetilde{\Theta}_\lambda^*) = 1$ and $\widetilde{\Theta}_\lambda$ is unitarily equivalent to $[R_1, \ldots, R_n]$.

We define the right creation operators $R_{i,j}$ acting on the Fock space $F^2(H_{n_i})$ and the ampliations $\mathbf{R}_{i,j}$ acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$.

THEOREM 3.9. *Let $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$, where $\mathbf{n} = (n_1, \ldots, n_k)$ is in $\mathbb{N}^k$, and let $\widehat{\boldsymbol{\Psi}} = (\widehat{\Psi}_1, \ldots, \widehat{\Psi}_k)$ be the boundary function with respect to the universal model $\mathbf{S} = \{\mathbf{S}_{i,j}\}$. The following statements hold:*

(i) *$\boldsymbol{\Psi}$ is a free holomorphic function on the regular polyball $\gamma\mathbf{B_n}$ for some $\gamma > 1$.*

(ii) *The boundary function $\widehat{\boldsymbol{\Psi}}$ with respect to $\mathbf{S}$ is a pure element in the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$ and $\widehat{\boldsymbol{\Psi}} := \lim_{r\to 1} \boldsymbol{\Psi}(r\mathbf{S}) = \boldsymbol{\Psi}(\mathbf{S})$. Each $\widehat{\Psi}_i = (\widehat{\Psi}_{i,1}, \ldots, \widehat{\Psi}_{i,n_i})$ is an isometry with entries in the noncommutative disk algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$ and the identity.*

(iii) *$\boldsymbol{\Psi}$ is a homeomorphism of $\mathbf{B_n}(\mathcal{H})^-$ onto $\mathbf{B_n}(\mathcal{H})^-$.*

(iv) *If $\Psi \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, then the identity*

$$\boldsymbol{\Delta}_{\boldsymbol{\Psi}(\mathbf{X})}(I) = \boldsymbol{\Delta}_{\boldsymbol{\lambda}} \Big[\prod_{i=1}^k \Big(I_{\mathcal{H}} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} X_{i,j} \Big)^{-1} \Big] \boldsymbol{\Delta}_{\mathbf{X}}(I) \Big[\prod_{i=1}^k \Big(I_{\mathcal{H}} - \sum_{j=1}^{n_i} \lambda_{i,j} X_{i,j}^* \Big)^{-1} \Big]$$

holds for any $\mathbf{X} = \{X_{i,j}\} \in \mathbf{B_n}(\mathcal{H})^-$, where $\boldsymbol{\Delta}_{\boldsymbol{\lambda}} = \prod_{i=1}^k (1 - \|\lambda_i\|_2^2)$.

(v) *The defect of the boundary function of $\boldsymbol{\Psi}$ with respect to the universal model $\mathbf{R} = \{\mathbf{R}_{i,j}\}$ satisfies the relation*

$$\boldsymbol{\Delta}_{\boldsymbol{\Psi}(\mathbf{R})}(I) = \mathbf{K}_{\boldsymbol{\Psi}^{-1}(0)} \mathbf{K}_{\boldsymbol{\Psi}^{-1}(0)}^*,$$

where $\mathbf{K}_{\boldsymbol{\Psi}^{-1}(0)}$ is the noncommutative Berezin kernel at $\boldsymbol{\Psi}^{-1}(0) \in \mathbf{B_n}(\mathbb{C})$.

(vi) rank $\boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}}(I) = 1$ *and $\widehat{\boldsymbol{\Psi}}$ is unitarily equivalent to the universal model $\mathbf{S}$.*

Proof. Due to Theorem 3.6, if $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, then there are unique unitary operators $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, and a unique permutation $\sigma \in \mathcal{S}_k$ with $n_{\sigma(i)} = n_i$ such that

$$\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}},$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$ and $\boldsymbol{\Psi}_{\boldsymbol{\lambda}} := (\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_k})$. Since $\Psi_{\lambda_i} := -\Theta_{\lambda_i}$ is a free holomorphic function on the open ball $[B(\mathcal{H})^{n_i}]_{\gamma_i}$, where $\gamma_i := \frac{1}{\|\lambda_i\|_2}$ if $\lambda_i \neq 0$ and $\gamma_i = \infty$, otherwise, Proposition 3.1 part (iii) implies that $\boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ is a free holomorphic function on the regular polyball $\gamma\mathbf{B_n}$ for $\gamma := \min\{\gamma_i : i \in \{1, \ldots, k\}\}$. Using Theorem 2.4 and Proposition 3.1, one can complete the proof of item (i).

The first part of item (ii) follows from (i) and the continuity of the $\boldsymbol{\Psi}$ on $\gamma\mathbf{B_n}$. On the other hand, due to the remarks preceding the theorem, we know that $\widehat{\Psi}_{\lambda_i} := \lim_{r\to 1} \Psi_{\lambda_i}(rS_i) = \Psi_{\lambda_i}(S_i)$ is a pure row isometry with entries in the noncommutative disc algebra generated by $S_{i,1}, \ldots, S_{i,n_i}$ and the identity, on the full Fock space $F^2(H_{n_i})$. If $U_i \in B(\mathbb{C}^{n_i})$ are unitary operators, it is clear that the components of the boundary function

$$\widehat{\boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}} = (\Psi_{\lambda_1}(\mathbf{S})U_1, \ldots, \Psi_{\lambda_k}(\mathbf{S})U_k)$$

are isometries. On the other hand, set $(\xi_{i,1}, \ldots, \xi_{i,n_i}) := \mathbf{S}_i U_i$ and note that each $\xi_{i,j}$ is a linear combination of $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$. Note that $\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha|=p} \xi_{i,\alpha} \xi_{i,\alpha}^*(\mathrm{e}_\beta^{\mathrm{i}}) = 0$ for any $\beta \in \mathbb{F}_{n_i}^+$ and $p > |\beta|$. Since $\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha|=p} \xi_{i,\alpha} \xi_{i,\alpha}^* \leqslant I$, we deduce that

$$\lim_{p\to\infty} \sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha|=p} \xi_{i,\alpha} \xi_{i,\alpha}^* x = 0, \quad x \in F^2(H_{n_i}),$$

which proves that $\widehat{\boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}}$ is a pure element in $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. For any permutation $\sigma \in \mathcal{S}_k$ with $n_{\sigma(i)} = n_i$, the boundary function $\widehat{p}_\sigma = (\mathbf{S}_{\sigma(1)}, \ldots, \mathbf{S}_{\sigma(k)})$ has the entries pure row isometries. Now, using Lemma 2.1, we deduce that the boundary function of the composition $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ satisfies the required properties of item (ii).

According to the remarks preceding Theorem 3.6, each Ψ_λ is a homeomorphism of $[B(\mathcal{H})^{n_i}]_1^-$ and $\Psi_{\lambda_i}(\Psi_{\lambda_i}(X_i)) = X_i$ for any $X_i \in [B(\mathcal{H})^{n_i}]_1^-$. This implies that

$$\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\mathbf{X})) = \mathbf{X}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-,$$

which proves that $\boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ is a homeomorphism of $\mathbf{B_n}(\mathcal{H})^-$. According to Proposition 3.1, $\boldsymbol{\Phi}_{\boldsymbol{\lambda}}$ and p_σ are also homeomorphisms of $\mathbf{B_n}(\mathcal{H})^-$. Since, due to Theorem 3.6, each $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ has the representation $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$, we conclude that $\boldsymbol{\Psi}$ is a homeomorphism of $\mathbf{B_n}(\mathcal{H})^-$, which proves item (iii).

For each $i \in \{1, \ldots, k\}$, let $S_i = (S_{i,1}, \ldots, S_{i,n_i})$ be the n_i-tuple of left creation operators on the full Fock space $F^2(H_{n_i})$. According to the remarks preceding Theorem 3.6, we have

$$(\mathrm{id} - \Phi_{\psi_{\lambda_i}(S_i)})(I) = (1 - \|\lambda_i\|_1^2)\Big(I - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} S_{i,j}\Big)^{-1} (\mathrm{id} - \Phi_{S_i})(I) \Big(I - \sum_{j=1}^{n_i} \lambda_{i,j} S_{i,j}^*\Big)^{-1}.$$

Taking the tensor product of these relations when $i \in \{1, \ldots, k\}$, and using the definition of the universal model $\mathbf{S}$, we obtain

$$\begin{aligned}
&(\mathrm{id} - \Phi_{\psi_{\lambda_1}(\mathbf{S}_i)}) \circ \cdots \circ (\mathrm{id} - \Phi_{\psi_{\lambda_k}(\mathbf{S}_k)})(I_{\otimes_{i=1}^k F^2(H_{n_i})}) \\
&\quad = \prod_{i=1}^k (1 - \|\lambda_i\|_1^2) \prod_{i=1}^k \Big(I_{\otimes_{i=1}^k F^2(H_{n_i})} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} \mathbf{S}_{i,j}\Big)^{-1} \\
&\qquad \times (\mathrm{id} - \Phi_{\mathbf{S}_1}) \circ \cdots \circ (\mathrm{id} - \Phi_{\mathbf{S}_k})(I_{\otimes_{i=1}^k F^2(H_{n_i})}) \prod_{i=1}^k \Big(I_{\otimes_{i=1}^k F^2(H_{n_i})} - \sum_{j=1}^{n_i} \lambda_{i,j} \mathbf{S}_{i,j}^*\Big)^{-1}.
\end{aligned}$$

Note that both sides of the relation above, as well as the factors involved, are in the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_n$. Applying the Berezin transform at any element $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-$, we obtain

$$\Delta_{\boldsymbol{\Psi}_\lambda(\mathbf{X})}(I) = \Delta_\lambda \Big[\prod_{i=1}^k \Big(I_\mathcal{H} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} X_{i,j} \Big)^{-1} \Big] \Delta_\mathbf{X}(I) \Big[\prod_{i=1}^k \Big(I_\mathcal{H} - \sum_{j=1}^{n_i} \lambda_{i,j} X_{i,j}^* \Big)^{-1} \Big], \tag{3.6}$$

where $\Delta_\lambda = \prod_{i=1}^k (1 - \|\lambda_i\|_2^2)$. Since each $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ has the representation $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_\mathbf{U} \circ \boldsymbol{\Psi}_\lambda$, one can easily see that $\Delta_{\boldsymbol{\Psi}(\mathbf{X})}(I) = \Delta_{\boldsymbol{\Psi}_\lambda(\mathbf{X})}(I)$, which shows that item (iv) holds.

Now, we prove item (v). If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, the Berezin kernel $\mathbf{K}_\lambda : \mathbb{C} \to \bigotimes_{i=1}^k F^2(H_{n_i})$ is defined by

$$\mathbf{K}_\lambda(1) = \sum_{(\alpha) \in F_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \Delta_\lambda^{1/2} \overline{\boldsymbol{\lambda}}_{(\alpha)} \otimes e_{(\alpha)}.$$

It is easy to see that $\mathbf{K}_\lambda^*(e_{(\alpha)}) = \Delta_\lambda^{1/2} \boldsymbol{\lambda}_{(\alpha)}$ and

$$\mathbf{K}_\lambda \mathbf{K}_\lambda^* = \mathbf{K}_\lambda(\Delta_\lambda^{1/2} \boldsymbol{\lambda}_{(\alpha)}) = \Delta_\lambda^{1/2} \boldsymbol{\lambda}_{(\alpha)} \sum_{(\beta) \in F_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \overline{\boldsymbol{\lambda}}_{(\beta)} \otimes e_{(\beta)}.$$

On the other hand, relation (3.6) written for the universal model $\mathbf{R} = \{\mathbf{R}_{i,j}\}$ implies

$$\begin{aligned}
\Delta_{\boldsymbol{\Psi}(\mathbf{R})}(I)(e_{(\alpha)}) &= \Delta_\lambda \Big[\prod_{i=1}^k \Big(I_\mathcal{H} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} \mathbf{R}_{i,j} \Big)^{-1} \Big] P_\mathbb{C} \Big[\prod_{i=1}^k \Big(I_\mathcal{H} - \sum_{j=1}^{n_i} \lambda_{i,j} \mathbf{R}_{i,j}^* \Big)^{-1} \Big] (e_{(\alpha)}) \\
&= \Delta_\lambda \Big[\prod_{i=1}^k \Big(I_\mathcal{H} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} \mathbf{R}_{i,j} \Big)^{-1} \Big] (\boldsymbol{\lambda}_{(\alpha)}) \\
&= \Delta_\lambda^{1/2} \boldsymbol{\lambda}_{(\alpha)} \sum_{(\beta) \in F_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \overline{\boldsymbol{\lambda}}_{(\beta)} \otimes e_{(\beta)}.
\end{aligned}$$

Therefore, item (v) follows. The fact that rank $\Delta_{\widehat{\boldsymbol{\Psi}}}(I) = 1$ is a simple consequence of item (iv) or (v). Since the boundary function $\widehat{\boldsymbol{\Psi}} = (\widehat{\Psi}_1, \ldots, \widehat{\Psi}_k)$, with respect to the universal model $\mathbf{S} = \{\mathbf{S}_{i,j}\}$, is a pure element in the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$ and each $\widehat{\Psi}_i = (\widehat{\Psi}_{i,1}, \ldots, \widehat{\Psi}_{i,n_i})$ is an isometry with entries in the noncommutative disk algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$ and the identity, we deduce that $\widehat{\boldsymbol{\Psi}} = (\widehat{\Psi}_{i,1}, \ldots, \widehat{\Psi}_{i,n_i})$ is a pure doubly commuting tuple of isometries with rank $\Delta_{\widehat{\boldsymbol{\Psi}}}(I) = 1$. Now, using the Wold type decomposition for nondegenerate $*$-representations of the C^*-algebra $C^*(\mathbf{S})$ from [24] (see Corollary 7.3 and its consequences), we conclude that $\widehat{\boldsymbol{\Psi}}$ is unitarily equivalent to the universal model $\mathbf{S}$. The proof is complete. ∎

THEOREM 3.10. *The map* $\Lambda : \mathrm{Aut}(\mathbf{B_n}) \to \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1)$ *defined by*

$$\Lambda(\mathbf{\Psi})(\mathbf{z}) := (\mathbf{B_z}[\widehat{\mathbf{\Psi}}_1], \ldots, \mathbf{B_z}[\widehat{\mathbf{\Psi}}_k]) \quad \mathbf{z} \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1,$$

is a group isomorphism, where $\widehat{\mathbf{\Psi}}$ *is the boundary function of* $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$ *with respect to the universal model* $\mathbf{S}$ *and* $\mathbf{B_z}$ *is the noncommutative Berezin transform at* $\mathbf{z}$.

Proof. Fix $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$ and $\boldsymbol{\lambda} = \{\lambda_{i,j}\} = \mathbf{\Psi}^{-1}(0) \in \mathbf{B_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$. Then, due to Theorem 3.6, there are unique unitary operators $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, and a unique permutation $\sigma \in \mathcal{S}_k$ with $n_{\sigma(i)} = n_i$ such that

$$\mathbf{\Psi} = p_\sigma \circ \mathbf{\Phi_U} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}}, \tag{3.7}$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$. According to Theorem 3.9, Each $\widehat{\Psi}_i = (\widehat{\Psi}_{i,1}, \ldots, \widehat{\Psi}_{i,n_i})$ is a pure row isometry with entries in the noncommutative disk algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$ and the identity. Note that if $\mathbf{z} = \{z_{i,j}\} \in \mathbf{B_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$, then the Berezin kernel $\mathbf{K_z} : \mathbb{C} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ is an isometry and $z_{i,j} = \mathbf{K_z}^* \mathbf{S}_{i,j} \mathbf{K_z}$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Hence, using the continuity of the noncommutative Berezin transform in the operator norm topology and relation (3.7), we deduce that

$$[\Lambda(\mathbf{\Psi})](\mathbf{z}) : = (\mathbf{B_z}[\widehat{\mathbf{\Psi}}_1], \ldots, \mathbf{B_z}[\widehat{\mathbf{\Psi}}_k]) = (p_\sigma \circ \mathbf{\Phi_U} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}})(\mathbf{z})$$

for any $\mathbf{z} \in \mathbf{B_n}(\mathbb{C})$. Due to [10], [27], [29], each automorphism of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$ has the form $\mathbf{z} \mapsto (p_\sigma \circ \mathbf{\Phi_U} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}})(\mathbf{z})$. Therefore, $\Lambda(\mathbf{\Psi}) \in \mathrm{Aut}(\mathbf{B_n}(\mathbb{C}))$, which proves the surjectivity of Λ. Moreover, we have $[\Lambda(\mathbf{\Psi})](\mathbf{z}) = \mathbf{\Psi}(\mathbf{z})$, $\mathbf{z} \in \mathbf{B_n}(\mathbb{C})$, which clearly implies that Λ is a homomorphism. To prove injectivity of Λ, assume that $\Lambda(\mathbf{\Psi}) = \mathrm{id}$, where $\mathbf{\Psi} = p_\sigma \circ \mathbf{\Phi_U} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}}$. Using the calculations above, we have $p_\sigma \circ \mathbf{\Phi_U} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}}(\mathbf{z}) = \mathbf{z}$ for any $\mathbf{z} \in \mathbf{B_n}(\mathbb{C})$. Hence, one can easily deduce that $\boldsymbol{\lambda} = 0$, $U = -I$, and $\sigma = \mathrm{id}$, which implies $\Psi = \mathrm{id}$. Therefore, Λ is a group isomorphism. This completes the proof. ∎

4. AUTOMORPHISMS OF CUNTZ–TOEPLITZ ALGEBRAS

In this section, we determine the group of automorphisms of the Cuntz–Toeplitz C^*-algebra $C^*(\mathbf{S})$ which leave invariant the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$, and the group of unitarily implemented automorphisms of the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$ (respectively Hardy algebra $\mathbf{F}_{\mathbf{n}}^\infty$). As a consequence, we obtain a concrete description for the group of automorphisms of the tensor product $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ of Cuntz–Toeplitz algebras which leave invariant the tensor product $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$ of noncommutative disc algebras, which extends Voiculescu's result when $k = 1$.

PROPOSITION 4.1. *A free holomorphic function $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})^-$ has a continuous extension (also denoted by F) to the closed polyball $\mathbf{B_n}(\mathcal{H})^-$ if and only if the boundary function $\widehat{F}$ has the entries in the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ and $\widehat{F} \in \mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Moreover, the noncommutative Berezin transform has the property that*

$$\boldsymbol{\mathcal{B}}_{F(\mathbf{X})}[g] = \boldsymbol{\mathcal{B}}_\mathbf{X}[\boldsymbol{\mathcal{B}}_{\widehat{F}}[g]]$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and $g \in C^*(\mathbf{S})$. If, in addition, $\widehat{F}$ is a pure element of the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$, then the same relation holds for any pure element $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and $g \in \mathbf{F}_\mathbf{n}^\infty$.

Proof. The first part of the proposition follows from [24] (Corollary 4.3). To prove the second part, let $F = (F_1, \ldots, F_k)$, with $F_i = (F_{i,1}, \ldots, F_{i,n_i})$. Note that the boundary function $\widehat{F} = (\widehat{F}_1, \ldots, \widehat{F}_k)$, with $\widehat{F}_i = (\widehat{F}_{i,1}, \ldots, \widehat{F}_{i,n_i})$, is an element of the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$ and the entries $\widehat{F}_{i,j} := \lim_{r\to 1} F_{i,j}(r\mathbf{S})$ are in the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$. Let $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and set $\mathbf{A} := (A_1, \ldots, A_k)$, with $A_i = (A_{i,1}, \ldots, A_{i,n_i})$, where

$$A_{i,j} := F_{i,j}(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{F}_{i,j}] := \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\mathbf{X}}[\widehat{F}_{i,j}].$$

We recall that the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_\mathbf{X} : C^*(\mathbf{S}) \to B(\mathcal{H})$, which is defined by $\boldsymbol{\mathcal{B}}_\mathbf{X}(f) := \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\mathbf{X}}[g]$, is a completely contractive linear map such that

$$\boldsymbol{\mathcal{B}}_\mathbf{X}[fg^*] = \boldsymbol{\mathcal{B}}_\mathbf{X}[f]\boldsymbol{\mathcal{B}}_\mathbf{X}[g]^*, \quad f, g \in \mathcal{A}_\mathbf{n},$$

and the restriction $\boldsymbol{\mathcal{B}}_\mathbf{X}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$ is a unital contractive homomorphism from $\boldsymbol{\mathcal{A}}_\mathbf{n}$ to $B(\mathcal{H})$. Now, note that $A_{(\alpha)} = F_{(\alpha)}(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{F}_{(\alpha)}]$ and

$$\begin{aligned}\boldsymbol{\mathcal{B}}_\mathbf{X}[\boldsymbol{\mathcal{B}}_{\widehat{F}}[\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^*] &= \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{F}_{(\alpha)}\widehat{F}_{(\beta)}^*] = \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{F}_{(\alpha)}]\boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{F}_{(\beta)}^*] \\ &= F_{(\alpha)}(\mathbf{X})F_{(\beta)}(\mathbf{X})^* = A_{(\alpha)}A_{(\beta)}^* = \boldsymbol{\mathcal{B}}_{F(\mathbf{X})}[\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^*]\end{aligned}$$

for any $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. Since the linear span of the monomials $\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^*$ is dense in the C^*-algebra $C^*(\mathbf{S})$ and the Berezin transform is continuous in the operator norm topology, we deduce that $\boldsymbol{\mathcal{B}}_{\widehat{F}}[g]$ is in $C^*(\mathbf{S})$ for any $g \in C^*(\mathbf{S})$, and $\boldsymbol{\mathcal{B}}_{F(\mathbf{X})}[g] = \boldsymbol{\mathcal{B}}_\mathbf{X}[\boldsymbol{\mathcal{B}}_{\widehat{F}}[g]]$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and $g \in C^*(\mathbf{S})$.

Now, we assume, in addition, that $\widehat{F}$ is a pure element of the regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Let $f \in \mathbf{F}_\mathbf{n}^\infty$ have the Fourier representation $\sum_{(\alpha)} a_{(\alpha)}\mathbf{S}_{(\alpha)}$ and set

$$f_r(\mathbf{S}) = \sum_{q=0}^\infty \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}\mathbf{S}_{(\alpha)}, \quad r \in [0, 1),$$

where the convergence is in the operator norm topology. Since $F(\mathbf{X})$ is pure for any pure element $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$, we can use the $\mathbf{F}_\mathbf{n}^\infty$-functional calculus for pure elements in the regular polyball to deduce that

$$\begin{aligned}\boldsymbol{\mathcal{B}}_{F(\mathbf{X})}[f] &= \text{SOT-}\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{rF(\mathbf{X})}[f] \\ &= \text{SOT-}\lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} F_{(\alpha)}(\mathbf{X}).\end{aligned}$$

On the other hand, since the boundary function $\widehat{F} = (\widehat{F}_1, \ldots, \widehat{F}_n)$ is a pure element in the polyball, we have

$$\boldsymbol{\mathcal{B}}_{\widehat{F}}[f] = \text{SOT-}\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{\widehat{F}}[f_r] = \text{SOT-}\lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} \widehat{F}_{(\alpha)}.$$

Now, since $\mathbf{X}$ is pure, the Berezin transform $\boldsymbol{\mathcal{B}}_{\mathbf{X}} : \mathbf{F}_{\mathbf{n}}^\infty \to B(\mathcal{H})$ is SOT-continuous on bounded sets, and it coincides with the $\mathbf{F}_{\mathbf{n}}^\infty$-functional calculus. Hence, using the calculations above and the fact that $\boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widehat{F}_{(\alpha)}] = F_{(\alpha)}(\mathbf{X})$ for any $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we deduce that

$$\begin{aligned}\boldsymbol{\mathcal{B}}_{\mathbf{X}}[\boldsymbol{\mathcal{B}}_{\widehat{F}}[f]] &= \text{SOT-}\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{\mathbf{X}}\Big[\sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} \widehat{F}_{(\alpha)}\Big] \\ &= \text{SOT-}\lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)} \widehat{F}_{(\alpha)} = \boldsymbol{\mathcal{B}}_{F(\mathbf{X})}[f]\end{aligned}$$

for any $f \in \mathbf{F}_{\mathbf{n}}^\infty$. This completes the proof. ∎

A consequence of Proposition 4.1 is the following.

COROLLARY 4.2. *If $\Psi, \Phi \in \mathrm{Aut}(\mathbf{B}_{\mathbf{n}})$, then $\boldsymbol{\mathcal{B}}_{\widehat{\Psi\circ\Phi}}[g] = (\boldsymbol{\mathcal{B}}_{\widehat{\Phi}}\boldsymbol{\mathcal{B}}_{\widehat{\Psi}})[g]$ for any g in the Cuntz–Toeplitz algebra $C^*(\mathbf{S})$, or any $g \in \mathbf{F}_{\mathbf{n}}^\infty$.*

Proof. Note that $\widehat{\Psi\circ\Phi} = (\Psi\circ\Phi)(\mathbf{S}) = \Psi(\widehat{\Phi})$. Taking $\mathbf{X} = \widehat{\Phi}$ in Proposition 4.1, the result follows. ∎

THEOREM 4.3. *Let $\mathbf{T} = \{T_{i,j}\} \in \mathbf{B}_{\mathbf{n}}(\mathcal{H})^-$ and let $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ be the universal model of the regular polyball. Then $\mathbf{T}$ is unitarily equivalent to $\mathbf{S} \otimes I_{\mathcal{K}}$, where $\mathcal{K}$ is a Hilbert space, if and only if $\dim \mathcal{D}_{\mathbf{T}} = \dim \mathcal{K}$, where $\mathcal{D}_{\mathbf{T}} = \overline{\mathbf{\Delta}_{\mathbf{T}}(I)(\mathcal{H})}$, and the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{T}}$ is a unitary operator. Moreover, in this case,*

$$T_{i,j} = \mathbf{K}_{\mathbf{T}}^*(\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_{\mathbf{T}}})\mathbf{K}_{\mathbf{T}} = \mathbf{K}_{\mathbf{T}}^*(I \otimes W)(\mathbf{S}_{i,j} \otimes I_{\mathcal{K}})(I \otimes W^*)\mathbf{K}_{\mathbf{T}}$$

for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, where $W : \mathcal{K} \to \mathcal{D}_{\mathbf{T}}$ is a unitary operator.

Proof. First, we assume that $\mathbf{T}$ is unitarily equivalent to $\mathbf{S} \otimes I_{\mathcal{K}} := \{\mathbf{S}_{i,j} \otimes I_{\mathcal{K}}\}$, i.e., there is a unitary operator $U : (\bigotimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K} \to \mathcal{H}$ such that

$$T_{i,j} = U(\mathbf{S}_{i,j} \otimes I_{\mathcal{K}})U^*, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}.$$

We show that the noncommutative Berezin kernel $\mathbf{K_T}$ satisfies the relation

$$\mathbf{K_T} = (I \otimes W)U^*,$$

where $W : \mathcal{K} \to \mathcal{D}_\mathbf{T}$ is a unitary operator. Indeed, note that we have $\mathbf{\Delta_T}(I) = U\mathbf{\Delta}_{\mathbf{S}\otimes I_\mathcal{K}}(I)U^* = U(P_{\mathbb{C}\otimes\mathcal{K}})U^*$ and $\mathbf{\Delta_T}(I)^{1/2} = U\mathbf{\Delta}_{\mathbf{S}\otimes I_\mathcal{K}}(I)^{1/2}U^*$. Consequently, we have $\dim \overline{\mathbf{\Delta_T}(I)(\mathcal{H})} = \dim \mathcal{K}$ and $U(1 \otimes \mathcal{K}) = \overline{\mathbf{\Delta_T}(I)(\mathcal{H})}$. Using the definition of the noncommutative Berezin kernel, we deduce that

$$\begin{aligned}
\mathbf{K_T}h :&= \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes \mathbf{\Delta_T}(I)^{1/2} T^*_{1,\beta_1} \cdots T^*_{k,\beta_k} h \\
&= \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes U\mathbf{\Delta}_{\mathbf{S}\otimes I_\mathcal{K}}(I)^{1/2}U^*U(\mathbf{S}^*_{1,\beta_1} \cdots \mathbf{S}^*_{k,\beta_k} \otimes I_\mathcal{K})U^*h \\
&= \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes U(P_{\mathbb{C}\otimes\mathcal{K}})(\mathbf{S}^*_{1,\beta_1} \cdots \mathbf{S}^*_{k,\beta_k} \otimes I_\mathcal{K})U^*h.
\end{aligned}$$

Consider the unitary operator $W : \mathcal{K} \to \mathcal{D}_\mathbf{T}$ defined by $Wy := U(1 \otimes y)$, $y \in \mathcal{K}$. For any vector $g = \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes y_{(\beta)}$ in $(\bigotimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}$, the computations above imply

$$\begin{aligned}
\mathbf{K_T}Ug &= \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes U(P_{\mathbb{C}\otimes\mathcal{K}})(\mathbf{S}^*_{1,\beta_1} \cdots \mathbf{S}^*_{k,\beta_k} \otimes I_\mathcal{K})g \\
&= \sum_{\beta_i\in\mathbb{F}^+_{n_i}, i=1,\dots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes Wy_{(\beta)} = (I \otimes W)g.
\end{aligned}$$

Hence, $\mathbf{K_T} = (I \otimes W)U^*$ is a unitary operator. On the other hand, we have

$$\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_\mathbf{T}} = (I \otimes W)(\mathbf{S}_{i,j} \otimes I_\mathcal{K})(I \otimes W^*)$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Due to the properties of the noncommutative Berezin kernel, we have $\mathbf{K_T}T^*_{i,j} = (\mathbf{S}^*_{i,j} \otimes I_{\mathcal{D}_\mathbf{T}})\mathbf{K_T}$. Since $\mathbf{K_T}$ is a unitary operator, we deduce that

$$T_{i,j} = \mathbf{K}^*_\mathbf{T}(I \otimes W)(\mathbf{S}_{i,j} \otimes I_\mathcal{K})(I \otimes W^*)\mathbf{K_T}.$$

Conversely, if the noncommutative Berezin kernel $\mathbf{K_T}$ is a unitary operator, then, due to the fact that $\mathbf{T}$ is a pure element in $\mathbf{B_n}(\mathcal{H})^-$ and $T_{i,j} = \mathbf{K}^*_\mathbf{T}(\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_\mathbf{T}})\mathbf{K_T}$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, we complete the proof. ∎

COROLLARY 4.4. *Let $\mathbf{T} = \{T_{i,j}\} \in \mathbf{B_n}(\mathcal{H})^-$ and let $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ be the universal model of the regular polyball. Then $\mathbf{T}$ is unitarily equivalent to $\mathbf{S}$ if and only if $\dim \mathcal{D}_\mathbf{T} = 1$ and the noncommutative Berezin kernel $\mathbf{K_T}$ is a unitary operator. Moreover, in this case, the defect space $\mathcal{D}_\mathbf{T} = \mathbb{C}v_0$ for some vector $v_0 \in \mathcal{H}$ with $\|v_0\| = 1$, and*

$$T_{i,j} = \mathbf{K}^*_\mathbf{T}(\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_\mathbf{T}})\mathbf{K_T} = \mathbf{K}^*_\mathbf{T}\widetilde{W}\mathbf{S}_{i,j}\widetilde{W}^*\mathbf{K_T}, \quad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\},$$

where $\widetilde{W}:\bigotimes_{i=1}^k F^2(H_{n_i})\to(\bigotimes_{i=1}^k F^2(H_{n_i}))\otimes\mathbb{C}v_0$ *is the unitary operator defined by*

$$\widetilde{W}g:=g\otimes v_0,\quad g\in\bigotimes_{i=1}^k F^2(H_{n_i}).$$

We denote by $\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}(C^*(\mathbf{S}))$ the group of automorphisms of the Cuntz–Toeplitz algebra $C^*(\mathbf{S})$ such that $\Gamma(\boldsymbol{\mathcal{A}}_{\mathbf{n}})=\boldsymbol{\mathcal{A}}_{\mathbf{n}}$.

THEOREM 4.5. *Any automorphism Γ of the Cuntz–Toeplitz C^*-algebra $C^*(\mathbf{S})$ which leaves invariant the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$, i.e. $\Gamma(\boldsymbol{\mathcal{A}}_{\mathbf{n}})=\boldsymbol{\mathcal{A}}_{\mathbf{n}}$, has the form*

$$\Gamma(g):=\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[g]=\mathbf{K}_{\widehat{\Psi}}[g\otimes I_{\mathcal{D}_{\widehat{\Psi}}}]\mathbf{K}_{\widehat{\Psi}}^*,\quad g\in C^*(\mathbf{S}),$$

where $\Psi\in\mathrm{Aut}(\mathbf{B_n})$ and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is the noncommutative Berezin transform at the boundary function $\widehat{\Psi}$. In this case, the noncommutative Berezin kernel $\mathbf{K}_{\widehat{\Psi}}$ is a unitary operator and Γ is a unitary implemented automorphism of $C^*(\mathbf{S})$. Moreover, we have

$$\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}(C^*(\mathbf{S}))\simeq\mathrm{Aut}(\mathbf{B_n}).$$

Proof. Let $\Gamma\in\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}(C^*(\mathbf{S}))$, i.e., Ψ is an automorphism of the Cuntz–Toeplitz algebra $C^*(\mathbf{S})$ such that $\Gamma(\boldsymbol{\mathcal{A}}_{\mathbf{n}})=\boldsymbol{\mathcal{A}}_{\mathbf{n}}$. For each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, set $\widetilde{\varphi}_{i,j}:=\Gamma(\mathbf{S}_{i,j})$. If $\widetilde{\varphi}_i:=(\widetilde{\varphi}_{i,1},\dots,\widetilde{\varphi}_{i,n_i})$, then, using the fact that Γ is a morphism of C^*-algebras, we have

$$(\mathrm{id}-\Phi_{\widetilde{\varphi}_i})(I)=\Gamma\Big(I-\sum_{j=1}^{n_i}\mathbf{S}_{i,j}\mathbf{S}_{i,j}^*\Big)\geqslant 0$$

and, similarly,

$$(\mathrm{id}-\Phi_{\widetilde{\varphi}_1})^{p_1}\circ\cdots\circ(\mathrm{id}-\Phi_{\widetilde{\varphi}_k})^{p_k}=\Gamma[(\mathrm{id}-\Phi_{\mathbf{S}_i})^{p_1}\circ\cdots\circ(\mathrm{id}-\Phi_{\mathbf{S}_i})^{p_k}(I)]\geqslant 0$$

for any $p_i\in\{0,1\}$. On the other hand, if $s,t\in\{1,\dots,k\}$, $s\neq t$, then

$$\widetilde{\varphi}_{s,j}\widetilde{\varphi}_{t,p}=\Gamma(\mathbf{S}_{s,j}\mathbf{S}_{t,p})=\Gamma(\mathbf{S}_{t,p}\mathbf{S}_{s,j})=\widetilde{\varphi}_{t,p}\widetilde{\varphi}_{s,j}$$

for any $j\in\{1,\dots,n_s\}$ and $p\in\{1,\dots,n_t\}$. Consequently, the k-tuple $\widetilde{\varphi}:=(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_k)$ is in the closed regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Now, using the noncommutative Berezin transform, we define $\varphi_{i,j}(\mathbf{X}):=\boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\varphi}_{i,j}]$ for $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})$, and remark that, due to Proposition 4.1, the mapping $\varphi:\mathbf{B_n}(\mathcal{H})\to\mathbf{B_n}(\mathcal{H})^-$ defined by $\varphi(\mathbf{X}):=(\varphi_1(\mathbf{X}),\dots,\varphi_k(\mathbf{X}))$ and $\varphi_i(\mathbf{X}):=(\varphi_{i,1}(\mathbf{X}),\dots,\varphi_{i,n_i}(\mathbf{X}))$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ which has a continuous extension to the closed polyball $\mathbf{B_n}(\mathcal{H})^-$. This extension is also denoted by φ.

Now, note that $\Gamma^{-1}(\boldsymbol{\mathcal{A}}_{\mathbf{n}})=\boldsymbol{\mathcal{A}}_{\mathbf{n}}$. For each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, let $\widetilde{\xi}_{i,j}:=\Gamma^{-1}(\mathbf{S}_{i,j})$. As in the first part of the proof, one can show that the k-tuple $\widetilde{\xi}:=(\widetilde{\xi}_1,\dots,\widetilde{\xi}_k)$, with $\widetilde{\xi}_i:=(\widetilde{\xi}_{i,1},\dots,\widetilde{\xi}_{i,n_i})$, is in the closed regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Using the noncommutative Berezin transform, we define $\xi_{i,j}(\mathbf{X}):=\boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\xi}_{i,j}]$ for $\mathbf{X}\in\mathbf{B_n}(\mathcal{H})$, and using again Proposition 4.1 we deduce that the map $\xi:\mathbf{B_n}(\mathcal{H})\to\mathbf{B_n}(\mathcal{H})^-$ defined by $\xi(\mathbf{X}):=(\xi_1(\mathbf{X}),\dots,\xi_k(\mathbf{X}))$ and

$\xi_i(\mathbf{X}) := (\xi_{i,1}(\mathbf{X}), \ldots, \xi_{i,n_i}(\mathbf{X}))$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$ which has a continuous extension to $\mathbf{B_n}(\mathcal{H})^-$, which is also denoted by ξ.

According to the results preceding Lemma 2.1, each $\widetilde{\xi}_{i,j} \in \boldsymbol{\mathcal{A}}_\mathbf{n}$ has a unique formal Fourier type representation $\sum_{(\alpha)} a_{(\alpha)}^{(i,j)} \mathbf{S}_{(\alpha)}$ such that

$$\widetilde{\xi}_{i,j} = \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \mathbf{S}_{(\alpha)},$$

where the limit is in the operator norm topology. Using the continuity of Γ in the norm topology, we deduce that

$$\begin{aligned}\mathbf{S}_{i,j} = \Gamma(\widetilde{\xi}_{i,j}) &= \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \Gamma(\mathbf{S}_{(\alpha)})\\ &= \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \widetilde{\varphi}_{1,\alpha_1}\cdots\widetilde{\varphi}_{k,\alpha_k}.\end{aligned}$$

Due to the continuity in norm of the Berezin transform $\boldsymbol{\mathcal{B}}_\mathbf{X}$, where $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, we have

$$\begin{aligned}X_{i,j} = \boldsymbol{\mathcal{B}}_\mathbf{X}[\mathbf{S}_{i,j}] &= \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \boldsymbol{\mathcal{B}}_\mathbf{X}[\widetilde{\varphi}_{1,\alpha_1}\cdots\widetilde{\varphi}_{k,\alpha_k}]\\ &= \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \varphi_{1,\alpha_1}(\mathbf{X})\cdots\varphi_{k,\alpha_k}(\mathbf{X})\\ &= \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{\varphi(\mathbf{X})}\Big[\sum_{q=0}^{\infty} \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|=q} r^q a_{(\alpha)}^{(i,j)} \mathbf{S}_{(\alpha)}\Big] = \boldsymbol{\mathcal{B}}_{\varphi(\mathbf{X})}[\widetilde{\xi}_{i,j}] = \xi_{i,j}(\varphi(\mathbf{X}))\end{aligned}$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, $i \in \{1, \ldots, k\}$, and $j \in \{1, \ldots, n_i\}$. Consequently, using the continuity in norm of φ and ξ on the closed polyball $\mathbf{B_n}(\mathcal{H})^-$, we deduce that $(\xi \circ \varphi)(\mathbf{X}) = \mathbf{X}$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$. Similarly, one can prove that $(\varphi \circ \xi)(\mathbf{X}) = \mathbf{X}$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$. Therefore, $\varphi : \mathbf{B_n}(\mathcal{H})^- \to \mathbf{B_n}(\mathcal{H})^-$ is a homeomorphism such that φ and $\varphi^{-1} = \xi$ are free holomorphic functions on $\mathbf{B_n}(\mathcal{H})$.

The next step is to prove that $\varphi(\mathbf{X}) \in \mathbf{B_n}(\mathcal{H})$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. Indeed, due to Corollary 1.7, the scalar representations of φ and ξ are holomorphic functions on $\mathbf{B_n}(\mathbb{C})$ with values in the closed polyball $\mathbf{B_n}(\mathbb{C})^-$. Applying the open mapping theorem from complex analysis to the scalar representations of φ and ξ, we deduce that $\varphi(\mathbf{B_n}(\mathbb{C})) = \mathbf{B_n}(\mathbb{C})$ and $\xi(\mathbf{B_n}(\mathbb{C})) = \mathbf{B_n}(\mathbb{C})$. In particular, for each $i \in \{1, \ldots, k\}$, $\varphi_i : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^{n_i}$ is a free holomorphic function with the properties: $\|\varphi_i\|_\infty = 1$ and $\|\varphi_i(0)\| < 1$. Applying the maximum principle of Theorem 2.10, we conclude that $\|\varphi_i(\mathbf{X})\| < 1$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. Hence, and using Proposition 1.3 from [24], we deduce that $\varphi(\mathbf{X}) \in \mathbf{B_n}(\mathcal{H})$, which proves our assertion. Similarly, one proves that $\xi(\mathbf{X}) \in \mathbf{B_n}(\mathcal{H})$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. Therefore, $\varphi \in \mathrm{Aut}(\mathbf{B_n})$.

Now, we apply Theorem 3.9 and deduce that rank $\boldsymbol{\Delta}_{\widetilde{\varphi}} = 1$ and $\widetilde{\varphi}$ is unitarily equivalent to the universal model $\mathbf{S}$. Combining this with Theorem 4.3 and Corollary 4.4, we deduce that the noncommutative Berezin transform $\mathbf{K}_{\widetilde{\varphi}}$ is a unitary operator and

$$\Gamma(\mathbf{S}_{i,j}) = \widetilde{\varphi}_{i,j} = \mathbf{K}_{\widetilde{\varphi}}^*(\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_{\widetilde{\varphi}}})\mathbf{K}_{\widetilde{\varphi}} = \mathbf{K}_{\widetilde{\varphi}}^* \widetilde{W} \mathbf{S}_{i,j} \widetilde{W}^* \mathbf{K}_{\widetilde{\varphi}},$$

where $\widetilde{W} : \bigotimes_{i=1}^{k} F^2(H_{n_i}) \to (\bigotimes_{i=1}^k F^2(H_{n_i})) \otimes \mathbb{C}v_0$ is the unitary operator defined by

$$\widetilde{W}g := g \otimes v_0, \quad g \in \bigotimes_{i=1}^{k} F^2(H_{n_i}),$$

where $\mathcal{D}_{\widetilde{\varphi}} = \mathbb{C}v_0$ for some vector $v_0 \in \bigotimes_{i=1}^{k} F^2(H_{n_i})$ with $\|v_0\| = 1$. Hence, we also have

$$\Gamma(g) = \mathbf{K}_{\widetilde{\varphi}}^*(g \otimes I_{\mathcal{D}_{\widetilde{\varphi}}})\mathbf{K}_{\widetilde{\varphi}}, \quad g \in C^*(\mathbf{S}).$$

Conversely, assume that $\Gamma : C^*(\mathbf{S}) \to C^*(\mathbf{S})$ is defined by

$$\Gamma(g) := \boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[g] := \mathbf{K}_{\widehat{\Psi}}[g \otimes I_{\mathcal{D}_{\widehat{\Psi}}}]\mathbf{K}_{\widehat{\Psi}}^*, \quad g \in C^*(\mathbf{S}), \tag{4.1}$$

where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$ and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is the Berezin transform at the boundary function $\widehat{\Psi}$. As above, due to Theorem 3.9, Theorem 4.3, and Corollary 4.4, the noncommutative Berezin transform $\mathbf{K}_{\widehat{\Psi}}$ is a unitary operator and Γ is a unitarily implemented automorphism of $C^*(\mathbf{S})$.

Now, note that each $\Gamma \in \mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S}))$ corresponds to a unique $\Psi \in \mathrm{Aut}(\mathbf{B_n})$ such that relation (4.1) holds. Indeed, if $\Psi_1, \Psi_2 \in \mathrm{Aut}(\mathbf{B_n})$ and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}_1} = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}_2}$, then $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}_1}[\mathbf{S}_{i,j}] = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}_2}[\mathbf{S}_{i,j}]$, which shows that $(\widehat{\Psi}_1)_{i,j} = (\widehat{\Psi}_2)_{i,j}$. Applying the Berezin transform at $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, we obtain $(\Psi_1)_{i,j}(\mathbf{X}) = (\widehat{\Psi}_2)_{i,j}(\mathbf{X})$, which implies $\Psi_1 = \Psi_2$.

Define $\Lambda : \mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S})) \to \mathrm{Aut}(\mathbf{B_n})$ by setting $\Lambda(\Gamma) = \Psi$. As we have seen above, Λ is a bijection. Let $\Gamma_1, \Gamma_2 \in \mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S}))$ and $\Psi_1, \Psi_2 \in \mathrm{Aut}(\mathbf{B_n})$ be such that $\Lambda(\Gamma_j) = \Psi_j$, $j = 1, 2$. Using Proposition 4.1 and Corollary 4.2, we deduce that

$$\Gamma_1[\Gamma_2(g))] = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}_1}[\Gamma_2(g)] = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}_1}[\boldsymbol{\mathcal{B}}_{\widehat{\Psi}_2}[g]] = \boldsymbol{\mathcal{B}}_{\Psi_2(\widehat{\Psi})}[g] = \boldsymbol{\mathcal{B}}_{\widehat{\Psi_2 \circ \Psi_1}}[g] = \Lambda^{-1}(\Psi_2 \circ \Psi_1)(g)$$

for any $g \in C^*(\mathbf{S})$. Hence, we obtain $\Lambda(\Gamma_1\Gamma_2) = \Psi_2 \circ \Psi_1 = \Lambda(\Gamma_2) \circ \Lambda(\Gamma_1)$. The proof is complete. ∎

We remark that, in the particular case when $k = 1$, the result of Theorem 4.5 is contained in Theorem 3.4 from [19], which extends one of Voiculescu's results from [30]. In [24], we proved that the C^*-algebra $C^*(\mathbf{S})$ is irreducible and contains the compact operators in $B(\bigotimes_{i=1}^k F^2(H_{n_i}))$. Standard results in representation theory of C^*-algebras (see e.g. [1]), imply that any automorphism of $C^*(\mathbf{S})$ is a unitarily implemented automorphism. Having this result at hand, we remark that an alternative proof of the fact that $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ in Theorem 4.5 can be obtained using some ideas from the proof of Theorem 5.5 and avoiding the use of the open mapping theorem from complex analysis.

The Cuntz–Toeplitz algebra $\mathcal{T}_n$ is the unique unital C^*-algebra generated by $n \in \mathbb{N}$ isometries $s_1, \dots, s_n$ satisfying relations $s_i^* s_j = \delta_{ij} 1$ and $s_1 s_1^* + \cdots + s_n s_n^* < 1$. The noncommutative disc algebra $\mathcal{A}_n$ (see [15], [16]) is the unique non-self-adjoint closed algebra generated by $s_1, \dots, s_n$ and the identity. We also recall [5] that the Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C^*-algebra generated by $n \geqslant 2$ isometries satisfying relations $\sigma_i^* \sigma_j = \delta_{ij} 1$ and $\sigma_1 \sigma_1^* + \cdots + \sigma_n \sigma_n^* = 1$. In [5], Cuntz showed that if $\boldsymbol{\mathcal{K}} \subset \mathcal{T}_n$ denotes the algebra of compact operators, then

$$0 \to \boldsymbol{\mathcal{K}} \to \mathcal{T}_n \to \mathcal{O}_n \to 0$$

is a short exact sequence of C^*-algebras. Since the Cuntz algebra $\mathcal{O}_n$ and the algebra of compact operators $\boldsymbol{\mathcal{K}}$ are nuclear, so is the Cuntz–Toeplitz algebra $\mathcal{T}_n$. This implies that the tensor products of C^*-algebras $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ and $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ have a unique C^*-norm. The C^*-algebra $C^*(\mathbf{S})$ generated by the universal model $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ is $*$-isomorphic to $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ (see [17]). According to the definition of the min norm on tensor products of operator algebras ([12]) and since $\mathcal{A}_{n_i}$ can be seen as a subalgebra of $\mathcal{T}_{n_i}$ (see [16]), we also have that $\boldsymbol{\mathcal{A}}_{\mathbf{n}} \simeq \mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$.

Using the short exact sequence obtained by Cuntz [5], one can deduce that there is a a surjective $*$-representation $\chi : C^*(\mathbf{S}) \to \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ such that $\chi(\mathbf{S}_{i,j}) = \boldsymbol{\sigma}_{i,j}$, where

$$\boldsymbol{\sigma}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes \sigma_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, where $\{\sigma_{i,j}\}_{j=1}^{n_i}$ is a set of generators of the Cuntz algebra $\mathcal{O}_{n_i}$. We also remark (see [16]) that the closed non-seladjoint algebra $\mathrm{Alg}(1, \sigma_i)$ generated by $\{\sigma_{i,j}\}_{j=1}^{n_i}$ and the identity is completely isometric isomorphic to the noncommutative disc algebra $\mathcal{A}_{n_i}$. Consequently, one can see $\boldsymbol{\mathcal{A}}_{\mathbf{n}} \simeq \mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$ as a subalgebra of $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$.

COROLLARY 4.6. *Let* $\mathbf{n} = (n_1, \dots n_k) \in \mathbb{N}^k$. *Each holomorphic automorphism of the regular polyball* $\mathbf{B_n}$ *induces an automorphism of the* C^**-algebra* $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ *which leaves invariant the non-self-adjoint subalgebra* $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$.

5. AUTOMORPHISMS OF THE POLYBALL ALGEBRA $\mathcal{A}(\mathbf{B_n})$ AND THE HARDY ALGEBRA $H^\infty(\mathbf{B_n})$

In this section, we determine the group of unitarily implemented automorphisms of the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$ and Hardy algebra $\mathbf{F}_{\mathbf{n}}^\infty$ and show that they are isomorphic to the group $\mathrm{Aut}(\mathbf{B_n})$. We also present the corresponding results for the Hardy algebra of all bounded free holomorphic functions on the regular polyball $H^\infty(\mathbf{B_n})$ and the polyball algebra $\mathcal{A}(\mathbf{B_n})$.

PROPOSITION 5.1. *Let $f : \mathbf{B_m}(\mathcal{H}) \to B(\mathcal{H})$ and $g : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H})$ be free holomorphic functions. Then the following statements hold:*

(i) *If f and g have continuous extensions to the closed polyballs $\mathbf{B_m}(\mathcal{H})^-$ and $\mathbf{B_n}(\mathcal{H})^-$, respectively, then $f \circ g \in A(\mathbf{B_n})$.*

(ii) *If $f \in H^\infty(\mathbf{B_m})$ then $f \circ g \in H^\infty(\mathbf{B_n})$ and $\|f \circ g\|_\infty \leqslant \|f\|_\infty$.*

(iii) *If $f \in H^\infty(\mathbf{B_m})$ and $\widehat{g} = (\widehat{g}_1, \dots, \widehat{g}_m)$ is a pure element of the polyball $\mathbf{B_m}(\bigotimes_{i=1}^k F^2(H_{n_i}))$ with entries $\widehat{g}_j \in \boldsymbol{\mathcal{A}}_\mathbf{n}$, then $(f \circ g)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\boldsymbol{\mathcal{B}}_{\widehat{g}}[\widehat{f}]]$ for $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$.*

Proof. Using Theorem 2.4, part (i) and (ii) are obvious. Since $\operatorname{range} g \subset \mathbf{B_m}(\mathcal{H})$, Proposition 2.2 implies $g(r\mathbf{S}) \in \mathbf{B_m}(\bigotimes_{i=1}^k F^2(H_{n_i}))$, $r \in [0, 1)$, where $\mathbf{S}$ is the universal model of $\mathbf{B_n}$. Since $f \circ g \in H^\infty(\mathbf{B_n})$ its boundary function $\widehat{f \circ g}$ exists and

$$\widehat{f \circ g} = \text{SOT-}\lim_{r \to 1} f(g(r\mathbf{S})). \tag{5.1}$$

According to the second part of Proposition 4.1, we have

$$f(g(r\mathbf{S})) = \boldsymbol{\mathcal{B}}_{g(r\mathbf{S})}[\widehat{f}] = \boldsymbol{\mathcal{B}}_{r\mathbf{S}}[\boldsymbol{\mathcal{B}}_{\widehat{g}}[\widehat{f}]]. \tag{5.2}$$

Due to Theorem 3.5 and Lemma 3.3 from [24], if $\Psi \in \mathbf{F}_\mathbf{n}^\infty$, then we have $\psi = \text{SOT-}\lim_{r \to 1} \boldsymbol{\mathcal{B}}_{r\mathbf{S}}[\psi]$. Applying this result to $\boldsymbol{\mathcal{B}}_{\widehat{g}}[\widehat{f}] \in \mathbf{F}_\mathbf{n}^\infty$ and using relations (5.1) and (5.2), we deduce that $\widehat{f \circ g} = \boldsymbol{\mathcal{B}}_{\widehat{g}}[\widehat{f}]$. Since $(f \circ g)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\widehat{f \circ g}]$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, we complete the proof. ∎

COROLLARY 5.2. *Let $f \in \mathrm{Hol}(\mathbf{B_n})$ and $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Then the following statements hold:*

(i) *$f \circ \Psi \in A(\mathbf{B_n})$ for any $f \in A(\mathbf{B_n})$.*

(ii) *$f \circ \Psi \in H^\infty(\mathbf{B_n})$ for any $f \in H^\infty(\mathbf{B_n})$.*

(iii) *If $f \in H^\infty(\mathbf{B_n})$, then $\|f \circ \Psi\|_\infty = \|f\|_\infty$ and*

$$(f \circ \Psi)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[\widehat{f}]], \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

We remark that there are operator-valued coefficient versions of the previous two results and the proofs are similar.

THEOREM 5.3. *Any unitarily implemented automorphism of the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ is the Berezin transform $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$ of a boundary function $\widehat{\Psi}$, where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Moreover, we have*

$$\mathrm{Aut_u}(\boldsymbol{\mathcal{A}}_\mathbf{n}) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

Proof. First, assume that $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Due to Theorem 4.5, the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is a unitarily implemented automorphism of the Cuntz–Toeplitz algebra $C^*(\mathbf{S})$ such that $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}(\boldsymbol{\mathcal{A}}_\mathbf{n}) = \boldsymbol{\mathcal{A}}_\mathbf{n}$. Consequently, $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$

is a unitarily implemented automorphism of the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$.

Now, we assume that Γ is a unitarily implemented automorphism of $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$, i.e., there exists a unitary operator $U \in B(\bigotimes_{i=1}^k F^2(H_{n_i}))$ such that $\Gamma(Y) = U^*YU$ for any $Y \in \boldsymbol{\mathcal{A}}_{\mathbf{n}}$. As in the proof of Theorem 4.5, we deduce that there is $\Psi \in \mathrm{Aut}(\mathbf{B_n})$ such that $\Gamma = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}}|_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}$ and $\mathrm{Aut}(\mathbf{B_n}) \simeq \mathrm{Aut}_{\mathrm{u}}(\boldsymbol{\mathcal{A}}_{\mathbf{n}})$. The proof is complete. ∎

We remark that Theorem 4.5 and Theorem 5.3 reveal that each unitarily implemented automorphism of $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$ has a unique extension to an automorphism of the C^*-algebra $C^*(\mathbf{S})$. Moreover, the mappings $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}|_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}} \mapsto \boldsymbol{\mathcal{B}}_{\widehat{\Psi}} \mapsto \Psi$ are group isomorphisms, showing that

$$\mathrm{Aut}_{\mathrm{u}}(\boldsymbol{\mathcal{A}}_{\mathbf{n}}) \simeq \mathrm{Aut}_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}(C^*(\mathbf{S})) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

If $\Lambda : A(\mathbf{B_n}) \to A(\mathbf{B_n})$ is an algebraic homomorphism, it induces a unique homomorphism $\widetilde{\Lambda} : \boldsymbol{\mathcal{A}}_n \to \boldsymbol{\mathcal{A}}_n$ such that $\Lambda\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}\widetilde{\Lambda}$. The homomorphisms Λ and $\widetilde{\Lambda}$ uniquely determine each other by the formulas:

$$(\Lambda f)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\Lambda}(\widehat{f})], \quad f \in A(\mathbf{B_n}),\ \mathbf{X} \in \mathbf{B_n}(\mathcal{H}), \quad \text{and} \quad \widetilde{\Lambda}(\widehat{f}) = \widehat{\Lambda(f)}, \quad \widehat{f} \in \boldsymbol{\mathcal{A}}_{\mathbf{n}}.$$

We say that a unital completely contractive homomorphism $\widetilde{\Lambda} : \boldsymbol{\mathcal{A}}_{\mathbf{n}} \to \boldsymbol{\mathcal{A}}_{\mathbf{n}}$ has a completely contractive hereditary linear extension to $C^*(\mathbf{S})$ if the linear maps defined by

$$\begin{aligned} &\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^* \mapsto \widetilde{\Lambda}(\mathbf{S}_{(\alpha)})\widetilde{\Lambda}(\mathbf{S}_{(\beta)})^*, \quad (\alpha),(\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, \quad \text{and} \\ &\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^* \mapsto \widetilde{\Lambda}^{-1}(\mathbf{S}_{(\alpha)})\widetilde{\Lambda}^{-1}(\mathbf{S}_{(\beta)})^*, \quad (\alpha),(\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, \end{aligned}$$

are completely contractive.

THEOREM 5.4. *Let $\Lambda : A(\mathbf{B_n}) \to A(\mathbf{B_n})$ be a unital algebraic automorphism. Then the following statements are equivalent:*

(i) *$\widetilde{\Lambda}$ is a unitarily implemented automorphism of $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$.*

(ii) *There is $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ such that*

$$\Lambda(f) = f \circ \varphi, \quad f \in A(\mathbf{B_n}).$$

(iii) *$\widetilde{\Lambda}$ is a completely contractive automorphism of $\boldsymbol{\mathcal{A}}_{\mathbf{n}}$ with completely contractive hereditary linear extension to $C^*(\mathbf{S})$.*

(iv) *$\widetilde{\Lambda}$ is continuous and $\{\widetilde{\Lambda}(\mathbf{S}_{i,j})\}$ and $\{\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ are in the regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$, where $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ is the universal model of the regular polyball $\mathbf{B_n}$.*

Proof. Assume that (i) holds. According to Theorem 5.3, there is $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ such that $\widetilde{\Lambda} = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}|_{\boldsymbol{\mathcal{A}}_{\mathbf{n}}}$. Consequently, using Proposition 4.1 we obtain

$$\Lambda(f)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\Lambda}(\widehat{f})] = \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\boldsymbol{\mathcal{B}}_{\widehat{\varphi}}[\widehat{f}]] = \boldsymbol{\mathcal{B}}_{\varphi(\mathbf{X})}[\widehat{f}] = f(\varphi(\mathbf{X})) = (f \circ \varphi)(\mathbf{X})$$

for any $f \in A(\mathbf{B_n})$, therefore item (ii) holds. Now, we prove that (ii) $\Longrightarrow$ (iii). Note that we have

$$\widetilde{\Lambda}(\widehat{f}) = \widehat{\Lambda(f)} = \widehat{f \circ \varphi} = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}[\widehat{f}]$$

for any $f \in A(\mathbf{B_n})$. Hence $\widetilde{\Lambda} = \boldsymbol{\mathcal{B}}_{\widehat{\Psi}}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$, which is a completely contractive automorphism and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is a completely contractive hereditary linear extension to $C^*(\mathbf{S})$ (see Theorem 4.5). Let us prove that (iii) $\Longrightarrow$ (iv). Assume that (iii) holds. For each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, set $\widehat{\varphi}_{i,j} := \widetilde{\Lambda}(\mathbf{S}_{i,j}) \in \boldsymbol{\mathcal{A}}_\mathbf{n}$. We need to show that $\widehat{\varphi} := (\widehat{\varphi}_1, \dots, \widehat{\varphi}_k)$, with $\widehat{\varphi}_{i,1}, \dots, \widehat{\varphi}_{i,n_i})$, is in the noncommutative polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i})^-$. Since $\Phi_{\mathbf{S}_i}(I) \leqslant I$ and $\widetilde{\Lambda}$ is completely contractive, we deduce that $\Phi_{\widehat{\varphi}_i}(I) \leqslant I$ for $i \in \{1, \dots, k\}$. Let $1 \leqslant p \leqslant k$ and $i_1 < \cdots < i_p$ with $i_1, \dots, i_p \in \{1, \dots, k\}$. We have

$$\begin{aligned} 0 &\leqslant (\mathrm{id} - \Phi_{\mathbf{S}_{i_1}}) \circ \cdots \circ (\mathrm{id} - \Phi_{\mathbf{S}_{i_p}})(I) \\ &= I - \sum_{q_j \in \{0,1\},\, q_1 + \cdots + q_p > 0} (-1)^{q_1 + \cdots + q_k + 1} \Phi_{\mathbf{S}_{i_1}} \cdots \Phi_{\mathbf{S}_{i_p}}, \end{aligned}$$

which is equivalent to

$$\left\| \sum_{q_j \in \{0,1\},\, q_1 + \cdots + q_p > 0} (-1)^{q_1 + \cdots + q_k + 1} \Phi_{\mathbf{S}_{i_1}} \cdots \Phi_{\mathbf{S}_{i_p}} \right\| \leqslant 1.$$

Since $\widetilde{\Lambda}$ has completely contractive hereditary linear extension, we deduce that

$$\left\| \sum_{q_j \in \{0,1\},\, q_1 + \cdots + q_p > 0} (-1)^{q_1 + \cdots + q_k + 1} \Phi_{\widehat{\varphi}_{i_1}} \cdots \Phi_{\widehat{\varphi}_{i_p}} \right\| \leqslant 1.$$

Taking into account that the operator under the norm is self-adjoint, we deduce that

$$\sum_{q_j \in \{0,1\},\, q_1 + \cdots + q_p > 0} (-1)^{q_1 + \cdots + q_k + 1} \Phi_{\widehat{\varphi}_{i_1}} \cdots \Phi_{\widehat{\varphi}_{i_p}} \leqslant I,$$

which is equivalent to

$$(\mathrm{id} - \Phi_{\widehat{\varphi}_{i_1}}) \circ \cdots \circ (\mathrm{id} - \Phi_{\widehat{\varphi}_{i_p}})(I) \geqslant 0.$$

This shows that $\widehat{\varphi} := (\widehat{\varphi}_1, \dots, \widehat{\varphi}_k)$ is in the noncommutative regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i})^-$. Similarly, we can show that $\{\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ is in the regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Therefore, item (iv) holds.

It remains to prove that (iv) $\Longrightarrow$ (i). Assume that $\widetilde{\Lambda}(\mathbf{S}) := \{\widetilde{\Lambda}(\mathbf{S}_{i,j})\} \in \mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Due to the noncommutative von Neumann type inequality [24], we have

$$\|[\widetilde{\Lambda}(p_{i,j}(\mathbf{S}))]_{m\times m}\| = \|[p_{i,j}(\widetilde{\Lambda}(\mathbf{S}))]_{m\times m}\| \leqslant \|[p_{i,j}(\mathbf{S})]_{m\times m}\|$$

for any operator matrix $[p_{i,j}(\mathbf{S})]_{m\times m} \in \boldsymbol{\mathcal{A}}_\mathbf{n} \otimes M_{m\times m}(\mathbb{C})$. Since $\widetilde{\Lambda}$ is continuous and $\boldsymbol{\mathcal{A}}_\mathbf{n}$ is the norm closed self-adjoint algebra generated by $\{\mathbf{S}_{i,j}\}$ and the identity, we deduce that $\widetilde{\Lambda} : \boldsymbol{\mathcal{A}}_\mathbf{n} \to \boldsymbol{\mathcal{A}}_\mathbf{n}$ is a completely contractive homomorphism.

Similarly, using the fact that $\{\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ is in the polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$, one can prove that $\widetilde{\Lambda}^{-1}:\boldsymbol{\mathcal{A}}_\mathbf{n}\to\boldsymbol{\mathcal{A}}_\mathbf{n}$ is also a completely contractive homomorphism. Now, as in the proof of Theorem 4.5, one can show that $\widetilde{\Lambda}$ is a unitarily implemented automorphism of $\boldsymbol{\mathcal{A}}_\mathbf{n}$. This completes the proof. ∎

We remark that if $\Lambda: A(\mathbf{B_n})\to A(\mathbf{B_n})$ is a unital algebraic homomorphism and at least one of $n_1,\dots,n_k$ is greater than or equal 2, then $\widetilde{\Lambda}$ is automatically continuous. Indeed, assume that there is $i_0\in\{1,\dots,k\}$ such that $n_{i_0}\geqslant 2$ and $\widetilde{\Lambda}$ is not continuous in the operator norm. Then there is a sequence $\{g_p\}_{p=1}^\infty$ of elements in the polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ such that $\widetilde{\Lambda}(g_p)\geqslant p$ and $\|g_p\|\leqslant\frac{1}{M^{p+2}}$ for any $p\in\mathbb{N}$, for some constant $M>1$ with $M>\|\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\|$ for any $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$. Note that $g:=\sum_{p=1}^\infty\widetilde{\Lambda}^{-1}(\mathbf{S}_{i_0,1})^p\widetilde{\Lambda}^{-1}(\mathbf{S}_{i_0,2})g_p$ is convergent in norm and, consequently, it is in the polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$. For each $q\in\mathbb{N}$, we have $\widetilde{\Lambda}(g)=\sum_{p=1}^q\mathbf{S}_{i_0,1}^p\mathbf{S}_{i_0,2}\widetilde{\Lambda}(g_p)+\mathbf{S}_{i_0,1}^{q+1}\widetilde{\Lambda}(\xi_q)$ for some $\xi_q\in\boldsymbol{\mathcal{A}}_\mathbf{n}$. Since $\mathbf{S}_{i_0,1}$ and $\mathbf{S}_{i_0,2}$ are isometries with orthogonal ranges, we have $\mathbf{S}_{i_0,2}^*(\mathbf{S}_{i_0,1}^*)^q\widetilde{\Lambda}(g)=\widetilde{\Lambda}(g_q)$ and, consequently, $\|\widetilde{\Lambda}(g)\|\geqslant\|\widetilde{\Lambda}(g_q)\|\geqslant q$ for $q\in\mathbb{N}$, which is a contradiction. Therefore, $\widetilde{\Lambda}$ is continuous.

THEOREM 5.5. *Any unitarily implemented automorphism of the noncommutative Hardy algebra $\mathbf{F_n^\infty}$ is the Berezin transform $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ of a boundary function $\widehat{\Psi}$, where $\Psi\in\mathrm{Aut}(\mathbf{B_n})$. Moreover, we have*

$$\mathrm{Aut_u}(\mathbf{F_n^\infty})\simeq\mathrm{Aut}(\mathbf{B_n}).$$

Proof. Let $\boldsymbol{\Psi}=(\Psi_1,\dots,\Psi_k)\in\mathrm{Aut}(\mathbf{B_n})$. According to Theorem 3.9, each $\widehat{\Psi}_i=(\widehat{\Psi}_{i,1},\dots,\widehat{\Psi}_{i,n_i})$ is a pure row isometry with entries in the noncommutative disk algebra generated by $\mathbf{S}_{i,1},\dots,\mathbf{S}_{i,n_i}$ and the identity. Consider the Berezin transform $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}:\mathbf{F_n^\infty}\to B(F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k}))$ defined by

$$\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[f]:=\mathbf{K}_{\widehat{\Psi}}[f\otimes I_{\mathcal{D}_{\widehat{\Psi}}}]\mathbf{K}_{\widehat{\Psi}}^*,\quad f\in\mathbf{F_n^\infty}.$$

Due to Theorem 3.9 and Corollary 4.4, the noncommutative Berezin kernel $\mathbf{K}_{\widehat{\Psi}}$ is a unitary operator. We recall that if $f\in\mathbf{F_n^\infty}$, then $f_r\in\boldsymbol{\mathcal{A}}$, $\|f_r\|\leqslant\|f\|$ and SOT-$\lim_{r\to1}f_r=f$. Since $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[\mathbf{S}_{(\alpha)}]=\widehat{\Psi}_{(\alpha)}$ is in $\boldsymbol{\mathcal{A}}_\mathbf{n}$ for any $(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, and $\mathbf{F_n^\infty}$ is the WOT-closed non-selfadjoint algebra generate by the operators $\{\mathbf{S}_{(\alpha)}\}_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}$, we deduce that $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}[\mathbf{F_n^\infty}]\subseteq\mathbf{F_n^\infty}$. On the other hand, $\boldsymbol{\mathcal{B}}_{\widehat{\Psi^{-1}}}$ has similar properties and, due to Proposition 4.1, we have $(\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}\boldsymbol{\mathcal{B}}_{\widehat{\Psi^{-1}}})[f]=f$ for any $f\in\mathbf{F_n^\infty}$. Therefore $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}(\mathbf{F_n^\infty})=\mathbf{F_n^\infty}$ and $\boldsymbol{\mathcal{B}}_{\widehat{\Psi}}$ is a unitarily implemented automorphism of $\mathbf{F_n^\infty}$.

Now, we assume that Γ is a unitarily implemented automorphism of $\mathbf{F_n^\infty}$, i.e., there exists a unitary operator $U\in B(\otimes_{i=1}^k F^2(H_{n_i}))$ such that $\Gamma(Y)=U^*YU$

for any $Y \in \mathbf{F}_{\mathbf{n}}^\infty$. For each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, set $\widetilde{\varphi}_{i,j} := \Gamma(\mathbf{S}_{i,j}) \in \mathbf{F}_{\mathbf{n}}^\infty$. Since

$$(\mathrm{id} - \Phi_{\widetilde{\varphi}_1})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\widetilde{\varphi}_k})^{p_k} = U^*[(\mathrm{id} - \Phi_{\mathbf{S}_i})^{p_1} \circ \cdots \circ (\mathrm{id} - \Phi_{\mathbf{S}_i})^{p_k}(I)]U \geqslant 0$$

for any $p_i \in \{0, 1\}$, and

$$\widetilde{\varphi}_{s,j}\widetilde{\varphi}_{t,p} = U^*(\mathbf{S}_{s,j}\mathbf{S}_{t,p})U = U(\mathbf{S}_{t,p}\mathbf{S}_{s,j})U = \widetilde{\varphi}_{t,p}\widetilde{\varphi}_{s,j}$$

for $s, t \in \{1, \dots, k\}$, $s \neq t$, and any $j \in \{1, \dots, n_s\}$, $p \in \{1, \dots, n_t\}$, we deduce that the k-tuple $\widetilde{\varphi} := (\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_k)$ is in the closed regular polyball $\mathbf{B}_{\mathbf{n}}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. On the other hand, each n_i-tuple $\widetilde{\varphi}_i := (\widetilde{\varphi}_{i,1}, \dots, \widetilde{\varphi}_{i,n_i})$ is a row isometry with entries in the Hardy algebra $\mathbf{F}_{\mathbf{n}}^\infty$, and

$$\Phi_{\widetilde{\varphi}_i}^p(I) = \sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \widetilde{\varphi}_{i,\alpha}\widetilde{\varphi}_{i,\alpha}^* = U^*\Big(\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^*\Big)U$$

for any $p \in \mathbb{N}$. Consequently, $\Phi_{\widetilde{\varphi}_i}^p(I) \to 0$ strongly as $p \to \infty$. Setting

$$\varphi_{i,j}(\mathbf{X}) := \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\varphi}_{i,j}], \quad \mathbf{X} \in \mathbf{B}_{\mathbf{n}}(\mathcal{H}),$$

we deduce that the map φ defined by $\varphi(\mathbf{X}) := (\varphi_1(\mathbf{X}), \dots, \varphi_k(\mathbf{X}))$ is a free holomorphic function on $\mathbf{B}_{\mathbf{n}}(\mathcal{H})$ with values in $\mathbf{B}_{\mathbf{n}}(\mathcal{H})^-$. If $\mathbf{X} \in \mathbf{B}_{\mathbf{n}}(\mathcal{H})$, we can use the Berezin transform at $\mathbf{X}$ and obtain

$$\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \varphi_{i,\alpha}(\mathbf{X})\varphi_{i,\alpha}(\mathbf{X})^* = \mathbf{K}_{\mathbf{X}}^*\Big(\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \widetilde{\varphi}_{i,\alpha}\widetilde{\varphi}_{i,\alpha}^* \otimes I_{\mathcal{H}}\Big)\mathbf{K}_{\mathbf{X}}.$$

Since $\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \widetilde{\varphi}_{i,\alpha}\widetilde{\varphi}_{i,\alpha}^* \leqslant I$ for any $p \in \mathbb{N}$ and $\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \widetilde{\varphi}_{i,\alpha}\widetilde{\varphi}_{i,\alpha}^* \to 0$ strongly as $p \to \infty$, we deduce that $\sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \varphi_{i,\alpha}(\mathbf{X})\varphi_{i,\alpha}(\mathbf{X})^* \to 0$ strongly as $p \to \infty$. Therefore, each $\varphi_i(\mathbf{X})$ is a pure row contraction for any $\mathbf{X} \in \mathbf{B}_{\mathbf{n}}(\mathcal{H})$. In particular, $\varphi_i(0) = \lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,n_i}) \in (\mathbb{C}^{n_i})_1^-$. Hence, we deduce that

$$\Big(\sum_{j=1}^{n_i} |\lambda_{i,j}|^2\Big)^p = \sum_{\alpha \in \mathbb{F}_{n_i}^+, |\alpha| = p} \varphi_{i,\alpha}(0)\varphi_{i,\alpha}(0)^* \to 0, \quad \text{as } p \to \infty.$$

This implies $\|\lambda_i\|_2 < 1$ and $\varphi(0) = (\varphi_1(0), \dots, \varphi_k(0)) \in \mathbf{B}_{\mathbf{n}}(\mathbb{C})$. Therefore, for each $i \in \{1, \dots, k\}$, $\varphi_i : \mathbf{B}_{\mathbf{n}}(\mathcal{H}) \to B(\mathcal{H})^{n_i}$ is a free holomorphic function with the properties: $\|\varphi_i\|_\infty = 1$ and $\|\varphi_i(0)\| < 1$. Applying Theorem 2.10, we conclude that $\|\varphi_i(\mathbf{X})\| < 1$ for any $\mathbf{X} \in \mathbf{B}_{\mathbf{n}}(\mathcal{H})$. Hence, and using Proposition 1.3 from [24], we deduce that $\varphi(\mathbf{X}) \in \mathbf{B}_{\mathbf{n}}(\mathcal{H})$.

Now, note that $\Gamma^{-1}(Y) = UYU^*$ for any $Y \in \mathbf{F}_{\mathbf{n}}^\infty$. For each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, let $\widetilde{\xi}_{i,j} := \Gamma^{-1}(\mathbf{S}_{i,j}) \in \mathbf{F}_{\mathbf{n}}^\infty$. As in the first part of the proof, one can show that the k-tuple $\widetilde{\xi} := (\widetilde{\xi}_1, \dots, \widetilde{\xi}_k)$, with $\widetilde{\xi}_i := (\widetilde{\xi}_{i,1}, \dots, \widetilde{\xi}_{i,n_i})$, is in the closed regular polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$. Using the noncommutative Berezin transform, we define

$$\xi_{i,j}(\mathbf{X}) := \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\xi}_{i,j}], \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

and using again Proposition 4.1 we deduce that the map ξ defined by $\xi(\mathbf{X}) := (\xi_1(\mathbf{X}), \dots, \xi_k(\mathbf{X}))$ and $\xi_i(\mathbf{X}) := (\xi_{i,1}(\mathbf{X}), \dots, \xi_{i,n_i}(\mathbf{X}))$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$. As above, one can prove that $\xi(\mathbf{X}) \in \mathbf{B_n}(\mathcal{H})$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$. As in the proof of Theorem 4.5, we have $(\xi \circ \varphi)(\mathbf{X}) = (\varphi \circ \xi)(\mathbf{X}) = \mathbf{X}$ for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, which shows $\varphi \in \mathrm{Aut}(\mathbf{B_n})$. Moreover, one can show that $\Gamma|_{\boldsymbol{\mathcal{A}}_\mathbf{n}} = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$. Since $\boldsymbol{\mathcal{A}}_\mathbf{n}$ is w^*-dense in $\mathbf{F}_\mathbf{n}^\infty$ and Γ and $\boldsymbol{\mathcal{B}}_{\widehat{\varphi}}$ are unitarily implemented (therefore w^*-continuous), we deduce that $\Gamma = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}$. The fact that $\mathrm{Aut}(\mathbf{B_n}) \simeq \mathrm{Aut_u}(\mathbf{F}_\mathbf{n}^\infty)$ can be proved as in Theorem 4.5. The proof is complete. ∎

If $\Lambda : H^\infty(\mathbf{B_n}) \to H^\infty(\mathbf{B_n})$ is an algebraic homomorphism, it induces a unique homomorphism $\widetilde{\Lambda} : \mathbf{F}_\mathbf{n}^\infty \to \mathbf{F}_\mathbf{n}^\infty$ such that $\Lambda\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}\widetilde{\Lambda}$. The homomorphisms Λ and $\widetilde{\Lambda}$ uniquely determine each other by the formulas:

$$(\Lambda f)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_{\mathbf{X}}[\widetilde{\Lambda}(\widehat{f})], \quad f \in H^\infty(\mathbf{B_n}),\ \mathbf{X} \in \mathbf{B_n}(\mathcal{H}), \quad \text{and} \quad \widetilde{\Lambda}(\widehat{f}) = \widehat{\Lambda(f)}, \quad \widehat{f} \in \mathbf{F}_\mathbf{n}^\infty.$$

THEOREM 5.6. *Let $\Lambda : H^\infty(\mathbf{B_n}) \to H^\infty(\mathbf{B_n})$ be a unital algebraic automorphism. Then the following statements are equivalent:*

(i) *$\widetilde{\Lambda}$ is a unitarily implemented automorphism of $\mathbf{F}_\mathbf{n}^\infty$.*

(ii) *There is $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ such that*

$$\Lambda(f) = f \circ \varphi, \quad f \in H^\infty(\mathbf{B_n}).$$

(iii) *$\widetilde{\Lambda}$ is a* WOT*-continuous, completely contractive automorphism of $\mathbf{F}_\mathbf{n}^\infty$ with completely contractive hereditary linear extension.*

(iv) *$\widetilde{\Lambda}$ is norm-continuous and* WOT*-continuous with the property that $\{\widetilde{\Lambda}(\mathbf{S}_{i,j})\}$ and $\{\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ are in the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$, where $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ is the universal model of the regular polyball $\mathbf{B_n}$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) follow from Theorem 5.5 and Proposition 4.1. Now, assume that item (iii) holds. As in the proof of Theorem 5.4 (implication (iii) $\Longrightarrow$ (iv)), one can prove that $\{\widetilde{\Lambda}(\mathbf{S}_{i,j})\}$ and $\{\widetilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ are in the polyball $\mathbf{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))^-$, hence, item (iv) holds. If we assume that (iv) holds, then, due to the continuity in norm of $\widetilde{\Lambda}$, we deduce, according to Theorem 5.4, that $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ and $\widetilde{\Lambda}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}} = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}|_{\boldsymbol{\mathcal{A}}_\mathbf{n}}$. Recall that $\widehat{\varphi}$ is pure (see Theorem 3.9) and $\boldsymbol{\mathcal{B}}_{\widehat{\varphi}}$ is a unitarily implemented automorphism of $\mathbf{F}_\mathbf{n}^\infty$. Since $\boldsymbol{\mathcal{A}}_\mathbf{n}$ is WOT-dense in $\mathbf{F}_\mathbf{n}^\infty$ and $\widetilde{\Lambda}$ and $\boldsymbol{\mathcal{B}}_{\widehat{\varphi}}$ are WOT-continuous on $\mathbf{F}_\mathbf{n}^\infty$, we deduce that $\widetilde{\Lambda} = \boldsymbol{\mathcal{B}}_{\widehat{\varphi}}$. Therefore, item (i) holds. The proof is complete. ∎

6. THE AUTOMORPHISM GROUP $\mathrm{Aut}(\mathbf{B_n})$ AND UNITARY PROJECTIVE REPRESENTATIONS

In this section, we prove that, under a natural topology, the free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n}))$ is a metrizable, σ-compact, locally compact group, and provide a concrete unitary projective representation of it in terms of noncommutative Berezin kernels associated with regular polyballs.

According to Section 3, any $\Phi \in \mathrm{Aut}(\mathbf{B_n})$, it is uniformly continuous on $\mathbf{B_n}(\mathcal{H})^-$. Using standard arguments, one can easily prove the following result.

LEMMA 6.1. *Let Φ_m, Φ, Γ_p, and Γ be in the automorphism group $\mathrm{Aut}(\mathbf{B_n})$, where $m, p \in \mathbb{N}$. If $\Phi_m \to \Phi$ and $\Gamma_p \to \Gamma$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$, then $\Phi_m \circ \Gamma_p \to \Phi \circ \Gamma$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$, as $m, p \to \infty$.*

Let $\phi, \psi \in \mathrm{Aut}(\mathbf{B_n})$ and define

$$d_{\mathbf{B_n}}(\phi, \psi) := \|\phi - \psi\|_\infty + \|\phi^{-1}(0) - \psi^{-1}(0)\|.$$

It is clear that $d_{\mathbf{B_n}}$ is a metric on $\mathrm{Aut}(\mathbf{B_n})$.

LEMMA 6.2. *Let $\boldsymbol{\Psi}_m = p_{\sigma^{(m)}} \circ \boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}$, $m \in \mathbb{N}$, and $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ be free holomorphic automorphisms of the noncommutative polyball $\mathbf{B_n}(\mathcal{H})$ in standard form, where $\sigma^{(m)}, \sigma \in \mathcal{S}_k$, with $n_{\sigma^{(m)}(i)} = n_{\sigma(i)} = n_i$ for $i \in \{1, \ldots, k\}$,*

$$\mathbf{U}^{(m)} = U_1^{(m)} \oplus \cdots \oplus U_k^{(m)} \quad \textit{and} \quad \mathbf{U} = U_1 \oplus \cdots \oplus U_k \quad \textit{with } U_i^{(m)}, U_i \in \mathcal{U}(\mathbb{C}^{n_i}),$$

and

$$\boldsymbol{\lambda}^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_k^{(k)}) \quad \textit{and} \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \quad \textit{with } \lambda_i^{(k)}, \lambda_i \in (\mathbb{C}^{n_i})_1.$$

Then the following statements are equivalent:

(i) *For each $i \in \{1, \ldots, k\}$, $U_i^{(m)} \to U_i$ in $B(\mathbb{C}^{n_i})$ and $\lambda_i^{(m)} \to \lambda_i$ in the Euclidean norm of $\mathbb{C}^{n_i}$, and there is $N \in \mathbb{N}$ such that $\sigma^{(m)} = \sigma$ for any $m \geqslant N$.*

(ii) *$p_{\sigma^{(m)}} \to p_\sigma$, $\boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to \boldsymbol{\Phi}_{\mathbf{U}}$, and $\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}} \to \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$.*

(iii) *$\boldsymbol{\Psi}_m \to \boldsymbol{\Psi}$ in the metric $d_{\mathbf{B_n}}$.*

Proof. First, we prove that (i) is equivalent to (ii). Assume that $U_i^{(m)} = [u_{st}^{(m)}]_{n_i \times n_i}$, $m \in \mathbb{N}$, and $U_i = [u_{st}]_{n_i \times n_i}$ are unitary matrices with scalar entries, and $\boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to \boldsymbol{\Phi}_U$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$, as $m \to \infty$. For each $j = 1, \ldots, n_i$, denote $\mathbf{E}_{ij} := [0, \ldots, E_{ij}, \ldots, 0]$, where E_{ij} is on the i-position, and $E_{ij} = [0, \ldots, I, \ldots, 0]$, where the identity is on the j-position. Note that

$$\|\boldsymbol{\Phi}_{\mathbf{U}^{(m)}}(\mathbf{E}_{i,j}) - \boldsymbol{\Phi}_{\mathbf{U}}(\mathbf{E}_{ij})\| = \Big(\sum_{j=1}^{n_i} |u_{ij}^{(m)} - u_{ij}|^2 \Big)^{1/2}.$$

Consequently, if $\boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to \boldsymbol{\Phi}_{\mathbf{U}}$, then, for each $i \in \{1, \ldots, k\}$, we have $u_{ij}^{(m)} \to u_{ij}$ as $m \to \infty$. Hence, $U_i^{(m)} \to U_i$ in $B(\mathbb{C}^{n_i})$. Conversely, assume that the latter condition holds. Since

$$\|\boldsymbol{\Phi}_{\mathbf{U}^{(m)}}(\mathbf{X}) - \boldsymbol{\Phi}_{\mathbf{U}}(\mathbf{X})\| \leqslant k\|\mathbf{X}\| \max_{i\in\{1,\dots,k\}} \|U_i^{(m)} - U_i\|$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$, we deduce that $\boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to \boldsymbol{\Phi}_{\mathbf{U}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$.

Now we prove that $\lambda_i^{(m)} \to \lambda_i$ in the Euclidean norm of $\mathbb{C}^{n-i}$ if and only if $\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}} \to \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$. Since $\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}(0) = \boldsymbol{\lambda}^{(m)}$ and $\boldsymbol{\Psi}_{\boldsymbol{\lambda}}(0) = \boldsymbol{\lambda}$, one implication is clear. To prove the converse, assume that, for each $i \in \{1,\dots,k\}$, $\lambda_i^{(m)} \to \lambda_i$ in the Euclidean norm of $\mathbb{C}^{n_i}$. Since the left creation operators $\mathbf{S}_{i,1},\dots,\mathbf{S}_{i,n_i}$ are isometries with orthogonal ranges on the full Fock space $F^2(H_{n_i})$, we have

$$\Big\| \sum_{j=1}^{n_i} \overline{\lambda}_{i,j}\mathbf{S}_{i,j} \Big\| = \Big(\sum_{j=1}^{n_i} |\lambda_{i,j}|^2 \Big)^{1/2} < 1.$$

Consequently, $\Big(\sum_{j=1}^{n_i} \overline{\lambda}_{i,j}^{(m)}\mathbf{S}_{i,j} \Big)^{-1}$ converges to $\Big(\sum_{j=1}^{n_i} \overline{\lambda}_{i,j}\mathbf{S}_{i,j} \Big)^{-1}$, as $m \to \infty$, in the operator norm. Taking into account that

$$\widehat{\Psi}_{\lambda_i} = \lambda_i - \Delta_{\lambda_i}\Big(I - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j}\mathbf{S}_{i,j} \Big)^{-1} [\mathbf{S}_{i,1},\dots,\mathbf{S}_{i,n_i}]\Delta_{\lambda_i^*}$$

and a similar relation holds for $\widehat{\Psi}_{\lambda_i^{(m)}}$, we deduce that $\widehat{\Psi}_{\lambda_i^{(m)}} \to \widehat{\Psi}_{\lambda_i}$ in the operator norm. Due to the noncommutative von Neumann inequality [17], we have

$$\|\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}(X) - \boldsymbol{\Psi}_{\boldsymbol{\lambda}}(X)\| \leqslant k \max_{i\in\{1,\dots,k\}} \|\widehat{\Psi}_{\lambda_i^{(m)}} - \widehat{\Psi}_{\lambda_i}\|$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$. Hence, $\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}} \to \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$, which proves our assertion.

If $\sigma^{(m)} \neq \sigma$, then there is $i_0 \in \{1,\dots,k\}$ such that $\sigma^{(m)}(i_0) \neq \sigma(i_0)$. Hence

$$\begin{aligned}\|p_{\sigma^{(m)}} - p_\sigma\|_\infty &\geqslant \sup_{\mathbf{X}=(X_1,\dots,X_k)\in\mathbf{B_n}(\mathcal{H})} \|X_{\sigma^{(m)}(i_0)} - X_{\sigma(i_0)}\| \\ &\geqslant \sup_{X_{\sigma(i_0)}\in[B(\mathcal{H})^{n_{i_0}}]_1} \|X_{\sigma(i_0)}\| = 1.\end{aligned}$$

Therefore, $p_{\sigma^{(m)}} \to p_\sigma$ as $m \to \infty$ if and only if there is $N \in \mathbb{N}$ such that $\sigma^{(m)} = \sigma$ for any $m \geqslant N$. In conclusion, (i) is equivalent to (ii).

Now, we prove that (iii) $\Longrightarrow$ (i). Assume that $d_{\mathbf{B_n}}(\boldsymbol{\Psi}_m, \boldsymbol{\Psi}) \to 0$ as $k \to \infty$. Hence, $\boldsymbol{\Psi}_m \to \boldsymbol{\Psi}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$ and $\boldsymbol{\lambda}^{(m)} = \boldsymbol{\Psi}_m^{-1}(0) \to \boldsymbol{\lambda} = \boldsymbol{\Phi}^{-1}(0)$ in $\mathbf{P_n}(\mathbb{C})$. Consequently, as proved above, we have that $\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}} \to \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$. Using Lemma 6.1 and the fact that $\boldsymbol{\Psi}_m = p_{\sigma^{(m)}} \circ \boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}$, $m \in \mathbb{N}$, and $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$, we deduce that

$$p_{\sigma^{(m)}} \circ \boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}}, \quad \text{as } m \to \infty, \tag{6.1}$$

uniformly on $\mathbf{B_n}(\mathcal{H})^-$. Note that we have $\Phi_{\mathbf{U}^{(m)}}(\mathbf{X}) = (X_1U_1^{(m)}, \dots, X_kU_k^{(m)})$ and $\Phi_{\mathbf{U}}(\mathbf{X}) = (X_1U_1, \dots, X_kU_k)$ for any $\mathbf{X} = (X_1, \dots, X_k) \in \mathbf{B_n}(\mathcal{H})$. If $\sigma^{(m)} \neq \sigma$, then there is $i_0 \in \{1, \dots, k\}$ such that $\sigma^{(m)}(i_0) \neq \sigma(i_0)$. Consequently, we have

$$\begin{aligned}\|p_{\sigma^{(m)}} \circ \boldsymbol{\Psi}_{\mathbf{U}^{(m)}} - p_\sigma \circ \boldsymbol{\Psi}_{\mathbf{U}}\|_\infty &\geqslant \sup_{\mathbf{X}=(X_1,\dots,X_k)\in\mathbf{B_n}(\mathcal{H})} \|X_{\sigma^{(m)}(i_0)}U_{\sigma^{(m)}(i_0)} - X_{\sigma(i_0)}U_{\sigma(i_0)}\| \\ &\geqslant \sup_{X_{\sigma(i_0)}\in[B(\mathcal{H})^{n_{i_0}}]_1} \|X_{\sigma(i_0)}U_{\sigma(i_0)}\| = 1.\end{aligned}$$

Hence, we deduce that relation (6.1) holds if and only if there is $N \in \mathbb{N}$ such that $\sigma^{(m)} = \sigma$ for any $m \geqslant N$, and $\boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \to \boldsymbol{\Phi}_{\mathbf{U}}$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$. Due to the equivalence of (i) with (ii), the latter convergence is equivalent to $U_i^{(m)} \to U_i$ in $B(\mathbb{C}^{n_i})$ for each $i \in \{1, \dots, k\}$.

The implication (i) $\Longrightarrow$ (iii) is straightforward if one uses the equivalence of (i) with (ii) and Lemma 6.1. The proof is complete. ∎

After these preliminaries, we can prove the following.

THEOREM 6.3. *The free holomorphic automorphism group* $\mathrm{Aut}(\mathbf{B_n})$ *is a* σ*-compact, locally compact topological group with respect to the topology induced by the metric* $d_{\mathbf{B_n}}$.

Proof. First, we prove that the map

$$\mathrm{Aut}(\mathbf{B_n}) \times \mathrm{Aut}(\mathbf{B_n}) \ni (\boldsymbol{\Psi}, \boldsymbol{\Gamma}) \mapsto \boldsymbol{\Psi} \circ \boldsymbol{\Gamma} \in \mathrm{Aut}(\mathbf{B_n})$$

is continuous when $\mathrm{Aut}(\mathbf{B_n})$ has the topology induced by the metric $d_{\mathbf{B_n}}$. For $m, p \in \mathbb{N}$, let

$$\begin{aligned}\boldsymbol{\Psi}_m &= p_{\sigma^{(m)}} \circ \Phi_{\mathbf{U}^{(m)}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}, \quad \boldsymbol{\Psi} = p_\sigma \circ \Phi_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}, \\ \boldsymbol{\Gamma}_p &= p_{\omega^{(p)}} \circ \Phi_{\mathbf{W}^{(p)}} \circ \boldsymbol{\Psi}_{\boldsymbol{\mu}^{(p)}}, \quad \boldsymbol{\Gamma} = p_\omega \circ \Phi_{\mathbf{W}} \circ \boldsymbol{\Psi}_{\boldsymbol{\mu}},\end{aligned}$$

be free holomorphic automorphisms of $\mathbf{B_n}$, in standard decomposition. Then

$$\begin{aligned}\mathbf{U}^{(m)} &= U_1^{(m)} \oplus \cdots \oplus U_k^{(m)}, \quad \mathbf{U} = U_1 \oplus \cdots \oplus U_k, \\ \mathbf{W}^{(p)} &= W_1^{(p)} \oplus \cdots \oplus W_k^{(p)}, \quad \mathbf{W} = W_1 \oplus \cdots \oplus W_k,\end{aligned}$$

where $U_i^{(m)}, W_i^{(p)}, U_i, W_i$ are unitary operators on $\mathbb{C}^{n_i}$ and $\boldsymbol{\lambda}^{(m)}, \boldsymbol{\mu}^{(p)}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ are in $\mathbf{P_n}(\mathbb{C})$ satisfying relations

$$\boldsymbol{\lambda}^{(m)} = \boldsymbol{\Psi}_k^{-1}(0), \quad \boldsymbol{\mu}^{(p)} = \boldsymbol{\Gamma}_p^{-1}(0), \quad \boldsymbol{\lambda} = \boldsymbol{\Psi}^{-1}(0), \quad \text{and} \quad \boldsymbol{\mu} = \boldsymbol{\Gamma}^{-1}(0).$$

Assume that $d_{\mathbf{B_n}}(\boldsymbol{\Psi}_m, \boldsymbol{\Psi}) \to 0$ as $m \to \infty$ and $d_{\mathbf{B_n}}(\boldsymbol{\Gamma}_p, \boldsymbol{\Gamma}) \to 0$ as $p \to \infty$. Using Lemma 6.2, we deduce that $\boldsymbol{\Psi}_m \circ \boldsymbol{\Gamma}_p \to \boldsymbol{\Psi} \circ \boldsymbol{\Gamma}$ uniformly on $\mathbf{B_n}(\mathcal{H})$. Note that

$$\begin{aligned}(\boldsymbol{\Psi}_m \circ \boldsymbol{\Gamma}_p)^{-1}(0) &= (\boldsymbol{\Psi}_{\mu^{(p)}}^{-1} \circ \Phi_{\mathbf{W}^{(p)}}^{-1} \circ p_{\omega^{(p)}}^{-1} \circ \boldsymbol{\Psi}_m^{-1})(0) \\ &= (\boldsymbol{\Psi}_{\mu^{(p)}} \circ \Phi_{(\mathbf{W}^{(p)})^*} \circ p_{(\omega^{(p)})^{-1}})(\boldsymbol{\lambda}^{(m)}).\end{aligned}$$

Similarly, we have

$$(\boldsymbol{\Psi}\circ\boldsymbol{\Gamma})^{-1}(0)=(\boldsymbol{\Psi}_\mu^{-1}\circ\Phi_{\mathbf{W}}^{-1}\circ p_\omega^{-1}\circ\boldsymbol{\Psi}^{-1})(0)=(\boldsymbol{\Psi}_\mu\circ\Phi_{\mathbf{W}^*}\circ p_{\omega^{-1}})(\boldsymbol{\lambda}).$$

According to Lemma 6.2, $\boldsymbol{\lambda}^{(m)}\to\boldsymbol{\lambda}$ in $\mathbf{P_n}(\mathbb{C})$, $W_i^{(p)}\to W_i$ in $B(\mathbb{C}^n)$, $p^{-1}_{\omega^{(p)}}\to p_{\omega^{-1}}$ and $\boldsymbol{\Psi}_{\mu^{(p)}}\to\boldsymbol{\Psi}_\mu$ uniformly on $\mathbf{B_n}(\mathcal{H})^-$. Consequently, $(\boldsymbol{\Psi}_m\circ\boldsymbol{\Gamma}_p)^{-1}(0)\to(\boldsymbol{\Psi}\circ\boldsymbol{\Gamma})^{-1}(0)$ as $m,p\to\infty$. Therefore, $\boldsymbol{\Psi}_m\circ\boldsymbol{\Gamma}_p\to\boldsymbol{\Psi}\circ\boldsymbol{\Gamma}$ in the topology induced by the metric $d_{\mathbf{B_n}}$.

In what follows, we show that the map $\boldsymbol{\Psi}\mapsto\boldsymbol{\Psi}^{-1}$ is continuous on $\mathrm{Aut}(\mathbf{B_n})$ with the topology induced by $d_{\mathbf{B_n}}$. Assume that $d_{\mathbf{B_n}}(\boldsymbol{\Psi}_m,\boldsymbol{\Psi})\to 0$ as $k\to\infty$. Using the same notations as above, we have

$$\boldsymbol{\Psi}_m^{-1}=\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}\circ\Phi_{(\mathbf{U}^{(m)})^*}\circ p_{(\sigma^{(m)})^{-1}}\quad\text{and}\quad\boldsymbol{\Psi}_m^{-1}=\boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}\circ\Phi_{(\mathbf{U}^{(m)})^*}\circ p_{(\sigma^{(m)})^{-1}}.$$

Using Lemma 6.1 and Lemma 6.2, one can easily see that $d_{\mathbf{B_n}}(\boldsymbol{\Psi}_m^{-1},\boldsymbol{\Psi}^{-1})\to 0$ as $m\to\infty$. Therefore, $\mathrm{Aut}(\mathbf{B_n})$ is a topological group with respect to the topology induced by the metric $d_{\mathbf{B_n}}$.

On the other hand, each free holomorphic automorphism $\boldsymbol{\Psi}\in\mathrm{Aut}(\mathbf{B_n})$ has a unique representation $\boldsymbol{\Psi}=p_\sigma\circ\Phi_{\mathbf{U}}\circ\boldsymbol{\Psi}_{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda}:=\boldsymbol{\Phi}^{-1}(0)$ and $\mathbf{U}=U_1\oplus\cdots\oplus U_k$ with $U_i\in\mathcal{U}(\mathbb{C}^{n_i})$, the unitary group on $\mathbb{C}^{n_i}$. This generates a bijection

$$\chi:\mathrm{Aut}(\mathbf{B_n})\to\Sigma\times\mathcal{U}(\mathbb{C}^{n_1})\times\cdots\times\mathcal{U}(\mathbb{C}^{n_k})\times\mathbf{P_n}(\mathbb{C}),$$

by setting $\chi(\boldsymbol{\Psi}):=(\sigma,U_1,\cdots,U_k,\boldsymbol{\lambda})$, where Σ is the discrete subgroup

$$\Sigma:=\{\sigma\in\mathcal{S}_k:(n_{\sigma(1)},\ldots,n_{\sigma(k)})=(n_1,\ldots,n_k)\}.$$

According to Lemma 6.2, the map χ is a homeomorphism of topological spaces, where $\mathrm{Aut}(\mathbf{B_n})$ has the topology induced by the metric $d_{\mathbf{B_n}}$ and $\mathcal{U}(\mathbb{C}^{n_i})$ and $\mathbf{P_n}(\mathbb{C})$ have the natural topology. Since $\Sigma\times\mathcal{U}(\mathbb{C}^{n_1})\times\cdots\times\mathcal{U}(\mathbb{C}^{n_k})\times\mathbf{P_n}(\mathbb{C})$ is a σ-compact, locally compact topological space, so is the automorphism group $\mathrm{Aut}(\mathbf{B_n})$. The proof is complete. ∎

COROLLARY 6.4. *Let* $\mathbf{n}=(n_1,\ldots,n_k)\in\mathbb{N}^k$ *and*

$$\Sigma:=\{\sigma\in\mathcal{S}_k:(n_{\sigma(1)},\ldots,n_{\sigma(k)})=(n_1,\ldots,n_k)\}.$$

The free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n})$ *has* $\mathrm{card}(\Sigma)$ *path connected components.*

Proof. We saw in the proof of Theorem 6.3 that the map

$$\chi:\mathrm{Aut}(\mathbf{B_n})\to\Sigma\times\mathcal{U}(\mathbb{C}^{n_1})\times\cdots\times\mathcal{U}(\mathbb{C}^{n_k})\times(\mathbb{C}^{n_1})_1\times\cdots\times(\mathbb{C}^{n_k})_1$$

is a homeomorphism. Since $\mathcal{U}(\mathbb{C}^{n_i})$ and $(\mathbb{C}^{n_i})_1$ are path connected and Σ has $\mathrm{card}(\Sigma)$ path connected components, the result follows. ∎

Let $\mathrm{Aut}(\mathbf{B_n})$ be the free holomorphic automorphism group of the noncommutative polyball $\mathbf{B_n}$ and let $\mathcal{U}(\mathcal{K})$ be the unitary group on the Hilbert space $\mathcal{K}$. According to Theorem 6.3, $\mathrm{Aut}(\mathbf{B_n})$ is a topological group with respect to the metric $d_{\mathbf{B_n}}$. A map $\pi : \mathrm{Aut}(\mathbf{B_n}) \to \mathcal{U}(\mathcal{K})$ is called a (unitary) projective representation if the following conditions are satisfied:

(i) $\pi(\mathrm{id}) = I$, where id is the identity on $\mathbf{B_n}(\mathcal{H})$;

(ii) $\pi(\mathbf{\Phi})\pi(\mathbf{\Psi}) = c_{(\mathbf{\Phi},\mathbf{\Psi})}\pi(\mathbf{\Phi} \circ \mathbf{\Psi})$, for any $\mathbf{\Phi}, \mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n})$, where $c_{(\mathbf{\Phi},\mathbf{\Psi})}$ is a complex number with $|c_{(\mathbf{\Phi},\mathbf{\Psi})}| = 1$;

(iii) the map $\mathrm{Aut}(\mathbf{B_n}) \ni \mathbf{\Phi} \mapsto \langle \pi(\mathbf{\Phi})\xi, \eta\rangle \in \mathbb{C}$ is continuous for each $\xi, \eta \in \mathcal{K}$.

THEOREM 6.5. *If $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$ with $\Psi_i = (\Psi_{i,1}, \ldots, \Psi_{i,n_i})$, then there is a unitary operator $U_{\mathbf{\Psi}} \in B(F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}))$ satisfying the relations*

$$\begin{aligned} &\Psi_{i,j}(\mathbf{S}) = U_{\mathbf{\Psi}}^* \mathbf{S}_{i,j} U_{\mathbf{\Psi}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}, \quad \text{and} \\ &U_{\mathbf{\Psi}} U_{\mathbf{\Phi}} = c_{(\mathbf{\Psi},\mathbf{\Phi})} U_{\mathbf{\Psi}\circ\mathbf{\Phi}}, \quad \mathbf{\Phi}, \mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n}), \end{aligned}$$

for some complex number $c_{(\mathbf{\Phi},\mathbf{\Psi})} \in \mathbb{T}$. Moreover, the map $\mathbf{\Psi} \to U_{\mathbf{\Psi}}^*$ is continuous from the uniform topology to the strong operator topology, and the map

$$\pi : \mathrm{Aut}(\mathbf{B_n}) \to B(F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})) \quad \text{defined by } \pi(\mathbf{\Psi}) := U_{\mathbf{\Psi}}$$

is a projective representation of the automorphism group $\mathrm{Aut}(\mathbf{B_n})$.

Proof. Let $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_n) \in \mathrm{Aut}(\mathbf{B_n})$ and let $\widehat{\mathbf{\Psi}} = (\widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)$ be its boundary function with respect to the universal model $\mathbf{S}$. According to Theorem 3.9, $\widehat{\mathbf{\Psi}}$ is a pure element in the regular polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$ and $\widehat{\Psi}_i = (\widehat{\Psi}_{i,1}, \ldots, \widehat{\Psi}_{i,n_i})$ is an isometry with entries in the noncommutative disk algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$ and the identity. Moreover, rank $\mathbf{\Delta}_{\widehat{\mathbf{\Psi}}} = 1$ and $\widehat{\mathbf{\Psi}}$ is unitarily equivalent to the universal model $\mathbf{S}$. Combining these results with Theorem 4.3 and Corollary 4.4, we deduce that the noncommutative Berezin kernel $\mathbf{K}_{\widehat{\mathbf{\Psi}}}$ is a unitary operator. Moreover, in this case, we have

$$\widehat{\Psi}_{i,j} = \mathbf{K}_{\widehat{\mathbf{\Psi}}}^* (\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_{\widehat{\mathbf{\Psi}}}}) \mathbf{K}_{\widehat{\mathbf{\Psi}}} = \mathbf{K}_{\widehat{\mathbf{\Psi}}}^* \widetilde{W} \mathbf{S}_{i,j} \widetilde{W}^* \mathbf{K}_{\widehat{\mathbf{\Psi}}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}, \tag{6.2}$$

where $\widetilde{W}_{\mathbf{\Psi}} : \bigotimes_{i=1}^k F^2(H_{n_i}) \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathbb{C}v_0$ is the unitary operator defined by

$$\widetilde{W}_{\mathbf{\Psi}} g := g \otimes v_0, \quad g \in \bigotimes_{i=1}^k F^2(H_{n_i}),$$

and the defect space $\mathcal{D}_{\widehat{\mathbf{\Psi}}} = \mathbb{C}v_0$ for some vector $v_0 \in \bigotimes_{i=1}^k F^2(H_{n_i})$ with $\|v_0\| = 1$.

According to Theorem 3.6, if $\mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n}(\mathcal{H}))$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \mathbf{\Psi}^{-1}(0)$, then there are unique unitary operators $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, and a unique permutation $\sigma \in \mathcal{S}_k$ with $n_{\sigma(i)} = n_i$ such that

$$\mathbf{\Psi} = p_\sigma \circ \mathbf{\Phi}_{\mathbf{U}} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}},$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$ and $\boldsymbol{\Psi}_{\boldsymbol{\lambda}} := (\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_k})$. Moreover, we have

$$\boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}}(I) = \boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}_{\boldsymbol{\lambda}}}(I) = \boldsymbol{\Delta}_{\boldsymbol{\lambda}} \Big[\prod_{i=1}^{k} \Big(I_{\mathcal{H}} - \sum_{j=1}^{n_i} \overline{\lambda}_{i,j} \mathbf{S}_{i,j} \Big)^{-1} \Big] P_{\mathbb{C}} \Big[\prod_{i=1}^{k} \Big(I_{\mathcal{H}} - \sum_{j=1}^{n_i} \lambda_{i,j} \mathbf{S}_{i,j}^* \Big)^{-1} \Big],$$

where $\boldsymbol{\Delta}_{\boldsymbol{\lambda}} = \prod_{i=1}^{k} (1 - \|\lambda_i\|_2^2)$. Hence we deduce that

$$\|\boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}}(I)^{1/2}(1)\|^2 = \Big\| \boldsymbol{\Delta}_{\boldsymbol{\lambda}}^{1/2} P_{\mathbb{C}} \Big[\prod_{i=1}^{k} \Big(I_{\mathcal{H}} - \sum_{j=1}^{n_i} \lambda_{i,j} \mathbf{S}_{i,j}^* \Big)^{-1} \Big](1) \Big\|^2 = \boldsymbol{\Delta}_{\boldsymbol{\lambda}}.$$

Let $v_0 := \boldsymbol{\Delta}_{\boldsymbol{\lambda}}^{-1/2} \boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}}(I)^{1/2}(1) \in \bigotimes_{i=1}^{k} F^2(H_{n_i})$ and note that $\|v_0\| = 1$. Now, relation (6.2) becomes

$$\widehat{\Psi}_{i,j} = \Psi_{i,j}(\mathbf{S}) = U_{\boldsymbol{\Psi}}^* \mathbf{S}_{i,j} U_{\boldsymbol{\Psi}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\},$$

where $U_{\boldsymbol{\Psi}} := \widetilde{W}^* \mathbf{K}_{\widehat{\boldsymbol{\Psi}}}$. Note that if $\boldsymbol{\Phi} \in \mathrm{Aut}(\mathbf{B_n})$ with $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_k)$ and $\Phi_i = (\Phi_{i,1}, \ldots, \Phi_{i,n_i})$, then the relation above written for $\boldsymbol{\Psi} \circ \boldsymbol{\Phi}$ shows that

$$(\boldsymbol{\Psi}_{i,j} \circ \boldsymbol{\Phi})(\mathbf{S}) = (\widehat{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}})_{i,j} = U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}} \mathbf{S}_{i,j} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}. \tag{6.3}$$

On the other hand, due to Corollary 4.2,

$$\boldsymbol{\mathcal{B}}_{\widehat{\Psi \circ \Phi}}[g] = (\boldsymbol{\mathcal{B}}_{\widehat{\Phi}} \boldsymbol{\mathcal{B}}_{\widehat{\Psi}})[g]$$

for any g in the Cuntz–Toeplitz algebra $C^*(\mathbf{S})$. In particular, when $g = \mathbf{S}_{i,j}$, we obtain

$$\mathbf{K}_{\widehat{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}}^* (\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_{\widehat{\Psi \circ \Phi}}}) \mathbf{K}_{\widehat{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}} = \mathbf{K}_{\widehat{\boldsymbol{\Phi}}}^* \{ [\mathbf{K}_{\widehat{\boldsymbol{\Psi}}}^* (\mathbf{S}_{i,j} \otimes I_{\mathcal{D}_{\widehat{\boldsymbol{\Psi}}}}) \mathbf{K}_{\widehat{\boldsymbol{\Psi}}}] \otimes I_{\mathcal{D}_{\widehat{\boldsymbol{\Phi}}}} \} \mathbf{K}_{\widehat{\boldsymbol{\Phi}}}.$$

Hence, and using relation (6.3), we deduce that

$$(\boldsymbol{\Psi}_{i,j} \circ \boldsymbol{\Phi})(\mathbf{S}) = (\widehat{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}})_{i,j} = U_{\boldsymbol{\Phi}}^* U_{\boldsymbol{\Psi}}^* \mathbf{S}_{i,j} U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}.$$

Combining this relation with (6.3), we deduce that

$$U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}} \mathbf{S}_{i,j} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}} = U_{\boldsymbol{\Phi}}^* U_{\boldsymbol{\Psi}}^* \mathbf{S}_{i,j} U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}}$$

which is equivalent to

$$U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}^* \mathbf{S}_{i,j} = \mathbf{S}_{i,j} U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}^*.$$

Since $C^*(\mathbf{S})$ is irreducible and $U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}^*$ is a unitary operator, we deduce that $U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}^* = c_{(\boldsymbol{\Psi}, \boldsymbol{\Phi})} I$ for some complex number with $|c_{(\boldsymbol{\Psi}, \boldsymbol{\Phi})}| = 1$. Hence, we deduce that $U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} = c_{(\boldsymbol{\Psi}, \boldsymbol{\Phi})} U_{\boldsymbol{\Psi} \circ \boldsymbol{\Phi}}$ for any $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n})$.

Let $\boldsymbol{\Psi}^{(m)} = (\Psi_1^{(m)}, \ldots, \Psi_k^{(m)})$, $m \in \mathbb{N}$, with $\Psi_i^{(m)} = (\Psi_{i,1}^{(m)}, \ldots, \Psi_{i,n_i}^{(m)})$ and $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_k)$, $m \in \mathbb{N}$, with $\Psi_i = (\Psi_{i,1}, \ldots, \Psi_{i,n_i})$ be free holomorphic automorphisms of the noncommutative polyball $\mathbf{B_n}(\mathcal{H})$. Assume that $\boldsymbol{\Psi}^{(m)} \to \boldsymbol{\Psi}$ in the uniform norm. Then, for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, $\widehat{\Psi}_{i,j}^{(m)} \to \widehat{\Psi}_{i,j}$ in the operator norm topology.

Now consider the standard representations $\boldsymbol{\Psi}^{(m)} = p_{\sigma^{(m)}} \circ \boldsymbol{\Phi}_{\mathbf{U}^{(m)}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}^{(m)}}$ and $\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}}$. Since $\boldsymbol{\Psi}^{(m)}(0) = \boldsymbol{\lambda}^{(m)}$ and $\boldsymbol{\Psi}(0) = \boldsymbol{\lambda}$, we deduce that $\|\boldsymbol{\lambda}^{(m)}\|_2 \to \|\boldsymbol{\lambda}\|_2$ as $m \to \infty$. Given $\varepsilon > 0$ and $x = \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\alpha)} e_{(\alpha)} \in \bigotimes_{i=1}^k F^2(H_{n_i})$, let $q \in \mathbb{N}$ be such that

$$\left\| x - \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} e_{(\alpha)} \right\| < \frac{\varepsilon}{4}. \tag{6.4}$$

Since $U_{\boldsymbol{\Psi}^{(m)}} := \widetilde{W}^*_{\boldsymbol{\Psi}^{(m)}} \mathbf{K}_{\widehat{\boldsymbol{\Psi}}^{(m)}}$ and $\widetilde{W}_{\boldsymbol{\Psi}^{(m)}} : \bigotimes_{i=1}^k F^2(H_{n_i}) \to (\bigotimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}_{\widehat{\boldsymbol{\Psi}}^{(m)}}$ is the unitary operator defined by

$$\widetilde{W}_{\boldsymbol{\Psi}^{(m)}} g := g \otimes \boldsymbol{\Delta}^{-1/2}_{\boldsymbol{\lambda}^{(m)}} \boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}^{(m)}}(I)^{1/2}(1), \quad g \in \bigotimes_{i=1}^k F^2(H_{n_i}),$$

we can use the properties of the noncommutative Berezin kernel to deduce that

$$\begin{aligned}
\sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} U^*_{\boldsymbol{\Psi}^{(m)}} e_{(\alpha)} &= \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} \mathbf{K}^*_{\widehat{\boldsymbol{\Psi}}^{(m)}} \widetilde{W}_{\boldsymbol{\Psi}^{(m)}} e_{(\alpha)} \\
&= \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} \mathbf{K}^*_{\widehat{\boldsymbol{\Psi}}^{(m)}} (e_{(\alpha)} \otimes \boldsymbol{\Delta}^{-1/2}_{\boldsymbol{\lambda}^{(m)}} \boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}^{(m)}}(I)^{1/2}(1)) \\
&= \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} [\widehat{\boldsymbol{\Psi}}^{(m)}]_{(\alpha)} \boldsymbol{\Delta}^{-1/2}_{\boldsymbol{\lambda}^{(m)}} \boldsymbol{\Delta}_{\widehat{\boldsymbol{\Psi}}^{(m)}}(I)(1).
\end{aligned}$$

A similar relation holds if we replace $\boldsymbol{\Psi}^{(m)}$ with $\boldsymbol{\Psi}$. Since, for each $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, $\widehat{\Psi}^{(m)}_{i,j} \to \widehat{\Psi}_{i,j}$ in the operator norm topology, and $\|\boldsymbol{\lambda}^{(m)}\|_2 \to \|\boldsymbol{\lambda}\|_2$ as $m \to \infty$, there is $N \in \mathbb{N}$ such that

$$\left\| \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} U^*_{\boldsymbol{\Psi}^{(m)}} e_{(\alpha)} - \sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,\ |\alpha_1|+\cdots+|\alpha_k|\leqslant q} a_{(\alpha)} U^*_{\boldsymbol{\Psi}} e_{(\alpha)} \right\| < \frac{\varepsilon}{2} \tag{6.5}$$

for all $q \geqslant N$. Using relations (6.4), (6.5) and the fact that $U_{\boldsymbol{\Psi}^{(m)}}$ and $U_{\boldsymbol{\Psi}}$ are unitary operators, one can easily deduce that

$$\|U^*_{\boldsymbol{\Psi}^{(m)}} x - U^*_{\boldsymbol{\Psi}} x\| < \varepsilon$$

for any $q \geqslant N$. Therefore the map $\boldsymbol{\Psi} \to U^*_{\boldsymbol{\Psi}}$ is continuous from the uniform topology to the strong operator topology.

To prove the last part of this theorem, note that if $\boldsymbol{\Psi}^{(m)} \to \boldsymbol{\Psi}$ in the metric $d_{\mathbf{B_n}}$, then $\boldsymbol{\Psi}^{(m)} \to \boldsymbol{\Psi}$ in the uniform norm and, using the first part of the theorem, we can complete the proof. ∎

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